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Note on the Permutations which Preserve Buck's Measure Density

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Abstract. In the first part some conditions under which a permutation preserves Buck's measure density, are derived. In the second part is constructed a countable system of permutations which has the ergodic property on the system of Buck's measurable sets.

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1. Introduction

Let $\mathbb{N}$ be the set of natural numbers. For any subset $A \subseteq \mathbb{N}$ and $x > 0$ let $A(x)$ be the cardinality of $A \cap [0, x)$. If the limit $\lim\limits_{x \to +\infty} x^{-1}A(x) := d(A)$ exists, then we say that A has the *asymptotic density*. For more details on asymptotic density we refer to the paper [2]. By aD we denote the set of all subsets of $\mathbb{N}$ which have the asymptotic density. Let the conditions

(a) $\forall A \subseteq \mathbb{N}$: $A \in aD \Longleftrightarrow g(A) \in aD$,
(b) $\forall A \in aD$: $d\big(g(A)\big) = d(A)$,

hold for a permutation $g : \mathbb{N} \to \mathbb{N}$. Then we say that *g preserves the asymptotic density*. Denote by G the set of all permutations $g : \mathbb{N} \to \mathbb{N}$ such that

$$\lim_{N \to +\infty} \frac{1}{N} |\{j \leq N : g(j) > N\}| = 0.$$

This set is a group with respect to the composition and is called Lévy's group, originated in [3]. It can be easilly proved that the permutations from G preserve the asymptotic density. This group is the object of observations in [5].

The aim of this paper is to study the permutations from the point of view of other type of finitely additive measure on the set of positive integers – the Buck's measure density. This notion was introduced in 1946 by R.C. Buck in [1]

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as follows: For any $r, m \in \mathbb{N}$, $m \neq 0$, let $r + m\mathbb{N} = \{r + m \cdot n : \ n \in \mathbb{N}\}$ denote the arithmetic progression with modulus m and first element r. Let $S \subset \mathbb{N}$, then the value $\mu^*(S) = \inf\Big\{\frac{1}{m_1} + \cdots + \frac{1}{m_k} : \ S \subseteq \bigcup_{i=1}^{k} r_i + m_i\mathbb{N}, \ k \in \mathbb{N}\Big\}$ is called the *measure density* of S. It is proved

(i) $\mu^*\big(S_1 \cup S_2\big) \leq \mu^*(S_1) + \mu^*\big(S_2\big) \ \ \forall S_1, S_2 \subseteq \mathbb{N}$.

The set $S \subseteq \mathbb{N}$ is called *Buck's measurable* if and only if $\mu^*(S) + \mu^*(\mathbb{N} \setminus S) = 1$. We denote the class of all these sets by D_μ. It is known that

(ii) *D_μ is an algebra of sets.*

We can consider the restriction of μ^* on D_μ, i.e.,

$$\mu = \mu^*\big|_{D_\mu},$$

and then the following holds:

(iii) *The set function μ is a finitely additive probability measure on D_μ.*

For the proofs we refer to [1], [6].

Remark that there are some analogies between the Buck's measure density and Buck's measurability on the set of positive integers and the Jordan measure and the Jordan measurability on the unit interval.

It is easy to see that a set S belongs to D_μ if and only if $\forall\varepsilon > 0$ there exists two sets S_1, S_2 which are union of finite number of arithmetic progressions such that $S_1 \subset S \subset S_2$ and $\mu(S_2) - \mu(S_1) < \varepsilon$ (see [1]). Let m be a positive integer divisible by all its moduli, for instance the product of all moduli. Then, every one from these arithmetic progressions can be represented as the union of finite number of arithmetic progressions with the modulus m. And so we obtain:

(iv) $S \in D_\mu \iff \forall\varepsilon > 0 \ \exists m \in \mathbb{N}$ and $a_1, \ldots, a_k, b_1, \ldots, b_l \in \mathbb{N}$ such that

$$\bigcup_{i=1}^{k} a_i + m\mathbb{N} \subseteq S \subseteq \bigcup_{j=1}^{l} b_j + m\mathbb{N} \quad \text{if} \quad \frac{l-k}{m} < \varepsilon$$

(a_i incongruent $\mod m$, b_j incongruent $\mod m$).

In this paper we study the permutations $g : \mathbb{N} \to \mathbb{N}$ such that

(c) $g(S) \in D_\mu \ \ \forall S \in D_\mu$,
(d) $\mu\big(g(S)\big) = \mu(S) \ \ \forall S \in D_\mu$.

In the first part we prove some criterions and give an example that these permutations do not form a group. In the second part we prove the ergodic property for a subgroup of these permutations (see Proposition 3.5). A similar result for G is proved in [5].

Recently, in [4] it has been proved that $(a) \implies (b)$. We give an example that this does not hold for Buck's measurability.

2. Buck's Measure Density

Let $g : \mathbb{N} \to \mathbb{N}$ be a permutation which satisfies (c) and (d). Then we say that g *preserves Buck's measure density.*

Proposition 2.1. *Let $g : \mathbb{N} \to \mathbb{N}$ be a permutation. The following properties are equivalent:*

a) *g preserves Buck's measure density;*
b) *For $A \subseteq \mathbb{N}$ it holds that $\mu^*\big(g(A)\big) \le \mu^*(A)$;*
c) *For every arithmetic progression $a + m\mathbb{N}$, $a, m \in \mathbb{N}$, $m \neq 0$, it holds that $\mu^*\big(g(a + m\mathbb{N})\big) \le \frac{1}{m}$.*

Proof. *b) $\Rightarrow$ a)* Let b) hold. Then for $A \subseteq D_\mu$ we have $\mathbb{N} \setminus g(A) = g(\mathbb{N} \setminus A)$ and so

$$1 \le \mu^*\big(g(A)\big) + \mu^*\big(\mathbb{N} \setminus g(A)\big) \le \mu^*(A) + \mu^*(\mathbb{N} \setminus A) = 1,$$

thus, $g(A) \in D_\mu$ and $\mu^*(A) = \mu^*\big(g(A)\big)$.
a) $\Rightarrow$ b) Let g preserves Buck's measure density. Then $\mu^*\big(g(a + m\mathbb{N})\big) = \frac{1}{m}$. For $A \subseteq \mathbb{N}$ and $\varepsilon > 0$ there exist arithmetic progressions $a_1 + m_1\mathbb{N}, \dots, a_s + m_s\mathbb{N}$ such that

$$A \subseteq (a_1 + m_1\mathbb{N}) \cup \cdots \cup (a_s + m_s\mathbb{N}) \tag{2.1}$$

and

$$\sum_{i=1}^{s} \frac{1}{m_i} \le \mu^*(A) + \varepsilon. \tag{2.2}$$

From (2.1) we obtain

$$g(A) \subset g(a_1 + m_1\mathbb{N}) \cup \cdots \cup g(a_s + m_s\mathbb{N})$$

and so

$$\mu^*\big(g(A)\big) \le \sum_{i=1}^{s} \frac{1}{m_i} \le \mu^*(A) + \varepsilon.$$

For $\varepsilon \to 0^+$ we obtain $\mu^*\big(g(A)\big) \le \mu^*(A)$.
a) $\Leftrightarrow$ c) If g preserves Buck's measure density, then c) holds. Vice versa, if c) holds, then we can consider the covering (2.1) and (2.2). We obtain $\mu^*\big(g(A)\big) \le \mu^*(A)$, thus g preserves Bucks's measure density. □

Let S be the set of all permutations from $\mathbb{N}$ to $\mathbb{N}$ which preserve Buck's measure density. Clearly, S contains the identical permutation and Proposition 2.1 implies that with two permutations S contains its composition. Thus S is a semigroup with identity.

In the paper [6] a limit formula for the Buck's measure density is proved. In the following we shall use this statement. First of all we recall the notation, which will be used also in the proofs: For any $A \subset \mathbb{N}, B \in \mathbb{N}, B \neq 0$, let $R(A : B)$ be the maximal number of elements of A which are incongruent modulo B.

Theorem 2.2. *Suppose that $\{B_n\}$ is a sequence of positive integers such that*

(v) *$\forall\, d \in \mathbb{N}\ \exists n_0$ such that it holds $d | B_n$ for $n > n_0$.*

Then, for an arbitrary $A \subset \mathbb{N}$ we have

$$\mu^*(A) = \lim_{n\to+\infty} \frac{R(A : B_n)}{B_n}. \tag{2.3}$$

For the proof, we refer to [6, Theorem 1].

Remark 2.3. If for some permutation g the value $|g(n) - n|$ is bounded then g belongs to G and so it preserves the asymptotic density. The following example shows that this is not true for the case of Buck's measure density.

Example 2.4. Let $A = \{k + k! : k = 2, 3, ...\}$. For $m \geq 2$, $0 \leq a < m$ it holds $(m + a) + (m + a)! \in a + m\mathbb{N}$ thus A has non empty intersection with every arithmetic progression and so (2.3) yields $\mu^*(A) = 1$(see [6]). Let the permutation $g : \mathbb{N} \to \mathbb{N}$ be defined as follows:

$$g(2n + (2n + 1)!) = 2n + 1 + (2n + 1)!,\ g(2n + 1 + (2n + 1)!) = 2n + (2n + 1)!)$$

and $g(a) = a$ in other cases. Now, consider the set $B = \{2n+(2n+1)!, 2n+(2n)! : n = 1, 2, ...\}$. Clearly, it holds $B \subset 2\mathbb{N}$ and so $\mu^*(B) \leq \frac{1}{2}$. But $g(B) = A$ thus Proposition 2.1 a) yields that g does not preserve the Buck's measure density. Trivially we have that $|g(n) - n| \leq 1$.

Now, we give an example which shows that S is not a group.

Example 2.5. Let P be the set of all prime numbers. Then it is known that $\mu(P) = 0$ (see [1], [7]. Let $A = \{a + a! : a \in \mathbb{N} \setminus P\}$ and $B = \mathbb{N} \setminus A = \{b_1 < b_2 < \dots\}$. Let us consider the arbitrary arithmetic progression $r + m\mathbb{N}$. This set contains some number $r+mk$ which is not a prime. Then the number $r+mk+(r+mk)!$ belongs to the intersection $(r+m\mathbb{N})\cap A$. Thus A has the nonempty intersection with arbitrary arithmetic progression, therefore for arbitrary sequence of positive integers $\{B_n\}$ satisfying the condition (v) we have $R(A : B_n) = B_n$ and so (2.3) yields $\mu^*(A) = 1$. But the asymptotic density of A is zero, thus the asymptotic density of B is 1. Easily, it can be derived that the asymptotic density does not exceed the measure density (see [6]) and so $\mu^*(B) = 1$. Therefore, we have $A, B \notin D_\mu$.
Define $g : \mathbb{N} \to \mathbb{N}$ as follows: $g(B) = P$ and g is one to one on B and $g(a + a!) = a, a \notin P$. Then $g(A) = \mathbb{N} \setminus P$ and g is one to one on A. Let us denote for two sets $C_1, C_2 : C_1 \doteq C_2$ if and only if their symmetric difference

$$C_1 \ominus C_2 := (C_1 \setminus C_2) \cup (C_2 \setminus C_1)$$

is a finite set (for this symbol we refer to [1]). Now, consider an arithmetic progression $r + m\mathbb{N}$. Then for $a \geq m \quad a + a! \equiv r \pmod{m}$ if and only if it is $a \equiv r \pmod{m}$. We see that

$$A \cap (r + m\mathbb{N}) \doteq \big\{(r + ms) + (r + ms)! :\ s = 1, 2, \dots,\ \text{and } r + m \cdot s \in P\big\}.$$

And so $g\big(A \cap (r + m\mathbb{N})\big) \doteq (r + m\mathbb{N}) \setminus P$. Moreover $g\big(B \cap (r + m\mathbb{N})\big) \subseteq P$ and so $\mu\big(g(B \cap (r + m\mathbb{N})\big) = 0$, thus $\mu^*\big(g(r + m\mathbb{N})\big) = \frac{1}{m}$. From Proposition 2.1 we get that g preserves Buck's measure density. But evidently $g^{-1}(P) = B$ and so g^{-1} does not preserve Buck's measure density.

Remark 2.6. a) If we consider in Example 2.5 all possibilities of bijective mapping $g(B) = P$ we obtain that the cardinality of S is continuum.
b) Proposition 2.1 yields immediately that $g, g^{-1} \in S$ if and only if $\mu^*\big(g(A)\big) = \mu^*(A)$ for an arbitrary $A \subseteq \mathbb{N}$.

Now, we use Proposition 2.1 and (2.3) for the proof of the criterions in the limit form. Let $\{a(n)\}$, $\{b(n)\}$ be two sequences of positive integers. As usual, we shall write $a(n) = o(b(n))$ if and only if $\frac{a(n)}{b(n)} \to 0$ for $n \to +\infty$.

Corollary 2.7. *Suppose that $\{B_n\}$ is a sequence of positive integers which satisfies the condition* (v)*. Then, for a permutation $g : \mathbb{N} \to \mathbb{N}$ the following conditions are equivalent:*

- a) *g preserves Buck's measure density,*
- b) *for every $m \in \mathbb{N}$, $m \neq 0$, $a \in \mathbb{N}$, the set $g(a+m\mathbb{N})$ contains at most $\frac{B_n}{m}+o(B_n)$ elements incongruent modulo B_n, where the therm $o(B_n)$ depends on m and a modulo m.*

Proof. *a)* $\implies$ *b)* Consider an arithmetic progression $a + m\mathbb{N}, a, m \in \mathbb{N}, m \neq 0$. Denote by a' the rest of a after division by m. Then $a + m\mathbb{N} \subset a' + m\mathbb{N}$, therefore the set $g(a + m\mathbb{N})$ contains at most $R(g(a' + m\mathbb{N}) : B_n)$ elements incongruent modulo B_n. But (2.3) yields

$$R(g(a' + m\mathbb{N}) : B_n) = \mu^*(g(a' + m\mathbb{N}))B_n + o(B_n).$$

And so Proposition 2.1 *b)* implies *b)*.
b) $\implies$ *a)* From *b)* we deduce that $R(g(a + m\mathbb{N}) : B_n) \le \frac{B_n}{m} + o(B_n)$. Now, from Proposition 2.1 *b)* we obtain that g preserves Buck's measure density. □

Proposition 2.8. *Suppose that $\{B_n\}$ is a sequence of positive integers which satisfies the condition* (v)*. Then, a permutation $g : \mathbb{N} \to \mathbb{N}$ preserves Buck's measure density if and only if for every $m \in \mathbb{N}$, $m \neq 0$, and an arbitrary sequence of finite sets of positive integers $\{k_1^{(n)}, \dots, k_{r(n)}^{(n)}\}$, $r(n) \ge 1, n = 1, 2, \dots,$ which are incongruent modulo B_n, the set of co-images $\{g^{-1}\big(k_1^{(n)}\big), \dots, g^{-1}\big(k_{r(n)}^{(n)}\big)\}$ contains at least*

$$T_n = \frac{r(n)}{B_n}\left(\frac{1}{m} + o(1)\right)^{-1} \tag{2.4}$$

elements incongruent modulo m, for $n = 1, 2, \dots$.

Proof. Let g preserve the Buck's measure density. Let $g^{-1}(k_j^{(n)}) = l_j^{(n)}$, $j = 1, \dots, r(n)$. Suppose that the numbers $l_j^{(n)}$, $j = 1, \dots, r(n)$ contain W_n elements incongruent modulo m. Then, these numbers belong to W_n arithmetic progressions $a_1 + m\mathbb{N}, \dots, a_s + m\mathbb{N}, s = W_n$. By applying Corollary 2.7 to the arithmetic progression $a_j + m\mathbb{N}$ we obtain that the set $g(a_j + m\mathbb{N})$ contains at most $\frac{B_n}{m} + \delta_j(B_n)$ elements incongruent modulo B_n, where $\delta_j(B_n) = o(B_n), j = 1, \dots, s$.
Therefore, the sequence $k_j^{(n)} = g\big(l_j^{(n)}\big)$, $j = 1, \dots, r(n)$, contains at most $W_n \cdot \frac{B_n}{m} +$

$\delta_1(B_n) + \cdots + \delta_s(B_n)$ elements incongruent modulo B_n. Clearly, $1 \le s = W_n \le m$ thus $\delta_1(B_n) + \cdots + \delta_s(B_n) = o(B_n)$. Therefore,

$$r(n) \le W_n \cdot \frac{B_n}{m} + o(B_n) = W_n(\cdot \frac{B_n}{m} + o(B_n))$$

and this implies $T_n \le W_n$.
Let us suppose that (2.4) holds. Let $g(a + ms_1), \ldots, g(a + ms_{r_{(n)}})$, $s_j \in \mathbb{N}$ for $j = 1, ..., r(n)$, be the representatives of $g(a+m\mathbb{N})$ modulo B_n. Then these numbers are incongruent modulo B_n and $r(n) = R\big(g(a + m\mathbb{N}) : B_n\big)$.
From (2.4) we see that the sequence $a + ms_1, \ldots, a + ms_{r_{(n)}}$ contains at least T_n elements incongruent modulo m, but it contains only 1 element incongruent modulo m, thus

$$\frac{r(n)}{B_n}\left(\frac{1}{m} + o(1)\right)^{-1} \le 1$$

and so

$$\frac{r(n)}{B_n} \le \frac{1}{m} + o(1).$$

Therefore, (2.3) yields that

$$\mu^*\big(g(a + m\mathbb{N})\big) = \lim_{n\to+\infty} \frac{R\big(g(a + m\mathbb{N}) : B_n\big)}{B_n} \le \frac{1}{m}.$$

Now, Proposition 2.1 implies that g preserves Buck's measure density. □

Corollary 2.9. *If g preserves Buck's measure density then for every $m \in \mathbb{N}$, $m \ne 1$, and every sequence of complete remainder systems modulo B_n, $n = 1, 2, ...$ $k_1^{(n)}, \ldots, k_{B_n}^{(n)}$, the sequence $g^{-1}\big(k_1^{(n)}\big), \ldots, g^{-1}\big(k_{B_n}^{(n)}\big)$ contains at least $\frac{m}{1+o(1)}$ elements incongruent modulo m.*

3. A Countable System of Permutations

In this part we construct a sufficiently small set, countable, of permutations which preserve Buck's measure density, but which is sufficiently rich from the ergodic point of view (see Proposition 3.5). We shall use the following result.

Proposition 3.1. *Suppose that $\{B_n\}$ is a sequence of positive integers which satisfies the condition* (v). *Let $g : \mathbb{N} \to \mathbb{N}$ be such a permutation that for every sequence of complete remainder systems $\{k_1^{(n)}, \ldots, k_{B_n}^{(n)}\}$ modulo B_n, the set of the co-images $\{g^{-1}\big(k_1^{(n)}\big), \ldots, g^{-1}\big(k_{B_n}^{(n)}\big)\}$ contains at least $\frac{B_n}{1+o(1)}$ elements incongruent modulo B_n, $n = 1, 2, \ldots$. Then g preserves Buck's measure density.*

Proof. Let $A \subseteq \mathbb{N}$ and let $g\big(l_1^{(n)}\big), \ldots, g\big(l_{r(n)}^{(n)}\big)$, for $l_j^{(n)} \in A$, are incongruent modulo B_n. Then, this sequence can be completed building the complete remainder system modulo B_n, by the elements $g\big(l_j^{(n)}\big)$, $j = r(n)+1, \ldots, B_n$. Thus, $l_1^{(n)}, \ldots, l_{r_{(n)}}^{(n)}$,

$l^{(n)}_{r(n)+1}, \dots, l^{(n)}_{B_n}$ contains at least $\frac{B_n}{1+o(1)}$ elements incongruent modulo B_n.
Let $l^{(n)}_1, \dots, l^{(n)}_{r(n)}$ contain at most $p(n)$ elements incongruent modulo B_n. Then,

$$\frac{B_n}{1+o(1)} \le p(n) + \big(B_n - r(n)\big),$$

and so

$$r(n) - p(n) \le B_n\left(1 - \frac{1}{1+o(1)}\right).$$

Therefore,

$$\lim_{n\to+\infty} \frac{r(n)-p(n)}{B_n} = 0. \tag{3.1}$$

Let $r(n) = R\big(g(A) : B_n\big)$, then A contains at least $p(n)$ elements incongruent modulo B_n and so

$$R(A : B_n) \ge p(n) = R\big(g(A) : B_n\big) + p(n) - r(n),$$

thus from (3.1) and (2.3) we have

$$\mu^*(A) \ge \mu^*\big(g(A)\big)$$

and Proposition 2.1 implies the assertion. □

Now, we construct a system of quite trivial permutations $g \in S$ such that $g^{-1} \in S$.

Proposition 3.2. *Suppose that m is a positive integer. Let Z_m be two complete reminder systems modulo m and $\pi : Z_m \to Z_m$ a permutation. Then the mapping $g_\pi : \mathbb{N} \to \mathbb{N}$, defined as*

$$g_\pi(a + jm) = \pi(a) + j \cdot m, \tag{3.2}$$

is a permutation which preserves Buck's measure density.

Proof. It is trivial that g_π defined by (3.2) is a permutation. Suppose that $\{B_n\}$ is a sequence of positive integers which fulfils the condition (v) and $m|B_n, n = 1, \dots$. Let $\{k^{(n)}_1, \dots, k^{(n)}_{B_n}\}$ be a complete reminder system modulo B_n. Put $k^{(n)}_j = a_j + ml_j$. Then, $g^{-1}_\pi\big(k^{(n)}_j\big) = \pi^{-1}(a_j) + ml_j$. If $g^{-1}_\pi\big(k^{(n)}_j\big) \equiv g^{-1}_\pi\big(k^{(n)}_i\big) \pmod{B_n}$, then $\pi^{-1}(a_j) + ml_j \equiv \pi^{-1}(a_i) + ml_i \pmod{B_n}$, and so $\pi^{-1}(a_j) \equiv \pi^{-1}(a_i) \pmod{m}$, thus $a_j = a_i$. Therefore, $ml_j \equiv ml_i \pmod{B_n}$, this implies $k^{(n)}_j \equiv k^{(n)}_i \pmod{B_n}$: a contradiction. So, Proposition 3.1 yields the assertion. □

Remark 3.3. It is easy to see that the value $|g_\pi(n) - n|$ is bounded and so g_π preserves the asymptotic density. Moreover, these permutations form a countable set.

Proposition 3.4. *Let $A, B \in D_\mu$ and $\mu(A) = \mu(B)$. Then, for $\varepsilon > 0$ there exist the sets $A_1 \subseteq A$, $B_1 \subseteq B$, $A_1, B_1 \in D_\mu$ and a permutation g_π (given by (3.2)) such that $\mu(A) - \mu(A_1) < \varepsilon$, $\mu(B) - \mu(B_1) < \varepsilon$ and $g_\pi(A_1) = B_1$.*

Proof. If $\mu(A) = 0 = \mu(B)$, we can consider $A_1 = \emptyset = B_1$ and g_π-identic permutation. Suppose that $\mu(A) > 0$. Then, from (iv) it follows that there exists $m \in \mathbb{N}$ and $a_1, \dots, a_s$, $b_1, \dots, b_q \in \mathbb{N}$ such that

$$(a_1 + m\mathbb{N}) \cup \cdots \cup (a_s + m\mathbb{N}) \subseteq A, \quad (b_1 + m\mathbb{N}) \cup \cdots \cup (b_q + m\mathbb{N}) \subseteq B$$

and $\mu(A) - \frac{s}{m} < \varepsilon, \mu(B) - \frac{q}{m} < \varepsilon$. If $\mu(A) = \mu(B)$ we can assume that $s = q$. Now, if we consider $A_1 = \bigcup a_i + m\mathbb{N}$, $B_1 = \bigcup b_i + m\mathbb{N}$, and a permutation π of complete reminder system modulo m such that $\pi(a_i \pmod m) = b_i \pmod m$, then g_π satisfies the assertion. □

Proposition 3.5. *Let $A \in D_\mu$ be a set such that*

$$\mu\big(A \ominus g_\pi(A)\big) = 0 \tag{3.3}$$

($\ominus$ is the symmetric difference), for every permutation g_π given by (3.2)*. Then $\mu(A) = 1$ or $\mu(A) = 0$.*

Proof. Suppose that $0 < \mu(A) < 1$. Clearly, (3.3) holds also for $\mathbb{N} \setminus A$ and so we can suppose that $0 < \mu(A) \le \frac{1}{2}$, $\frac{1}{2} \le \mu(\mathbb{N} \setminus A) < 1$. The Darboux property of Buck's measure density (cf. [6]) implies that there exists a set $B \subset \mathbb{N} \setminus A$ such that $\mu(B) = \mu(A)$. Let $\mu(A) > \varepsilon > 0$ and $A_1 \subset A$, $B_1 \subset B$ be the sets from Proposition 3.4. Then $g_\pi(A_1) = B_1$ for a suitable permutation g_π. Clearly,

$$\mu\big(A \ominus g_\pi(A)\big) = \mu\big(A \setminus g_\pi(A)\big) + \mu\big(g_\pi(A) \setminus A\big).$$

Put $A = A_1 \cup \widetilde{A}$, where $A_1 \cap \widetilde{A} = \emptyset$, then $\widetilde{A} \in D_\mu$ and $\mu(\widetilde{A}) < \varepsilon$, thus

$$A \setminus g_\pi(A) = A \setminus g_\pi\big(A_1 \cup \widetilde{A}\,\big) = A \setminus \big(g_\pi(A_1) \cup g(\widetilde{A}\,)\big) = A \setminus \big(B_1 \cup g_\pi(\widetilde{A}\,)\big) = A \setminus g_\pi\big(\widetilde{A}\,\big)$$

and so

$$\mu\big(A \setminus g_\pi(A)\big) = \mu(A) - \mu\big(g_\pi(\widetilde{A}\,)\big) \ge \mu(A) - \varepsilon.$$

Considering $\varepsilon < \frac{1}{2}\mu(A)$ we obtain $\mu(A \ominus g_\pi(A)) > 0$: a contradiction. □

4. An Example

Now, we construct an example of permutation $g : \mathbb{N} \to \mathbb{N}$ which preserves Buck's measurability but does not preserve Buck's measure density. Put

$$g(2n) = 4n, \quad g(4n+1) = 4n+2, \quad g(4n+3) = 2n+1.$$

Clearly, this mapping is a permutation from $\mathbb{N}$ to $\mathbb{N}$. Now, consider an arithmetic progression of the form $a + 4m\mathbb{N}$.

Suppose that $2|a$. Then $a = 2a_1$, thus for $k \in a + 4m\mathbb{N}$ we have $k + 2a_1 + 4m_j$, and so $g(k) = 4a_1 + 8m_j$, therefore it holds $g(a + 4m\mathbb{N}) = 2a + 8m\mathbb{N}$.

Now, suppose that $a \equiv 1 \pmod 4$. Then $a + 4m\mathbb{N} = 1 + 4a_1 + 4m\mathbb{N}$ and so for $k \in a + 4m\mathbb{N}$ we have $k = 4a_1 + 1 + 4m_j$, thus $g(k) = 4a_1 + 2 + 4m_j$, therefore in this case $g(a + 4m\mathbb{N}) = a + 1 + 4m\mathbb{N}$.

Finally, consider $a \equiv 3 \pmod 4$. Then, for $k \in a + 4m\mathbb{N}$ we have $k = 4a_1 + 3 + 4m_j$, and so $g(k) = 1 + 2a_1 + 2m_j$, thus $g(a + 4m\mathbb{N}) = \frac{a-1}{2} + 2m\mathbb{N}$.

Let $A \subseteq 3+4\mathbb{N}$ be a Buck's measurable set. Then, for $\varepsilon > 0$ there exists such $m \in \mathbb{N}$ and $a_1, \dots, a_s$, $b_1, \dots, b_r \in \mathbb{N}$ such that

$$(a_1 + 4m\mathbb{N}) \cup \cdots \cup (a_s + 4m\mathbb{N}) \subseteq A \subseteq \bigcup_{i=1}^{r} (b_i + 4m\mathbb{N})$$

and $\frac{r-s}{4m} < \varepsilon$. As $A \subseteq 3 + 4\mathbb{N}$, we can consider $a_i \equiv 3 \pmod 4$, $b_j \equiv 3 \pmod 4$. Thus,

$$\bigcup_{i=1}^{s} g(a_i + 4m\mathbb{N}) \subseteq g(A) \subseteq \bigcup_{i=1}^{r} g(b_i + 4m\mathbb{N}).$$

Clearly, it results

$$\bigcup_{i=1}^{s} a'_i + 2m\mathbb{N} \subseteq g(A) \subseteq \bigcup_{i=1}^{r} b'_i + 2m\mathbb{N}, \quad \text{with } a'_i = \frac{a_i - 1}{2},\ b'_i = \frac{b_i - 1}{2},$$

moreover,

$$\frac{r-s}{2m} < 2\varepsilon$$

and so $g(A)$ is a Buck's measurable set, and from the previous considerations it follows that $\mu\big(g(A)\big) = 2\mu(A)$.

Analogously we can prove that for $B \subseteq 1 + 4\mathbb{N}$ the image $g(B)$ is Buck's measurable and $\mu\big(g(B)\big) = \mu(B)$, and for $C \subseteq 2\mathbb{N}$ the set $g(C)$ is Buck's measurable, and $\mu\big(g(C)\big) = \frac{1}{2}\mu(C)$. If $S \subseteq \mathbb{N}$ is a Buck's measurable set then it can be represented in the form

$$S = A \cup B \cup C, \quad A \subseteq 3 + 4\mathbb{N}, \quad B \subseteq 1 + 4\mathbb{N}, \quad C \subseteq 2\mathbb{N}.$$

From (ii) it follows that the sets A, B, C are Buck's measurable and so $g(S) = g(A) \cup g(B) \cup g(C)$ is a Buck's measurable set. Similarly it can be proved that g^{-1} preserves Buck's measurability. It is easy to see that g does not preserve Buck's measure density.

References

[1] R.C. Buck, *The measure theoretic approach to density.* Amer. J. Math. **68** (1946), 560-580.

[2] G. Grekos, *On various definitions of density (survey).* Tatra Mount. Math. Publ. **31** (2005), 17-27.

[3] P. Lévy, *Problemes Concrets d'Analyse Fonctionelle. Avec un Complement sur les Fonctionelles Analytiques par F. Pellegrino.* Gouthiers-Villars, Paris, 1951.

[4] M.B. Nathanson, R. Parikh, *Density of sets of natural numbers and the Lévy group.* J. Number Theory **124** (2007), 151-158.

[5] N. Obata, *Density of natural numbers and the Lévy group.* J. Number Theory **30** (1988), 288-297.

[6] M. Paštéka, *Some properties of Buck's measure density.* Math. Slovaca **42** (1992), 15-32.

[7] M. Paštéka, *A note about the submeasures and Fermat last theorem*. Ricerche Mat. **XLIII** (1994), 145-152.

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