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Recent Advances for Meson Algebras and their Lipschitz Monoids

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Abstract. This article has two purposes. After a short reminder of classical properties of meson algebras (also called Duffin-Kemmer algebras), Sects. 4 to 7 present recent advances in the study of their algebraic structure. Then Sects. 8 to 11 explain that each meson algebra contains a Lipschitz monoid with properties quite similar to those of Lipschitz monoids in Clifford algebras.

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1. Introduction

Whereas Clifford algebras first interested mathematicians before drawing the attention of physicists, meson algebras were first introduced by physicists before interesting mathematicians. For the concerned physicists, meson algebras should play the same role for some particles called mesons as Clifford algebras for electrons. Therefore, it is sensible to predict that many ideas involved in the study of Clifford algebras should be suitable also for meson algebras, and the present work shall corroborate this prediction.

The present work has two purposes. Firstly, it presents the recent advances in the treatment of meson algebras. Some of the treatments that are effective for Clifford algebras, are equally effective for meson algebras. The mesonic versions of these treatments are presented in Sects. 4, 5, 6 and 7, but I have not recalled their Cliffordian versions because I suppose that all acquainted readers will guess them without any reminder being necessary. Secondly, I recently published two works [5,6] about Lipschitz monoids in Clifford algebras where I explained that Lipschitz monoids must play a capital role in the study of Clifford algebras; but among the arguments justifying

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my conviction, I could not recall that a similar treatment of Lipschitz monoids was valid for meson algebras. Now I want to do it here, especially since this treatment has recently been improved. Sections 8, 9, 10 and 11 are devoted to Lipschitz monoids in meson algebras.

In all the present article, V is a vector space of finite dimension n over a field K, and $F : V\times V \to K$ is a symmetric bilinear form on V. The meson algebra $\mathcal{B}(V,F)$ is the (unital and associative) algebra generated by V with the relations $uvu = F(u,v)\,u$ (for all u, $v \in V$). The dimension of this algebra is $\binom{2n+1}{n}$; it is 126 in the case $n = 4$ which especially concerns physicists. Like every Clifford algebra, it is provided with a parity gradation $\mathcal{B}(V,F) = \mathcal{B}_0(V,F) \oplus \mathcal{B}_1(V,F)$, with a grade automorphism σ, and with a reversion ρ (the involutive anti-automorphism that extends the identity mapping $\mathbf{1}_V$ of V). But a new feature appears in Sect. 4: the even subalgebra is provided with an involutive automorphism τ such that $\tau(uv) = F(u,v) - vu$. When $\mathrm{char}(K)$ (the characteristic of K) is not equal to 2, it determines a subparity gradation of the even subalgebra: $\mathcal{B}_0(V,F) = \mathcal{B}_{00}(V,F) \oplus \mathcal{B}_{01}(V,F)$. As far as I know, the automorphism τ (which leads to interior multiplications and deformations in Sects. 5, 6 and 7) is my contribution to the theory of meson algebras. When $\mathrm{char}(K) \neq 2$, the orthogonal group $\mathrm{O}(V,F)$ has two components which, in my opinion, must be related to some parity gradation; but the odd component $\mathcal{B}_1(V,F)$ (of dimension $\binom{2n}{n-1}$) contains no invertible elements; therefore, I conjectured the presence of a subparity gradation in the even subalgebra $\mathcal{B}_0(V,F)$ (of dimension $\binom{2n}{n}$); and eventually, my conjecture proved to be correct. When the treatment of meson algebras imitates the treatment of Clifford algebras, the role of the Cliffordian grade automorphism is played sometimes by σ, but more often by τ.

Sections 2 and 3 are reminders: they recall already known things without proofs, but with the necessary explanations. Proofs are given only when advances are presented in the following sections. The missing proofs, together with more detailed information, can be found in [2–4, 7].

1.1. Quadratic Forms and Symmetric Bilinear Forms

With every quadratic form Q on V is associated the symmetric bilinear form A defined by $A(u,v) = Q(u{+}v) - Q(u) - Q(v)$, and with every symmetric bilinear form F is associated the quadratic form $v \longmapsto F(v,v)$. When both operations are performed successively, the quadratic form Q becomes $2Q$, and the symmetric bilinear form F becomes $2F$; indeed, it is well known that $A(v,v) = 2\,Q(v)$, and that

$$F(u{+}v,\ u{+}v) = F(u,u) + F(v,v) + 2\,F(u,v). \tag{1.1}$$

When $\mathrm{char}(K) \neq 2$, this fact confirms that a quadratic form gives the same information as the associated symmetric bilinear form, and conversely. But when $\mathrm{char}(K) = 2$, quadratic forms do not give the same information as symmetric bilinear forms. Fields of characteristic 2 let us realize that Clifford algebras must be derived from quadratic forms, and meson algebras from symmetric bilinear forms.

Let us assume $\mathrm{char}(K) = 2$. The space of all symmetric bilinear forms on V contains the space of all alternating bilinear forms because the word "symmetric" has the same meaning as "skew symmetric" when $\mathrm{char}(K) = 2$. We must also notice that the quadratic form associated with a symmetric bilinear form F is an *additive quadratic form:* $F(u+v, u+v) = F(u,u) + F(v,v)$ because of (1.1). Consequently, the subset V_0 of all $v \in V$ such that $F(v,v) = 0$ is a vector subspace of V; it is the largest subspace on which the restriction of F is alternating. The additive quadratic forms constitute a subspace of dimension n whereas F has been chosen in a space of dimension $n(n+1)/2$; therefore, a great part of the information given by F has been lost by the quadratic form $v \longmapsto F(v,v)$.

When F is not alternating, there are always orthogonal bases in V; but in Sect. 11 (where $\mathrm{char}(K) = 2$), these orthogonal bases will not at all help us to study the automorphisms of (V,F). We shall use the above subspace V_0 and some results of symplectic geometry.

To emphasize the discrepancy between quadratic forms and symmetric bilinear forms, I recall this theorem of Witt: if the quadratic form Q is non-degenerate on V, and if U is a subspace of V, every injective linear mapping $g : U \longmapsto V$ such that $Q(g(u)) = Q(u)$ for all $u \in U$ extends to an automorphism (an orthogonal transformation) of (V,Q). When $\mathrm{char}(K) = 2$, this theorem is not valid for a non-degenerate symmetric bilinear form F on V, and a injective mapping $g : U \to V$ such that $F(g(u), g(v)) = F(u,v)$ for all u, $v \in U$. For instance, let us assume that there is a basis (e_1, e_2, e_3) of V such that

$$F(\lambda_1 e_1+\lambda_2 e_2+\lambda_3 e_3,\ \mu_1 e_1+\mu_2 e_2+\mu_3 e_3) = \lambda_1\mu_1 + \lambda_2\mu_3 + \lambda_3\mu_2$$

for all λ_1, λ_2, λ_3, μ_1, μ_2, $\mu_3 \in K$. In this example, F is alternating on the subspace V_0 spanned by (e_2, e_3), every automorphism of (V,F) leaves V_0 invariant and determines a symplectic transformation of V_0. Consequently, it leaves invariant the vector e_1 orthogonal to V_0. We obtain an orthogonal basis (e'_1, e'_2, e'_3) if we set

$$e'_1 = e_1 + e_2 + e_3, \quad e'_2 = e_1 + e_2, \quad e'_3 = e_1 + e_3.$$

If U is the line spanned by e'_1, and if $g : U \to V$ is the linear mapping defined by $g(e'_1) = e_1$, then $F(g(u), g(v)) = F(u,v)$ for all u, $v \in U$ because $F(e'_1, e'_1) = F(e_1, e_1) = 1$. But g does not extend to an automorphism of (V,F) although F is non-degenerate.

Although the core of the theory of Clifford algebras can be developed independently of the characteristic of the field, the theory of meson algebras cannot ignore the characteristic so extensively. As soon as automorphisms of (V,F) are under consideration, we must distinguish two cases, either $\mathrm{char}(K) \neq 2$, or $\mathrm{char}(K) = 2$, and the latter case will be the more toilsome. Nevertheless, I shall also consider the latter case because I want to prove that my ideas about Lipschitz monoids remain relevant even in this rather disconcerting case.

1.2. Historical Information

Serious historical information can be found at the beginning of [7]; this information and the corresponding bibliography was collected by Artibano Micali. Moreover, he also acquainted me with meson algebras. Here, I just recall some essential facts. Meson algebras are also called *Duffin-Kemmer algebras*. The physicist Duffin imagined a wave equation for some particles which were first conjectured, and later confirmed by experiments. The physicist Kemmer studied the algebras that were involved in this wave equation. His work did not always comply with strict mathematical rigor; for instance, he admitted without proof that his algebras were semi-simple. Nevertheless, it was a great achievement since he correctly listed all the irreducible representations of his algebras. Rigorous mathematical studies came later. Littlewood [10] began with the calculation of the centers of $\mathcal{B}_0(V,F)$ and $\mathcal{B}(V,F)$, and continued with the determination of the ideals of $\mathcal{B}_0(V,F)$ and $\mathcal{B}(F)$. In Sect. 3, I propose the inverse procedure: first, the ideals, and afterwards, the centers. Jacobson met the meson algebras because they were the associative hulls of some Jordan algebras; he started his study with the injectivity of the algebra homomorphism $\Delta : \mathcal{B}(V,F) \to \mathrm{Cl}(V,\bar{F})\otimes\mathrm{Cl}(V,\bar{F})$ which is recalled in Sect. 2.1, and which exists only when $\mathrm{char}(K) \neq 2$.

In recent times, meson algebras seem not to have drawn much interest, neither among physicists, nor among mathematicians.

2. Definition and First Properties

The meson algebra $\mathcal{B}(V,F)$ can be defined when V is a module over a commutative, associative and unital ring K, and F a symmetric bilinear form $V\times V \to K$: it is the quotient of the tensor algebra $\mathrm{T}(V)$ by the ideal generated by all $u\otimes v\otimes u - F(u,v)\,u$ with $u,\, v \in V$. This definition has the advantage of immediately proving the existence of $\mathcal{B}(V,F)$ under very weak hypotheses. But here, V shall be a vector space of finite and non-zero dimension n over a field K.

This definition implies that $\mathcal{B}(V,F)$ is provided with a parity gradation. An element x is said to be homogeneous if it is even or odd, and its degree (or parity) is denoted by ∂x. The grade automorphism σ maps every homogeneous element x to $(-1)^{\partial x}x$. The homogeneous components $\mathcal{B}_0(V,F)$ and $\mathcal{B}_1(V,F)$ are the eigenspaces of σ only when $\mathrm{char}(K) \neq 2$. The existence of the reversion ρ is also an easy consequence of the above definition.

Let $\mathrm{T}^+(V)$ be the sum of all $\mathrm{T}^k(V)$ with $k > 0$; it is an ideal of $\mathrm{T}(V)$ which contains all elements $u\otimes v\otimes u - F(u,v)\,u$. Consequently, $\mathcal{B}(V,F)$ is the direct sum of K and the ideal $\mathcal{B}^+(V,F)$ generated by the image of V in this quotient algebra. The projection $\mathcal{B}(V,F) \to K$ will be denoted by Scal; it is an algebra homomorphism. Its presence in meson algebras is an important discrepancy between meson algebras and Clifford algebras.

When $F = 0$, $\mathcal{B}(V,0)$ is the quotient of $\mathrm{T}(V)$ by the ideal generated by the elements $u\otimes v\otimes u$ of $\mathrm{T}^3(V)$. Thus, $\mathcal{B}(V,0)$ inherits the $\mathbb{Z}$-gradation of $\mathrm{T}(V)$: it is the direct sum of the $\mathbb{Z}$-homogeneous components $\mathcal{B}^k(V,0)$.

The injectivity of the canonical mapping $V \to \mathcal{B}(V,F)$ can be proved at once by means of suitable algebra homomorphisms defined on $\mathcal{B}(V,F)$. It will allow us to identify every element of V with its image in $\mathcal{B}(V,F)$, and to claim that the following relations hold in $\mathcal{B}(V,F)$ for all u, v, $w \in V$:

$$uvu = F(u,v)\,u \qquad \text{and} \qquad uvw + wvu = F(u,v)\,w + F(w,v)\,u. \tag{2.1}$$

2.1. Jacobson's Homomorphism $\Delta : \mathcal{B}(V,F) \to \mathrm{Cl}(V,\bar{F})\otimes\mathrm{Cl}(V,\bar{F})$

Here, $\bar{F}$ is the quadratic form on V such that $\bar{F}(v) = F(v,v)$ for all $v \in V$. The relations $v^2 = F(v,v)$ and $uv + vu = 2\,F(u,v)$ hold in the Clifford algebra $\mathrm{Cl}(V,\bar{F})$. Nathan Jacobson concerned himself with meson algebras when he studied Jordan algebras, and these algebras required $\mathrm{char}(K) \neq 2$. He discovered the following relation in $\mathrm{Cl}(V,\bar{F})\otimes\mathrm{Cl}(V,\bar{F})$ (an ordinary tensor product of algebras, without twisting):

$$(u\otimes 1 + 1\otimes u)\,(v\otimes 1 + 1\otimes v)\,(u\otimes 1 + 1\otimes u) = 4\,F(u,v)\,(u\otimes 1 + 1\otimes u). \tag{2.2}$$

This equality proves that the mapping $v \longmapsto (v\otimes 1 + 1\otimes v)/2$ extends to an algebra homomorphism Δ from $\mathcal{B}(V,F)$ into $\mathrm{Cl}(V,\bar{F})\otimes\mathrm{Cl}(V,\bar{F})$. Obviously, $\Delta(v) \neq 0$ for every non-zero $v \in V$. Jacobson even proved the injectivity of Δ when F was non-degenerate. But it immediately follows that Δ is still injective when F is degenerate, because (V,F) can always be embedded in a larger space (V'',F'') where F'' is non-degenerate.

For the usual parity gradation of $\mathrm{Cl}(V,\bar{F})\otimes\mathrm{Cl}(V,\bar{F})$, $x\otimes y$ is even if x and y have the same parity in $\mathrm{Cl}(V,\bar{F})$, odd if they have different parities. This parity gradation turns Δ into a graded algebra homomorphism.

Jacobson's homomorphism will much help us to study mesonic Lipschitz monoids. Unfortunately, it does not exist when $\mathrm{char}(K) = 2$; and it is difficult to determine the image of $\mathcal{B}(V,F)$ in $\mathrm{Cl}(V,\bar{F})\otimes\mathrm{Cl}(V,\bar{F})$. Yet it is clear that this image is invariant under the automorphism that maps every $x\otimes y$ (with x, $y \in \mathrm{Cl}(V,\bar{F})$) to $y\otimes x$.

2.2. The Homomorphism $\mathcal{B}(V,F) \to \mathrm{End}\big(\bigwedge(V)\times\bigwedge(V)\big)$

The bilinear form F allows every vector v of V to operate on the exterior algebra $\bigwedge(V)$ as a twisted derivation on the left or right side. These operations are announced by the symbols $\rfloor$ and $\lfloor$ (or $\rfloor_F$ and ${}_F\lfloor$ if more precision is necessary), and they satisfy the following equalities (where u, v, $w \in V$, x, y, $z \in \bigwedge(V)$, y is homogeneous and $\hat{y} = (-1)^{\partial y} y$):

$$\begin{aligned} v\rfloor w &= F(v,w), \qquad & v\rfloor(y\wedge z) &= (v\rfloor y)\wedge z + \hat{y}\wedge(v\rfloor z),\\ u\lfloor v &= F(u,v), \qquad & (x\wedge y)\lfloor v &= x\wedge(y\lfloor v) + (x\lfloor v)\wedge\hat{y}. \end{aligned}$$

Since $v\rfloor(v\rfloor y) = (y\lfloor v)\lfloor v) = 0$ for all $v \in V$ and all $y \in \bigwedge(V)$, we can define the interior products $x\rfloor y$ and $y\lfloor z$ for all x, y, $z \in \bigwedge(V)$. The interior multiplications comply with these properties of associativity:

$$(x\wedge y)\rfloor z = x\rfloor(y\rfloor z), \quad (x\rfloor y)\lfloor z = x\rfloor(y\lfloor z), \quad (x\lfloor y)\lfloor z = x\lfloor(y\wedge z).$$

Now we consider the vector space $\bigwedge(V)\times\bigwedge(V)$ provided with the following parity gradation: $\bigwedge(V)\times 0$ is the even component, and $0\times\bigwedge(V)$ is the odd component. With every vector v of V, we associate the following operator Θ_v on $\bigwedge(V)\times\bigwedge(V)$:

$$\Theta_v(x,y) = (v\wedge y,\ v\rfloor x). \tag{2.3}$$

It is an odd operator because it maps each homogeneous component of $\bigwedge(V)\times\bigwedge(V)$ into the other one. A direct calculation proves that, for all u, $v\in V$, and all x, $y\in\bigwedge(V)$,

$$\Theta_u\circ\Theta_v\circ\Theta_u(x,y) = F(u,v)\,\Theta_u(x,y).$$

Consequently, the mapping $v\longmapsto\Theta_v$ extends to a graded algebra homomorphism from $\mathcal{B}(V,F)$ into $\mathrm{End}(\bigwedge(V)\times\bigwedge(V))$. Since $\Theta_v(0,1)=(v,0)$, we have $\Theta_v\neq 0$ if $v\neq 0$. In Sect. 3, where F is non-degenerate, it is stated that it is an isomorphism from $\mathcal{B}(V,F)$ onto a subalgebra of dimension $\binom{2n+1}{n}$. Nevertheless, this homomorphism cannot be injective for all symmetric bilinear forms F since $\Theta_u\circ\Theta_v=0$ when $F=0$. Let us also notice that Θ_v leaves invariant each subspace $\bigwedge^k(V)\times\bigwedge^{k-1}(V)$ for $k=0,1,\ldots,n,n{+}1$, and that this subspace has dimension $\binom{n+1}{k}$.

Representations in spaces of dimension $\binom{n+1}{k}$ were already discovered by Kemmer when F was non-degenerate. They were irreducible except when $2k{-}1=n$. His followers soon identified these spaces with $\bigwedge^k(V)\times\bigwedge^{k-1}(V)$. But the above representation of $\mathcal{B}(V,F)$ in $\bigwedge(V)\times\bigwedge(V)$ has the advantage of giving more information than the collection of all representations in the spaces $\bigwedge^k(V)\times\bigwedge^{k-1}(V)$.

Among the $n{+}2$ subspaces $\bigwedge^k(V)\times\bigwedge^{k-1}(V)$, the subspaces $\bigwedge^0(V)\times 0$ and $0\times\bigwedge^n(V)$ (corresponding to $k=0$ and $k=n{+}1$) have dimension 1. They are annihilated by every Θ_v, therefore, by every element of $\mathcal{B}^+(V,F)$. With the above parity gradation of $\bigwedge(V)\times\bigwedge(V)$, the former is totally even and the latter is totally odd; they are not isomorphic as graded spaces.

After the proof of the injectivity of the mapping $V\to\mathcal{B}(V,F)$, the next concern is the dimension of $\mathcal{B}(V,F)$.

2.3. Let us Prove that $\dim(\mathcal{B}(V,F))\leqslant\binom{2n+1}{n}$

As an algebra generated by V, $\mathcal{B}(V,F)$ is provided with an increasing filtration by subspaces $\mathcal{B}^{\leqslant k}(V,F)$: $\mathcal{B}^{\leqslant k}(V,F)=0$ if $k<0$, $\mathcal{B}^{\leqslant 0}(V,F)=K$, $\mathcal{B}^{\leqslant 1}(V,F)=K+V$, and when $k\geqslant 2$, $\mathcal{B}^{\leqslant k}(V,F)$ is spanned by $\mathcal{B}^{<k}(V,F)$ and the products of k factors in V.

Let $(e_1,e_2,\ldots,e_n)$ and $(e'_1,\ldots,e'_n)$ be two bases of V (which may be equal). The even subalgebra $\mathcal{B}_0(V,F)$ (resp. the odd component $\mathcal{B}_1(V,F)$) is spanned by 1 (resp. by $(e_1,\ldots,e_n)$) and by all products

$$e_{i_1}e'_{j_1}e_{i_2}e'_{j_2}\cdots e_{i_r}e'_{j_r}\qquad\big(\text{resp.}\quad e_{i_1}e'_{j_1}e_{i_2}e'_{j_2}\cdots e_{i_r}e'_{j_r}e_{i_{r+1}}\big) \tag{2.4}$$

where r is any positive integer. The relations (2.1) show that a permutation in the sequence $(i_1,i_2,\ldots)$ or in the sequence $(j_1,j_2,\ldots)$ does not modify the product under consideration modulo $\mathcal{B}^{\leqslant 2r-2}(V,F)$ (resp. modulo $\mathcal{B}^{\leqslant 2r-1}(V,F)$) and up to a factor ± 1. Moreover, this product falls into $\mathcal{B}^{\leqslant 2r-2}(V,F)$ (resp. into $\mathcal{B}^{\leqslant 2r-1}(V,F)$) if two indices are equal in the sequence $(i_1,i_2,\ldots)$ or $(j_1,j_2,\ldots)$. It follows (by induction on r) that every such product is a linear combination of similar products associated with strictly increasing sequences of indices. In other words, $\mathcal{B}_0(V,F)$ (resp. $\mathcal{B}_1(V,F)$) is spanned by 1 (resp. $(e_1,\ldots,e_n)$) and all products in (2.4) such that $i_1 < i_2 < \cdots$ and $j_1 < j_2 < \cdots$. All these elements (including 1 and the vectors e_i) will be called the *regular products derived from the bases* $(e_1,\ldots,e_n)$ *and* $(e'_1,\ldots,e'_n)$; and when these bases are equal, they are the regular products derived from $(e_1,\ldots,e_n)$.

The number of the regular products that span $\mathcal{B}_0(V,F)$ is the sum of all squares $\binom{n}{r}^2$, and the number of the regular products that span $\mathcal{B}_1(V,F)$ is the sum of all products $\binom{n}{r}\binom{n}{r+1}$. The notation $\binom{n}{r}$ is meaningful whenever $n \geqslant 0$ and $r \in \mathbb{Z}$: it is the number of subsets of cardinality r in a set of cardinality n, and it means 0 if $r < 0$ or $r > n$. We have

$$\sum_{r\in\mathbb{Z}} \binom{n}{r}^2 = \binom{2n}{n} \qquad \text{and} \qquad \sum_{r\in\mathbb{Z}} \binom{n}{r}\binom{n}{r+1} = \binom{2n}{n-1}. \tag{2.5}$$

Therefore, the dimensions of $\mathcal{B}_0(V,F)$, $\mathcal{B}_1(V,F)$ and $\mathcal{B}(V,F)$ are not greater than $\binom{2n}{n}$, $\binom{2n}{n-1}$ and $\binom{2n+1}{n}$. In fact, they are always equal to these maximal values. In other words, the regular products derived from a pair of bases are always linearly independent. This assertion will be proved in this way: Sect. 3 shall recall that it is true when F is non-degenerate; besides, when (V,F) is embedded in a larger space (V'',F''), this assertion is true for (V,F) if it is true for (V'',F'') because every basis of V can be extended to a basis of V''; since (V,F) can be embedded in a space (V'',F'') where F'' is non-degenerate, this assertion is true for (V,F).

2.4. The Homomorphism $\mathcal{B}(V,0) \to \bigwedge(V)\otimes\bigwedge(V)$

Let $\mathcal{A}$ be a (unital and associative) algebra provided with a parity gradation $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$, and let $\top$ be a graded involutive automorphism of $\mathcal{A}$. Thus the expression $\top^d$ is meaningful when the exponent d is a parity. With the $\top$-twisted multiplication $(a,b) \longmapsto a \odot b$ defined by

$$a \odot b = a\,\top^{\partial a}(b) \qquad \text{(when } a \text{ is homogeneous)},$$

$\mathcal{A}$ is still a unital and associative algebra. Indeed, $(a \odot b) \odot c$ and $a \odot (b \odot c)$ are both equal to $a\,\top^{\partial a}(b)\,\top^{\partial a+\partial b}(c)$.

Here $\mathcal{A}$ is $\bigwedge(V)\otimes\bigwedge(V)$ provided with its usual parity gradation: $x\otimes y$ is even if x and y have the same parity in $\bigwedge(V)$, odd if they have different parities. And $\top(x\otimes y) = y\otimes x$. For all u, $v \in V$, we can write the following equalities in the $\top$-twisted algebra $\bigwedge(V)\otimes\bigwedge(V)$:

$$(u\otimes 1) \odot (v\otimes 1) \odot (u\otimes 1) = (u\otimes 1)\,(1\otimes v)\,(u\otimes 1) = (u \wedge u)\otimes v = 0.$$

Therefore, the mapping $v \longmapsto v\otimes 1$ extends to a graded algebra homomorphism from the neutral meson algebra $\mathcal{B}(V,0)$ into the $\top$-twisted algebra $\bigwedge(V)\otimes\bigwedge(V)$. Every even (resp. odd) product of vectors $u_1v_1u_2v_2\cdots u_rv_r$ (resp. $u_1v_1u_2v_2\cdots u_rv_ru_{r+1}$) is mapped to

$$(u_1\wedge u_2\wedge\cdots\wedge u_r)\otimes(v_1\wedge v_2\wedge\cdots\wedge v_r) \quad \big(\text{resp. } (u_1\wedge\cdots\wedge u_{r+1})\otimes(v_1\wedge\cdots\wedge v_r)\big).$$

Thus we know the images of the $\mathbb{Z}$-homogeneous components of $\mathcal{B}(V,0)$: the image of $\mathcal{B}^{2r}(V,0)$ is $\bigwedge^r(V)\otimes\bigwedge^r(V)$, and the image of $\mathcal{B}^{2r+1}(V,0)$ is

$\bigwedge^{r+1}(V)\otimes\bigwedge^{r}(V)$. Because of (2.5), the image of $\mathcal{B}_0(V,0)$ in $\bigwedge(V)\otimes\bigwedge(V)$ is a subalgebra of dimension $\binom{2n}{n}$, and the image of $\mathcal{B}_1(V,0)$ is a subspace of dimension $\binom{2n}{n-1}$. Thus, the image of $\mathcal{B}(V,0)$ in $\bigwedge(V)\otimes\bigwedge(V)$ has dimension $\binom{2n+1}{n}$, and the dimension of $\mathcal{B}(V,0)$ is at least $\binom{2n+1}{n}$. Because of Sect. 2.3, we can conclude that $\dim(\mathcal{B}(V,0)) = \binom{2n+1}{n}$, and that the homomorphism $\mathcal{B}(V,0) \to \bigwedge(V)\otimes\bigwedge(V)$ induces an isomorphism from $\mathcal{B}(V,0)$ onto a subalgebra of $\bigwedge(V)\otimes\bigwedge(V)$ that is well described.

The above $\top$-twisting of $\bigwedge(V)\otimes\bigwedge(V)$ does not affect the subalgebra $\bigoplus_r \bigwedge^r(V)\otimes\bigwedge^r(V)$ which is the image of $\mathcal{B}_0(V,0)$. The following properties of $\mathcal{B}_0(V,0)$ can be deduced either from the relations (2.1), or from the isomorphism $\mathcal{B}_0(V,0) \to \bigoplus_r \bigwedge^r(V)\otimes\bigwedge^r(V)$. This algebra $\mathcal{B}_0(V,0)$ is commutative. There is a linear bijection $V\otimes V \longmapsto \mathcal{B}^2(V,0)$ which maps every $u\otimes v$ to uv, and for each product uv, we have $(uv)^2 = 0$.

Every element of $\mathcal{B}^2(V,0)$ has an exponential in $\mathcal{B}_0(V,0)$ in the same way as every element of $\bigwedge^2(V)$ has an exponential in $\bigwedge_0(V)$. Every element of $\mathcal{B}^2(V,0)$ can be written as a sum $u_1v_1 + u_2v_2 + \cdots + u_kv_k$ with an arbitrary number k of terms, and by definition,

$$\exp\left(\sum_{i=1}^{k} u_i v_i\right) = \prod_{i=1}^{k}(1 + u_i v_i). \tag{2.6}$$

This definition implies that

$$\forall \eta,\, \theta \in \mathcal{B}^2(V,0), \quad \exp(\eta + \theta) = \exp(\eta)\, \exp(\theta). \tag{2.7}$$

But it is legitimate only after the following lemma has been proved.

Lemma 2.1. *If* $\sum_i u_i v_i = 0$, *then* $\prod_i (1 + u_i v_i) = 1$.

Proof. Let us carry the problem from $\mathcal{B}_0(V,0)$ into the isomorphic algebra $\bigoplus_r \bigwedge^r(V)\otimes\bigwedge^r(V)$. We assume that $\sum_i u_i\otimes v_i = 0$, and we must prove that the product of all factors $(1\otimes 1 + u_i\otimes v_i)$ is equal to 1. Besides the ordinary tensor product $\bigwedge(V)\otimes\bigwedge(V)$, there is the twisted tensor product $\bigwedge(V)\hat{\otimes}\bigwedge(V)$ which is isomorphic to $\bigwedge(V\times V)$, and by this isomorphism, each $u_i\otimes v_i$ corresponds to an element of $\bigwedge^2(V\times V)$. Since $\sum_i u_i\otimes v_i = 0$, the twisted product of all $(1\otimes 1 + u_i\otimes v_i)$ in $\bigwedge(V)\hat{\otimes}\bigwedge(V)$ is equal to 1. For each degree $r = 0, 1, \ldots, n$, the ordinary product of all $(1\otimes 1 + u_i\otimes v_i)$ and their twisted product have the same component in $\bigwedge^r(V)\otimes\bigwedge^r(V)$ up to a factor ± 1. Therefore, their ordinary product is also equal to 1. □

3. The Structure of Non-degenerate Meson Algebras

We continue the study of the homomorphism presented in Sect. 2.2: with every $b \in \mathcal{B}(V,F)$ is associated an operator Θ_b on $\bigwedge(V)\times\bigwedge(V)$. For all the proofs, I refer to [7].

When x and y belong to the same $\bigwedge^r(V)$ for some exponent r, both interior products $x\rfloor y$ and $x\lfloor y$ belong to $\bigwedge^0(V) = K$, and they are equal. If

ω is a non-zero element of $\bigwedge^n(V)$, there is a basis $(e_1,\dots,e_n)$ of V such that $\omega = e_1\wedge e_2\wedge\cdots\wedge e_n$, and we have

$$\omega\rfloor\omega = \omega\lfloor\omega = (-1)^{n(n-1)/2}\ \det\big(F(e_i,e_j)\big)_{i,j\,\in\{1,2,\dots,n\}};$$

thus F is non-degenerate if and only if $\omega\rfloor\omega \neq 0$. Besides, we consider $\Omega = K\times\bigwedge^n(V)$ (a graded subspace of $\bigwedge(V)\times\bigwedge(V)$) and we turn it into a graded commutative, associative and unital algebra over K by setting (for all λ, μ, λ', $\mu' \in K$)

$$(\lambda,\,\mu\omega)(\lambda',\,\mu'\omega) = \big(\lambda\lambda' + \mu\mu'\,\omega\rfloor\omega,\ (\lambda\mu' + \mu\lambda')\,\omega\big).$$

The odd component of Ω is $0\times\bigwedge^n(V)$, whatever the parity of n may be.

Lemma 3.1. *The algebra Ω operates on $\bigwedge(V)\times\bigwedge(V)$ in this way:*

$$(\lambda,\mu\omega)\,(x,y) = \big(\lambda x + \mu y\rfloor\omega,\ \lambda y + \mu x\rfloor\omega\big),$$

and the operation Θ_b of every $b\in\mathcal{B}(V,F)$ is Ω-linear.

Lemma 3.1 means that the following two equalities are true for all x, $y\in\bigwedge(V)$ and for all $v\in V$:

$$\big((x\rfloor\omega)\rfloor\omega,\ (y\rfloor\omega)\rfloor\omega\big) = \big((\omega\rfloor\omega)\,x,\ (\omega\rfloor\omega)\,y\big),$$
$$\big(v\wedge(x\rfloor\omega),\ v\rfloor(y\rfloor\omega)\big) = \big((v\rfloor x)\rfloor\omega,\ (v\wedge y)\rfloor\omega\big).$$

On another side, the action of $(0,\omega)$ maps every subspace $\bigwedge^k(V)\times 0$ into $0\times\bigwedge^{n-k}(V)$, and every $0\times\bigwedge^k(V)$ into $\bigwedge^{n-k}(V)\times 0$. Each of the $n{+}2$ subspaces $\bigwedge^k(V)\times\bigwedge^{k-1}(V)$ is invariant under the action of $\mathcal{B}(V,F)$, and $(0,\omega)$ maps it into $\bigwedge^{n-k+1}(V)\times\bigwedge^{n-k}(V)$ which is also invariant under the action of $\mathcal{B}(V,F)$, but different from $\bigwedge^k(V)\times\bigwedge^{k-1}(V)$, except when $n=2k{-}1$. When $n=2k{-}1$, $\bigwedge^k(V)\times\bigwedge^{k-1}(V)$ is invariant under the actions of Ω and $\mathcal{B}(V,F)$, and the notation $\mathrm{End}_\Omega(\bigwedge^k(V)\times\bigwedge^{k-1}(V))$ is meaningful.

3.1. The Structure of $\mathcal{B}(V,F)$ when F is Non-degenerate

Up to the end of Sect. 3, we assume that F is non-degenerate.

Theorem 3.2. *The mapping $b\longmapsto\Theta_b$ is a graded isomorphism from $\mathcal{B}(V,F)$ onto the subalgebra of all elements of* $\mathrm{End}(\bigwedge(V)\times\bigwedge(V))$ *satisfying these two properties: they are Ω-linear, and they leave $\bigwedge^k(V)\times\bigwedge^{k-1}(V)$ invariant for $k=0,1,\dots,n{+}1$.*

Theorem 3.3. *When $n=2r$, the restrictions of the operators Θ_b to the direct sum of the subspaces $\bigwedge^k(V)\times\bigwedge^{k-1}(V)$ with $k=0,1,\dots,r$ give a graded algebra isomorphism*

$$\mathcal{B}(V,F)\ \longrightarrow\ \prod_{k=0}^{r}\ \mathrm{End}\big(\bigwedge^k(V)\times\bigwedge^{k-1}(V)\big).$$

When $n=2r-1$, the restrictions of the operators Θ_b to the direct sum of the subspaces $\bigwedge^k(V)\times\bigwedge^{k-1}(V)$ with $k=0,1,\dots,r$ give a graded isomorphism

$$\mathcal{B}(V,F)\longrightarrow \mathrm{End}_\Omega\big(\bigwedge^r(V)\times\bigwedge^{r-1}(V)\big)\ \times\ \prod_{k=0}^{r-1}\mathrm{End}\big(\bigwedge^k(V)\times\bigwedge^{k-1}(V)\big).$$

Theorem 3.4. *The restrictions of the even operators* Θ_b *to the even subspace* $\bigwedge(V)\times 0$ *give an isomorphism*

$$\mathcal{B}_0(V,F) \longrightarrow \prod_{k=0}^{n} \mathrm{End}\big(\bigwedge^k(V)\big).$$

The linear mappings $\bigwedge(V)\times 0 \to 0\times\bigwedge(V)$ *induced by the odd operators* Θ_b *give a linear bijection*

$$\mathcal{B}_1(V,F) \longrightarrow \prod_{k=1}^{n} \mathrm{Hom}\big(\bigwedge^k(V),\, \bigwedge^{k-1}(V)\big).$$

Theorem 3.4 shows that the dimension of $\mathcal{B}_0(B;F)$ is the sum of all squares $\binom{n}{k}^2$, which is $\binom{2n}{n}$, and that the dimension of $\mathcal{B}_1(V,F)$ is the sum of all products $\binom{n}{k}\binom{n}{k-1}$, which is $\binom{2n}{n-1}$ (see (2.5)). Thus the dimension of $\mathcal{B}(V,F)$ is the maximal possible value $\binom{2n+1}{n}$.

Theorems 3.3 and 3.4 look like corollaries of Theorem 3.2, but in [7], the argument begins with Theorem 3.4, continues with Theorem 3.2, and ends with Theorem 3.3. As far as I know, Theorem 3.2 appeared in [7] for the first time, but a great part of this theorem is a synthesis of the results of Kemmer and his followers.

3.2. The Irreducible Graded Modules over $\mathcal{B}(V,F)$

Let M be a graded left module over $\mathcal{B}(V,F)$: thus $M = M_0 \oplus M_1$, and every even (resp. odd) element of $\mathcal{B}(V,F)$ maps each component M_0 and M_1 into itself (resp. into the other one). With M is associated a module M^s *with shifted gradation:* the even (resp. odd) elements of M^s are the odd (resp. even) elements of M. It may happen that M and M^s are isomorphic as graded modules, but often they are not. The spinor spaces used by the physicists are graded modules over Clifford algebras, and the homogeneous spinors are called Weyl spinors, but the physicists do not say that Weyl spinors may be even or odd, they say that they are left-hand or right-hand spinors. I suppose that the physicists prefer chiralities rather than parities because the gradation of a module is something different from the gradation of an algebra: we may shift the gradation of a module, but not the gradation of an algebra.

The $n+2$ graded modules $\bigwedge^k(V)\times\bigwedge^{k-1}(V)$ are all the irreducible graded modules over $\mathcal{B}(V,F)$ up to isomorphy. The odd element $(0,\omega)$ of Ω maps $\bigwedge^k(V)\times\bigwedge^{k-1}(V)$ onto $\bigwedge^{n-k+1}(V)\times\bigwedge^{n-k}(V)$. If $n \neq 2k-1$, we obtain a pair of two non-isomorphic graded modules where each module is isomorphic to the other one with shifted gradation. But if $n = 2k-1$, we obtain an irreducible graded module that is isomorphic to the module with shifted gradation.

All these irreducible graded modules are still irreducible without their gradation, except when $n = 2k-1$ and Ω is not a field (in other words, $\omega\rfloor\omega$ admits a square root in K). In this exceptional case, every non-trivial ideal of Ω gives a non-trivial submodule of $\bigwedge^k(V)\times\bigwedge^{k-1}(V)$. If $\mathrm{char}(K) \neq 2$ and Ω is not a field, then Ω is isomorphic to $K\times K$. If $\mathrm{char}(K) = 2$ and Ω is not a field, then Ω contains a non-zero element ϵ such that $\epsilon^2 = 0$. In Kemmer's work, $K = \mathbb{C}$ and $\Omega \cong \mathbb{C}\times\mathbb{C}$.

3.3. The Center of $\mathcal{B}_0(V, F)$

According to Theorem 3.4, the center of $\mathcal{B}_0(V, F)$ has dimension $n+1$, and it is spanned by $n+1$ primitive idempotents ε_p with $p = 0, 1, \ldots, n$. The image of ε_p in $\mathrm{End}(\bigwedge(V)\times\bigwedge(V))$ projects $\bigwedge(V)\times 0$ onto $\bigwedge^p(V)\times 0$. The sum of all idempotents ε_p is 1, and $\varepsilon_p\varepsilon_q = 0$ whenever $p \neq q$.

Let $(e_1, e_2, \ldots, e_n)$ be a basis of V. Since F is non-degenerate, there is a basis $(e'_1, \ldots, e'_n)$ of V such that $F(e_i, e'_i) = 1$ for $i = 1, 2, \ldots, n$, and $F(e_i, e'_j) = 0$ whenever $i \neq j$. The relations (2.1) show that the n products $e_ie'_i$ are pairwise commuting idempotents. For every subset P of $N = \{1, 2, \ldots, n\}$, let $e^\wedge_P$ be the product in $\bigwedge(V)$ of all e_i such that $i \in P$, in the increasing order of the indices i. The operation of $e_ie'_i$ on the even component $\bigwedge(V)\times 0$ leaves $(e^\wedge_P, 0)$ invariant if $i \in P$, but maps it to 0 if $i \notin P$. With P, we also associate the following idempotent in $\mathcal{B}_0(V, F)$:

$$\varepsilon(P) = \prod_{i\in P} e_ie'_i \prod_{j\notin P}(1 - e_je'_j). \tag{3.1}$$

Since $\varepsilon(P)\,\varepsilon(Q) = 0$ whenever $P \neq Q$, the idempotents $\varepsilon(P)$ are the primitive idempotents in the commutative subalgebra generated by all idempotents $e_ie'_i$. The operation of $\varepsilon(P)$ in $\bigwedge(V)\times 0$ leaves $(e^\wedge_P, 0)$ invariant, and maps $(e^\wedge_Q, 0)$ to 0 if Q is another subset of N. Consequently, *ε_p is the sum of all idempotents $\varepsilon(P)$ such that P is a subset of cardinality p in N.*

Of course, ε_p does not depend on the choice of the basis $(e_1, \ldots, e_n)$. If we replace this basis with $(e'_1, \ldots, e'_n)$, the idempotents $e_ie'_i$ are replaced with e'_ie_i, and since $e'_ie_i = \rho(e_ie'_i)$, we realize that the reversion ρ leaves invariant every idempotent ε_p.

The following formulas deserve to be recalled: for all $v \in V$,

$$v\,\varepsilon_p = \varepsilon_{n-p+1}\,v \qquad \text{and} \qquad \varepsilon_p\,v = v\,\varepsilon_{n-p+1}. \tag{3.2}$$

It suffice to prove them when v is one of the vectors e_i, and to observe that

$$\begin{aligned} &e_i(e'_ie_i) = (e_ie'_i)e_i = e_i\,, \qquad && e_i(1 - e'_ie_i) = (1 - e_ie'_i)e_i = 0,\\ &e_i(e'_je_j) = (1 - e_je'_j)e_i\,, \qquad && e_i(1 - e'je_j) = (e_je'_j)e_i \qquad \text{if } i \neq j. \end{aligned}$$

All idempotents ε_p are in the ideal $\mathcal{B}^+(V, F)$ except ε_0. Moreover, when $p = 0$, (3.2) means $v\varepsilon_0 = \varepsilon_0 v = 0$, because ε_{n+1} means 0.

3.4. The Center of $\mathcal{B}(V, F)$

As in Theorem 3.3, we must distinguish two cases according to the parity of n, either $n = 2r$, or $n = 2r-1$ (with $r \geqslant 1$).

When $n = 2r$, the center of $\mathcal{B}(V, F)$ has dimension $r+1$, and it is spanned by $r+1$ primitive idempotents. From Theorem 3.2, or from the formulas (3.2), we can deduce which are these idempotents:

$$\varepsilon_0, \quad \varepsilon_1+\varepsilon_n, \quad \varepsilon_2+\varepsilon_{n-1}, \quad \ldots, \quad \varepsilon_r+\varepsilon_{r+1}.$$

The center of $\mathcal{B}(V,F)$ is more complicated when $n = 2r-1$. It has an even component of dimension $r+1$ which is spanned by the following primitive idempotents:

$$\varepsilon_0, \quad \varepsilon_1+\varepsilon_n, \quad \varepsilon_2+\varepsilon_{n-1}, \quad \ldots, \quad \varepsilon_{r-1}+\varepsilon_{r+1}, \quad \varepsilon_r.$$

The center of $\mathcal{B}(V,F)$ has also an odd component which is contained in the ideal generated by ε_r; it is spanned by the element η that operates on $\bigwedge^r(V)\times\bigwedge^{r-1}(V)$ in the same way as the element $(0,\omega)$ of Ω, and that maps all the other submodules $\bigwedge^k(V)\times\bigwedge^{k-1}(V)$ to 0. To calculate η, we suppose that $(e_1,\ldots,e_n)$ is an orthogonal basis of (V,F). Orthogonal bases always exist except when $\mathrm{char}(K) = 2$ and F is alternating, but this exception never occurs when n is odd, and F non-degenerate. We suppose that $\omega = e_1\wedge e_2\wedge\cdots\wedge e_n$, and consequently,

$$\omega\rfloor\omega = (-1)^{r-1}\, F(e_1,e_1)\, F(e_2,e_2)\, \cdots\, F(e_n,e_n).$$

Let P be a subset of cardinality r in $N = \{1,2,\ldots,n\}$, let $\{i_1,i_2,\ldots,i_r\}$ be the list of its elements, and $\{j_1,j_2,\ldots,j_{r-1})$ the list of the elements of the complementary subset $N-P$. Moreover, let $\mathrm{sgn}(i_1,j_1,\ldots,j_{r-1},i_r)$ be the signature of the permutation $(i_1,j_1,i_2,j_2,\ldots,i_{r-1},j_{r-1},i_r)$ of N. With this notation, let us set

$$\eta(P) = \mathrm{sgn}(i_1,j_1,\ldots,j_{r-1},i_r)\; e_{i_1}e_{j_1}e_{i_2}e_{j_2}\,\cdots\, e_{i_{r-1}}e_{j_{r-1}}e_{i_r}. \tag{3.3}$$

In [7], Sect. 7, it is proved that

$$\varepsilon(P)\,\eta(P) = \eta(P)\,\varepsilon(P) = \eta(P) \quad \text{and} \quad \eta(P)^2 = (\omega\rfloor\omega)\,\varepsilon(P).$$

The wanted element η is the sum of all odd elements $\eta(P)$ associated with the subsets of cardinality r in N. Moreover, $\rho(\eta) = (-1)^{r-1}\eta$.

4. The Involutive Automorphism τ of $\mathcal{B}_0(V,F)$

As it is explained in Sect. 2.3, we have proved that the dimension of $\mathcal{B}(V,F)$ is $\binom{2n+1}{n}$ (even if F is degenerate), and we can state the following theorem.

Theorem 4.1. *Let $(e_1,\ldots,e_n)$ and $(e'_1,\ldots,e'_n)$ be two bases of V (which may be equal). The regular products derived from these bases constitute a basis of $\mathcal{B}(V,F)$. They are the even (resp. odd) products*

$$e_{i_1}e'_{j_1}e_{i_2}e'_{j_2}\cdots e_{i_r}e'_{j_r} \qquad \big(\textit{resp.}\quad e_{i_1}e'_{j_1}e_{i_2}e'_{j_2}\cdots e_{i_r}e'_{j_r}e_{i_{r+1}}\,\big)$$

such that $i_1 < i_2 < \cdots$ and $j_1 < j_2 < \cdots$. When $r = 0$, this product means 1 (resp. e_{i_1}).

A classical argument deduces from this theorem that *every element $x \in \mathcal{B}(V,F)$ has a support* $\mathrm{Sup}(x)$ *in V*: it is the smallest subspace S of V such that x belongs to the (unital) subalgebra $\mathcal{B}(S,F)$ generated by S in $\mathcal{B}(V,F)$. As a matter of fact, every homogeneous element x has two partial supports in V, and $\mathrm{Sup}(x)$ is the sum of the two partial supports as it is explained in [3]; but here, this refinement may be forgotten.

4.1. Elementary Properties of the Automorphism τ

Theorem 4.2. *There is a unique automorphism τ of $\mathcal{B}(V,F)$ that maps every product uv (with u, $v \in V$) to $F(u,v) - vu$, and it is involutive.*

Proof. If τ exists, it is clear that it is unique. To prove its existence, I propose a new proof where (V,F) is embedded in a space (V',F') of dimension $n+1$, spanned by V and a vector e_0 orthogonal to V. Let $(e_1,\dots,e_n)$ be a basis of V. By means of the basis $(e_0,e_1,\dots,e_n)$ (resp. $(e_1,\dots,e_n,e_0)$), we deduce from Theorem 4.1 the following assertion: if an element of $\mathcal{B}_1(V',F')$ is equal to e_0x (resp. xe_0) for some $x \in \mathcal{B}_0(V,F)$, this element x is unique. On another side, the relations (2.1) allow us to write, for all u, $v \in V$,

$$e_0(uv) = (F(u,v) - vu)e_0 \qquad \text{and} \qquad (uv)e_0 = e_0(F(u,v) - vu).$$

It follows that, for every $x \in \mathcal{B}_0(V,F)$, there is a unique $\tau(x)$ (resp. $\tau'(x)$) in $\mathcal{B}_0(V,F)$ such that $e_0x = \tau(x)\,e_0$ (resp. $xe_0 = e_0\tau'(x)$), and that $\tau(x)$ and $\tau'(x)$ are equal; indeed, if $x = u_1v_1u_2v_2\cdots u_kv_k$, then

$$\tau(x) = \tau'(x) = (F(u_1,v_1) - v_1u_1)(F(u_2,v_2) - v_2u_2)\cdots(F(u_k,v_k) - v_ku_k).$$

The definitions of τ and τ' imply that they are algebra homomorphisms from $\mathcal{B}_0(V,F)$ into itself, and that $\tau\tau'$ and $\tau'\tau$ are the identity mapping of $\mathcal{B}_0(V,F)$. Since $\tau = \tau'$, we conclude that τ is an involutive automorphism of $\mathcal{B}_0(V,F)$. □

The relations (2.1) mean $u\,\tau(uv) = 0$, $\tau(vu)\,u = 0$, $u\,\tau(wv) = -w\tau(uv)$ and $\tau(vu)\,w = -\tau(vw)\,u$. It is easy to generalize these equalities.

Lemma 4.3. *For all $x_1 \in \mathcal{B}_1(V,F)$, and for all u, $v \in V$, we have*

$$u\,\tau(ux_1) = 0, \qquad u\,\tau(vx_1) = -v\,\tau(ux_1),$$
$$\tau(x_1v)\,v = 0, \qquad \tau(x_1u)\,v = -\tau(x_1v)\,u.$$

The next lemma comes from [2], Sect. 6.

Lemma 4.4. *Let X and Y be two subspaces of V that are orthogonal to each other, and let x_0, x_1, y_0 and y_1 be elements of $\mathcal{B}_0(X,F)$, $\mathcal{B}_1(X,F)$, $\mathcal{B}_0(Y,F)$ and $\mathcal{B}_1(Y,F)$. We have:*

$$y_0x_0 = x_0y_0\,, \qquad y_0x_1 = x_1\,\tau(y_0),$$
$$y_1x_1 = -\tau(x_1y_1), \qquad y_1x_0 = \tau(x_0)\,y_1\,.$$

When $F = 0$, we can apply Lemma 4.4 with $X = Y = V$; it proves again that the even subalgebra $\mathcal{B}_0(V,0)$ is commutative.

4.2. The Automorphism τ and the Homomorphisms of Sect. 2

By Jacobson's homomorphism Δ, the image of $\mathcal{B}_0(V,F)$ is contained in the even subalgebra of $\mathrm{Cl}(V,\bar{F})\otimes\mathrm{Cl}(V,\bar{F})$, which is

$$\big(\mathrm{Cl}(V,\bar{F})\otimes\mathrm{Cl}(V,\bar{F})\big)_0 = \big(\mathrm{Cl}_0(V,\bar{F})\otimes\mathrm{Cl}_0(V,\bar{F})\big) \oplus \big(\mathrm{Cl}_1(V,\bar{F})\otimes\mathrm{Cl}_1(V,\bar{F})\big).$$

This even subalgebra is provided with an automorphism $\ddot{\tau}$ that maps $y\otimes z$ to itself (resp. to $-y\otimes z$) if y and z are both even (resp. odd). For every $x \in \mathcal{B}_0(V,F)$, we have

$$\Delta(\tau(x)) = \ddot{\tau}(\Delta(x)). \tag{4.1}$$

Indeed, a direct calculation shows that

$$\begin{aligned} 4\,\Delta(uv) &= (u\otimes 1 + 1\otimes u)\,(v\otimes 1 + 1\otimes v) \\ &= uv\otimes 1 + 1\otimes uv \;+\; u\otimes v + v\otimes u\,, \\ 4\,\Delta(F(u,v){-}vu) &= (2F(u,v){-}vu)\otimes 1 + 1\otimes(2F(u,v){-}vu) - u\otimes v - v\otimes u \\ &= uv\otimes 1 + 1\otimes uv \;-\; u\otimes v - v\otimes u\,. \end{aligned}$$

On another side, let T be the element of $\mathrm{End}(\bigwedge(V)\times\bigwedge(V))$ defined by $T(y,z) = (z,y)$. For every $x \in \mathcal{B}_0(V,F)$, we have:

$$\Theta_{\tau(x)} = T \circ \Theta_x \circ T. \tag{4.2}$$

Indeed, we have $u\rfloor(v\wedge z) = F(u,v)z - v\wedge(u\rfloor z)$ and $v\rfloor(u\wedge y) = F(u,v)y - u\wedge(v\rfloor y)$ for all y, $z \in \bigwedge(V)$, and consequently,

$$\begin{aligned} \Theta_u \circ \Theta_v(y,z) &= (u\wedge(v\rfloor y),\; F(u,v)z - v\wedge(u\rfloor z)), \\ F(u,v)(z,y) - \Theta_v \circ \Theta_u(z,y) &= (F(u,v)z - v\wedge(u\rfloor z),\; u\wedge(v\rfloor y)). \end{aligned}$$

4.3. The Dimension of the Subalgebra $\ker(\tau - \mathbf{1})$

I believed that the dimension of the eigenspace $\ker(\tau{-}\mathbf{1})$ was always the half of $\dim(\mathcal{B}_0(V,F))$ until the precise examination of the trivial case $n = 1$ made me realize my error. The proof of the next theorem uses the following consequence of Theorem 7.5 (stated farther): *if F and F' are two symmetric bilinear forms on V, and if there is a bilinear form β on V such that*

$$\forall u,\, v \in V, \quad F'(u,v) = F(u,v) + \beta(u,v) + \beta(v,u), \tag{4.3}$$

then the dimension of the eigenspace $\ker(\tau{-}\mathbf{1})$ is the same in the algebra $\mathcal{B}_0(V,F')$ as in the algebra $\mathcal{B}_0(V,F)$. When $\mathrm{char}(K) \neq 2$, such a bilinear form β always exists; for instance, $\beta = (F'{-}F)/2$. But if $\mathrm{char}(K) = 2$, it exits if and only if F and F' give the same quadratic form on V: $F'(v,v) = F(v,v)$ for all $v \in V$.

Theorem 4.5. *When $\mathrm{char}(K) \neq 2$, the dimension of the subalgebra $\ker(\tau{-}\mathbf{1})$ is always $\binom{2n-1}{n-1}$ (the half of $\dim(\mathcal{B}_0(V,F))$). When $\mathrm{char}(K) = 2$, the same is true if F is not alternating. But if F is alternating, then the dimension of $\ker(\tau{-}\mathbf{1})$ is $\binom{2n-1}{n-1} + 2^{n-1}$.*

Proof. When $\mathrm{char}(K) \neq 2$ and $F \neq 0$, there is $v \in V$ such that $F(v,v) \neq 0$. Let us set $b = F(v,v) - 2v^2$. This element b belongs to the other eigenspace $\ker(\tau{+}\mathbf{1})$, and it is invertible because $b^2 = F(v,v)^2$. Since τ is an involutive automorphism, the multiplication $x \longmapsto bx$ permutes the two eigenspaces $\ker(\tau{-}\mathbf{1})$ and $\ker(\tau{+}\mathbf{1})$. Therefore, these eigenspaces have the same dimension, and the conclusion follows. When $F = 0$, the direct calculation of the image of $\tau{-}\mathbf{1}$ gives the same result, but it is easier to use the consequence of Theorem 7.5 that is stated just above.

When $\mathrm{char}(K) = 2$ and F is alternating, there is a bilinear form β on V such that $F(u,v) = \beta(u,v) + \beta(v,u)$ for all u, $v \in V$. Therefore, it suffices to treat the case $F = 0$. Let $(e_1,\ldots,e_n)$ be a basis of V. Let P and Q be two subsets of the same cardinality in $N = \{1,2,\ldots,n\}$, let $(i_1,i_2,\ldots,i_k)$

(resp. $(j_1, j_2, \ldots, j_k)$) be the increasing sequence of the elements of P (resp. Q), and let us set

$$e_{PQ} = e_{i_1} e_{j_1} e_{i_2} e_{j_2} \cdots e_{i_k} e_{j_k};$$

moreover, $e_{PQ} = 1$ if $P = Q = \emptyset$. It is clear that $\tau(e_{PQ}) = e_{QP}$. Therefore, $\tau - \mathbf{1}$ maps e_{PP} to 0 for every subset P of N. It follows that the dimension of the image of $\tau - \mathbf{1}$ is the number of all sets $\{P, Q\}$ where $P \neq Q$ and $\text{Card}(P) = \text{Card}(Q)$, that is

$$\sum_{k=0}^{n} \frac{1}{2} \binom{n}{k} \left(\binom{n}{k} - 1 \right) = \frac{1}{2} \left(\binom{2n}{n} - 2^n \right).$$

Now it is easy to calculate the dimension of $\ker(\tau - \mathbf{1})$.

At last, we suppose that $\text{char}(K) = 2$ and that F is not alternating. We may assume that $F(e_1, e_1) \neq 0$. The dimension of $\ker(\tau - \mathbf{1})$ remains the same after every field extension of K. By adjoining suitable square roots to the field K, we can reduce the problem to the case $F(e_i, e_i) \in \{0, 1\}$ for $i = 1, 2, \ldots, n$. If $F(e_i, e_i) = 1$ for some $i \in \{2, 3, \ldots, n\}$, we replace e_i with $e_i + e_1$ such that $F(e_i + e_1,\, e_i + e_1) = 0$. Thus we may assume that $F(e_1, e_1) = 1$ and $F(e_i, e_i) = 0$ whenever $i \geqslant 2$. Finally, we introduce the bilinear form β such that $\beta(e_i, e_j) = 0$ if $i \leqslant j$, and $\beta(e_i, e_j) = F(e_i, e_j)$ if $i > j$; it allows us to reduce the problem to the case $F(e_1, e_1) = 1$ and $F(e_i, e_j) = 0$ whenever $i + j \geqslant 3$. We define e_{PQ} as above; moreover, if P contains 1, then $P^\flat = P - \{1\}$, and if P does not, then $P^\sharp = P \cup \{1\}$. In the present case,

$$\tau(e_{PQ}) = e_{QP} + e_{Q^\flat P^\flat} \qquad \text{if } \ 1 \in P \cap Q,$$
$$\tau(e_{PQ}) = e_{QP} \qquad \text{if } \ 1 \notin P \cap Q.$$

The image of $\tau - \mathbf{1}$ contains all $e_{PQ} + e_{QP} + e_{Q^\flat P^\flat}$ such that $1 \in P \cap Q$, and their number is the sum of all squares $\binom{n-1}{k}^2$ which is $\binom{2n-2}{n-1}$. It also contains all $e_{PQ} + e_{QP}$ such that $1 \in P$ and $1 \notin Q$, and their number is the sum of all products $\binom{n-1}{k-1}\binom{n-1}{k}$ which is $\binom{2n-2}{n-2}$. All these elements are linearly independent, and all other elements that may still emerge are in the subspace that they span. Indeed, if neither P nor Q contains 1, then $e_{PQ} + e_{QP}$ is the image by $\tau - \mathbf{1}$ of $e_{P^\sharp Q^\sharp} + e_{Q^\sharp P^\sharp}$. Finally, the dimension of the image of $\tau - \mathbf{1}$ is $\binom{2n-2}{n-1} + \binom{2n-2}{n-2} = \binom{2n-1}{n-1}$. This is the half of $\dim(\mathcal{B}_0(V, F))$, in agreement with Theorem 4.5. $\square$

5. Interior Multiplications by Vectors

The imitation of Cliffordian arguments has already begun in Sect. 4, and shall become yet more demonstrative in the following sections.

5.1. First Properties

Theorem 5.1. *Let $\varphi : U \times V \to K$ and $\psi : V \times W \to K$ be two bilinear mappings. They allow every vector $u \in U$ (resp. $w \in W$) to operate on $\mathcal{B}(V, F)$ on the left (resp. right) side by an interior multiplication denoted by $\rfloor$ or $\rfloor_\varphi$ (resp. $\lfloor$ or ${}_\psi\lfloor$). The interior multiplications by u and w are determined* *by the following six properties, valid for all* $v,\ v' \in V$, $y_0 \in \mathcal{B}_0(V,F)$ *and* $y' \in \mathcal{B}(V,F)$*:*

$$u\rfloor v = \varphi(u,v), \quad u\rfloor(vv') = \varphi(u,v)\,v', \quad v\lfloor w = \psi(v,w), \quad (v'v)\lfloor w = v'\psi(v,w),$$
$$u\rfloor(y_0y') = (u\rfloor y_0)\,y' + \tau(y_0)(u\rfloor y'), \qquad (y'y_0)\lfloor w = y'(y_0\lfloor w) + (y'\lfloor w)\,\tau(y_0).$$

The associativity property $(u\rfloor y)\lfloor w = u\rfloor(y\lfloor w)$ *holds for all* $y \in \mathcal{B}(V,F)$*.*

The equalities involving y_0 and y' imply $u\rfloor 1 = 1\lfloor w = 0$ when $y_0 = y' = 1$. Of course, if we need only interior multiplications on the left (resp. right) side, we apply Theorem 5.1 with $W = 0$ (resp. $U = 0$). I will propose a new proof of Theorem 5.1 which requires that it be proved together with the following two lemmas.

Lemma 5.2. *The interior multiplications presented in Theorem 5.1 also satisfy the following properties for all* $u \in U$, $v \in V$, $w \in W$, $y_0 \in \mathcal{B}_0(V,F)$, $y_1 \in \mathcal{B}_1(V,F)$ *and* $y \in \mathcal{B}(V,F)$*:*

$$\begin{aligned} &u\rfloor(y_0v) = (u\rfloor y_0)\,v + \varphi(u,v)\,\tau(y_0), && (vy_0)\lfloor w = v\,(y_0\lfloor w) + \psi(v,w)\,\tau(y_0),\\ &u\rfloor(y_1v) = (u\rfloor y_1)\,v, && (vy_1)\lfloor w = v\,(y_1\lfloor w),\\ &\mathrm{Sup}(u\rfloor y) \subset \mathrm{Sup}(y), && \mathrm{Sup}(y\lfloor w) \subset \mathrm{Sup}(y). \end{aligned}$$

I rely on the readers for understanding that the six properties required in Theorem 5.1 determine the interior multiplications in a unique way, and that they imply the six properties stated in Lemma 5.2. The properties involving supports justify the word "interior": if an interior multiplication is inflicted on y, it operates inside the (unital) subalgebra generated by the support of y in V. The proof shall begin after the second lemma (where $U = V = W$ and $\varphi = \psi = F$).

Lemma 5.3. *The interior multiplications determined by* F *itself satisfy the following properties for all* $v \in V$, $y_0 \in \mathcal{B}_0(V,F)$ *and* $y_1 \in \mathcal{B}_1(V,F)$*:*

$$\begin{aligned} &v\rfloor\, y_0 = v\,y_0 - \tau(y_0)\,v, && y_0\,\lfloor v = y_0\,v - v\,\tau(y_0),\\ &v\rfloor\, y_1 = v\,y_1 + \tau(y_1\,v), && y_1\,\lfloor v = y_1\,v + \tau(v\,y_1). \end{aligned}$$

Proof. Let us begin with Lemma 5.3. Let us take the properties stated in Lemma 5.3 as the definition of the interior multiplications determined by F, and let us verify that this definition implies all the properties mentioned in Theorem 5.1. It suffices to consider interior multiplications on the left side since interior multiplications on the right side involve symmetric calculations. It is clear that, for all $u,\ v,\ v' \in V$,

$$u\rfloor v = uv + \tau(vu) = F(u,v) \quad \text{and} \quad u\rfloor(vv') = uvv' - \tau(vv')\,u = F(u,v)\,v'$$

because of the relations (2.1). Then we must distinguish two cases: y' may be an even element y'_0 or an odd element y'_1; in both cases, the verification is obvious:

$$\begin{aligned} u\,y_0y'_0 - \tau(y_0y'_0)\,u &= (uy_0 - \tau(y_0)u)\,y'_0 + \tau(y_0)(uy'_0 - \tau(y'_0)u),\\ u\,y_0y'_1 + \tau(y_0y'_1u) &= (uy_0 - \tau(y_0)u)\,y'_1 + \tau(y_0)(uy'_1 + \tau(y'_1u)). \end{aligned}$$

If $y = v_1 v_2 v_3 \cdots v_r$, then $u\rfloor y$ is the sum of all products

$$\tau(v_1 \cdots v_{2k})\, F(u, v_{2k+1})\, v_{2k+2} \cdots v_r \quad \text{where } 0 \leqslant 2k \leqslant r{-}1;$$

of course, $v_1 \cdots v_{2k}$ means 1 if $k = 0$, and $v_{2k+2} \cdots v_r$ means 1 if $2k = r{-}1$. Thus it is clear that, for every $y \in \mathcal{B}(V, F)$, the support of y in V contains the support of $u\rfloor y$. Now let us prove the associativity property. For every $y_0 \in \mathcal{B}_0(V, F)$, $(u\rfloor y_0)\lfloor w$ and $u\rfloor(y_0\lfloor w)$ are both equal to

$$uy_0w + \tau(w\,\tau(y_0)\,u) + F(u, w)\,\tau(y_0) - \tau(y_0)\,uw - uw\,\tau(y_0).$$

And for every $y_1 \in \mathcal{B}_1(V, F)$, $(u\rfloor y_1)\lfloor w$ and $u\rfloor(y_1\lfloor w)$ are both equal to

$$uy_1w - wy_1u + \tau(y_1u)\,w + u\,\tau(wy_1) \qquad \text{(see Lemma 4.3)}.$$

The equalities $u\rfloor y_1\lfloor u = 0$ and $u\rfloor y_1\lfloor w = -w\rfloor y_1\lfloor u$ are certainly worth a notice, but here they will never serve.

The general case is an easy consequence of this particular case: it suffices to embed (V, F) in the larger space (V'', F'') where $V'' = U{\times}V{\times}W$, and

$$F''((u, v, w), (u', v', w')) = \varphi(u, v'){+}\varphi(u', v) + F(v, v') + \psi(v, w'){+}\psi(v', w).$$

The mapping $v \longmapsto (0, v, 0)$ extends to a homomorphism J from $\mathcal{B}(V, F)$ into $\mathcal{B}(V'', F'')$, and J is injective because of Theorem 4.1. It maps $\mathcal{B}(V, F)$ onto the (unital) subalgebra of $\mathcal{B}(V'', F'')$ generated by $0{\times}V{\times}0$. To construct the interior product $u\rfloor y$ (determined by φ), we let the vector $(u, 0, 0)$ operate on $J(y)$; since their interior product has a support contained in $0{\times}V{\times}0$, it is the image by J of some element of $\mathcal{B}(V, F)$ which is $u\rfloor y$. And $y\lfloor w$ is defined in the same way. The interior multiplications determined by F'' satisfy all wanted properties, and the homomorphism J allows us to validate them for the interior multiplications determined by φ and ψ. □

In Sect. 6, it shall be proved that the interior multiplications by vectors of U (resp. W) extends to an action of the neutral algebra $\mathcal{B}(U, 0)$ (resp. $\mathcal{B}(W, 0)$) on the left (resp. right) side. In other words, $x\rfloor y$ and $y\lfloor z$ shall be defined for all $x \in \mathcal{B}(U, 0)$, $y \in \mathcal{B}(V, F)$ and $z \in \mathcal{B}(W, 0)$.

5.2. Properties Involving Jacobson's Homomorphism Δ

When $\mathrm{char}(K) \neq 2$, Jacobson's homomorphism (Sect. 2.1) has a nice behavior with respect to the interior multiplications, and later it will much help us in the study of mesonic Lipschitz monoids. The bilinear mapping φ (resp. ψ) allows the exterior algebra $\bigwedge(U)$ (resp. $\bigwedge(W)$) to operate on $\mathrm{Cl}(V, \bar{F})$ on the left (resp. right) side; the operations of the vectors of U (resp. W) satisfy these properties: $u\rfloor v = \varphi(u, v)$, $v\lfloor w = \psi(v, w)$ and for all y, $y' \in \mathrm{Cl}(V, \bar{F})$,

$$u\rfloor(yy') = (u\rfloor y)y' + \hat{y}\,(u\rfloor y') \quad \text{and} \quad (y'y)\lfloor w = y'(y\lfloor w) + (y'\lfloor w)\,\hat{y}$$

if $\hat{y}$ means y or $-y$ according as y is even or odd. Consequently, the algebras $\bigwedge(U){\otimes}\bigwedge(U)$ and $\bigwedge(W){\otimes}\bigwedge(W)$ (tensor products without twisting) operate on $\mathrm{Cl}(V, \bar{F}){\otimes}\mathrm{Cl}(V, \bar{F})$:

$$(x{\otimes}x')\rfloor(y{\otimes}y') = (x\rfloor y){\otimes}(x'\rfloor y') \quad \text{and} \quad (y{\otimes}y')\lfloor(z{\otimes}z') = (y\lfloor z){\otimes}(y'\lfloor z')$$

for all x, $x' \in \bigwedge(U)$, y, $y' \in \mathrm{Cl}(V, \bar{F})$ and z, $z' \in \bigwedge(W)$.

Lemma 5.4. *For all $u \in U$, $w \in W$ and $y \in \mathcal{B}(B, F)$, we have*

$$\Delta(u\rfloor y) = (u\otimes 1 + 1\otimes u)\rfloor \Delta(y),$$
$$\Delta(y\lfloor w) = \Delta(y)\lfloor (w\otimes 1 + 1\otimes w).$$

Proof. Since Lemma 5.4 follows form [2], Sect. 11, a sketch of proof is now sufficient. The above equalities are obviously true when y is a vector v of V, and a direct calculation shows that they are still true when $y = vv'$. Then we need the automorphism $\ddot{\tau}$ of the even subalgebra of $\mathrm{Cl}(V, \bar{F})\otimes\mathrm{Cl}(V, \bar{F})$ (see Sect. 4.2): it maps $y_1\otimes y_2$ to itself or to $-y_1\otimes y_2$ according as y_1 and y_2 are both even or both odd in $\mathrm{Cl}(V, \bar{F})$. It suffices to prove that the interior multiplications by $u\otimes 1$ and $1\otimes u$ are $\ddot{\tau}$-twisted derivations of $\mathrm{Cl}(V, \bar{F})\otimes\mathrm{Cl}(V, \bar{F})$ in the same way as the interior multiplication by u is a τ-twisted derivation of $\mathcal{B}(V, F)$. It is clear that

$$(u\otimes 1)\rfloor \big((y_1\otimes y_2)\,(y_1'\otimes y_2')\big) = \big((u\otimes 1)\rfloor\,(y_1\otimes y_2)\big)\,(y_1'\otimes y_2') \\ \pm (y_1\otimes y_2)\,\big((u\otimes 1)\rfloor\,(y_1'\otimes y_2')\big)$$

where $\pm$ means $+$ or $-$ according as y_1 is even or odd. If y_1 and y_2 have the same parity in $\mathrm{Cl}(V, \bar{F})$, we obtain the same $\pm$ when $u\otimes 1$ is replaced with $1\otimes u$. A symmetric argument works for $w\otimes 1$ and $1\otimes w$. □

5.3. Other Properties

From the information presented in [2], I recall only what is indispensable in the present study.

Lemma 5.5. *Let ψ^o be the mapping $W\times V \to K$ defined by $\psi^o(w, v) = \psi(v, w)$. For all $w \in W$, $y_0 \in \mathcal{B}_0(V, F)$ and $y_1 \in \mathcal{B}_1(V, F)$, we have*

$$y_0\ {}_{\psi}\lfloor w = -w\rfloor_{\psi^o}\,\tau(y_0), \qquad y_1\ {}_{\psi}\lfloor w = \tau\big(w\rfloor_{\psi^o}\, y_1\big).$$

Proof. When $V = W$ and $\psi = \psi^o = F$, this lemma is an easy consequence of Lemma 5.3. For the general case, we embed (V, F) into the space (V'', F'') where $V'' = V\times W \cong W\times V$, and F'' is defined in an obvious way. □

If V^* is the dual space of V, every $\ell \in V^*$ operates on $\mathcal{B}(V, F)$ on both sides: $\ell\rfloor v = v\lfloor \ell = \ell(v)$.

Lemma 5.6. *The following three assertions (a), (b) and (c) are equivalent for all $y_0 \in \mathcal{B}_0(V, F)$ and all $\ell \in V^*$:*

(a) $\mathrm{Sup}(y_0) \subset \ker(\ell)$. (b) $\ell\rfloor\, y_0 = y_0\lfloor \ell = 0$. (c) $y_0\lfloor \ell = \tau(y_0)\lfloor \ell = 0$.

Proof. The equivalence of (b) and (c) follows from Lemma 5.5, and we may assume $\ell \neq 0$. Let $(e_1, \ldots, e_n)$ be a basis of V such that $\ell(e_1) = 1$ and $\ell(e_i) = 0$ when $i \geqslant 2$. Let us apply Theorem 4.1 with the two bases $(e_1, e_2, \ldots, e_n)$ and $(e_2, \ldots, e_n, e_1)$: in the (unital) subalgebra generated by $\ker(\ell)$, there are elements y, y', y'' and y''' such that

$$y_0 = e_1 y e_1 + e_1 y' + y'' e_1 + y''',$$
$$\text{whence}\quad \ell\rfloor\, y_0 = y e_1 + y' \quad\text{and}\quad y_0\lfloor \ell = e_1 y + y''.$$

Both assertions (a) and (b) are equivalent to $y = y' = y'' = 0$. □

Exponentials of elements of $\mathcal{B}^2(V,0)$ have been presented in Sect. 2.4. For every $\eta \in \mathcal{B}^2(V,0)$, we have $\tau(\exp(\eta) = \exp(\tau(\eta))$. Because of Lemma 4.4, $\exp(\eta)\, v = v\, \tau(\exp(\eta))$ for every $v \in V$. Besides, $u\rfloor\eta$ and $\eta\lfloor w$ are in V for all $u \in U$ and $w \in W$.

Lemma 5.7. *For all $u \in U$, $w \in W$ and $\eta \in \mathcal{B}^2(V,0)$, we have:*

$$u\rfloor \exp(\eta) = (u\rfloor\eta)\ \exp(\eta) \quad \textit{and} \quad \exp(\eta)\lfloor w = \exp(\eta)\,(\eta\lfloor w).$$

Proof. These equalities are obvious if $\eta = vv'$. If they are true for η and θ, they are also true for $\eta+\theta$. Indeed, because of (2.7),

$$\begin{aligned} u\rfloor\ \exp(\eta{+}\theta) &= \big(u\rfloor \exp(\eta)\big)\ \exp(\theta) + \tau\big(\exp(\eta)\big)\ \big(u\rfloor \exp(\theta)\big) \\ &= (u\rfloor\eta)\ \exp(\eta)\ \exp(\theta) + \tau(\exp(\eta))\,(u\rfloor\theta)\ \exp(\theta) \\ &= (u\rfloor\eta)\ \exp(\eta)\ \exp(\theta) + (u\rfloor\theta)\ \exp(\eta)\ \exp(\theta) \\ &= \big(u\rfloor(\eta{+}\theta)\big)\ \exp(\eta{+}\theta), \end{aligned}$$

and the same with $\exp(\eta{+}\theta)\lfloor w$. □

6. A Theorem Involving a Totally Isotropic Subspace

6.1. Statement of the Theorem

Let U be a subspace of V. The (unital) subalgebra generated by U in $\mathcal{B}(V,F)$ will be denoted by $\mathcal{B}(U,F)$; it is isomorphic to the meson algebra of the restriction of F to U. Let us suppose that V contains a totally isotropic subspace T complementary to U: $V = U \oplus T$ and $F(u,v) = 0$ for all u, $v \in T$. As it happens with Clifford algebras, T determines a left ideal complementary to $\mathcal{B}(U,F)$ in $\mathcal{B}(V,F)$; it turns $\mathcal{B}(U,F)$ into a left module over $\mathcal{B}(V,F)$ because there is a linear bijection from $\mathcal{B}(U,F)$ onto the quotient of $\mathcal{B}(V,F)$ by this left ideal. It also determines a right ideal complementary to $\mathcal{B}(U,F)$ which turns $\mathcal{B}(U,F)$ into a right module. These assertions shall now be explained and proved in detail.

Let us set $m = \dim(U)$, and let $(e_1,\dots,e_n)$ be a basis of V such that $(e_1,\dots,e_m)$ is a basis of U, and $(e_{m+1},\dots,e_n)$ a basis of T. Since T is totally isotropic, the subalgebra $\mathcal{B}(T,F)$ (isomorphic to the neutral algebra $\mathcal{B}(T,0)$) is provided with a $\mathbb{Z}$-gradation: the component $\mathcal{B}^k(T,F)$ of degree k is not reduced to 0 for $k = 0,1,\dots,2(n{-}m)$, and we are especially interested in the subspace $\mathcal{B}^{2(n-m)}(T,F)$ of dimension 1. Let ω be an element that spans this subspace, for instance

$$\omega = e_{m+1}^2\, e_{m+2}^2\ \cdots\ e_n^2. \tag{6.1}$$

Since $\mathcal{B}_0(T,0)$ is a commutative algebra, the squares written in the right side of (6.1) are pairwise commuting. Consequently, $t\omega = \omega t = 0$ for every $t \in T$; indeed, it suffices to prove this when $t = e_i$ with $m < i \leqslant n$, and to recall that $e_i^3 = 0$ because of (2.1). Moreover, $\tau(x_1 t)\,\omega = \omega\,\tau(tx_1) = 0$ for all $t \in T$ and all $x_1 \in \mathcal{B}_1(V,F)$; it suffices to prove this when $t = e_i$ with $m < i \leqslant n$, and to recall Lemma 4.3.

Theorem 6.1. *Let us assume that $V = U \oplus T$ with a totally isotropic subspace T (as above). The left ideal of all $x \in \mathcal{B}(V,F)$ such that $x\omega = 0$ is the left ideal generated by all $t \in T$ and all $\tau(vt)$ with $t \in T$ and $v \in V$. It is complementary to the subalgebra $\mathcal{B}(U,F)$. Similarly, the right ideal of all $x \in \mathcal{B}(V,F)$ such that $\omega x = 0$ is the right ideal generated by all $t \in T$ and all $\tau(tv)$ with $t \in T$ and $v \in V$. It is also complementary to $\mathcal{B}(U,F)$.*

Proof. Let $\mathcal{J}$ be the left ideal of all x such that $x\omega = 0$, and $\mathcal{J}'$ the left ideal generated by all $t \in T$ and all $\tau(vt)$. If we manage to prove that

$$\mathcal{J}' \subset \mathcal{J}, \qquad \mathcal{J} \cap \mathcal{B}(U,F) = 0 \quad \text{and} \quad \mathcal{B}(V,F) = \mathcal{J}' + \mathcal{B}(U,F),$$

it follows immediately that $\mathcal{J} = \mathcal{J}'$ and that $\mathcal{B}(V,F)$ is the direct sum of $\mathcal{J}$ and $\mathcal{B}(U,F)$. Firstly, it has been explained just above that $\mathcal{J}$ contains all $t \in T$ and all $\tau(vt)$. Secondly, $x\omega \neq 0$ for every non-zero $x \in \mathcal{B}(U,F)$ because Theorem 4.1 proves that the multiplication $x \longmapsto x\omega$ is injective from $\mathcal{B}(U,F)$ into $\mathcal{B}(V,F)$. Thirdly, let us prove by induction on k that every element of $\mathcal{B}^{\leqslant k}(V,F)$ is in $\mathcal{J}' + \mathcal{B}(U,F)$. This is true for $\mathcal{B}^{\leqslant 1}(V,F) = K \oplus V$. Because of Theorem 4.1, every element of $\mathcal{B}^{\leqslant k}(V,F)$ is a sum of terms of the following three kinds: either an element of $\mathcal{B}^{\leqslant k}(U,F)$, or a product yt with $y \in \mathcal{B}^{\leqslant k-1}(V,F)$ and $t \in T$, or a product ytu with $y \in \mathcal{B}^{\leqslant k-2}(V,F)$, $t \in T$ and $u \in U$. Only ytu raises a problem, but it is an easy problem: ytu is the sum of $-y\,\tau(ut)$ and $F(t,u)\,y$ which falls into $\mathcal{J}' + \mathcal{B}(U,F)$ because the induction hypothesis may be used for $y \in \mathcal{B}^{\leqslant k-2}(V,F)$. Now Theorem 6.1 has been proved for the left ideal. For the right ideal, we may use either the basis $(e_{m+1}, \dots, e_n, e_1, \dots, e_m)$, or the reversion ρ in $\mathcal{B}(V,F)$. □

Theorem 6.1 lets the algebra $\mathcal{B}(V,F)$ act on the space $\mathcal{B}(U,F)$ on the left side and on the right side: with every $y \in \mathcal{B}(V,F)$ is associated and operator L_y such that $(yx - L_y(x))\,\omega = 0$ for all $x \in \mathcal{B}(U,F)$, and an operator R_y such that $\omega\,(xy - R_y(x)) = 0$ for all $x \in \mathcal{B}(U,F)$.

Corollary 6.2. *When $\mathcal{B}(V,F)$ acts on $\mathcal{B}(U,F)$ on the left (resp. right) side, the elements of U operate by ordinary multiplication on the left (resp. right) side, and the elements of T operate by interior multiplications on the left (resp. right) side. These interior multiplications are determined by the bilinear mappings $T \times U \to K$ and $U \times T \to K$ induced by F.*

Proof. When y is in $\mathcal{B}(U,F)$ it is clear that $L_y(x) = yx$ and $R_y(x) = xy$ for all $x \in \mathcal{B}(U,F)$. When y is a vector t of T, Lemma 5.3 shows that

$$tx_0 - t\rfloor x_0 = \tau(x_0)\,t, \qquad x_0 t - x_0\lfloor t = t\,\tau(x_0),$$
$$tx_1 - t\rfloor x_1 = -\tau(x_1 t), \qquad x_1 t - x_1\lfloor t = -\tau(tx_1)$$

for all $x_0 \in \mathcal{B}_0(U,F)$ and all $x_1 \in \mathcal{B}_1(U,F)$. Therefore, $L_t(x) = t\rfloor x$ and $R_t(x) = x\lfloor t$ for all $x \in \mathcal{B}(U,F)$. □

Yet $\mathcal{B}(U,F)$ is not a bimodule over $\mathcal{B}(V,F)$ because the operations on the left side do not always commute with the operations on the right side. Indeed, Lemma 5.2 shows that $t\rfloor(x_0 u) = (t\rfloor x_0)u + F(t,u)\tau(x_0)$ for all $t \in T$, $u \in U$ and $x_0 \in \mathcal{B}_0(U,F)$.

Besides, the left or right ideal mentioned in Theorem 6.1 is a graded subspace of $\mathcal{B}(V,F)$, and τ leaves invariant its even component. Consequently, for all $x \in \mathcal{B}_0(U,F)$ and all $y \in \mathcal{B}_0(V,F)$, we have

$$L_{\tau(y)}(\tau(x)) = \tau(L_y(x)) \quad \text{and} \quad R_{\tau(y)}(\tau(x)) = \tau(R_y(x)). \tag{6.2}$$

6.2. Consequences for Interior Multiplications

As in Sect. 5, we consider two bilinear mappings $\varphi : U\times V \to K$ and $\psi : V\times W \to K$. A decisive step was achieved when (V,F) was embedded in a larger space (V'',F'') with $V'' = U\times V\times W$; but this space was only necessary to prove the associativity property. At all other places, it was possible to reach the wanted result with $V'' = U\times V$ or with $V'' = V\times W$. When $V'' = U\times V$ and

$$F''((u,v),(u',v')) = \varphi(u,v') + \varphi(u',v) + F(v,v'),$$

V'' is the direct sum of $0\times V$ and the totally isotropic subspace $U\times 0$, and because of Theorem 6.1, $\mathcal{B}(V'',F'')$ acts on $\mathcal{B}(V,F)$ on the left side. When $V'' = V\times W$ and

$$F''((v,w),(v',w')) = F(v,v') + \psi(v,w') + \psi(v',w),$$

V'' is the direct sum of $V\times 0$ and the totally isotropic subspace $0\times W$, and $\mathcal{B}(V'',F'')$ acts on $\mathcal{B}(V,F)$ on the right side. In both cases, the action of $\mathcal{B}(V'',F'')$ is described by Corollary 6.2, and thus we realize that the following equalities hold for all u, $u' \in U$, all $v \in V$, all w, $w' \in W$ and all $y \in \mathcal{B}(V,F)$:

$$u\rfloor(u'\rfloor(u\rfloor y)) = 0, \qquad ((y\lfloor w)\lfloor w')\lfloor w = 0, \tag{6.3}$$

$$u\rfloor(v\,(u\rfloor y)) = \varphi(u,v)\,u\rfloor y, \qquad ((y\lfloor w)\,v)\lfloor w = \psi(v,w)\,y\lfloor w, \tag{6.4}$$

$$v\,(u\rfloor(vy)) = \varphi(u,v)\,vy, \qquad ((yv)\lfloor w)\,v = \psi(v,w)\,yv. \tag{6.5}$$

Because of (6.3), the neutral algebras $\mathcal{B}(U,0)$ and $\mathcal{B}(W,0)$ act on $\mathcal{B}(V,F)$ and turn it into a bimodule:

$$x\rfloor(x'\rfloor y) = (xx')\rfloor y, \quad (x\rfloor y)\lfloor z = x\rfloor(y\lfloor z), \quad (y\lfloor z)\lfloor z' = y\lfloor(zz') \tag{6.6}$$

for all x, $x' \in \mathcal{B}(U,0)$, all $y \in \mathcal{B}(V,F)$ and all z, $z' \in \mathcal{B}(W,0)$. The second equality in (6.6) follows from the associativity property in Theorem 5.1 without any new intervention of $U\times V\times W$.

The bilinear mapping $\varphi : U\times V \to K$ allows every vector v of V to act on the neutral algebra $\mathcal{B}(U,0)$ on the right side, and the mapping $\psi : V\times W \to K$ allows v to act on $\mathcal{B}(W,0)$ on the left side. These interior multiplications by v appear in the next lemma.

Lemma 6.3. *For all $x_0 \in \mathcal{B}_0(U,0)$, $y \in \mathcal{B}(V,F)$, $z_0 \in \mathcal{B}_0(W,0)$ and $v \in V$,*

$$x_0\rfloor(vy) - v\,(\tau(x_0)\rfloor\, y) = (x_0\lfloor v)\rfloor\, y \quad \textit{and} \quad (yv)\lfloor z_0 - (y\,\lfloor\tau(z_0))\,v = y\,\lfloor(v\rfloor z_0).$$

Proof. For the first equality, we use $V'' = U\times V$ and the homomorphism $J : \mathcal{B}(U,F) \to \mathcal{B}(V'',F'')$ that extends $u \longmapsto (u,0)$. The left side of the equality is the operation of $J(x_0)\,(0,v) - (0,v)\,J(\tau(x_0))$ on y. Because of Lemma 5.3, this element is equal to $J(x_0)\,{}_{F''}\lfloor(0,v)$ which is the same thing as $J(x_0\,{}_{\varphi}\lfloor v)$. A symmetric argument proves the second equality. □

All results presented in Sect. 6.2 appeared already in [2], but since [2] ignored Theorem 6.1, all proofs were based on Theorem 5.1. Thus many pages of calculations were necessary, with many inductions on filtering degrees. Theorem 6.1 achieves an amazing simplification of the proofs. Theorem 6.1 is yet more wonderful than the Cliffordian theorem that it imitates.

6.3. An Important Particular Case

Section 7 shall use the space $V^{\oplus} = V^* \oplus V$. The notation $\ell + v$ (where $\ell \in V^*$ and $v \in V$) will be used for the elements of $V^{\oplus}$, so that V and V^* may be identified with subspaces of $V^{\oplus}$. Let $F^{\oplus}$ be the symmetric bilinear form on $V^{\oplus}$ defined in this way (for all ℓ, $\ell' \in V^*$ and all v, $v' \in V$):

$$F^{\oplus}(\ell + v,\, \ell' + v') = \ell(v') + \ell'(v) + F(v, v'). \tag{6.7}$$

The algebra $\mathcal{B}(V^{\oplus}, F^{\oplus})$ acts on $\mathcal{B}(V, F)$ on the left side and on the right side. Since $F^{\oplus}$ is non-degenerate, $\mathcal{B}(V^{\oplus}, F^{\oplus})$ is a semi-simple algebra; it is the direct sum of $n+1$ ideals of dimensions $\binom{2n+1}{k}^2$ with $0 \leqslant k \leqslant n$ (see Theorem 3.3). Since the dimension of the largest ideal is the square of $\dim(\mathcal{B}(V, F))$, the next theorem is not a surprise.

Theorem 6.4. *When $\mathcal{B}(V^{\oplus}, F^{\oplus})$ acts on $\mathcal{B}(V, F)$ on the left or right side, only the largest ideal has a non-trivial action, and it is mapped bijectively onto* $\mathrm{End}(\mathcal{B}(V, F))$.

Proof. Let us prove Theorem 6.4 for the action on the left side. Following Sect. 3.3, we use too bases $(f_1, \ldots, f_{2n})$ and $(f'_1, \ldots, f'_{2n})$ of $V^{\oplus}$ such that $F^{\oplus}(f_i, f'_i) = 1$ for all i, and $F^{\oplus}(f_i, f'_j) = 0$ if $i \neq j$. If $(e_1, \ldots, e_n)$ is a basis of V, and $(e^*_1, \ldots, e^*_n)$ the dual basis of V^*, we can choose

$$f_i = e^*_i, \qquad f_{n+i} = e_i \qquad \text{for } i = 1, 2, \ldots, n,$$
$$f'_i = -\ell_i + e_i, \qquad f'_{n+i} = e^*_i,$$

where each ℓ_i is defined by $\ell_i(v) = F(e_i, v)$. The operation of the idempotents $f_i f'_i$ and $f_{n+i} f'_{n+i}$ in $\mathcal{B}(V, F)$ map 1 respectively to 1 and to 0. Let us set

$$\varepsilon = \prod_{1 \leqslant i \leqslant n} f_i f'_i \; (1 - f_{n+i} f'_{n+i}).$$

This idempotent ε belongs to the largest ideal of $\mathcal{B}(V^{\oplus}, F^{\oplus})$, and its operation maps 1 to itself. Therefore, there is a submodule of $\mathcal{B}(V, F)$ on which this largest ideal does not act trivially, whereas all other ideals act trivially, and its dimension is at least $\binom{2n+1}{n}$. This submodule must be equal to $\mathcal{B}(V, F)$. □

7. Deformations

Let β be a bilinear form $V \times V \to K$, and β^o the bilinear form defined by $\beta^o(u, v) = \beta(v, u)$. Besides the given symmetric bilinear form F on V, we will also use the symmetric bilinear form $F' = F + \beta + \beta^o$. This relation between F and F' is equivalent to (4.3). Moreover, β_u and β^o_u are the linear forms such that $\beta_u(v) = \beta(u, v)$ and $\beta^o_u(v) = \beta(v, u)$ for all $v \in V$. The spaces

$(V^\oplus, F^\oplus)$ and $(V^\oplus, F'^\oplus)$ have been defined in Sect. 6.3. For all ℓ, $\ell' \in V^*$ and all v, $v‘ \in V$, we have:

$$F^\oplus(\ell + \beta_v + v,\ \ell' + \beta_{v'} + v') = F'^\oplus(\ell + v,\ \ell' + v'),$$
$$F^\oplus(\ell + \beta^o_v + v,\ \ell' + \beta^o_{v'} + v') = F'^\oplus(\ell + v,\ \ell' + v').$$

Therefore, β determines two isomorphisms Φ and Ψ from $\mathcal{B}(V^\oplus, F'^\oplus)$ onto $\mathcal{B}(V^\oplus, F^\oplus)$; the former maps every $\ell + v$ to $\ell + \beta_v + v$, and the latter maps every $\ell + v$ to $\ell + \beta^o_v + v$.

The neutral algebra $\mathcal{B}(V^*, 0)$ may be identified with a subalgebra of $\mathcal{B}(V^\oplus, F^\oplus)$ and with a subalgebra of $\mathcal{B}(V^\oplus, F'^\oplus)$. Let ω be a non-zero element of $\mathcal{B}^{2n}(V^*, 0)$; it belongs to $\mathcal{B}(V^\oplus, F^\oplus)$ and to $\mathcal{B}(V^\oplus, F'^\oplus)$, and $\Phi(\omega) = \Psi(\omega) = \omega$. Let $\mathcal{J}_l$ (resp. $\mathcal{J}'_l$) be the left ideal of all elements z of $\mathcal{B}(V^\oplus, F^\oplus)$ (resp. $\mathcal{B}(V^\oplus, F'^\oplus)$) such that $z\omega = 0$, and let $\mathcal{J}_r$ (resp. $\mathcal{J}'_r$) be the right ideal of all elements z of $\mathcal{B}(V^\oplus, F^\oplus)$ (resp. $\mathcal{B}(V^\oplus, F'^\oplus)$) such that $\omega z = 0$. It is clear that $\Phi(\mathcal{J}'_l) = \mathcal{J}_l$ and $\Psi(\mathcal{J}'_r) = \mathcal{J}_r$. In the algebra $\mathcal{B}(V^\oplus, F^\oplus)$, both subalgebras $\mathcal{B}(V, F)$ and $\Phi(\mathcal{B}(V, F'))$ are complementary to $\mathcal{J}_l$; therefore, there is a natural linear bijection $\Phi(\mathcal{B}(V, F')) \to \mathcal{B}(V, F)$ (the parallel projection with respect to $\mathcal{J}_l$), and finally a linear bijection J_l from $\mathcal{B}(V, F')$ onto $\mathcal{B}(V, F)$. Similarly, both subalgebras $\mathcal{B}(V, F)$ and $\Psi(\mathcal{B}(V, F'))$ are complementary to $\mathcal{J}_r$ in $\mathcal{B}(V^\oplus, F^\oplus)$, and this results in a linear bijection $J_r : \mathcal{B}(V, F') \to \mathcal{B}(V, F)$. Fortunately, we shall soon realize that $J_l = J_r$ This bijection $J = J_l = J_r$ allows us to carry the multiplication of $\mathcal{B}(V, F')$ onto the space $\mathcal{B}(V, F)$:

$$\forall x,\, y \in \mathcal{B}(V, F), \quad x \star y = J(J^{-1}(x)\, J^{-1}(y)). \tag{7.1}$$

The deformation $\mathcal{B}(V, F; \beta)$ is the space $\mathcal{B}(V, F)$ provided with the $\star$-multiplication defined by (7.1). Let us now detail this construction.

7.1. Preliminary Lemmas

The algebra $\mathcal{B}(V^\oplus, F^\oplus)$ acts on the space $\mathcal{B}(V, F)$ on both sides. As it is explained in Sect. 6.1, every $y \in \mathcal{B}(V^\oplus, F^\oplus)$ determines an operation L_y on the left side and an operation R_y on the right side; for every $x \in \mathcal{B}(V, F)$, $L_y(x)$ (resp. $R_y(x)$) is the parallel projection of yx (resp. xy) onto $\mathcal{B}(V, F)$ with respect to $\mathcal{J}_l$ (resp. $\mathcal{J}_r$); and we have $L_{yz} = L_y \circ L_z$ and $R_{yz} = R_z \circ R_y$ for all y, $z \in \mathcal{B}(V^\oplus, F^\oplus)$. From the definition of L_y (resp. R_y), it follows that $L_y(1)$ (resp. $R_y(1)$) is the projection of y onto $\mathcal{B}(V, F)$ with respect to $\mathcal{J}_l$ (resp. $\mathcal{J}_r$). Consequently, for all $z \in \mathcal{B}(V, F')$, $L_{\Phi(z)}(1)$ and $R_{\Psi(z)}(1)$ are the elements of $\mathcal{B}(V, F)$ that have been called $J_l(z)$ and $J_r(z)$ just above. On another side, Corollary 6.2 shows that, for all $v \in V$ and all $x \in \mathcal{B}(V, F)$, we have

$$L_{\Phi(v)}(x) = vx + v \rfloor_\beta x \quad \text{and} \quad R_{\Psi(v)}(x) = xv + x_\beta\lfloor v. \tag{7.2}$$

Lemma 7.1. *For all y, $z \in \mathcal{B}(V, F')$, $L_{\Phi(y)}$ and $R_{\Psi(z)}$ commute.*

Proof. It suffices to prove that $L_{\Phi(v)}$ and $R_{\Psi(w)}$ commute for all v, $w \in V$; in other words, for all $x \in \mathcal{B}(V, F)$,

$$v(xw + x\lfloor w) + v\rfloor(xw + x\lfloor w) = (vx + v\rfloor x)w + (vx + v\rfloor x)\lfloor w$$

if $\rfloor$ and $\lfloor$ mean $\rfloor_\beta$ and ${}_\beta\lfloor$. When x is an even element x_0 (resp. an odd element x_1), Lemma 5.2 shows that both sides are equal to

$$\begin{gathered} vx_0w + v\rfloor x_0\lfloor w + v(x_0\lfloor w) + (v\rfloor x_0)w + \beta(v, w)\,\tau(x_0) \\ \big(\text{resp. } \; vx_1w + v\rfloor x_1\lfloor w + v(x_1\lfloor w) + (v\rfloor x_1)w\,\big). \end{gathered}$$

The conclusion follows. □

Lemma 7.2. *For every $z \in \mathcal{B}(V, F')$, $L_{\Phi(z)}(1) = R_{\Psi(z)}(1)$.*

Proof. This is true if $z = 1$, and if z is in V (recall (7.2)). Let us suppose that $L_{\Phi(y)}(1) = R_{\Psi(y)}(1)$ and $L_{\Phi(z)}(1) = R_{\Psi(z)}(1)$; then,

$$\begin{aligned} L_{\Phi(yz)}(1) &= L_{\Phi(y)} \circ L_{\Phi(z)}(1) = L_{\Phi(y)} \circ R_{\Psi(z)}(1) \\ &= R_{\Psi(z)} \circ L_{\Phi(y)}(1) = R_{\Psi(z)} \circ R_{\Psi(y)}(1) = R_{\Psi(yz)}(1). \end{aligned}$$

Since the algebra $\mathcal{B}(V, F')$ is generated by V, the conclusion follows. □

If we set $J(z) = L_{\Phi(z)}(1) = R_{\Psi(z)}(1)$ for every $z \in \mathcal{B}(V, F')$, then J is a linear bijection $\mathcal{B}(V, F') \to \mathcal{B}(V, F)$ as it has been explained above. We have $J(1) = 1$, and for all u, v, $w \in V$,

$$J(v) = v, \quad J(uv) = uv + \beta(u, v), \quad J(uvw) = uvw + \beta(u, v)w + \beta(v, w)u.$$

Now the definition (7.1) of $x \star y$ is meaningful.

Lemma 7.3. *If x and y are two elements of $\mathcal{B}(V, F)$, and if x' and y' are their images in $\mathcal{B}(V, F')$ by J^{-1}, then*

$$x \star y = L_{\Phi(x')}(y) = R_{\Psi(y')}(x).$$

Proof. Let us begin (for instance) with the second equality:

$$\begin{aligned} x \star y = J(x'y') &= R_{\Psi(x'y')}(1) = R_{\Psi(y')} \circ R_{\Psi(x')}(1) \\ &= R_{\Psi(y')}(J(x')) = R_{\Psi(y')}(x). \end{aligned}$$

A symmetric calculation gives the first equality. □

Because of Lemma 7.3, we have for all $v \in V$ and all $x \in \mathcal{B}(V, F)$,

$$1 \star x = x, \qquad v \star x = vx + v\,\rfloor_\beta\, x, \tag{7.3}$$

$$x \star 1 = x, \qquad x \star v = xv + x\,{}_\beta\lfloor\, v. \tag{7.4}$$

7.2. The Main Theorems

Theorem 7.4 is an immediate consequence of the preliminary lemmas.

Theorem 7.4. *There is a unique associative $\star$-multiplication on the space $\mathcal{B}(V,F)$ such that (7.3) holds for all $v \in V$ and all $x \in \mathcal{B}(V,F)$. It is also the unique associative $\star$-multiplication such that (7.4) holds for all v and x. With this $\star$-multiplication, the space $\mathcal{B}(V,F)$ becomes an associative and unital algebra $\mathcal{B}(V,F;\beta)$, and the identity mapping of V extends to an algebra isomorphism J from $\mathcal{B}(V,F')$ onto $\mathcal{B}(V,F;\beta)$.*

For all u, v, $w \in V$, we have

$$u \star v = uv + \beta(u,v), \qquad u \star v \star w = uvw + \beta(u,v)\,w + \beta(v,w)\,u.$$

This confirms that $u \star v \star u = F'(u,v)\,u$. On another side, the deformation $\mathcal{B}(V,F;\beta)$ has the same parity gradation as $\mathcal{B}(V,F)$. Thus it is sensible to ask what the automorphism τ_β of the even subalgebra $\mathcal{B}_0(V,F;\beta)$ may be. The following calculation allows us to guess the answer:

$$\begin{aligned}\tau_\beta(uv) &= \tau_\beta(u \star v - \beta(u,v)) = F'(u,v) - v \star u - \beta(u,v)\\ &= F'(u,v) - vu - \beta(v,u) - \beta(u,v) = F(u,v) - vu = \tau(uv).\end{aligned}$$

Theorem 7.5. *The automorphism τ_β of $\mathcal{B}_0(V,F;\beta)$ is the same as the automorphism τ of $\mathcal{B}_0(V,F)$.*

Proof. Let us denote by τ' and τ the automorphisms of the even subalgebras $\mathcal{B}_0(V^\oplus, F'^\oplus)$ and $\mathcal{B}_0(V^\oplus, F^\oplus)$. We must prove that $J(\tau'(x)) = \tau(J(x))$ for every $x \in \mathcal{B}_0(V,F')$. It is clear that $\Phi(\tau'(x)) = \tau(\Phi(x))$. Since $J(\tau'(x)) = L_{\Phi(\tau'(x))}(1)$, the conclusion follows from (6.2). $\square$

Theorem 7.5 was already involved in the proof of Theorem 4.5. Now we suppose that two mappings $\varphi : U \times V \to K$ and $\psi : V \times W \to K$ allow the neutral algebras $\mathcal{B}(U,0)$ and $\mathcal{B}(W,0)$ to act on the space $\mathcal{B}(V,F)$ by interior multiplications (see Sect. 6.2).

Theorem 7.6. *When $\mathcal{B}(U,0)$ and $\mathcal{B}(W,0)$ act on $\mathcal{B}(V,F)$ by interior multiplications, the interior products $x \rfloor y$ and $y \lfloor z$ (where $x \in \mathcal{B}(U,0)$, $y \in \mathcal{B}(V,F)$ and $z \in \mathcal{B}(W,0)$) are the same in the algebra $\mathcal{B}(V,F;\beta)$ as in the algebra $\mathcal{B}(V,F)$. In other words, the linear bijection $J : \mathcal{B}(V,F') \to \mathcal{B}(V,F)$ is an isomorphism of bimodules over the algebras $\mathcal{B}(U,0)$ and $\mathcal{B}(W,0)$.*

Proof. It suffices to prove that $u \rfloor y$ and $y \lfloor w$ (where $u \in U$ and $w \in W$) are the same in $\mathcal{B}(V,F;\beta)$ as in $\mathcal{B}(V,F)$. Yet $u \rfloor y = \varphi_u \rfloor y$ and $y \lfloor w = y \lfloor \psi_w^o$ if φ_u and ψ_w^o are the linear forms $v \longmapsto \varphi(u,v)$ and $v \longmapsto \psi(v,w)$. Therefore, it suffices to prove the following assertion:

$$\forall \ell \in V^*,\ \forall y \in \mathcal{B}(V,F'), \quad J(\ell \rfloor y) = \ell \rfloor J(y) \ \text{ and } \ J(y \lfloor \ell) = J(y) \lfloor \ell \tag{7.5}$$

if $\ell \rfloor y$ and $y \lfloor \ell$ are calculated in $\mathcal{B}(V,F')$ while $\ell \rfloor J(y)$ and $J(y) \lfloor \ell$ are calculated in $\mathcal{B}(V,F)$. In $\mathcal{B}(V^\oplus, F'^\oplus)$, we have $\ell \rfloor y \equiv \ell y$ modulo the left ideal $\mathcal{J}_l'$, and consequently, $\Phi(\ell \rfloor y) \equiv \Phi(\ell y)$ modulo $\mathcal{J}_l$ in $\mathcal{B}(V^\oplus, F^\oplus)$. Moreover, $\Phi(\ell y)$

$= \ell\,\Phi(y)$. We also have $\ell\rfloor J(y) \equiv \ell\, J(y)$ modulo $\mathcal{J}_l$, and $\Phi(z) \equiv J(z)$ for all $z \in \mathcal{B}(V, F')$. Consequently,

$$J(\ell\rfloor y) \equiv \Phi(\ell\rfloor y) \equiv \Phi(\ell y) \equiv \ell\,\Phi(y) \equiv \ell\, J(y) \equiv \ell\rfloor J(y) \qquad \text{modulo } \mathcal{J}_i.$$

Since $J(\ell\rfloor y)$ and $\ell\rfloor J(y)$ are two elements of $\mathcal{B}(V, F)$ such that $J(\ell\rfloor y) \equiv \ell\rfloor J(y)$, they are equal. A symmetric argument brings $J(y\lfloor\ell) = J(y)\lfloor\ell$. □

As it happens for Clifford algebras, a corollary immediately follows.

Corollary 7.7. *Let β and γ be two bilinear forms on V. The deformation of $\mathcal{B}(V, F; \beta)$ by means of γ is equal to the deformation $\mathcal{B}(V, F;\ \beta+\gamma)$.*

Therefore, $\mathcal{B}(V, F)$ is the deformation of $\mathcal{B}(V, F; \beta)$ by means of $-\beta$.

7.3. Isomorphic Deformations

There is a bijection from $V^*\otimes V^*$ onto the space of all bilinear forms on V which maps every $\ell\otimes\ell'$ to the bilinear form $(u, v) \longmapsto \ell'(u)\,\ell(v)$. There is also a bijection $\ell\otimes\ell' \longmapsto \ell\ell'$ from $V^*\otimes V^*$ onto $\mathcal{B}^2(V^*, 0)$ (see Sect. 2.4). These bijections lead to the next lemma.

Lemma 7.8. *There is a bijection from $\mathcal{B}^2(V^*, 0)$ onto the space of bilinear forms on V which maps every $\delta \in \mathcal{B}^2(V^*, 0)$ to the bilinear form α such that $\alpha(u, v) = \delta\rfloor uv$ for all $u, v \in V$. It maps $\tau(\delta)$ to the bilinear form $(u, v) \longmapsto -\alpha(v, u)$. If α_u and α_u^o are the linear forms $v \longmapsto \alpha(u, v)$ and $v \longmapsto \alpha(v, u)$, then $\alpha_u = \delta\lfloor u$ and $\alpha_u^o = u\rfloor\delta$.*

Proof. If $\delta = \ell\ell'$ (with $\ell, \ell' \in V^*$), then $\delta\rfloor(uv) = \ell\rfloor(\ell'\rfloor uv) = \ell'(u)\,\ell(v)$. Besides, $\tau(\delta) = -\ell'\ell$. These facts prove the first assertions in Lemma 7.8. If α is the bilinear form defined by $\alpha(u, v) = \ell'(u)\,\ell(v)$ (for all $u, v \in V$), then $\alpha_u = \ell'(u)\,\ell$ and $\alpha_u^o = \ell(u)\,\ell'$. Since $(\ell\ell')\lfloor u = \ell'(u)\,\ell$ and $u\rfloor(\ell\ell') = \ell(u)\,\ell'$, the last assertion is also proved. □

If β and γ are bilinear forms on V such that

$$\beta + \beta^o = \gamma + \gamma^o, \tag{7.6}$$

then $\mathcal{B}(V, F; \beta)$ and $\mathcal{B}(V, F; \gamma)$ are isomorphic to the same algebra $\mathcal{B}(V, F')$. Consequently, the identity mapping of V extends to an isomorphism from $\mathcal{B}(V, F; \beta)$ onto $\mathcal{B}(V, F; \gamma)$. The next theorem reveals it; it involves an exponential which is defined as it is explained in Sect. 2.4.

Theorem 7.9. *Let β and γ be two bilinear forms on V, and let δ be the element of $\mathcal{B}^2(V^*, 0)$ such that*

$$\forall u,\ v \in V, \quad \delta\rfloor uv = \gamma(u, v) - \beta(u, v).$$

If β and γ satisfy (7.6), then $\tau(\delta) = \delta$, and the mapping $x \longmapsto \exp(\delta)\rfloor x$ is the isomorphism $\mathcal{B}(V, F; \beta) \to \mathcal{B}(V, F; \gamma)$ that extends $\mathbf{1}_V$.

Proof. The relation (7.6) means that $\gamma - \beta$ is skew symmetric, and implies $\tau(\delta) = \delta$. Consequently, $\tau(\exp(\delta)) = \exp(\delta)$. The definition of δ also implies that $\delta\lfloor v = \gamma_v - \beta_v$ for all $v \in V$. Since $\exp(\delta) - 1$ belongs to $\mathcal{B}^{\geqslant 2}(V^*, 0)$, the interior multiplication by $\exp(\delta)$ leaves invariant every element of $\mathcal{B}^{\leqslant 1}(V, F)$.

Thus it suffices to prove that the mapping $x \longmapsto \exp(\delta)\rfloor x$ behaves correctly on a product $v \star x$ where the first factor v is in V:

$$\exp(\delta)\rfloor vx + \exp(\delta)\rfloor(v\rfloor_\gamma x) = v\,(\exp(\delta)\rfloor x) + v\rfloor_\beta(\exp(\delta)\rfloor x). \tag{7.7}$$

Since $\tau(\exp(\delta)) = \exp(\delta)$, Lemmas 6.3 and 5.7 allow us to write

$$\begin{aligned}\exp(\delta)\rfloor(vx) - v\,(\exp(\delta)\rfloor x) &= \big(\exp(\delta)\lfloor v)\rfloor x\\ &= \big(\exp(\delta)\,(\delta\lfloor v)\big)\rfloor x\\ &= \big(\exp(\delta)\,(\gamma_v - \beta_v)\big)\rfloor x.\end{aligned}$$

On another side, we may replace $v\rfloor_\gamma$ and $v\rfloor_\beta$ with $\gamma_v\rfloor$ and $\beta_v\rfloor$ in (7.7). Because of (6.6), we have

$$\begin{aligned}\exp(\delta)\rfloor(v\rfloor_\gamma x) &= \big(\exp(\delta)\,\gamma_v\big)\rfloor x,\\ v\rfloor_\beta(\exp(\delta)\rfloor x) &= \big(\beta_v\,\exp(\delta)\big)\rfloor x\\ &= \big(\exp(\delta)\,\beta_v\big)\rfloor x \quad \big(\text{see Lemma 4.4}\big).\end{aligned}$$

All this calculation proves (7.7). □

Unlike the automorphism τ of $\mathcal{B}_0(V,F)$, the reversion ρ is not invariant under deformation (see [2], Section 10). Let δ be the element of $\mathcal{B}^2(V^*,0)$ such that $\delta\rfloor uv = \beta(u,v) - \beta(v,u)$ for all u, $v \in V$ (whence $\tau(\delta)=\delta$). Since ρ is an anti-isomorphism from $\mathcal{B}(V,F;\beta)$ onto $\mathcal{B}(V,F;\beta^o)$, Theorem 7.9 allows us to calculate the reversion ρ_β in $\mathcal{B}(V,B;\beta)$:

$$\forall x \in \mathcal{B}(V,F), \qquad \rho_\beta(x) = \exp(\delta)\rfloor\rho(x) = \rho\big(\exp(-\delta)\rfloor x\big). \tag{7.8}$$

7.4. A Theorem Involving Jacobson's Homomorphism

The next theorem involves the deformation $\mathrm{Cl}(V,\bar F;2\beta)$ of the Clifford algebra $\mathrm{Cl}(V,\bar F)$ when $\mathrm{char}(K) \neq 2$. It is the space $\mathrm{Cl}(V,\bar F)$ provided with the associative $\star$-multiplication that admits the same unit element, and that satisfies the following properties for all $v \in V$ and all $z \in \mathrm{Cl}(V,\bar F)$:

$$v \star z = vz + 2\,v\rfloor_\beta z \quad \text{and} \quad z \star v = zv + 2\,z_\beta\lfloor v.$$

Instead of $v\rfloor_\beta z$ and $z_\beta\lfloor v$, we can write $\beta_v\rfloor z$ and $z\lfloor\beta^o_v$.

Theorem 7.10. *Jacobson's homomorphism* $\mathcal{B}(V,F) \to \mathrm{Cl}(V,\bar F)\otimes\mathrm{Cl}(V,\bar F)$ *is also an algebra homomorphism* $\mathcal{B}(V,F;\beta) \to \mathrm{Cl}(V,\bar F;2\beta)\otimes\mathrm{Cl}(V,\bar F;2\beta)$.

Proof. It suffices to verify that $\Delta(v\star x) = \Delta(v)\star\Delta(x)$ for all $v \in V$ and all $x \in \mathcal{B}(V,F)$. Because of Lemma 5.4, this means that

$$\frac{1}{2}(v\otimes 1 + 1\otimes v)\,\Delta(x) + (\beta_v\otimes 1 + 1\otimes\beta_v)\rfloor\Delta(x) = \frac{1}{2}(v\otimes 1 + 1\otimes v)\star\Delta(x).$$

It suffices to prove that, for all $v \in V$ and all y, $z \in \mathrm{Cl}(V,\bar F)$,

$$(v\otimes 1 + 1\otimes v)\,(y\otimes z) + 2\,(\beta_v\otimes 1 + 1\otimes\beta_v)\rfloor(y\otimes z) = (v\otimes 1 + 1\otimes v)\star(y\otimes z).$$

This last equality is true because $vy + 2\beta_v\rfloor y = v\star y$, and the same with z at the place of y. □

Sometimes, it is useful to use a deformation $\mathcal{B}(V,F;-\beta)$ that is isomorphic to a neutral algebra; this happens if $F=\beta+\beta^o$. When $\mathrm{char}(K)\neq 2$, there is a canonical choice (the only symmetric choice): $\beta=F/2$. Because of (7.8), this choice leaves the reversion ρ invariant. Jacobson's homomorphism is also a homomorphism from $\mathcal{B}(V,F;-F/2)$ into $\mathrm{Cl}(V,\bar{F};-F)\otimes \mathrm{Cl}(V,\bar{F};-F)$, which is isomorphic to $\bigwedge(V)\otimes\bigwedge(V)$.

8. Generalities about Lipschitz Monoids

Section 8 introduces the second part of this work which is devoted to Lipschitz monoids. Obviously, many methods employed in Sects. 4, 5, 6 and 7 were imitations of methods that give good results with Clifford algebras. The present study of meson algebras made a selection among the many methods that have been imagined for Clifford algebras, according as they could be adapted to meson algebras. Nevertheless, the imitations were never faithful copies, and the particular features of meson algebras left their mark on the imitations. Lipschitz monoids are an efficient tool in the study of Clifford algebras, and the next purpose is the research for an imitation in the frame of meson algebras.

The wanted Lipschitz monoid (or semi-group) $\mathrm{Lip}(V,F)$ must be a multiplicative monoid in the algebra $\mathcal{B}(V,F)$. Its elements will be called the Lipschitzian elements, and the group of its invertible elements will be denoted by $\mathrm{GLip}(V,F)$. The automorphisms of (V,F) (which I would call "autometries" if this word were accepted by everybody) are the linear transformations g of V such that $F(g(u),g(v))=F(u,v)$ for all $u,\,v\in V$. Every automorphism g of (V,F) extends to an automorphism $\mathcal{B}(g)$ of the algebra $\mathcal{B}(V,F)$. The automorphisms of (V,F) constitute a group $\mathrm{Aut}(V,F)$, and the wanted theory must bring a group homomorphism $\mathrm{GLip}(V,F)\to\mathrm{Aut}(V,F)$.

8.1. Twisted Inner Automorphisms

The wanted homomorphism $\mathrm{GLip}(V,f)\to\mathrm{Aut}(V,F)$ requires the preliminary definition of the twisted inner automorphisms of $\mathcal{B}(V,F)$. The odd component $\mathcal{B}_1(V,F)$ contains no invertible elements because it is contained in the ideal $\mathcal{B}^+(V,F)$ (the kernel of the algebra homomorphism $\mathrm{Scal}:\mathcal{B}(V,F)\to K$ defined in Sect. 2). Therefore, the twisted inner automorphisms shall be determined by elements of $\mathcal{B}_0(V,F)$ that are eigenvectors of τ, and the monoid $\mathrm{Lip}(V,F)$ shall contain only eigenvectors of τ. The following definition, which involves the grade automorphism σ (defined in Sect. 2), will prove to be the good one: when a is an invertible element of $\mathcal{B}_0(V,f)$ such that $\tau(a)=\pm a$, the twisted inner automorphism determined by a is $x\longmapsto axa^{-1}$ if $\tau(a)=a$, $x\longmapsto a\,\sigma(x)\,a^{-1}$ if $\tau(a)=-a$. Equivalently, every even x is mapped to axa^{-1}, and every odd x is mapped to $a\,x\,\tau(a)^{-1}$.

Theorem 8.1. *Let a be an invertible element of $\mathcal{B}_0(V,F)$ such that $\tau(a)=\pm a$, and such that $ava^{-1}\in V$ for all $v\in V$. If we set $G_a(v)=a\,v\,\tau(a)^{-1}$, then G_a is an automorphism of (V,F), and the kernel of $G_a-\mathbf{1}$ is $\mathrm{Sup}(a)^\perp$ (the subspace orthogonal to the support of a with respect to F). Moreover, the* *image of* $G_a - \mathbf{1}$ *is contained in* $\mathrm{Sup}(a)$*, and the equality* $\mathrm{im}(G_a - \mathbf{1}) = \mathrm{Sup}(a)$ *holds when* F *is non-degenerate.*

Proof. Since $\tau(a) = \pm a$, we have, for all $u, v \in V$,

$$(a\,u\,\tau(a)^{-1})\,(a\,v\,\tau(a)^{-1})\,(a\,u\,\tau(a)^{-1}) = a\,(uvu)\,\tau(a)^{-1}.$$

Consequently, $F(G_a(u), G_a(v))\,G_a(u) = F(u,v)\,G_a(u)$. Thus we have proved that $F(G_a(u), G_a(v)) = F(u,v)$ if $u \neq 0$, and this is sufficient to claim that $G_a \in \mathrm{Aut}(V,F)$. Now let us realize that, for all $v \in V$,

$$G_a(v) - v = (a_F\lfloor v)\;\tau(a)^{-1}. \tag{8.1}$$

Indeed, Lemma 5.3 allows us to write

$$av\,\tau(a)^{-1} - v = (av - v\,\tau(a))\,\tau(a)^{-1} = (a_F\lfloor v)\,\tau(a)^{-1}.$$

Thus the equality $G_a(v) = v$ is equivalent to $a_F\lfloor v = 0$. Since $\tau(a) = \pm a$, Lemma 5.6 proves that $G_a(v) = v$ if and only if $\mathrm{Sup}(a)$ is in the kernel of the linear form $u \longmapsto F(u,v)$. In other words, $\ker(G_a - \mathbf{1}) = \mathrm{Sup}(a)^{\perp}$. From (8.1) we can also deduce that $G_a(v) - v$ belongs to $\mathrm{Sup}(a)$ because $a_F\lfloor v$ and $\tau(a)^{-1}$ belong to the subalgebra $\mathcal{B}(\mathrm{Sup}(a), F)$; since this subalgebra has a finite dimension, it contains the inverses of all its invertible elements. Thus $\mathrm{im}(G_a - \mathbf{1}) \subset \mathrm{Sup}(a)$. When F is non-degenerate, we have

$$n = \dim(\mathrm{Sup}(a)) + \dim(\mathrm{Sup}(a)^{\perp}) = \dim(\mathrm{im}(G_a - \mathbf{1})) + \dim(\ker(G_a - \mathbf{1}));$$

consequently, $\mathrm{im}(G_a - \mathbf{1}) = \mathrm{Sup}(a)$. □

The above definition of the twisted inner automorphisms of $\mathcal{B}(V,F)$ shall be accepted as relevant only if every automorphism of (V,F) extends to a twisted inner automorphism of $\mathcal{B}(V,F)$ when F is non-degenerate. When F is degenerate, and when g is an automorphism of (V,F), Theorem 8.1 shows that $\mathcal{B}(g)$ cannot be a twisted inner automorphism of $\mathcal{B}(V,F)$ if $\ker(g - \mathbf{1})$ does not contain $V^{\perp} = \ker(F)$.

Every invertible Lipschitzian element a must determine a twisted inner automorphism of $\mathcal{B}(V,F)$ that leaves V invariant and induces an automorphism G_a of (V,F). It is sensible to hope that this requirement will be satisfied because every Lipschitzian element satisfies the following property (where ρ is the reversion defined in Sect. 1):

$$\begin{aligned} &\forall a \in \mathrm{Lip}(V,F), \qquad \rho(a) \in \mathrm{Lip}(V,F), \quad a\,\rho(a) = \rho(a)\,a \in K, \\ &\quad \text{and} \ \ \forall v \in V, \quad av\,\rho(a) \in V. \end{aligned} \tag{8.2}$$

The homomorphism $\mathrm{GLip}(V,F) \to \mathrm{Aut}(V,F)$ will follow from (8.2). It shall be accepted as relevant only if the following property is true: every automorphism g of (V,F) that extends to a twisted inner automorphism of $\mathcal{B}(V,F)$ is equal to G_a for some $a \in \mathrm{GLip}(V,F)$. Even if $g = G_b$ for some b that is not Lipschitzian, we may forget this b and replace it by a Lipschitzian a, because a satisfies many advantageous properties (for instance (8.2)) which are not always valid for b. Such inopportune elements b may exist when F is degenerate. For instance, when $F = 0$, we have $\mathbf{1}_V = G_b$ for every $b \in \mathcal{B}_0(V,0)$ such that $\tau(b) = b$ and $\mathrm{Scal}(b) \neq 0$.

Let us notice that ρ and τ commute on $\mathcal{B}_0(V,F)$; indeed, the mapping $\rho\circ\tau\circ\rho$ (defined on $\mathcal{B}_0(V,F)$) is equal to τ because it is an automorphism of $\mathcal{B}_0(V,F)$ that maps every uv to $F(u,v)-vu$.

8.2. Orthogonal Transformations

An automorphism g of (V,F) is called an orthogonal transformation if $\mathcal{B}(g)$ is a twisted inner automorphism of $\mathcal{B}(V,F)$, and the group of all orthogonal transformations is the orthogonal group $\mathrm{O}(V,F)$. Just above in Sect. 8.1, the presented concepts were accepted as relevant only if they complied with two requirements, and here, the two required properties can be stated in this way: $\mathrm{O}(V,F)$ is the image of the homomorphism $\mathrm{GLip}(V,F)\to\mathrm{Aut}(V,F)$, and $\mathrm{Aut}(V,F)=\mathrm{O}(V,F)$ whenever F is non-degenerate.

Some authors prefer to give the name "orthogonal transformation" to all automorphisms of (V,F), for which, anyway, I prefer the name "autometry". The purpose of new definitions is to state new theorems, but there is little to say about the "autometries" of (V,F). Almost all theorems about "autometries" are particular cases of more general theorems about "homometries". We need a special terminology only for the "autometries" g such that $\mathcal{B}(g)$ is a twisted inner automorphism.

The group $\mathrm{O}(V,F)$ shall be described in Sects. 9, 10 and 11. When $\mathrm{char}(K)\neq 2$, it is the group of all $g\in\mathrm{Aut}(V,F)$ such that $\ker(g-\mathbf{1})\supset V^\perp$. The same is true when $\mathrm{char}(K)=2$ and F is alternating. But when $\mathrm{char}(K)=2$ and F is not alternating, the subspace V_0 of all $v\in V$ such that $F(v,v)=0$ (see Sect. 1.1) shall play a capital role because $\mathrm{O}(V,F)$ is the group of all $g\in\mathrm{Aut}(V,F)$ such that $\ker(g-\mathbf{1})$ contains $V_0^\perp$ (the subspace orthogonal to V_0). In Sect. 11 (where $\mathrm{char}(K)=2$), the following lemma will help us to prove that $g(v)=v$ for all $v\in V_0^\perp$ and all $g\in\mathrm{O}(V,F)$; in this lemma, $V_0=V^\perp$, whence $V_0^\perp=V$, and we must prove that $\mathrm{O}(V,F)=\{\mathbf{1}_V\}$.

Lemma 8.2. *When* $\mathrm{char}(K)=2$, $\dim(V)=2$ *and* $\dim(V^\perp)=1$, *the only orthogonal transformation of* (V,F) *is the identity transformation.*

Proof. Let (u,v) be a basis of V such that u spans the line $V^\perp$. There is an invertible $f\in K$ such that

$$\forall\,\xi,\,\zeta,\,\xi',\,\zeta'\in K,\quad F(\xi u+\zeta v,\,\xi' u+\zeta' v)=f\,\zeta\zeta'. \tag{8.3}$$

Therefore, $V_0=V^\perp$. Let a be an element of $\mathcal{B}_0(V,F)$:

$$a=\kappa+\lambda u^2+\lambda' v^2+\mu uv+\mu' vu+\kappa' u^2v^2\quad\text{with } \kappa,\,\kappa',\,\lambda,\,\lambda',\,\mu,\,\mu'\in K,$$
$$\tau(a)=(\kappa+f\lambda')+(\lambda+f\kappa')\,u^2+\lambda' v^2+\mu' uv+\mu vu+\kappa' u^2v^2.$$

We have $\tau(a)=a$ if and only if $\kappa'=\lambda'=0$ and $\mu=\mu'$. If a is invertible, then $\kappa\neq 0$ because $\kappa=\mathrm{Scal}(a)$. If the twisted inner automorphism determined by a induces an orthogonal transformation G_a of (V,F), then $G_a(u)=u$ and $G(v)=v+\gamma u$ for some $\gamma\in K$. Consequently,

$$(v+\gamma u)\,(\kappa+\lambda u^2+\mu(uv+vu))=(\kappa+\lambda u^2+\mu(uv+vu))\,v.$$

A direct calculation (using $u^3 = uvu = vuv = 0$, $v^3 = fv$, $vu^2 = u^2v$ and $v^2u = uv^2 + fu$) deduces from this equality that

$$(\gamma\kappa + f\mu)\, u = \gamma\mu\, u^2 v, \quad \text{whence } \gamma\kappa + f\mu = \gamma\mu = 0.$$

Since κ and f are invertible, we have $\gamma = \mu = 0$ and $G_a = \mathbf{1}_V$.

A similar but longer argument can prove this stronger assertion: if g is an element of $\mathrm{Aut}(V, F)$ other than $\mathbf{1}_V$, there is no $a \in \mathcal{B}(V, F)$ such that $g(u)\, a = au$, $g(v)\, a = av$ and $\mathrm{Scal}(a) \neq 0$. □

When $\mathrm{char}(K) \neq 2$, the bilinear form F defined by (8.3) gives a quite different orthogonal group: if we set $a = f - 2(v + \theta u)^2$ for some $\theta \in K$, then $\tau(a) = -a$, a is invertible because $a^2 = f^2$, and a commutes with $v + \theta u$; consequently, $G_a(v + \theta u) = -v - \theta u$; and since $G_a(u) = u$, we have $G_a(v) = -v - 2\theta u$. The group $\mathrm{O}(V, F)$ contains all transformations such that $u \longmapsto u$ and $v \longmapsto \pm(v + \gamma u)$ for some $\gamma \in K$.

8.3. The Generators of $\mathrm{Lip}(V, F)$

For a first approach, it is sensible to define the monoid $\mathrm{Lip}(V, F)$ by means of a family of generators. These generators must be eigenvectors of τ, they must satisfy (8.2), and there must be enough generators to give a surjective homomorphism $\mathrm{GLip}(V, F) \to \mathrm{O}(V, F)$.

In a Clifford algebra $\mathrm{Cl}(V, Q)$, the Lipschitz monoid $\mathrm{Lip}(V, Q)$ is generated by the scalars, the vectors, and the elements $1 + uv$ where u and v span a totally isotropic plane P in V; since the subalgebra $\mathrm{Cl}(P, Q)$ is isomorphic to $\bigwedge(P)$, we can write $1 + uv = \exp(uv)$. Similarly, for the Lipschitz monoid $\mathrm{Lip}(V, F)$ in $\mathcal{B}(V, F)$, we will also obtain a satisfying result with the following three types of generators:

the scalars (the elements of K),

the elements $a \in \mathcal{B}_0(V, F)$ such that $\dim(\mathrm{Sup}(a)) = 1$ and $\tau(a) = -a$,

the elements $\exp(\eta)$ such that $\mathrm{Sup}(\eta)$ is a totally isotropic plane P in V, $\eta \in \mathcal{B}^2(P, F)$ and $\tau(\eta) = \eta$. See Sect. 2.4 for the definition of exp.

For a precise description of the generators of the second and third types, we must distinguish two cases, according to the characteristic of K.

Let us first assume that $\mathrm{char}(K) \neq 2$. If u is a non-zero vector in V, an easy calculation shows that the elements $a \in \mathcal{B}_0(V, F)$ such that $\mathrm{Sup}(a) = Ku$ and $\tau(a) = -a$ are the non-zero elements colinear to $F(u, u) - 2u^2$. If we choose $a = F(u, u) - 2u^2$, then $\rho(a) = a$, $a\,\rho(a) = a^2 = F(u, u)^2$, $au = ua = -F(u, u)\, u$, and for all $v \in V$,

$$\begin{aligned} av\,\rho(a) &= (va - \tau(a)\, v)\, a - va^2 = (v \rfloor_F a)\, a - va^2 \\ &= -2\, F(v, u)\, ua - va^2 = F(u, u)\, (2\, F(v, u)\, u - F(u, u)\, v). \end{aligned}$$

Thus the property (8.2) holds true. If this generator a is invertible, in other words, if $F(u, u) \neq 0$, the orthogonal transformation G_a is the *reflection* with respect to the hyperplane orthogonal to u; indeed, for all $v \in V$,

$$G_a(v) = -ava^{-1} = v - \frac{2\, F(v, u)}{F(u, u)}\, u. \tag{8.4}$$

Each generator of the third type is associated with a totally isotropic plane P. If $\eta \in \mathcal{B}^2(P,F)$ and $\tau(\eta)=\eta$, then there is a basis (u,u') of P such that $\eta = uu' - u'u$. If we set $a=\exp(\eta)$, then we have

$$a = \exp(uu' - u'u) = (1+uu')\,(1-u'u) = 1 + uu' - u'u + u^2u'^2.$$

Since $\rho(\eta) = u'u - uu' = -\eta$, we have $\rho(a) = \exp(-\eta) = a^{-1}$. For all $v \in V$,

$$\begin{aligned} av\,\rho(a) &= -(va - \tau(a)\,v)\,\rho(a) + va\,\rho(a) = -(v \rfloor_F a)\,\rho(a) + va\,\rho(a) \\ &= -(v \rfloor_F \eta)\,a\,\rho(a) + va\,\rho(a) \qquad \text{(see Lemma 5.7)} \\ &= -v \rfloor_F \eta + v \qquad \text{since } a\,\rho(a) = 1. \end{aligned}$$

Thus (8.2) holds true. The associated orthogonal transformation G_a maps every $v \in V$ to $v - v \rfloor_F \eta = v - F(v,u)\,u' + F(v,u')\,u$.

Section 9 shall prove that we have got enough generators. When $F \neq 0$, it shall also prove that every generator of the third type is a product of 4 generators of the second type and a scalar.

Now let us assume that $\mathrm{char}(K)=2$. If u is a non-zero vector in V, the elements $a \in \mathcal{B}_0(V,F)$ such that $\mathrm{Sup}(a) = Ku$ are the elements $a = \kappa + \lambda u^2$ with κ, $\lambda \in K$ and $\lambda \neq 0$. Therefore, $\rho(a) = a$. Since $\tau(a) = a + \lambda F(u,u)$, the condition $\tau(a) = -a$ (now equivalent to $\tau(a) = a$) means $F(u,u) = 0$. In other words, u must belong to the subspace V_0 of all $v \in V$ such that $F(v,v)=0$. If $F(u,u)=0$, then $a\,\rho(a) = a^2 = \kappa^2$, and for all $v \in V$,

$$\begin{aligned} av\,\rho(a) &= (va - \tau(a)v)\,a + va^2 = (v\rfloor_F a)\,a + \kappa^2 v \\ &= \lambda\,F(v,u)\,ua + \kappa^2 v = \kappa\,(\lambda\,F(v,u)\,u + \kappa v). \end{aligned}$$

Once again, (8.2) holds true. All invertible generators of the second type are colinear to $1+\lambda u^2$ for some non-zero $u \in V_0$ and some non-zero $\lambda \in K$. We can write $1+\lambda u^2 = \exp(\lambda u^2)$. Here is the orthogonal transformation determined by $a = 1+\lambda u^2$:

$$G_a(v) = ava^{-1} = v + \lambda\,F(v,u)\,u. \tag{8.5}$$

This transformation is called the *transvection* determined by (u,λ).

To study the generators of the third type, we consider a totally isotropic plane P in V, and an element $\eta \in \mathcal{B}^2(P,F)$ such that $\tau(\eta)=\eta$. If (u,u') is a basis of P, there are κ, λ, $\mu \in K$ such that

$$\begin{aligned} \eta &= \kappa u^2 + \lambda u'^2 + \mu(uu' + u'u) \\ &= (\kappa+\mu)\,u^2 + (\lambda+\mu)\,u'^2 + \mu\,(u+u')^2. \end{aligned}$$

In the commutative subalgebra $\mathcal{B}_0(P,F)$ (isomorphic to $\mathcal{B}_0(P,0)$), we have

$$\exp(\eta) = \exp((\kappa{+}\mu)u^2)\,\exp((\lambda{+}\mu)u'^2)\,\exp(\mu(u{+}u')^2);$$

consequently, $\exp(\eta)$ is the product of three generators of the second type. The generators of the third type are superfluous when $\mathrm{char}(K)=2$.

Sections 10 and 11 shall prove that we have got enough generators.

Although the list of Lipschitzian generators did not mention the characteristic of K, the result of this list is quite different according as $\mathrm{char}(K) \neq 2$ or $\mathrm{char}(K)=2$. But in all cases, all generators satisfy (8.2); consequently, all Lipschitzian elements satisfy (8.2), and all invertible Lipschitzian elements determine an orthogonal transformation of (V, F). Because of Theorem 8.1, the equality $G_a = \mathbf{1}_V$ is equivalent to $\mathrm{Sup}(a) \subset V^\perp$.

Theorem 8.3. *Every Lipschitzian element satisfies (8.2), and every invertible Lipschitzian element a determines an orthogonal transformation G_a: $v \longmapsto a v\, \tau(a)^{-1}$. The mapping $a \longmapsto G_a$ is a homomorphism from the Lipschitz group* $\mathrm{GLip}(V, F)$ *into the orthogonal group* $\mathrm{O}(V, F)$*. Its kernel is the subgroup of all $a \in \mathrm{GLip}(V, F)$ such that $\mathrm{Sup}(a) \subset V^\perp$. When F is non-degenerate, this kernel contains only scalars.*

It remains to prove that the homomorphism $\mathrm{GLip}(V, F) \to \mathrm{O}(V, F)$ is surjective, and that $\mathrm{O}(V, F) = \mathrm{Aut}(V, F)$ when F is non-degenerate.

8.4. About the Center of $\mathcal{B}_0(V, F)$ when F is Non-degenerate

I imagine that a contradictor might reproach me with having imprudently required that all Lipschitzian elements be eigenvectors of τ. What would happen if we accepted all invertible $a \in \mathcal{B}_0(V, F)$ such that $ava^{-1} \in V$ for all $v \in V$? Let us suppose that F is non-degenerate, and let us search for such elements a in the center of $\mathcal{B}_0(V, F)$. From Sect. 3.3, we know that such an element a can be written

$$a = \lambda_0 \varepsilon_0 + \lambda_1 \varepsilon_1 + \cdots + \lambda_n \varepsilon_n \quad \text{with invertible } \lambda_0, \lambda_1, \ldots, \lambda_n \in K.$$

If a determines an orthogonal transformation, it is in the center of $\mathrm{O}(V, F)$, and in most cases, this implies that it is $\pm \mathbf{1}_V$. Therefore, it is sensible to restrict the research by imposing the condition $av = va$ or $av = -va$ for all $v \in V$. The equalities (3.2) show that the first condition means $\lambda_p = \lambda_{n-p+1}$ for $p = 1, 2, \ldots, n$, and that the second condition means $\lambda_p = -\lambda_{n-p+1}$. These conditions are not drastic enough to prevent the irruption of a lot of unpleasant elements a in the resulting group. The idempotents ε_p have been calculated in Sect. 3.3, and they look rather daunting; if plenty of invertible combinations of these idempotents were accepted in the resulting group, this group would become terrifying.

The calculation of the idempotents ε_p shows that,

$$\text{for } p = 0, 1, \ldots, n, \qquad \tau(\varepsilon_p) = \varepsilon_{n-p}. \tag{8.6}$$

It is easy to find all invertible elements a in the center of $\mathcal{B}_0(V, F)$ such that $\tau(a) = \pm a$ and $av = va$ or $av = -va$ for all $v \in V$. If $\mathrm{char}(K) = 2$, such an element is a scalar. But if $\mathrm{char}(K) \neq 2$, such an element may be either in K, or in the line spanned by the element

$$c = \sum_{p=0}^{n} (-1)^p \varepsilon_p \quad \text{such that } c^2 = 1, \quad \tau(c) = (-1)^n c \text{ and } cv = (-1)^{n+1} vc.$$

These properties of c imply that $G_c = -\mathbf{1}_V$. The bases $(e_1, \ldots, e_n)$ and $(e'_1, \ldots, e'_n)$ used in Sect. 3.3 allow us to calculate c:

$$c = \prod_{i=1}^{n} (1 - 2\, e_i e'_i). \tag{8.7}$$

Indeed, the operation of $1-2e_ie'_i$ on the even component $\bigwedge(V)\times 0$ multiplies $(e_P^{\wedge},0)$ by -1 or $+1$ according as $i\in P$ or $i\notin P$, in agreement with the fact that Θ_c multiplies it by $(-1)^p$ if p is the cardinality of P. When the basis $(e_1,\ldots,e_n)$ is orthogonal and $e'_i=e_i/F(e_i,e_i)$ for $i=1,\ldots,p$, the equality (8.7) proves that c is Lipschitzian.

8.5. The Special Lipschitz Group

The algebra homomorphism Scal : $\mathcal{B}(V,F)\rightarrow K$ (which maps V to 0) arouses an important discrepancy between meson algebras and Clifford algebras. When a is Lipschitzian, (8.2) shows that $a\,\rho(a)=\mathrm{Scal}(a)^2$ because $\mathrm{Scal}(\rho(a))=\mathrm{Scal}(a)$. Consequently, the invertible Lipschitzian elements are the elements $a\in\mathrm{Lip}(V,F)$ such that $\mathrm{Scal}(a)\neq 0$. The mapping $a\longmapsto\mathrm{Scal}(a)$ is a surjective homomorphism from the group $\mathrm{GLip}(V,F)$ onto the group of invertible scalars, and it would be clumsy to emphasize the homomorphism $a\longmapsto a\,\rho(a)$ because this homomorphism is the only available homomorphism when we deal with Lipschitz groups in Clifford algebras.

If an orthogonal transformation is equal to G_a for some Lipschitzian element a, it is also equal to G_b if $b=a/\mathrm{Scal}(a)$, and b is a Lipschitzian element such that $\mathrm{Scal}(b)=1$. Therefore, it may be advantageous to replace the homomorphism $\mathrm{GLip}(V,F)\rightarrow\mathrm{O}(V,F)$ with the homomorphism $\mathrm{SLip}(V,F)\rightarrow\mathrm{O}(V,F)$ where $\mathrm{SLip}(V,F)$ (the special Lipschitz group) is the group of all $b\in\mathrm{Lip}(V,F)$ such that $\mathrm{Scal}(b)=1$. The latter homomorphism has the same image in $\mathrm{O}(V,F)$ as the former, but its kernel is smaller. When F is non-degenerate, the latter is injective.

However advantageous the special group $\mathrm{SLip}(V,F)$ may be, it may not challenge the leading role of the monoid $\mathrm{Lip}(V,F)$ because of the wonderful properties of $\mathrm{Lip}(V,F)$, for instance, the two properties that shall now be disclosed.

8.6. Two Wonderful Properties of Lipschitz Monoids

Sections 9, 10 and 11 have two purposes: firstly, they prove the statements that have not yet been proved in Sect. 8; secondly, they prove two wonderful properties of Lipschitz monoids. These two properties are valid for the monoids, not for the groups, and they are faithful imitations of two properties of Lipschitz monoids in Clifford algebras. When I discovered these two properties in Clifford algebras, I realized that Lipschitz monoids had to play a leading role in the study of Clifford algebras, and I devoted much time and energy to the research for other wonderful properties.

The first wonderful property is the invariance under deformation. Let $\mathcal{B}(V,F;\beta)$ be the deformation of $\mathcal{B}(V,F)$ by a bilinear form β on V (see Sect. 7); since $\mathcal{B}(V,F;\beta)$ is naturally isomorphic to a meson algebra $\mathcal{B}(V,F')$, the definition of the Lipschitz monoid $\mathrm{Lip}(V,F;\beta)$ is obvious, and the invariance theorem states that $\mathrm{Lip}(V,F)$ *and* $\mathrm{Lip}(V,F;\beta)$ *are equal as subsets of the space* $\mathcal{B}(V,F)$.

As it happens for Clifford algebras, the invariance theorem has two important corollaries. Firstly, let U be a subspace of V; since $\mathcal{B}(U,F)$ (the subalgebra of $\mathcal{B}(V,F)$ generated by U and 1) is isomorphic to the meson algebra of the restriction of F to U, the definition of $\mathrm{Lip}(U, F)$ is obvious. For every subspace U, we have

$$\mathrm{Lip}(U, F) \;=\; \mathrm{Lip}(V, F) \cap \mathcal{B}(U, F). \tag{8.8}$$

Secondly, every field extension $K \to K'$ allows us to define a K'-extension $K' \otimes (V, F)$ of (V, F), the algebra $\mathcal{B}(V, F)$ can be identified with a K-subalgebra of the K'-algebra $\mathcal{B}(K' \otimes (V, F))$, and we have

$$\mathrm{Lip}(V, F) \;=\; \mathrm{Lip}(K' \otimes (V, F)) \cap \mathcal{B}(V, F). \tag{8.9}$$

The proof of (8.8) and (8.9) is the same for meson algebras as for Clifford algebras: a good knowledge of the neutral monoid $\mathrm{Lip}(V, 0)$ settles the neutral case $F = 0$, and a deformation reduces the general case to the neutral case. With (8.8), we can precisely describe the kernel of the homomorphism $\mathrm{GLip}(V, F) \to \mathrm{O}(V, F)$ when F is degenerate: it is the neutral group $\mathrm{GLip}(V^{\perp}, F)$.

The second wonderful property involves the space $(V^{\oplus}, F^{\oplus})$ already defined in Sect. 6.3. An algebra homomorphism $z \longmapsto L_z$ (defined in Sect. 7) turns the space $\mathcal{B}(V, F)$ into a module over the algebra $\mathcal{B}(V^{\oplus}, F^{\oplus})$, and it is remarkable that $L_c(a)$ *belongs to* $\mathrm{Lip}(V, F)$ *for all* $c \in \mathrm{Lip}(V^{\oplus}, F^{\oplus})$ *and all* $a \in \mathrm{Lip}(V, F)$. *In this way, the group* $\mathrm{GLip}(V^{\oplus}, F^{\oplus})$ *acts in a transitive way on the set of all non-zero elements of* $\mathrm{Lip}(V, F)$.

9. Lipschitz Monoids when $\mathrm{char}(K) \neq 2$

The Lipschitz monoid $\mathrm{Lip}(V, F)$ is generated by the scalars, the elements $F(u, u) - 2u^2$ (where $u \in V$) and the elements $(1 + uu')(1 - u'u)$ where u and u' span a totally isotropic plane in V. The special group $\mathrm{SLip}(V, F)$ (defined in Sect. 8.5) is generated by all $1 - 2u^2/F(u, u)$ such that $F(u, u) \neq 0$, and by all $(1 + uu')(1 - u'u)$ such that $Ku + Ku'$ is a totally isotropic plane. Thus $\mathrm{Lip}(V, P)$ is generated by $\mathrm{SLip}(V, F)$, all scalars and all u^2 such that $F(u, u) = 0$.

9.1. The Homomorphism $\mathrm{GLip}(V, F) \to \mathrm{O}(V, F)$

When F is non-degenerate, it is well known that the group $\mathrm{Aut}(V, F)$ is generated by the reflections (see [1], Chapter III, Section 10). Because of (8.4), this implies the equality $\mathrm{O}(V, F) = \mathrm{Aut}(V, F)$, which was considered as indispensable (in Sect. 8.2) for the relevance of the wanted theory.

When F is degenerate, we must prove that the image of $\mathrm{GLip}(V, F)$ in $\mathrm{Aut}(V, F)$ is $\mathrm{O}(V, F)$. This is trivial when $F = 0$ because $\mathrm{O}(V, 0) = \{\mathbf{1}_V\}$. When $F \neq 0$, it suffices to prove that every orthogonal transformation of (V, F) is a product of reflections. To prove it, we consider the following exact sequence of groups where V_1 is any subspace complementary to $V^{\perp}$ in V:

$$\{0\} \to \mathrm{Hom}(V_1, V^{\perp}) \to \mathrm{Aut}(V, F) \to \mathrm{Aut}(V_1, F) \times \mathrm{GL}(V^{\perp}) \to \{1\}. \tag{9.1}$$

$\mathrm{Hom}(V_1, V^{\perp})$ is the additive group of all linear mappings $f : V_1 \to V^{\perp}$, and the image of f in $\mathrm{Aut}(V, F)$ maps every $v_1 \in V_1$ to $v_1 + f(v_1)$, and every $v_0 \in V^{\perp}$ to v_0. The third arrow in (9.1) maps every $g \in \mathrm{Aut}(V, F)$ to the pair (g_1, g_0) where $g_1(v_1)$ is the parallel projection of $g(v_1)$ in V_1 with respect to $V^\perp$, and g_0 is the restriction of g to $V^\perp$. This third arrow is surjective because of the homomorphism $\mathrm{Aut}(V_1,F)\times\mathrm{GL}(V^\perp)\to\mathrm{Aut}(V,F)$ which maps every (g_1,g_0) to the transformation $v_1+v_0\longmapsto g_1(v_1)+g_0(v_0)$. If g is an orthogonal transformation of (V,F), Theorem 8.1 shows that the image of g in $\mathrm{GL}(V^\perp)$ is the identity mapping. Therefore, we consider an element g of $\mathrm{Aut}(V,F)$ that has a trivial image in $\mathrm{GL}(V^\perp)$, and we prove that g is a product of reflections. Since F is non-degenerate on V_1, this is true when g is the image of an element of $\mathrm{Aut}(V_1,F)$. To complete the proof, it suffices to prove that the same is true when g is the image of an element f of $\mathrm{Hom}(V_1,V^\perp)$. It suffices to consider a linear mapping f of rank 1; then there is $(u_1,u_0)\in V_1\times V^\perp$ such that $f(v_1)=F(v_1,u_1)\,u_0$ for all $v_1\in V_1$. Since V_1 is spanned by the non-isotropic vectors, we may require $F(u_1,u_1)\neq 0$. A direct calculation shows that the automorphism of (V,F) determined by f is the product of the reflections determined by the two vectors u_1 and $u_1-F(u_1,u_1)\,u_0/2$:

$$v\ \longmapsto\ v-\frac{2\,F(v,u_1)}{F(u_1,u_1)}\left(u_1-\frac{F(u_1,u_1)}{2}u_0\right)\ \longmapsto\ v+F(v,u_1)\,u_0.$$

Theorem 9.1. *When* $\mathrm{char}(K)\neq 2$, *the orthogonal group* $\mathrm{O}(V,F)$ *is the subgroup of all* $g\in\mathrm{Aut}(V,F)$ *such that* $g(v)=v$ *for all* $v\in V^\perp$. *It is generated by the reflections, and it is equal to* $\mathrm{Aut}(V,F)$ *when* F *is non-degenerate. The homomorphism* $\mathrm{GLip}(V,F)\to\mathrm{O}(V,F)$ *is always surjective.*

9.2. Jacobson's Homomorphism and the Wonderful Properties

Jacobson's homomorphism Δ (see Sect. 2.1) brings a theorem that involves the Lipschitz monoid $\mathrm{Lip}(V,F)$ in $\mathcal{B}(V,F)$ and the Lipschitz monoid $\mathrm{Lip}(V,\bar F)$ in $\mathrm{Cl}(V,\bar F)$. For every $u\in V$, a direct calculation gives

$$\Delta(F(u,u)-2u^2)=-u\otimes u \tag{9.2}$$

If u and u' span a totally isotropic plane in V, another calculation gives

$$\Delta\left((1+uu')(1-u'u)\right)=\left(1+\frac{1}{2}uu'\right)\otimes\left(1+\frac{1}{2}uu'\right) \tag{9.3}$$

because the relations $u^2=u'^2=0$ and $u'u=-uu'$ are true in $\mathrm{Cl}(V,\bar F)$. Recall that the scalars, the vectors u and the elements $1+uu'/2$ (with $Ku\oplus Ku'$ totally isotropic) are the generators of $\mathrm{Lip}(V,\bar F)$. The resulting theorem will turn the study of $\mathrm{Lip}(V,F)$ much easier.

Theorem 9.2. *An element* $a\in\mathcal{B}_0(V,F)$ *is Lipschitzian if and only if there is a Lipschitzian* $b\in\mathrm{Lip}(V,\bar F)$ *such that* $\Delta(a)=\kappa\,b\otimes b$ *for some* $\kappa\in K$. *Moreover, for every* $b\in\mathrm{Lip}(V,\bar F)$, *there is* $a\in\mathrm{Lip}(V,F)$ *such that* $\Delta(a)$ $=\kappa\,b\otimes b$ *for some* $\kappa\in K$.

Theorem 9.2 affords an easy proof of the following lemma.

Lemma 9.3. *If* $F\neq 0$, *every generator* $(1+uu')(1-u'u)$ *of the third type is the product of 4 generators of the second type and a scalar.*

Proof. The same statement is valid for a generator of the third type in every Clifford algebra $\mathrm{Cl}(V,Q)$ where $Q \neq 0$, provided that the field K contains more than two elements. A generator of the third type in $\mathrm{Lip}(V,Q)$ is equal to $1+uu'$ with two vectors u and u' that span a totally isotropic plane in V. Let w be a vector such that $Q(w) \neq 0$, and let A be the bilinear form associated with Q (see Sect. 1.1). First we reduce the problem of the factorization of $1+uu'$ to the case $A(w,u)=0$: for this purpose, if $A(w,u) \neq 0$ and $A(w,u')=0$, we replace (u,u') with $(u',-u)$; and if $A(w,u)\,A(w,u') \neq 0$, we replace (u,u') with $(u-\kappa u',\, u')$ where $\kappa = A(w,u)/A(w,u')$. When $A(w,u)=0$, the equality

$$(w+\lambda u-\mu u')\, w\, (w-\lambda u)(w-\mu u') = \lambda\big(\lambda - A(w,u')\big)\, \mu^2\, (1+uu')$$

holds whenever $\lambda\mu = Q(w)$. It brings a factorization of $1+uu'$ into a product of 4 vectors if we choose λ different from 0 and $A(w,u')$, and $\mu = Q(w)/\lambda$. This procedure fails if $A(w,u') \neq 0$ and $K \cong \mathbb{Z}/2\mathbb{Z}$, and sometimes no such factorization exists. But since here we assume $\mathrm{char}(K) \neq 2$, this failure never occurs in the present context. □

Lemma 9.3 shows that the generators of the third type are indispensable only when $F=0$. If a contradictor retorts that neither are they indispensable when $F=0$ because $\mathrm{O}(V,0) = \{\mathbf{1}_V\}$, I shall explain that the purposes of the Lipschitz monoid $\mathrm{Lip}(V,F)$ are not limited to the construction of a surjective homomorphism $\mathrm{GLip}(V,F) \to \mathrm{O}(V,F)$. The wonderful properties mentioned in Sect. 8.6 are also included in the main purposes. These properties require the presence of all the generators mentioned in Sect. 8.3 (even the generators u^2 where $F(u,u)=0$). Besides, the neutral monoid $\mathrm{Lip}(V,0)$ often plays a capital role (see Sect. 9.3) despite the ludicrous triviality of the group $\mathrm{O}(V,0)$.

Theorem 9.2 enables us to prove that the wonderful properties are true for $\mathrm{Lip}(V,F)$ because they are true for $\mathrm{Lip}(V,\bar{F})$.

Theorem 9.4. *In case of a deformation of* $\mathcal{B}(V,F)$ *by a bilinear form* β, $\mathrm{Lip}(V,F)$ *and* $\mathrm{Lip}(V,B;\beta)$ *are equal as subsets of the space* $\mathcal{B}(V,F)$.

Proof. Theorem 7.10 states that Δ is also a homomorphism from $\mathcal{B}(V,F;\beta)$ into $\mathrm{Cl}(V,\bar{F};2\beta)\otimes\mathrm{Cl}(V,\bar{F};2\beta)$. An element $a \in \mathcal{B}(V,F)$ is in $\mathrm{Lip}(V,F)$ (resp. $\mathrm{Lip}(V,F;\beta)$) if and only if there is b in $\mathrm{Lip}(V,\bar{F})$ (resp. $\mathrm{Lip}(V,\bar{F};2\beta)$) such that $\Delta(a) = \kappa\, b\otimes b$ for some $\kappa \in K$. The conclusion follows from the fact that $\mathrm{Lip}(V,\bar{F})$ and $\mathrm{Lip}(V,\bar{F};2\beta)$ are equal as subsets of $\mathrm{Cl}(V,\bar{F})$. □

Theorem 9.5. *Let the algebra* $\mathcal{B}(V^\oplus,F^\oplus)$ *act on the space* $\mathcal{B}(V,F)$ *by the homomorphism* $z \longmapsto L_z$ *defined in Sect. 7. We have* $L_c(a) \in \mathrm{Lip}(V,F)$ *for all* $c \in \mathrm{Lip}(V^\oplus,F^\oplus)$ *and all* $a \in \mathrm{Lip}(V,F)$, *and the group* $\mathrm{GLip}(V^\oplus,F^\oplus)$ *acts transitively on the set of all non-zero elements of* $\mathrm{Lip}(V,F)$.

Proof. Let us recall that $L_{\ell+u}(x) = \ell\rfloor x + ux$ for all $x \in \mathcal{B}(V,F)$, all $u \in V$ and all $\ell \in V^*$; this formula describes the operation of an element $\ell{+}u \in V^\oplus$ on the space $\mathcal{B}(V,F)$. On another side, the quadratic form $\bar{F}$ on V gives a quadratic form $\bar{F}^\oplus$ on $V^\oplus$, and $\bar{F}^\oplus(\ell{+}u) = \ell(u) + \bar{F}(u)$. Consequently,

$$F^\oplus(\ell{+}u,\, \ell{+}u) = 2\,\ell(u) + F(u,u) = \bar{F}^\oplus(2\ell{+}u). \tag{9.4}$$

The Clifford algebra $\mathrm{Cl}(V^\oplus, \bar{F}^\oplus)$ acts on the space $\mathrm{Cl}(V, \bar{F})$ through an algebra isomorphism $z \longmapsto \underline{L}_z$ from $\mathrm{Cl}(V^\oplus, Q^\oplus)$ onto $\mathrm{End}(\mathrm{Cl}(V, Q))$, and we have $\underline{L}_{\ell+u}(x) = \ell \rfloor x + ux$ for all $x \in \mathrm{Cl}(V, Q)$. Therefore, the algebra

$$\mathrm{Cl}(V^\oplus, \bar{F}^\oplus) \otimes \mathrm{Cl}(V^\oplus, \bar{F}^\oplus) \ \text{ acts on } \ \mathrm{Cl}(V, \bar{F}) \otimes \mathrm{Cl}(V, \bar{F}),$$

and the operation of $y \otimes z$ (with y, $z \in \mathrm{Cl}(V^\oplus, \bar{F}^\oplus)$) is $\underline{L}_{y\otimes z} = \underline{L}_y \otimes \underline{L}_z$. Now let us introduce Jacobson's homomorphism. Lemma 5.4 (together with $\Delta(u) = (u\otimes 1 + 1 \otimes u)/2$) shows that, for all $x \in \mathcal{B}(V, F)$,

$$\Delta\big(L_{\ell+u}(x)\big) = \frac{1}{2}\, \underline{L}_{(2\ell+u)\otimes 1\, +\, 1\otimes(2\ell+u)}\big(\Delta(x)\big). \tag{9.5}$$

Both (9.4) and (9.5) show that the element $\ell{+}u$ of $(V^\oplus, F^\oplus)$ must be associated with the element $2\ell{+}u$ of $(V^\oplus, \bar{F}^\oplus)$. Let us consider a generator $F^\oplus(\ell{+}u,\, \ell{+}u) - 2(\ell{+}u)^2$ of $\mathrm{Lip}(V^\oplus, F^\oplus)$. We have

$$\Delta\big(L_{F^\oplus(\ell+u,\ell+u)-2(\ell+u)^2}(x)\big) = -\underline{L}_{(2\ell+u)\otimes(2\ell+u)}\big(\Delta(x)\big) \tag{9.6}$$

for all $x \in \mathcal{B}(V, F)$. Indeed, (9.5) proves the existence of an element $Z \in \mathrm{Cl}(V^\oplus, \bar{F}^\oplus) \otimes \mathrm{Cl}(V^\oplus, \bar{F}^\oplus)$ such that the left hand side of (9.6) is equal to $\underline{L}_Z(\Delta(x))$, and because of (9.4) and (9.5), we have

$$\begin{aligned} Z &= \bar{F}^\oplus(2\ell{+}u)\otimes 1 - \frac{1}{2}\big((2\ell{+}u)\otimes 1 \,+\, 1\otimes(2\ell{+}u)\big)^2 \\ &= -(2\ell{+}u)\otimes(2\ell{+}u) \qquad \text{because } (2\ell{+}u)^2 = \bar{F}^\oplus(2\ell{+}u). \end{aligned}$$

This proves (9.6). Since $\mathrm{Lip}(V^\oplus, F^\oplus)$ is generated by the scalars and all elements $F^\oplus(\ell{+}u,\, \ell{+}u) - 2(\ell{+}u)^2$, it follows from (9.6) that, for every $c \in \mathrm{Lip}(V^\oplus, F^\oplus)$, there is some $d \in \mathrm{Lip}(V^\oplus, \bar{F}^\oplus)$ such that

$$\Delta(L_c(x)) = \lambda\, \underline{L}_{d\otimes d}(\Delta(x)) \qquad \text{for some } \lambda \in K. \tag{9.7}$$

For every $a \in \mathrm{Lip}(V, F)$, we can write $\Delta(a) = \kappa b \otimes b$ for some $b \in \mathrm{Lip}(V, \bar{F})$ and some $\kappa \in K$; consequently,

$$\Delta(L_c(a)) = \kappa\lambda\, \underline{L}_d(b) \otimes \underline{L}_d(b). \tag{9.8}$$

Since $\underline{L}_d(b)$ is Lipschitzian in $\mathrm{Cl}(V, \bar{F})$ (see [5], Section 11.5), Theorem 9.2 proves that $L_c(a)$ is Lipschitzian in $\mathcal{B}(V, F)$. If $a \neq 0$, then $b \neq 0$, and there is an invertible d in $\mathrm{Lip}(V^\oplus, \bar{F}^\oplus)$ such that $\underline{L}_d(b) = 1$ (again [5]). Because of (9.6), there is $c \in \mathrm{GLip}(V^\oplus, F^\oplus)$ such that (9.7) is true. Therefore, (9.8) is true, and since $\kappa\lambda$ is in the group $\mathrm{GLip}(V^\oplus, F^\oplus)$, there is an element of this group that carries a to 1. Therefore, this group acts transitively in the set of all non-zero elements of $\mathrm{Lip}(V, F)$. □

9.3. The Neutral Monoid $\mathrm{Lip}(V, 0)$

The following argument is based on calculations in the commutative algebra $\mathcal{B}_0(V, 0)$ which can be justified either by a systematic and clever use of the relations $uvu = 0$ and $uvw = -wvu$ (see (2.1)), or by the isomorphism $\mathcal{B}_0(V, 0) \to \bigoplus_k \bigwedge^k(V) \otimes \bigwedge^k(V)$ (see Sect. 2.4). A non-zero element $a \in \mathrm{Lip}(V, 0)$ is the product of a scalar κ, of some squares of vectors $u_1^2, \ldots, u_k^2$ and of some exponential $\exp(\theta)$ with $\theta \in \mathcal{B}^2(V, 0)$ and $\tau(\theta) = \theta$:

$$a = \kappa\, u_1^2\, u_2^2 \cdots u_k^2\, \exp(\theta).$$

Of course, $\kappa \neq 0$, and the sequence $(u_1, u_2, \dots, u_k)$ is linearly independent, so that $a \neq 0$. Moreover, a is invertible if and only if $k = 0$ (in which case $u_1^2 u_2^2 \cdots u_k^2$ means 1). Let V' be a subspace of V complementary to the subspace spanned by $(u_1, u_2, \dots, u_k)$; for every $z \in \mathcal{B}_0(V, 0)$, there is a unique $z' \in \mathcal{B}_0(V', 0)$ such that $u_1^2 \cdots u_k^2\, z = u_1^2 \cdots u_k^2\, z'$, and z' is the image of z by the homomorphism $\mathcal{B}(V, 0) \to \mathcal{B}(V', 0)$ that extends the projector $V \to V'$. Therefore, we may require that θ belongs to $\mathcal{B}^2(V', 0)$. Since $\tau(\theta) = \theta$, the image of θ in $V' \otimes V'$ is a skew symmetric tensor, and there is a linearly independent family $(v_1, v_2, \dots, v_{2r})$ of vectors of V' such that

$$\theta = (v_1 v_2 - v_2 v_1) + (v_3 v_4 - v_4 v_3) + \cdots + (v_{2r-1} v_{2r} - v_{2r} v_{2r-1}).$$

Finally, there is a linearly independent sequence $(u_1, \dots, u_k, v_1, \dots, v_{2r})$ such that $k \geqslant 0$, $r \geqslant 0$ and

$$a = \kappa u_1^2 \,\cdots\, u_k^2\, (1 + v_1 v_2)(1 - v_2 v_1) \,\cdots\, (1 + v_{2r-1} v_{2r})(1 - v_{2r} v_{2r-1}). \tag{9.9}$$

This sequence $(u_1, \dots, v_{2r})$ spans the support of a, which has dimension $s = k+2r$, and we have $\tau(a) = a$ if s is even, but $\tau(a) = -a$ if s is odd. Moreover, a has a non-zero component in $\mathcal{B}^{2s}(V, 0)$, which is the (commutative) product of κ, all squares $u_1^2, \dots u_k^2$, and all squares $v_1^2, \dots v_{2r}^2$. It is also worth noticing that the subspace spanned by $(u_1, \dots, u_k)$ is the subspace $\mathrm{Ker}(a)$ of all $u \in V$ such that $ua = 0$.

As it happens with Clifford algebras, all this information allows us easily to prove that the properties (8.8) and (8.9) are true when $F = 0$. But when $F \neq 0$, a deformation $\mathcal{B}(V, F; -\beta)$ such that $F = \beta + \beta^o$ allows us to prove that (8.8) and (8.9) are also true for $\mathcal{B}(V, F)$ because $\mathcal{B}(V, F; -\beta) \cong \mathcal{B}(V, 0)$ and $\mathrm{Lip}(V, F) = \mathrm{Lip}(V, F; -\beta)$ (recall Theorem 9.4).

In the description (9.9) of a, there are yet other properties that are invariant under deformation. Although the algebra $\mathcal{B}(V, F)$ is not $\mathbb{Z}$-graded if $F \neq 0$, its filtration (recalled in Sect. 2.3) allows us to define a component of highest degree for every non-zero $x \in \mathcal{B}(V, F)$. If x belongs to $\mathcal{B}^{\leqslant k}(V, F)$ but not to $\mathcal{B}^{<k}(V, F)$, its component of highest degree is the element of $\mathcal{B}^k(V, 0)$ that follows from Lemma 9.6. This lemma is also valid for Sects. 10 and 11 where $\mathrm{char}(K) = 2$.

Lemma 9.6. *For $k = 0, 1, \dots, 2n$, there is a linear mapping $\mathcal{B}^{\leqslant k}(V, F) \to \mathcal{B}^k(V, 0)$ that maps every product $v_1 v_2 \cdots v_k$ in $\mathcal{B}(V, F)$ to the neutral product $v_1 v_2 \cdots v_k$ in $\mathcal{B}^k(V, 0)$, and that maps $\mathcal{B}^{<k}(V, F)$ to 0.*

Proof. The filtration of $\mathcal{B}(V, F)$ leads to a $\mathbb{Z}$-graded algebra $\mathrm{Gr}(\mathcal{B}(V, F))$ where the component $\mathrm{Gr}^k(\mathcal{B}(V, F)$ of degree k (for each $k \in \mathbb{Z}$) is the quotient $\mathcal{B}^{\leqslant k}(V, F)/\mathcal{B}^{<k}(V, F)$. If x and y are in $\mathcal{B}^{\leqslant j}(V, F)$ and $\mathcal{B}^{\leqslant k}(V, F)$, the product of their images x' and y' in $\mathrm{Gr}^j(\mathcal{B}(V, F))$ and $\mathrm{Gr}^k(\mathcal{B}(V, F))$ is the image of xy in $\mathrm{Gr}^{j+k}(\mathcal{B}(V, F))$. If u and v are in V, the relation $uvu = F(u, v)\, u$ gives the relation $u'v'u' = 0$ for their images u' and v' in $\mathrm{Gr}^1(\mathcal{B}(V, F))$. Therefore, the mapping $v \longmapsto v'$ extends to an algebra homomorphism from $\mathcal{B}(V, 0)$ into $\mathrm{Gr}(\mathcal{B}(V, F))$. It is a $\mathbb{Z}$-graded and surjective homomorphism. Since the two algebras have the same dimension, it is bijective. The mappings described in Lemma 9.6 follow from the inverse bijections $\mathrm{Gr}^k(\mathcal{B}(V, F)) \to \mathcal{B}^k(V, 0)$. $\square$

In meson algebras, as in Clifford algebras, the support of an element and its component of highest degree are invariant by deformation. Thus we obtain the next theorem, which is a consequence of (9.9) like (8.8) and (8.9). Theorem 9.4 allows us to deduce all these properties from (9.9), and this procedure explains the capital importance of the neutral monoid $\mathrm{Lip}(V,0)$ despite the ludicrous triviality of the orthogonal group $\mathrm{O}(V,0)$.

Theorem 9.7. *Let a be a non-zero element of $\mathrm{Lip}(V,F)$ and let s be the integer such that a belongs to $\mathcal{B}^{\leqslant 2s}(V,F)$ but not to $\mathcal{B}^{<2s}(V,F)$.*

(*a*) *We have $\dim(\mathrm{Sup}(a)) = s$.*

(*b*) *If $a^\dagger$ is the image of a in $\mathcal{B}^{2s}(V,0)$, then $\mathrm{Sup}(a) = \mathrm{Sup}(a^\dagger)$.*

(*c*) *We have $\tau(a) = a$ or $\tau(a) = -a$ according as s is even or odd.*

10. When $\mathrm{char}(K) = 2$ and F is Alternating

When $\mathrm{char}(K) = 2$ and F is alternating, the monoid $\mathrm{Lip}(V,F)$ is generated by all $\kappa + \lambda u^2$ with κ, $\lambda \in K$ and $u \in V$. The group $\mathrm{SLip}(V,F)$ (defined in Sect. 8.5) is generated by all $1 + \lambda u^2$. Thus $\mathrm{Lip}(V,F)$ is generated by $\mathrm{SLip}(V,F)$, all scalars κ and all squares u^2.

10.1. The Homomorphism $\mathrm{GLip}(V,F) \to \mathrm{O}(V,F)$

When F is alternating and non-degenerate, the group $\mathrm{Aut}(V,F)$, which is now a symplectic group, is generated by the transvections (see [1], Chapter II, Section 6). Each transvection $v \longmapsto v + \lambda\, F(v,u)\, u$ is determined by a pair $(u,\lambda) \in V \times K$; it is equal to $G_{1+\lambda u^2}$ (see (8.5)). Thus the indispensable equality $\mathrm{Aut}(V,F) = \mathrm{O}(V,F)$ is ensured.

When F is degenerate, we must prove the surjectivity of the homomorphism $\mathrm{GLip}(V,F) \to \mathrm{O}(V,F)$. We may suppose $F \neq 0$ because $\mathrm{O}(V,0) = \{\mathbf{1}_V\}$. As in Sect. 9.1, we choose a subspace V_1 complementary to $V^\perp$, the exact sequence (9.1) is still valid, and as it happened in Sect. 9.1, the argument ends with the proof of this fact: if an element $g \in \mathrm{Aut}(V,F)$ is the image of an element $f \in \mathrm{Hom}(V_1, V^\perp)$, then g is a product of transvections. It suffices to consider a linear mapping f of rank 1, determined by some $(u_1,u_0) \in V_1 \times V^\perp$: $f(v_1) = F(v_1,u_1)\, u_0$ for all $v_1 \in V_1$. A direct calculation shows that g is now the product of the transvections determined by $1 + u_1^2$ and $1 + (u_1 + u_0)^2$:

$$v \longmapsto v + F(v,u_1)(u_1 + u_0) \longmapsto v + F(v,u_1)\, u_0.$$

Theorem 10.1. *When $\mathrm{char}(K) = 2$ and F is alternating, the group $\mathrm{O}(V,F)$ is the subgroup of all $g \in \mathrm{Aut}(V,F)$ such that $g(v) = v$ for all $v \in V^\perp$. It is generated by the transvections, and it is equal to $\mathrm{Aut}(V,F)$ when F is non-degenerate. The homomorphism $\mathrm{GLip}(V,F) \to \mathrm{O}(V,F)$ is surjective.*

10.2. Preliminary Calculations

Since we can no longer use Jacobson's homomorphism to prove the wonderful properties of $\mathrm{Lip}(V, F)$, we shall imitate a method that has been successful for Lipschitz monoids in Clifford algebras. This imitation will give a more complicated argument, but the starting point is the same: we start with the neutral case $F = 0$, and we let the algebra $\mathcal{B}(V^{\oplus}, 0^{\oplus})$ act on the space $\mathcal{B}(V, 0)$. The bilinear form $0^{\oplus}$ on $V^{\oplus} = V^* \oplus V$ is alternating: $0^{\oplus}(\ell{+}u,\ \ell'{+}u') = \ell(u') + \ell'(u)$. The operation of an element $\ell + u \in V^{\oplus}$ on $\mathcal{B}(V, 0)$ is described by the formula $L_{\ell+u}(x) = \ell\rfloor x + ux$ (for all $x \in \mathcal{B}(V, 0)$).

Let a be a non-zero element of $\mathrm{Lip}(V, 0)$:

$$a = \kappa\, u_1^2\, u_2^2 \cdots u_k^2 \,\exp(\theta). \tag{10.1}$$

As in Sect. 9.3, κ is a scalar, the vectors $u_1, \ldots, u_k$ are linearly independent, the product $u_1^2 u_2^2 \cdots u_k^2$ means 1 if $k = 0$, and θ is an element of $\mathcal{B}^2(V, 0)$ such that $\tau(\theta) = \theta$. The subspace spanned by $(u_1, \ldots, u_k)$ is the subspace $\mathrm{Ker}(a)$ of all $u \in V$ such that $ua = 0$. If V' is any subspace complementary to $\mathrm{Ker}(a)$, we may require that θ belongs to $\mathcal{B}^2(V', 0)$. Since $\mathrm{char}(K) = 2$, the equality $\tau(\theta) = \theta$ means that the image of θ in $V' \otimes V'$ is a symmetric tensor which determines a symmetric bilinear form on the dual space V'^*. If this bilinear form is alternating, there is a linearly independent family $(v_1, \ldots, v_{2r})$ in V' such that

$$\theta = (v_1 v_2 + v_2 v_1) + (v_3 v_4 + v_4 v_3) + \cdots + (v_{2r-1} v_{2r} + v_{2r} v_{2r-1});$$

as in Sect. 9.3, this implies that the dimension of $\mathrm{Sup}(a)$ is $s = k{+}2r$, and that a has a non-zero component in $\mathcal{B}^{2s}(V, 0)$. But if this bilinear form on V'^* is not alternating, it admits orthogonal bases in V'^*, and from each such basis we can derive a linearly independent family $(v_1, \ldots, v_r)$ in V', and a sequence $(\kappa_1, \ldots, \kappa_r)$ of non-zero scalars, such that

$$\theta = \kappa_1 v_1^2 + \kappa_2 v_2^2 + \cdots + \kappa_r v_r^2;$$

now the dimension of $\mathrm{Sup}(a)$ is $s = k{+}r$, and it is still true that a has a non-zero component in $\mathcal{B}^{2s}(V, 0)$.

The precise description of the elements of $\mathrm{Lip}(V, 0)$ proves that (8.8) and (8.9) are true when $F = 0$.

Let us study the operation of a Lipschitzian generator

$$c = \lambda + \mu(\ell + w)^2 \qquad (\text{with } \lambda,\ \mu \in K,\ \ \mu \neq 0,\ \ \ell \in V^* \text{ and } w \in V) \tag{10.2}$$

on the above Lipschitzian element a. During the calculation of $L_c(a)$, we must recall that all squares $u_1^2, \ldots, u_k^2$ are in the center of the algebra $\mathcal{B}(V, 0)$ because of Lemma 4.4; and θ and $\exp(\theta)$ are also in the center because $\tau(\theta) = \theta$. Because of Lemma 5.7, we have

$$\ell\rfloor \exp(\theta) = (\ell\rfloor\theta)\, \exp(\theta) = \exp(\theta)\, (\ell\rfloor\theta).$$

Now we must distinguish two cases.

First case: $\mathrm{Ker}(a) \subset \ker(\ell)$.

When $k = 0$, or when $\ell(u_i) = 0$ for $i = 1, 2, \ldots, k$, the calculation of $L_c(a)$ is rather easy, and brings the vector $w + \ell\rfloor\theta \in V$:

$$L_c(a) \;=\; a\,\big(\lambda + \mu\,\ell(w + \ell\rfloor\theta) + \mu\,(w + \ell\rfloor\theta)^2\big). \tag{10.3}$$

Obviously, $L_c(a)$ is Lipschitzian too.

Second case: ℓ does not vanish everywhere on $\mathrm{Ker}(a)$.
In the subalgebra $\mathcal{B}(\mathrm{Ker}(a), 0)$, the component of highest degree $2k$ has dimension 1. If $(u'_1, \ldots, u'_k)$ is another basis of $\mathrm{Ker}(a)$, the product $u'^2_1 \cdots u'^2_k$ is colinear to $u_1^2 \cdots u_k^2$. Thus we can reduce the problem to the case

$$\ell(u_1) = 1 \quad \text{but} \quad \ell(u_2) = \cdots = \ell(u_k) = 0. \tag{10.4}$$

We continue the calculation under the assumption (10.4), and we set

$$\tilde{a} \;=\; \kappa\, u_2^2 \,\cdots\, u_k^2 \,\exp(\theta). \tag{10.5}$$

A direct calculation gives

$$L_{\ell+w}(a) = \tilde{a}\,\big(u_1 + u_1^2\,(w{+}\ell\rfloor\theta)\big)\,,$$
$$L_{(\ell+w)^2}(a) = \tilde{a}\,\big(1 + \ell(w{+}\ell\rfloor\theta))u_1^2 + u_1(w{+}\ell\rfloor\theta) + (w{+}\ell\rfloor\theta)u_1 + u_1^2(w{+}\ell\rfloor\theta)^2\big).$$

Since $u_1^3 = 0$, we come to the equality

$$L_c(a) = \mu\tilde{a}\,\exp\bigg(\big(\lambda\mu^{-1}{+}\ell(w{+}\ell\rfloor\theta)\big)u_1^2 + u_1(w{+}\ell\rfloor\theta) + (w{+}\ell\rfloor\theta)u_1\bigg) \tag{10.6}$$

under the assumptions (10.1), (10.2), (10.4) and (10.5). Again, we recognize that $L_c(a)$ is Lipschitzian.

The equalities (10.3) and (10.6) prove that the monoid $\mathrm{Lip}(V^{\oplus}, 0^{\oplus})$ acts on the set $\mathrm{Lip}(V, 0)$. They even prove that the group $\mathrm{GLip}(V^{\oplus}, 0^{\oplus})$ acts in a transitive way on the set of all non-zero elements of $\mathrm{Lip}(V, 0)$. In other words, every non-zero $a \in \mathrm{Lip}(V, 0)$ is mapped to 1 by some element of $\mathrm{GLip}(V^{\oplus}, 0^{\oplus})$. This is obvious if a is invertible, therefore in $\mathrm{GLip}(V^{\oplus}, 0^{\oplus})$. If a is not invertible, the equality (10.6) shows there is $c \in \mathrm{GLip}(V^{\oplus}, 0^{\oplus})$ such that the dimension of $\mathrm{Ker}(L_c(a))$ is strictly smaller than the dimension of $\mathrm{Ker}(a)$, and by repeating this process, we shall find an element of $\mathrm{GLip}(V^{\oplus}, 0^{\oplus})$ that maps a to an invertible element of $\mathrm{Lip}(V, 0)$.

10.3. The Invariance of $\mathrm{Lip}(V, F)$ under Deformation

The deformation $\mathcal{B}(V, F; \beta)$ (presented in Sect. 7) is isomorphic to $\mathcal{B}(V, F')$ if $F' = F{+}\beta{+}\beta^o$. Since F is alternating, F' is alternating too.

Theorem 10.2. *In case of a deformation of $\mathcal{B}(V, F)$ by a bilinear form β, $\mathrm{Lip}(V, F)$ and $\mathrm{Lip}(V, F; \beta)$ are equal as subsets of $\mathcal{B}(V, F)$.*

Proof. The proof shall be achieved in three steps.

First step: we reduce the problem to the case $F = 0$. Since F is alternating, there is a bilinear form γ such that $F = \gamma{+}\gamma^o$. Thus $\mathcal{B}(V, F; -\gamma)$ is isomorphic to $\mathcal{B}(V, 0)$. Because of Corollary 7.7, $\mathcal{B}(V, F)$ is the deformation of the neutral algebra $\mathcal{B}(V, F; -\gamma)$ by means of γ, and $\mathcal{B}(V, F; \beta)$ is the deformation of $\mathcal{B}(V, F; -\gamma)$ by means of $\beta{+}\gamma$. If we prove that Theorem 10.2 is true when $F = 0$, we may claim that $\mathrm{Lip}(V, F) = \mathrm{Lip}(V, F; -\gamma)$ and $\mathrm{Lip}(V, F; \beta) = \mathrm{Lip}(V, F; -\gamma)$; therefore, $\mathrm{Lip}(V, F; \beta) = \mathrm{Lip}(V, F)$.

Second step: we prove that $\mathrm{Lip}(V, 0; \beta) \subset \mathrm{Lip}(V, 0)$ when $F = 0$. As in Sect. 7, there is an algebra isomorphism Φ from $\mathcal{B}(V^{\oplus}, F'^{\oplus})$ onto $\mathcal{B}(V^{\oplus}, 0^{\oplus})$, such that $\Phi(\ell{+}v) = \ell{+}\beta_v{+}v$, and there is a bijection $J : \mathcal{B}(V, F') \to \mathcal{B}(V, 0)$ such that $J(z) = L_{\Phi(z)}(1)$. Since J is the isomorphism $\mathcal{B}(V, F') \to$

$\mathcal{B}(V,0;\beta)$ extending $\mathbf{1}_V$, the monoid $\mathrm{Lip}(V,0;\beta)$ is the set of all $J(z)$ with $z \in \mathrm{Lip}(V,F')$. Every such element z is in $\mathrm{Lip}(V^{\oplus},F'^{\oplus})$, and the isomorphism Φ maps it into $\mathrm{Lip}(V^{\oplus},0^{\oplus})$. In Sect. 10.2, it has been proved that $L_c(a) \in \mathrm{Lip}(V,0)$ for all $c \in \mathrm{Lip}(V^{\oplus},0^{\oplus})$ and all $a \in \mathrm{Lip}(V,0)$; consequently, $J(z) = L_{\Phi(z)}(1) \in \mathrm{Lip}(V,0)$.

Third step: we prove that $\mathrm{Lip}(V,0) \subset \mathrm{Lip}(V,0;\beta)$. As above, $F' = \beta+\beta^o$. Since $\mathrm{char}(K) = 2$, the word "symmetric" has the same meaning as "skew symmetric". If β is symmetric, then $F' = 0$, and since $\mathcal{B}(V,0)$ is the deformation of the neutral algebra $\mathcal{B}(V,0;\beta)$ by means of $-\beta$, the result of the second step gives $\mathrm{Lip}(V,0) \subset \mathrm{Lip}(V,0;\beta)$. Besides, if there is a symmetric bilinear form γ such that $\beta(v,v) = \gamma(v,v)$ for all $v \in V$, then β is symmetric too:

$$\begin{aligned}\beta(u,v)+\beta(v,u) &= \beta(u{+}v,u{+}v)-\beta(u,u)-\beta(v,v)\\ &= \gamma(u{+}v,u{+}v)-\gamma(u,u)-\gamma(v,v) = \gamma(u,v)+\gamma(v,u) = 0.\end{aligned}$$

Now let us consider the non-zero element $a \in \mathrm{Lip}(V,0)$ defined by (10.1). If the restriction of β to $\mathrm{Sup}(a)$ is symmetric, then $a \in \mathrm{Lip}(V,0;\beta)$. Indeed, a is in $\mathrm{Lip}(\mathrm{Sup}(a),0)$ because (8.8) is true when $F = 0$; and we have $\mathrm{Lip}(\mathrm{Sup}(a),0) \subset \mathrm{Lip}(\mathrm{Sup}(a),0;\beta) \subset \mathrm{Lip}(V,0;\beta)$.

If β is not symmetric on $\mathrm{Sup}(a)$, we prove by induction on the dimension s of $\mathrm{Sup}(a)$ that this element $a \in \mathrm{Lip}(V,0)$ belongs to $\mathrm{Lip}(V,0;\beta)$. The cases $s \leqslant 1$ are trivial since $v^{2\star} = v^2 + \beta(v,v)$ for all $v \in V$. Therefore, we suppose $s \geqslant 2$. To prove that a is in $\mathrm{Lip}(V,0;\beta)$, it suffices to prove that the $\star$-product of a and some element of $\mathrm{GLip}(V,0;\beta)$ belongs to $\mathrm{Lip}(V,0;\beta)$. This procedure allows us to reduce the problem to the case $\mathrm{Ker}(a) = 0$. Indeed, let us consider an invertible generator $x = \lambda + \mu w^{2\star}$ of $\mathrm{GLip}(V,0;\beta)$ such that $w \in \mathrm{Sup}(a)$; we have

$$x \star a = L_{\lambda+\mu(\beta_w+w)^2}(a).$$

If there is $u \in \mathrm{Ker}(a)$ such that $\beta(w,u) \neq 0$, we can apply (10.6), which shows that the dimension of $\mathrm{Ker}(x \star a)$ is strictly smaller than $\dim(\mathrm{Ker}(a))$, while $\mathrm{Sup}(x \star a) \subset \mathrm{Sup}(a)$. By iteration, we can reach the case $\mathrm{Ker}(a) = 0$ without increasing the support. If there is no $u \in \mathrm{Ker}(a)$ such that $\beta(w,u) \neq 0$ for some $w \in \mathrm{Sup}(a)$, the remedy is yet easier because

$$a = \kappa \exp(\theta)\, u_1^2\, u_2^2 \cdots u_k^2 = \kappa\, \exp(\theta) \star u_1^{2\star} \star u_2^{2\star} \star \cdots \star u_k^{2\star};$$

the induction hypothesis ensures that $\exp(\theta)$ is in $\mathrm{Lip}(V,0;\beta)$, and a too.

We continue with the hypothesis $\mathrm{Ker}(a) = 0$, which allows us to use the easy formula (10.3) instead of (10.6). We know that $2s$ is the smallest integer such that a has a non-zero component in $\mathcal{B}^{2s}(V,0)$. Let us search for an element $x = \lambda + \mu w^{2\star}$ (with $\lambda\mu \neq 0$ and $w \in \mathrm{Sup}(a)$) such that $x \star a$ has a null component in $\mathcal{B}^{2s}(V,0)$; if such an element x exists, $\mathrm{Sup}(x \star a)$ is strictly smaller than $\mathrm{Sup}(a)$ and the induction hypothesis implies that a is in $\mathrm{Lip}(V,0;\beta)$. The equality (10.3) shows that

$$x \star a = a\left(\lambda + \mu\,(\beta(w,w) + w^2\rfloor_\beta\,\theta) + \mu\,(w + w\rfloor_\beta\theta)^2\right).$$

Since the subspace $\mathcal{B}^{2s}(\mathrm{Sup}(a),0)$ has dimension 1, for all v and v' in $\mathrm{Sup}(a)$, the component of $vv'a$ in $\mathcal{B}^{2s}(V,0)$ is colinear to the component of a; the former is the product of the latter by some scalar $\delta(v,v')$, and δ is a bilinear form on $\mathrm{Sup}(a)$. The component of $x\star a$ in $\mathcal{B}^{2\,s}(V,F)$ vanishes if and only if

$$\lambda+\mu\,\big(\beta(w,w)+w^2\rfloor_\beta\,\theta\big)+\mu\,\delta\big(w+w\rfloor_\beta\theta,\;w+w\rfloor_\beta\theta\big)=0.$$

A suitable factor x exists when the quadratic form $w\longmapsto\beta(w,w)$ is not equal to the quadratic form $w\longmapsto w^2\rfloor\theta+\delta(w+w\rfloor\theta,\;w+w\rfloor\theta)$ on $\mathrm{Sup}(a)$. But when these quadratic forms are equal on $\mathrm{Sup}(a)$, the bilinear form β is symmetric on $\mathrm{Sup}(a)$, and we already know that a is in $\mathrm{Lip}(V,0;\beta)$. Indeed, the equality $\tau(\theta)=\theta$ implies that the bilinear form $(v,v')\longmapsto vv'\rfloor\theta$ is symmetric. Since τ leaves invariant every element of $\mathcal{B}^{2s}(\mathrm{Sup}(a),0)$, the bilinear form $(v,v')\longmapsto\delta(v,v')$ is also symmetric on $\mathrm{Sup}(a)$. Thus, if the above quadratic forms are equal, there is a symmetric bilinear form γ on $\mathrm{Sup}(a)$ such that $\beta(w,w)=\gamma(w,w)$ for all $w\in\mathrm{Sup}(a)$, and this fact implies that β is symmetric on $\mathrm{Sup}(a)$. □

10.4. Consequences of the Invariance Property

The alternating bilinear forms F on V are the bilinear forms such that $F=\beta+\beta^o$ for some bilinear form β on V. Therefore, the study of the corresponding meson algebras is equivalent to the study of the deformations $\mathcal{B}(V,0;\beta)$ of the neutral algebra $\mathcal{B}(V,0)$. It may be advantageous to study $\mathcal{B}(V,0;\beta)$ because the invariance property (Theorem 10.2) enables us to deduce properties of $\mathrm{Lip}(V,0;\beta)$ from properties of the easier monoid $\mathrm{Lip}(V,0)$. This fact explains the great importance of the neutral monoid $\mathrm{Lip}(V,0)$ despite the ludicrous triviality of the group $\mathrm{O}(V,0)$. This argument shows that the properties (8.8) and (8.9) are true for all the meson algebras which we deal with in the present section, and the same for the next theorem, which is a partial repetition of Theorem 9.7.

Theorem 10.3. *Let a be a non-zero element of* $\mathrm{Lip}(V,F)$*, and let s be the smallest integer such that a belongs to $\mathcal{B}^{\leqslant 2s}(V,F)$. We have* $\dim(\mathrm{Sup}(a))=s$*, and* $\mathrm{Sup}(a)$ *is also the support of the image of a in $\mathcal{B}^{2s}(V,0)$.*

Theorem 10.4 is the other wonderful property announced in Sect. 8.6.

Theorem 10.4. *Let the algebra $\mathcal{B}(V^\oplus,V^\oplus)$ act on the space $\mathcal{B}(V,F)$ by the homomorphism $z\longmapsto L_z$ defined in Sect. 7. We have $L_c(a)\in$* $\mathrm{Lip}(V,F)$ *for all $c\in$* $\mathrm{Lip}(V^\oplus,F^\oplus)$ *and all $a\in$* $\mathrm{Lip}(V,F)$*, and the group* $\mathrm{GLip}(V^\oplus,F^\oplus)$ *acts transitively on the set of all non-zero elements of* $\mathrm{Lip}(V,F)$*.*

Proof. With the same notation as in Sect. 7, let us consider an algebra $\mathcal{B}(V,F)$, a deformation $\mathcal{B}(V,F;\beta)$, and the two algebra isomorphisms

$$\Phi:\mathcal{B}(V^\oplus,F'^\oplus)\to\mathcal{B}(V^\oplus,F^\oplus)\quad\text{and}\quad J:\mathcal{B}(V,F')\to\mathcal{B}(V,F;\beta).$$

Because of (7.1) and (7.5), for all $x'\in\mathcal{B}(V,F')$, $v\in V$ and $\ell\in V^*$, we have

$$J(vx')=v\star J(x')=v\,J(x')+\beta_v\rfloor J(x')\quad\text{and}\quad J(\ell\rfloor x')=\ell\rfloor J(x').$$

Let the algebra $\mathcal{B}(V^{\oplus}, F'^{\oplus})$ act on $\mathcal{B}(V, F')$ on the left side, and for each $z' \in \mathcal{B}(V^{\oplus}, F'^{\oplus})$, let $L'_{z'}$ be its operation on $\mathcal{B}(V, F')$. We have

$$J(L'_{\ell+v}(x')) = L_{\ell+\beta_v+v}(J(x')) = L_{\Phi(\ell+v)}(J(x')).$$

Consequently, for all $z' \in \mathcal{B}(V^{\oplus}, F'^{,\oplus})$ and all $x' \in \mathcal{B}(V, F')$, we have

$$J(L'_{z'}(x')) = L_{\Phi(z')}(J(x')). \tag{10.7}$$

Now, let us consider these three assertions:

(a) $J(\mathrm{Lip}(V, F')) = \mathrm{Lip}(V, F)$ (or equivalently, $\mathrm{Lip}(V, F; \beta) = \mathrm{Lip}(V, F)$).

(b) $L_c(a) \in \mathrm{Lip}(V, F)$ for all $c \in \mathrm{Lip}(V^{\oplus}, F^{\oplus})$ and $a \in \mathrm{Lip}(V, F)$, and $\mathrm{GLip}(V^{\oplus}, F^{\oplus})$ acts transitively in the set of all non-zero $a \in \mathrm{Lip}(V, F)$.

(c) The same assertion as (b) is true for F' instead of F (and subsequently, L', c' and a' instead of L, c and a).

The assertions (a) and (b) imply (c) because of (10.7), and because Φ induces an isomorphism from $\mathrm{Lip}(V^{\oplus}, F'^{\oplus})$ onto $\mathrm{Lip}(V^{\oplus}, F^{\oplus})$. As it is explained at the end of Sect. 10.2, the assertion (b) is true when $F = 0$; consequently, it is true for all alternating bilinear forms F on V. □

In the above proof (which never uses the hypothesis $\mathrm{char}(K) = 2$), it is worth noticing that the assertions (b) and (c) imply (a). Therefore, if Theorem 10.4 can be proved independently of Theorem 10.2, Theorem 10.2 becomes a consequence of Theorem 10.4. Up to now, I have given the priority to Theorem 10.2 over Theorem 10.4 because I happened to discover Theorem 10.2 before Theorem 10.4. But it is not absurd to imagine that an inversion of priority might be advantageous. I know how to achieve this inversion of priority in the easier study of Lipschitz monoids in Clifford algebras.

11. When $\mathrm{char}(K) = 2$ and F is not Alternating

When $\mathrm{char}(K) = 2$ and F is not alternating, we need the subspace V_0 of all $v \in V$ such that $F(v, v) = 0$ because

$$\mathrm{Lip}(V, F) = \mathrm{Lip}(V_0, F) \qquad \text{and} \quad \mathrm{SLip}(V, F) = \mathrm{SLip}(V_0, F).$$

The main purpose of Sect. 11 is the following theorem.

Theorem 11.1. *When* $\mathrm{char}(K) = 2$ *and* F *is not alternating,* $\mathrm{O}(V, F)$ *is the subgroup of all* $g \in \mathrm{Aut}(V, F)$ *such that* $g(v) = v$ *for all* $v \in V_0^{\perp}$. *It is generated by the transvections, and it is equal to* $\mathrm{Aut}(V, F)$ *when* F *is nondegenerate. The homomorphism* $\mathrm{GLip}(V, F) \to \mathrm{O}(V, F)$ *is surjective.*

11.1. The Wonderful Properties of $\mathrm{Lip}(V, F)$

From the equality $\mathrm{Lip}(V, F) = \mathrm{Lip}(V_0, F)$, it follows that $\mathrm{Lip}(V, F)$ inherits all the wonderful properties of $\mathrm{Lip}(V_0, F)$ that have been established in Sect. 10: *Theorems 10.2, 10.3 and 10.4 are still true when* F *is not alternating, and the properties (8.8) and (8.9) too.*

When Theorem 10.2 involves the deformation of $\mathcal{B}(V, F)$ by a bilinear form β, the subspace V_0 is the same for F as for the new symmetric bilinear form $F+\beta+\beta^o$, and the equality $\mathrm{Lip}(V, F; \beta) = \mathrm{Lip}(V, F)$ occurs in the subalgebra $\mathcal{B}(V_0; F)$.

When Theorem 10.4 involves $(V^\oplus, F^\oplus)$, the largest subspace of $V^\oplus$ on which $F^\oplus$ is alternating is $V^*\oplus V_0$, and there is a canonical "homometry"

$$(V^*\oplus V_0,\, F^\oplus) \to (V_0^\oplus, F^\oplus), \qquad \ell + v \longmapsto \ell_0 + v$$

where ℓ_0 is the restriction of ℓ to V_0, and v must remain in V_0. It extends to an algebra homomorphism between the corresponding meson algebras, and determines a surjective monoid morphism

$$\mathrm{Lip}(V^\oplus, F^\oplus) \to \mathrm{Lip}(V_0^\oplus, F^\oplus).$$

If c is an element of $\mathrm{Lip}(V^\oplus, F^\oplus)$, its operation on $\mathcal{B}(V,F)$ leaves the subalgebra $\mathcal{B}(V_0,F)$ invariant, and the operation of c on $\mathcal{B}(V_0,F)$ is also the operation of the image of c in $\mathrm{Lip}(V_0^\oplus, F^\oplus)$.

Since the wonderful properties are obviously valid, we may forget them up to the end of Sect. 11, and focus our attention on Theorem 11.1.

11.2. The Decomposition $V = V^\perp \oplus P_1 \oplus P_2 \oplus P_3 \oplus P_4$

If U is a subspace of V containing $V^\perp$, the bilinear mapping $U\times V \to K$ determined by F induces a duality between the quotients $U/V^\perp$ and $V/U^\perp$. Let us consider $U = V_0 \cap V_0^\perp$ and $U^\perp = V_0 + V_0^\perp$, and let us choose subspaces P_1 and P_4 such that

$$V_0 \cap V_0^\perp = V^\perp \oplus P_1 \qquad \text{and} \quad V = (V_0 + V_0^\perp) \oplus P_4;$$

now F induces a duality between P_1 and P_4. Let us also choose subspaces P_2 and P_3 such that

$$V_0 \cap P_4^\perp = V^\perp \oplus P_2 \qquad \text{and} \quad V_0^\perp \cap P_4^\perp = V^\perp \oplus P_3;$$

since F induces a duality between P_4 and P_1, we have

$$V_0 = V^\perp \oplus P_1 \oplus P_2, \qquad V_0^\perp = V^\perp \oplus P_1 \oplus P_3,$$

and $V = V^\perp \oplus P_1 \oplus P_2 \oplus P_3 \oplus P_4$.

With respect to this decomposition, F can be described in this way:

F	$V^\perp$	P_1	P_2	P_3	P_4
$V^\perp$	0	0	0	0	0
P_1	0	0	0	0	F_{14}
P_2	0	0	F_{22}	0	0
P_3	0	0	0	F_{33}	0
P_4	0	F_{41}	0	0	F_{44}

(F_{14}, F_{41}) is a duality,
F_{22} is symplectic,
$F_{33}\oplus F_{44}$ is anisotropic.

It is clear that the restriction F_{22} of F to $P_2\times P_2$ is alternating and non-degenerate. The quadratic form $v \longmapsto F(v,v)$ is a homomorphism of additive groups $V \to K$; since $P_3 \oplus P_4$ is complementary to its kernel V_0, its restriction to $P_3 \oplus P_4$ is injective, therefore, anisotropic.

Let g be an automorphism of (V,F), and let (g_{ij}) be the family of its components $P_j \to V \to V \to P_i$ with $i, j \in \{0,1,2,3,4\}$ (if we identify P_0

with $V^\perp$). Let us prove that

$$g = \begin{bmatrix} g_{00} & g_{01} & g_{02} & g_{03} & g_{04} \\ 0 & \mathbf{1} & g_{12} & 0 & g_{14} \\ 0 & 0 & g_{22} & 0 & g_{24} \\ 0 & 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} \end{bmatrix} \tag{11.1}$$

Proof. It is clear that g leaves $V^\perp$, V_0 and $V_0^\perp$ invariant. Moreover, $g(v)-v \in V_0$ for every $v \in V$ because $F(g(v), g(v)) = F(v, v)$. These obvious facts prove the vanishing of g_{10}, g_{20}, g_{30}, g_{40}, g_{21}, g_{31}, g_{41}, g_{32}, g_{42}, g_{23}, g_{43} and g_{34}; they also prove that $g_{33} = \mathbf{1}$ and $g_{44} = \mathbf{1}$. It remains to prove that $g_{11} = \mathbf{1}$ and $g_{13} = 0$. The equality $g_{11} = \mathbf{1}$ follows from $g_{44} = \mathbf{1}$ because F determines a duality between P_1 and P_4. For all $v_3 \in P_3$ and $v_4 \in P_4$, we have $F(g(v_3), g(v_4)) = F(v_3, v_4) = 0$, therefore, $F(g_{13}(v_3), v_4) = 0$, and the duality between P_1 and P_4 implies $g_{13}(v_3) = 0$. □

Let g be a linear transformation like (11.1). Under which (necessary and sufficient) conditions is it an automorphism of (V, F)? A direct calculation brings the following three conditions:

$$\forall u,\ v \in P_2, \qquad F_{22}(g_{22}(u), g_{22}(v)) = F_{22}(u, v), \tag{11.2}$$

$$\forall u \in P_2,\ \forall v \in P_4, \qquad F_{22}(g_{22}(u), g_{24}(v)) = F_{14}(g_{12}(u), v), \tag{11.3}$$

$$\forall u,\ v \in P_4, \qquad F_{22}(g_{24}(u), g_{24}(v)) = F_{14}(g_{14}(u), v) + F_{41}(u, g_{14}(v)). \tag{11.4}$$

If Theorem 11.1 is true, the above automorphism g of (V, F) is an orthogonal transformation if and only if $g_{00} = \mathbf{1}$, $g_{01} = 0$ and $g_{03} = 0$. These equalities are trivially true when F is non-degenerate.

In the field K, the squares constitute a subfield K^{sq}, and the mapping $\kappa \longmapsto \kappa^2$ is a field isomorphism from K onto its subfield K^{sq}. When the dimension of K over K^{sq} is finite, the dimension of $P_3 \oplus P_4$ over K cannot be greater than the dimension of K over K^{sq}. Indeed, the quadratic form $v \longmapsto F(v, v)$ maps $P_3 \oplus P_4$ bijectively onto a K^{sq}-subspace of K, and the dimension of the latter over K^{sq} is the dimension of the former over K.

Let $(e_1, \ldots, e_n)$ be a basis of V such that $F(e_1, e_1) \neq 0$. The dimension of V_0 reaches the maximal possible value $n-1$ if and only if each scalar $F(e_1, e_1)\, F(e_i, e_i)$ (for $i = 2, 3, \ldots, n$) has a square root r_i in K (so that $F(e_1, e_1)e_i + r_i e_1 \in V_0$). When $\dim(V_0) = n-1$, the dimension of $P_3 \oplus P_4$ must be 1, and since $\dim(P_2)$ is even, we meet two cases: if $\dim(V/V^\perp)$ is even, then $P_3 = 0$ and $\dim(P_1) = \dim(P_4) = 1$; but if $\dim(V/V^\perp)$ is odd, then $P_1 = P_4 = 0$ and $\dim(P_3) = 1$.

The field K is said to be perfect if $K^{sq} = K$. Every finite field is perfect. When K is perfect, it is clear that V_0 is a hyperplane of V. When K is not perfect, a finite field extension $K \to K'$ suffices to bring an extension $K' \otimes (V, F)$ where the bilinear form is alternating on a hyperplane of $K' \otimes V$; it suffices to adjoin to K some square roots of elements of K, the number of which is smaller than n.

11.3. The Subgroup $\mathrm{O}^\dagger(V, F)$ Generated by the Transvections

Lemma 11.2. *The group* $\mathrm{O}^\dagger(V,F)$ *of all* $g \in \mathrm{Aut}(V;F)$ *such that* $\ker(g-\mathbf{1}) \supset V_0^\perp$ *is also the subgroup of* $\mathrm{O}(V,F)$ *generated by the transvections.*

Proof. Every transvection $v \longmapsto v + \lambda F(v,u)u$ is equal to $G_{1+\lambda u^2}$ for some $(u,\lambda) \in V_0 \times K$. It is an element of $\mathrm{O}(V,F)$ which leaves invariant every element of $V_0^\perp$. Conversely, every element of $\mathrm{O}^\dagger(V,F)$ is a product of transvections; this assertion shall be proved in two steps.

First step: $\mathrm{Hom}_S(P_4, P_1 \oplus V^\perp)$ and the mapping $h \longmapsto h^\dagger$. Let us consider the space $\mathrm{Hom}(P_4, P_1 \oplus V^\perp)$ of all linear mappings $h : P_4 \to P_1 \oplus V^\perp$. In the matrix (11.1), the pair (g_{14}, g_{04}) represents an element of this space. Every element h of this space determines a bilinear form on P_4 which maps every (v, v') to $F(h(v), v')$, and h is said to be symmetric if this bilinear form is symmetric; the subspace of symmetric elements is denoted by $\mathrm{Hom}_S(P_4, P_1 \oplus V^\perp)$; thus, by definition,

$$h \in \mathrm{Hom}_S(P_4, P_1 \oplus V^\perp) \iff \forall v, v' \in P_4, \quad F(h(v), v') = F(v, h(v')).$$

Every element of $\mathrm{Hom}(P_4, P_1 \oplus V^\perp)$ has a component in $\mathrm{Hom}(P_4, P_1)$ and a component in $\mathrm{Hom}(P_4, V^\perp)$; only the former is involved in the symmetry condition; thus we may write

$$\mathrm{Hom}_S(P_4,\ P_1 \oplus V^\perp) = \mathrm{Hom}_S(P_4, P_1) \oplus \mathrm{Hom}(P_4, V^\perp).$$

Since F determines a duality between P_4 and P_1, there is a linear bijection $P_1 \otimes P_1 \to \mathrm{Hom}(P_4, P_1)$ which maps every $u \otimes u'$ to $v \longmapsto F(v, u')\, u$, and by this bijection, the subspace $\mathrm{Hom}_S(P_4, P_1)$ is the image of the subspace of symmetric tensors. There is also a bijection $V^\perp \otimes P_1 \to \mathrm{Hom}(P_4, V^\perp)$ which maps every $u_0 \otimes u_1$ to $v \longmapsto F(v, u_1)\, u_0$. Nevertheless, we are rather concerned with the surjective mapping

$$(P_1 \oplus V^\perp) \otimes (P_1 \oplus V^\perp) \to \mathrm{Hom}(P_4, P_1 \oplus V^\perp), \quad u \otimes u' \longmapsto \big(v \longmapsto F(v, u')\, u\big).$$

This mapping maps $(P_1 \oplus V^\perp) \otimes V^\perp$ to 0, and it maps the subspace of symmetric tensors onto $\mathrm{Hom}_S(P_4, P_1 \oplus V^\perp)$. Indeed, the element of $\mathrm{Hom}(P_4, V^\perp)$ which is the image of $u_0 \otimes u_1$ (for some $u_0 \in V^\perp$ and $u_1 \in P_1$), is also the image of the symmetric tensor $u_0 \otimes u_1 + u_1 \otimes u_0$.

With every $h \in \mathrm{Hom}_S(P_4, P_1 \oplus V^\perp))$ is associated the element $h^\dagger$ of $\mathrm{O}^\dagger(V,F)$ that maps every vector in $V_0 + V_0^\perp$ to itself, and every vector $v \in P_4$ to $v + h(v)$. Indeed, the conditions (11.2), (11.3) and (11.4) are satisfied when $g_{22} = \mathbf{1}$, $g_{12} = 0$, $g_{24} = 0$ and g_{14} is the component of h in $\mathrm{Hom}_S(P_4, P_1)$.

It is clear that $(h + h')^\dagger = h^\dagger \circ h'^\dagger$ for all $h, h' \in \mathrm{Hom}_S(P_4, P_1 \oplus V^\perp)$.

Second step: the exact sequence

$$\{0\} \to \mathrm{Hom}_S(P_4, P_1 \oplus V^\perp) \to \mathrm{O}^\dagger(V,F) \to \mathrm{O}(V_0, F) \to \{1\}. \qquad (11.5)$$

The second arrow in (11.5) is the mapping $h \longmapsto h^\dagger$. The third arrow maps every transformation to its restriction to V_0; this restriction is an element of $\mathrm{Aut}(V_0, F)$ which belongs to $\mathrm{O}(V_0, F)$ because it leaves invariant every element of $V_0 \cap V_0^\perp$. The third arrow is surjective because every element of $\mathrm{O}(V_0, F)$ is a product of transvections, and every transvection of (V_0, F) extends naturally to a transvection of (V, F). We must also prove that $g = h^\dagger$ for some $h \in \mathrm{Hom}_s(P_4, P_1 \oplus V^\perp)$ if g leaves invariant every vector in V_0. Let us come back to the matrix (11.1), and suppose that $g_{22} = \mathbf{1}$ and $g_{12} = 0$. Since F_{22} is non-degenerate, the equality (11.3) implies $g_{24} = 0$. Consequently, (11.4) implies $g_{14} \in \mathrm{Hom}_S(P_4, P_1)$, and $g = h^\dagger$ if h is the element of $\mathrm{Hom}(P_4, P_1 \oplus V^\perp)$ determined by (g_{14}, g_{04}).

Since every element of $\mathrm{O}(V_0, F)$ is a product of transvections, which naturally extend to transvections of (V, F), the proof of Lemma 11.2 is now reduced to the proof of the following assertion: every transformation $h^\dagger$ is a product of transvections. Every $h \in \mathrm{Hom}_S(P_4, P_1 \oplus V^\perp)$ is the image of a symmetric tensor in $(P_1 \oplus V^\perp) \otimes (P_1 \oplus V^\perp)$, and every symmetric tensor is a sum of terms like $\lambda u \otimes u$ with $u \in P_1 \oplus V^\perp$ and $\lambda \in K$. Therefore, it suffices to consider the transformation that maps every element of $V_0 + V_0^\perp$ to itself, and every $v \in P_4$ to $v + \lambda F(v, u) u$. Actually, this transformation is exactly the transvection determined by (u, λ). □

11.4. The Group $\mathbf{O}(V, F)$ is Equal to $\mathbf{O}^\dagger(V, F)$

Lemma 11.3 shall complete the proof of Theorem 11.1.

Lemma 11.3. *If a is an invertible element of $\mathcal{B}_0(V, F)$ such that $\tau(a) = a$ and $ava^{-1} \in V$ for all $v \in V$, then* $\mathrm{Sup}(a) \subset V_0$ *and* $\ker(G_a - \mathbf{1}) \supset V_0^\perp$.

Proof. Since $\ker(G_a - \mathbf{1}) = \mathrm{Sup}(a)^\perp$ and $V_0 \supset V^\perp$, the inclusions $\mathrm{Sup}(a) \subset V_0$ and $\ker(G_a - \mathbf{1}) \supset V_0^\perp$ are equivalent to each other. We also know that $G_a(v) - v \in V^\perp$ for all $v \in V_0^\perp$ (recall (11.1)). Therefore, Lemma 11.3 is obvious when F is non-degenerate, and we may assume that $V^\perp \neq 0$. The proof of Lemma 11.3 shall be achieved in four steps.

First step: reduction to the case $\dim(V^\perp) = 1$. Let us suppose that $\dim(V^\perp) \geqslant 2$, and let H be a hyperplane of $V^\perp$. The quotient $V' = V/H$ may be identified with the direct sum of $V^\perp/H$ and $P_1 \oplus P_2 \oplus P_3 \oplus P_4$, and F induces a symmetric bilinear form F' on V'. Since $G_a(v) = v$ for all $v \in V^\perp$ (recall Theorem 8.1), G_a induces an automorphism of (V', F'). If this automorphism of (V', F') leaves invariant all elements of $P_1 \oplus P_3$, then $G_a(v) - v \in H$ for all $v \in P_1 \oplus P_3$. If this is true for every hyperplane H of $V^\perp$, it follows that $G_a(v) = v$ for all $v \in V_0^\perp$. Moreover, the quotient mapping $\pi : V \to V/H$ extends to an algebra homomorphism $\mathcal{B}(\pi)$ from $\mathcal{B}(V, F)$ onto $\mathcal{B}(V', F')$. If $a' = \mathcal{B}(\pi)(a)$, then the above automorphism of (V', F') is the orthogonal transformation $G_{a'}$. Consequently, it suffices to prove Lemma 11.3 when $\dim(V^\perp) = 1$.

Second step: reduction to the case $\dim(V_0) = n-1$. We must prove that $F(v, v) = 0$ for every $v \in \mathrm{Sup}(a)$. After a field extension $K \to K'$, the support of a in $K' \otimes V$ is equal to $K' \otimes \mathrm{Sup}(a)$. Therefore, it suffices to prove that the support of a in $K' \otimes V$ is contained in $(K' \otimes V)_0$. This subspace $(K' \otimes V)_0$ may be much larger than $K' \otimes V_0$. At the end of Sect. 11.2, it is explained that, after a suitable field extension $K \to K'$, we may obtain a bilinear form $K' \otimes F$ that is alternating on a hyperplane of $K' \otimes V$. Therefore, it suffices to treat the case $\dim(V_0) = n-1$.

In the third and fourth steps of the proof, we suppose that $\dim(V^\perp) = 1$, and that $\dim(V_0) = n-1$, or equivalently, $\dim(P_3 \oplus P_4) = 1$, which implies either $P_3 = 0$ or $P_4 = 0$.

Third step: we suppose that $P_4 = 0$, and consequently, $P_1 = 0$. Now $V^\perp$ is spanned by a vector u_0 and P_3 by a vector u_3. We can describe F and G_a in this way:

F	$V^\perp$	P_2	P_3
$V^\perp$	0	0	0
P_2	0	F_{22}	0
P_3	0	0	F_{33}

$$G_a = \begin{bmatrix} \mathbf{1} & g_{02} & g_{03} \\ 0 & g_{22} & 0 \\ 0 & 0 & \mathbf{1} \end{bmatrix}$$

The component g_{03} is determined by the scalar λ such that $G_a(u_3) = \lambda u_0$, and we must prove that $\lambda = 0$. Let g' be the automorphism of (V, F) that is represented by the same matrix as G_a but for one exception: $g'_{03} = 0$ (but $g'_{22} = g_{22}$ and $g'_{02} = g_{02}$). Since $\ker(g' - \mathbf{1}) \supset V_0^\perp$, Lemma 11.2 allows us to claim that $g' = G_b$ for some $b \in \mathrm{GLip}(V, F)$. Now let us consider $G_a^{-1} G_b$: this transformation leaves invariant every vector in $V^\perp \oplus P_2$, but maps u_3 to $u_3 - \lambda u_0$. Because of Theorem 8.1, $\mathrm{Sup}(a^{-1}b)$ is contained in the subspace orthogonal to $V^\perp \oplus P_2$, which is the plane $V^\perp \oplus P_3$. Thus $a^{-1}b$ induces an orthogonal transformation of this plane. But because of Lemma 8.2, this orthogonal transformation is the identity, and we conclude that $\lambda = 0$.

Fourth step: we suppose that $P_3 = 0$. We can describe F and G_a in this way:

F	$V^\perp$	P_1	P_2	P_4
$V^\perp$	0	0	0	0
P_1	0	0	0	F_{14}
P_2	0	0	F_{22}	0
P_4	0	F_{41}	0	F_{44}

$$G_a = \begin{bmatrix} \mathbf{1} & g_{01} & g_{02} & g_{04} \\ 0 & \mathbf{1} & g_{12} & g_{14} \\ 0 & 0 & g_{22} & g_{24} \\ 0 & 0 & 0 & \mathbf{1} \end{bmatrix}$$

Now $V^\perp$, P_1 and P_4 are lines spanned by vectors u_0, u_1 and u_4, and F_{44} and F_{14} are determined by the non-zero scalars $F(u_4, u_4)$ and $F(u_1, u_4)$. There is a scalar λ such that $g_{01}(u_1) = \lambda u_0$, and we must prove that $\lambda = 0$. Let g' be the automorphism of (V, F) that has the same matrix as G_a but for one exception, which is $g'_{01} = 0$. Since $\ker(g' - \mathbf{1}) \supset V_0^\perp$, we have $g' = G_b$ for some $b \in \mathrm{GLip}(V, F)$. Thus $G_a^{-1} G_b$ leaves invariant every vector in $V^\perp \oplus P_2 \oplus P_4$, but maps u_1 to $u_1 - \lambda u_0$. It follows that $\mathrm{Sup}(a^{-1}b)$ is contained in the subspace orthogonal to $V^\perp \oplus P_2 \oplus P_4$, which is the plane spanned by u_0 and $w = F(u_4, u_4)\, u_1 - F(u_1, u_4)\, u_4$. Consequently, $a^{-1}b$ determines an orthogonal transformation of this plane. Because of Lemma 8.2, it is the identity, and since $G_a^{-1} G_b(w) = w - \lambda F(u_4, u_4)\, u_0$, we conclude that $\lambda = 0$. □

It may be sensible to question whether the case $\mathrm{char}(K) = 2$ deserves the amount of efforts displayed in Sects. 10 and 11. The research about Lipschitz monoids in Clifford algebras has not yet reached a general agreement, and the research about meson algebras does not concern many people at present. Thus there may be severe disagreements about the conventions that lead these researches. The definitions are the most important part of these conventions. The relevance of a set of conventions much depends on its efficiency in the conception and in the proof of non-trivial new theorems. In the present domain of research, its relevance also depends on its compatibility with fields of characteristic 2. Since there is still a great uncertainty in the choice of the conventions, the confrontation with the characteristic 2 may help us to make the good decisions.

12. Final Comments

The present work confirms that it is advantageous to imitate the treatment of Clifford algebras when meson algebras are under consideration. Since the mesonic grade automorphism σ could not be a sufficient counterpart of the Cliffordian grade automorphism, the automorphism τ of $\mathcal{B}_0(V,F)$ came to light. Then it became possible to imagine mesonic counterparts of the interior multiplications and of the deformations, which are efficient tools in the study of Clifford algebras. The concept of Lipschitz monoid, which is still under discussion in the study of Clifford algebras, could be easily adapted to meson algebras, and Jacobson's homomorphism much helped to discover the mesonic adaptation (see Sect. 9.2). In my opinion, the existence of a mesonic adaptation strongly supports the relevance of this concept of Lipschitz monoid.

Nevertheless, there is a very important property of Clifford algebras for which there is no mesonic counterpart: if a space V provided with a quadratic form Q is the orthogonal direct sum of two subspaces U and U', there is an isomorphism $\mathrm{Cl}(V,Q) \to \mathrm{Cl}(U,Q)\hat{\otimes}\mathrm{Cl}(U',Q)$ which maps every $u+u'$ (with $u \in U$ and $u' \in U'$) to $u\otimes 1+1\otimes u'$. Although no mesonic counterpart has come to light, it is sensible to ask what may be said when (V,F) is the orthogonal direct sum of two subspaces U and U'. There is an answer in [4], where it is explained that $\mathcal{B}(V,F)$ admits a gradation over the group $\mathcal{D} = \mathbb{Z}\times(\mathbb{Z}/2\mathbb{Z})$ provided with the following non-commutative $*$-multiplication:

$$\forall k,\, l \in \mathbb{Z}, \quad \forall p,\, q \in \mathbb{Z}/2\mathbb{Z}, \qquad (k,p)*(l,q) = (k+(-1)^p l,\ p+q),$$

and that this gradation attributes the degree $(1,1)$ to all elements of U, and the degree $(0,1)$ to all elements of U'. If $\mathcal{B}_{k,p}(V,F)$ is the component of $\mathcal{D}$-degree (k,p), it is clear that $\mathcal{B}_{k,p}(V,F) \subset \mathcal{B}_p(V,F)$.

Lemma 4.4 implies that $\mathcal{B}(V,F)$ is spanned by the products

$$x\, u_1' u_1 u_2' u_2 \cdots u_k' u_k\, x' \quad \text{where} \quad x \in \mathcal{B}(U,F), \quad u_1,\ldots,u_k \in U,$$
$$x' \in \mathcal{B}(U',F), \quad u_1',\ldots,u_k' \in U'.$$

Of course, $u_1'u_1\cdots u_k'u_k$ means 1 if $k=0$, and we may require x and x' to be even or odd. The $\mathcal{D}$-degree of such a product depends on the parities ∂x and $\partial x'$. When $(\partial x, \partial x')$ is equal to $(0,0)$, resp. $(0,1)$, resp. $(1,0)$, resp. $(1,1)$, the $\mathcal{D}$-degree of this product is $(-k,0)$, resp. $(-k,1)$, resp. $(k+1,1)$, resp. $(k+1,0)$. Therefore, the image $\mathcal{B}(U,F)\,\mathcal{B}(U',F)$ of the injective linear mapping

$$\mathcal{B}(U,F)\otimes\mathcal{B}(U',F) \to \mathcal{B}(V,F), \qquad x\otimes x' \longmapsto xx',$$

is the direct sum of the four components of $\mathcal{D}$-degrees $(0,0)$, $(0,1)$, $(1,1)$ and $(1,0)$. Thus this $\mathcal{D}$-gradation affords a projector from $\mathcal{B}(V,F)$ onto this subspace $\mathcal{B}(U,F)\,\mathcal{B}(U',F)$. For instance, for u, $v\in U$ and u', $v'\in U'$, this projector annihilates $u'u$, $uu'v$ and $u'uv'$ because these elements are $\mathcal{D}$-homogeneous of degrees $(-1,0)$, $(2,1)$ and $(-1,1)$; but it leaves $u'uv$ and $u'v'u$ invariant because their $\mathcal{D}$-degrees are $(0,1)$ and $(1,1)$. In fact, Lemma 4.4 reveals that $u'uv=\tau(uv)\,u'$ and $u'v'u=u\,\tau(u'v')$.

When V is provided with symmetric bilinear forms F and F', a comultiplication $\mathcal{B}(V,F{+}F')\rightarrow\mathcal{B}(V,F)\otimes\mathcal{B}(V,F')$ can be defined in this way:

$$\begin{aligned}\mathcal{B}(V,F{+}F')\rightarrow\mathcal{B}(V{\times}V,\ F{\perp}F')\rightarrow\mathcal{B}(V{\times}0,\ F{\perp}F')\,\mathcal{B}(0{\times}V,\ F{\perp}F')\\ \rightarrow\mathcal{B}(V,F)\otimes\mathcal{B}(V,F')\,;\end{aligned}$$

$F{\perp}F'$ is defined by $(F{\perp}F')((u,u'),(v,v'))\ =\ F(u,v)+F'(u',v')$ for all u, u', v, $v'\in V$; the first arrow is the algebra homomorphism that extends the mapping $v\longmapsto(v,v)$; the second arrow is a projector defined as it is explained just above; the third arrow is a natural bijection.

The comultiplication $\mathcal{B}(V,0)\rightarrow\mathcal{B}(V,0)\otimes\mathcal{B}(V,0)$ turns $\mathcal{B}(V,0)$ into a coalgebra. It is not a bialgebra because the comultiplication is not an algebra homomorphism. Anyway, the dual space $\mathcal{B}^*(V,0)$ of $\mathcal{B}(V,0)$ is now an algebra, and the natural mapping $V^*\rightarrow\mathcal{B}^*(V,0)$ extends to an algebra isomorphism $\mathcal{B}(V^*,0)\rightarrow\mathcal{B}^*(V,0)$.

The comultiplication $\mathcal{B}(V,F)\rightarrow\mathcal{B}(V,0)\otimes\mathcal{B}(V,F)$ brings a new (and equivalent) definition of the interior products $\eta\rfloor x$ where $\eta\in\mathcal{B}(V^*,0)$ and $x\in\mathcal{B}(V,F)$. This definition refers to the mapping

$$\mathcal{B}(V^*,0)\otimes\mathcal{B}(V,F)\rightarrow\mathcal{B}^*(V,0)\otimes\mathcal{B}(V,0)\otimes\mathcal{B}(V,F)\rightarrow\mathcal{B}(V,F);$$

the first arrow involves the isomorphism $\mathcal{B}(V^*,0)\rightarrow\mathcal{B}^*(V,0)$ and the comultiplication; the second arrow involves the mapping $\mathcal{B}^*(V,0)\otimes\mathcal{B}(V,0)\rightarrow K$.

Similarly, the comultiplication $\mathcal{B}(V,F)\rightarrow\mathcal{B}(V,F)\otimes\mathcal{B}(V,0)$ brings a new definition of the interior products $x\lfloor\eta$.

It is true that the properties of meson algebras presented here (from Sects. 2 to 12) are not as easy and practical as the corresponding properties of Clifford algebras, but they are remarkable enough to catch attention. Although meson algebras are not of interest to many people at the present time, it is sensible to predict that their remarkable properties will make them become topical again in the future.

Data Availability The research presented in this article did neither generate nor analyze any data set.

Declarations

Conflict of interest The research presented in this article did not come into conflict of interest with any person or institution.

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