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Short Time Quaternion Quadratic Phase Fourier Transform and Its Uncertainty Principles

Bivek Gupta and Amit K. Verma*

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Abstract. In this paper, we extend the quadratic phase Fourier transform of a complex valued functions to that of the quaternion-valued functions of two variables. We call it the quaternion quadratic phase Fourier transform (QQPFT). Based on the relation between the QQPFT and the quaternion Fourier transform (QFT) we obtain the sharp Hausdorff–Young inequality for QQPFT, which in particular sharpens the constant in the inequality for the quaternion offset linear canonical transform (QOLCT). We define the short time quaternion quadratic phase Fourier transform (STQQPFT) and explore some of its properties including inner product relation and inversion formula. We find its relation with that of the $2D$ quaternion ambiguity function and the quaternion Wigner–Ville distribution associated with QQPFT and obtain the Lieb's uncertainty and entropy uncertainty principles for these three transforms.

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1. Introduction

In [11,12] authors have studied the quadratic phase Fourier transform (QPFT) defined as

$$(\mathcal{Q}^{\wedge} f)(\xi) = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{i\left(At^2+Bt\xi+C\xi^2+Dt+E\xi\right)} f(t)dt,\ \ \xi \in \mathbb{R}, \tag{1}$$

where $f \in L^2(\mathbb{R},\mathbb{C})$, $\wedge = (A,B,C,D,E)$, $B \neq 0$ which generalizes the classical Fourier transform (FT). Several other important integral transforms like fractional Fourier transform (FrFT) [1,41], linear canonical transform (LCT)

*Corresponding author.

[28], offset linear canonical transform (OLCT) [9,48], Fresnel transform [31] and Lorentz transform can be obtained by choosing $\wedge$ appropriately and amplifying (1) with suitable constants. Along with several important properties like the Riemann-Lebesgue lemma and Plancherel theorem, authors in [12] have given several convolutions and obtained the convolution theorem associated with the QPFT. Recently, Shah et al. [44] generalized several uncertainty principles for the FT, FrFT [46], and LCT for the QPFT defined in (1). Even though QPFT generalizes several integral transforms as mentioned above, due to the presence of a global kernel it fails to give the local quadratic phase spectrum content of non-transient signals. To overcome this, Shah et al. [43]) formulated a short time quadratic phase Fourier transform (STQPFT) and studied its important properties. They have generalized the Heisenberg's, logarithmic and local uncertainty principles (UPs) for FT and fractional FT [46,49] and Lieb's UP for short time FT [25] in the context of STQPFT. Apart from STQPFT, the wavelet transform and Wigner–Ville distribution associated with the QPFT have also been studied. Shah et al. [45] proposed a novel quadratic phase Wigner distribution by combining the advantages of Wigner distribution and the QPFT. They obtained several fundamental properties, including Moyal's formula and inversion formula. Prasad et al. [42] defined the wavelet transform associated with the QPFT and studied its properties like the inversion formula, Parseval's formula, and also its continuity on some function spaces.

In 1843, W.R. Hamilton first introduced the quaternion algebra. It is denoted by $\mathbb{H}$ in his honor. In Harmonic analysis and applied mathematics, the FT is an essential tool, so its extension to the quaternion-valued functions has become an interesting problem. The quaternion Fourier transform (QFT) was introduced by Ell [19] for the analysis of $2D$ linear time-invariant partial differential system and later applied in color image processing [20]. In the analysis of quaternion-valued functions, the quaternion Fourier transform plays a significant role. Because of the non-commutativity of the quaternion multiplication, the Fourier transform of the quaternion-valued function on $\mathbb{R}^2$ can be classified into various types, viz., right-sided, left-sided and two-sided Fourier transform [4,6,19]. Cheng et al. [14] gave the inversion theorem and the Plancherel theorem for the right-sided QFT and also obtained its relation with the left-sided and the two-sided QFT for the quaternion-valued square-integrable functions. It transforms a quaternion-valued $2D$ signal into a quaternion-valued frequency domain signal.

Lian [36] proved various inequalities like Pitt's inequality, logarithmic UP using the method adopted by Beckner [8] in the case of complex variables, entropy UP without using the sharp Hausdorff–Young inequality, for the two-sided QFT with optimal constants, which are same to those obtained in the complex case. The logarithmic UP obtained in [36] is different from that given in [13]. In [37], the author obtained the sharp Hausdorff–Young inequality, using the orthogonal plan split of the quaternion [30], for the two-sided QFT followed by the Hirschman's entropy UP using the standard differential approach. In [38], the author has extended the QFT to the Clifford valued function defined on $\mathbb{R}^n$, namely geometric FT, and derived several sharp inequalities including sharp Hausdorff-Young inequality and sharp Pitt's inequality, followed by the sharp entropy inequality for the Clifford ambiguity functions. Recently, QFT has been extended to the quaternion fractional Fourier transform (QFrFT) and quaternion linear canonical transform (QLCT).

Replacing the kernels $\mathcal{K}^i(t_1,\xi_1) = \frac{1}{\sqrt{2\pi}} e^{-it_1\xi_1}$ and $\mathcal{K}^j(t_2,\xi_2) = \frac{1}{\sqrt{2\pi}} e^{-jt_2\xi_2}$, in the definition

$$(\mathcal{F}_{\mathbb{H}}f)(\boldsymbol{\xi}) = \int_{\mathbb{R}^2} \mathcal{K}^i(t_1,\xi_1) f(\boldsymbol{t}) \mathcal{K}^j(t_2,\xi_2) d\boldsymbol{t},\ \boldsymbol{\xi} = (\xi_1,\xi_2) \in \mathbb{R}^2, \tag{2}$$

of the two-sided QFT [39], with that of the kernels of the FrFT [1,41,46] and LCT, respectively, results in the two-sided quaternion fractional Fourier transform (QFrFT) and the two-sided quaternion linear canonical transform (QLCT) [33]. Analogously, the right-sided and the left-sided QFrFT and QLCT have been defined in the literature (see [33,47]). Kou et al. [33] adopted the approach by Chen et al. [13] to obtain the energy theorem and proved the Heisenberg's UP for the QLCT. Using the orthogonal plan split method, authors in [35] have obtained the relation of the two-sided QLCT with that of the LCT and obtained some important inequalities and uncertainty principles of two-sided QLCT.

Bahri et al. [5] generalized the classical windowed Fourier transform to quaternion-valued functions of two variables. Using the machinery of the right-sided QFT [6], authors proved several important properties, including reconstruction formula, reproducing kernel, and orthogonality relation. Following the methods adopted by Wilczok [49], they also obtained the Heisenberg UP for the quaternion windowed Fourier transform (QWFT). In [3], authors gave the alternate proofs of the properties studied in [5]. They also studied the Pitt's inequality, Lieb's inequality, and the logarithmic UP for the two-sided QWFT studied in [5]. Including the orthogonality property, authors in [10,32] studied the local UP, logarithmic UP, Beckner's UP in terms of entropy, Lieb's UP, Amrein–Berthier UP for the two-sided QWFT. Replacing the Fourier kernel in the left-sided, right-sided, or two-sided QWFT by the kernels of the FrFT (or LCT) results in the left-sided, right-sided, and two-sided QWFrFT (or QWLCT), respectively. In [22], authors have studied the two-sided QWFT with the real-valued window function and studied its important properties and the associated Balian–Low theorem. In [23], authors studied the orthogonality relation along with Heisenberg's UP for the two-sided QWLCT, with a quaternion-valued window function. Bahri, in [2], has extended the classical ambiguity function (AF) and the Wigner–Ville distribution (WVD) to the quaternion algebra setting, namely, quaternion ambiguity function (QAF) and quaternion Wigner-Ville distribution (QWVD). They studied several important properties, including Moyal's principle and reconstruction formula for these two-sided QAF and QWVD. Authors in [21] have extended these two-sided QAF and QWVD in the linear canonical domain and obtained the relation among them. They have also studied their important properties like shifting, dilation, reconstruction formula, Moyal's theorem, etc.

Proposed problem: Several important properties, along with the UPs of the QPFT and the STQPFT, have been studied for the function of complex variables as mentioned above. The QPFT has more degrees of freedom and is more flexible with the parameters involved in the FT, FrFT, LCT, and the OLCT; with the same computational cost as the FT, it is natural to extend QPFT to a quaternion setting. To the best of our knowledge, the QPFT and the STQPFT have yet to be explored for the quaternion-valued functions. Due to the non-commutativity of the quaternion multiplication, we can define at least three different types of quaternion quadratic phase Fourier transform (QQPFT), viz., right-sided, left-sided, and two-sided.

A clear and insightful description of signals with several simultaneous components under control is offered by the theory of quaternions. Due to the distinguishing algebraic properties of the field of quaternions, the QQPFT is extended to the realm of quaternion algebra. The transforms play a vital role in the efficient representation of quaternion-valued signals, and as in [7,24], it can be applied in diverse areas of signal and image processing, such as color image processing, speech recognition, edge detection, and data compression. Despite the humongous merits of the QQPFT, it fails to provide an adequate time-frequency representation of non-stationary signals because of its global kernel. In the present study, our goal is to evade such limitations of the QQPFT by formulating a novel integral transform called the STQQPFT, which relies upon a sliding window to capture the localized spectral contents of non-stationary two-dimensional quaternion-valued signals.

This article concentrates on the two-sided QQPFT, which generalizes QOLCT [17] and obtains sharper bounds for some inequalities studied in [50]. Based on its relation with the quaternion Fourier transform (QFT), we obtain the sharp Hausdorff–Young inequality, which in particular sharpens the constant in the Hausdorff–Young inequality for quaternion OLCT [50]. Using the sharp Hausdorff–Young inequality, we obtain the Rènyi and Shannon entropy UP for QQPFT. We also define the STQQPFT and explore its important properties like boundedness, linearity, translation, scaling, inner product relation, and inversion formula. Based on the sharp Hausdorff-Young inequality we obtain the Lieb's uncertainty and entropy uncertainty principles of the STQQPFT followed by the same for the newly defined $2D$ quaternion quadratic phase ambiguity function (QQPAF) and $2D$ quaternion quadratic phase Wigner–Ville distribution (QQPWVD), using the relation of the later transforms with that of the STQQPFT. QQPWVD defined here generalizes the quaternion Wigner-Ville distribution associated with OLCT [16,18] and complements it with Lieb's and entropy UPs, in particular.

The organization of the paper is as follows: In Sect. 2, we recall some basic definitions and properties of quaternion algebra. In Sect. 3, we give the definition of two-sided QQPFT and study its important properties, like Parseval's identity, sharp Hausdorff–Young inequality, Rènyi, and Shannon entropy UPs. In Sect. 4, we generalize the two-sided quaternion windowed Fourier transform [22] to the two-sided STQQPFT and study its properties and its relations with that of the proposed two-sided QQPAF and the QQP-WVD, based on which we obtain the Lieb's and entropy UPs for these three transforms. Finally, in Sect. 5, we conclude our paper.

2. Preliminaries

The field of real and complex numbers are respectively denoted by $\mathbb{R}$ and $\mathbb{C}$. Let

$$\mathbb{H} = \{r = r_0 + ir_1 + jr_2 + kr_3 : r_0, r_1, r_2, r_3 \in \mathbb{R}\},$$

where i, j and k are the imaginary units such that they satisfy the following Hamilton's multiplication rule

$$ij = k = -ji,\ jk = i = -kj,\ ki = j = -ik,\ i^2 = j^2 = k^2 = -1.$$

For a quaternion $r = r_0 + ir_1 + jr_2 + kr_3$, we call r_0 the real scalar part of r, and denote it by $Sc(r)$. The scalar part satisfies the following cyclic multiplication symmetry [29]

$$Sc(pqr) = Sc(qrp) = Sc(rpq),\ \forall\ p, q, r \in \mathbb{H}. \tag{3}$$

We denote the quaternion conjugate of r as $\bar{r}$ and is defined as

$$\bar{r} = r_0 - ir_1 - jr_2 - kr_3.$$

The quaternion conjugate satisfy the following

$$\overline{qr} = \bar{r}\bar{q},\ \overline{q + r} = \bar{q} + \bar{r},\ \bar{\bar{q}} = q,\ \forall\ q, r \in \mathbb{H}. \tag{4}$$

The modulus of $r \in \mathbb{H}$ is defined as

$$|r| = \sqrt{r\bar{r}} = \left(\sum_{l=0}^{3} r_l^2\right)^{\frac{1}{2}}, \tag{5}$$

and it satisfies $|qr| = |q||r|,\ \forall\ q, r \in \mathbb{H}$.

A quaternion-valued function h defined on $\mathbb{R}^2$ can be written as

$$h(\boldsymbol{x}) = h_0(\boldsymbol{x}) + ih_1(\boldsymbol{x}) + jh_2(\boldsymbol{x}) + kh_3(\boldsymbol{x}),\ \boldsymbol{x} \in \mathbb{R}^2,$$

where h_0, h_1, h_2 and h_3 are real-valued function on $\mathbb{R}^2$.

If $1 \leq q < \infty$, then the L^q-norm of h is defined by

$$\begin{aligned}\|h\|_{L^q_{\mathbb{H}}(\mathbb{R}^2)} &= \left(\int_{\mathbb{R}^2} |h(\boldsymbol{x})|^q d\boldsymbol{x}\right)^{\frac{1}{q}} \\ &= \left\{\int_{\mathbb{R}^2} \left(\sum_{l=0}^{3} |h_l(\boldsymbol{x})|^2\right)^{\frac{q}{2}} d\boldsymbol{x}\right\}^{\frac{1}{q}}\end{aligned} \tag{6}$$

and $L^q_{\mathbb{H}}(\mathbb{R}^2)$ is a Banach space of all measurable quaternion-valued functions f having finite L^q-norm. $L^\infty_{\mathbb{H}}(\mathbb{R}^2)$ is the set of all essentially bounded quaternion-valued measurable functions with norm

$$\|f\|_{L^\infty_{\mathbb{H}}(\mathbb{R}^2)} = \text{ess sup}_{\boldsymbol{x} \in \mathbb{R}^2} |f(\boldsymbol{x})|. \tag{7}$$

Moreover, the quaternion-valued inner product

$$(f, g) = \int_{\mathbb{R}^2} f(\boldsymbol{x})\overline{g(\boldsymbol{x})}d\boldsymbol{x}, \tag{8}$$

with symmetric real scalar part

$$\begin{aligned}\langle f, g\rangle &= \frac{1}{2}[(f, g) + (g, f)] \\ &= \int_{\mathbb{R}^2} Sc\left[f(\boldsymbol{x})\overline{g(\boldsymbol{x})}\right] d\boldsymbol{x} \\ &= Sc\left(\int_{\mathbb{R}^2} f(\boldsymbol{x})\overline{g(\boldsymbol{x})}d\boldsymbol{x}\right) \end{aligned} \tag{9}$$

turns $L^2_{\mathbb{H}}(\mathbb{R}^2)$ to a Hilbert space, where the norm in Eq. (6) can be expressed as

$$\|f\|_{L^2_{\mathbb{H}}(\mathbb{R}^2)} = \sqrt{\langle f, f\rangle} = \sqrt{(f, f)} = \left(\int_{\mathbb{R}^2} |f(\boldsymbol{x})|^2 d\boldsymbol{x}\right)^{\frac{1}{2}}. \tag{10}$$

3. Quaternion quadratic phase Fourier transform (QQPFT)

In this section we give a definition of quaternion quadratic phase Fourier transform (QQPFT) and study its important properties.

Definition 3.1. Let $\wedge_l = (A_l, B_l, C_l, D_l, E_l), A_l, B_l, C_l, D_l, E_l \in \mathbb{R}$ and $B_l \neq 0$ for $l = 1, 2$. The quaternion quadratic phase Fourier transform (QQPFT) of $f(\boldsymbol{t}) \in L^2_{\mathbb{H}}(\mathbb{R}^2), \boldsymbol{t} = (t_1, t_2)$, is defined by

$$(\mathcal{Q}^{\wedge_1,\wedge_2}_{\mathbb{H}} f)(\boldsymbol{\xi}) = \int_{\mathbb{R}^2} \mathcal{K}^i_{\wedge_1}(t_1, \xi_1) f(\boldsymbol{t}) \mathcal{K}^j_{\wedge_2}(t_2, \xi_2) d\boldsymbol{t},\ \boldsymbol{\xi} = (\xi_1, \xi_2) \in \mathbb{R}^2 \tag{11}$$

where

$$\mathcal{K}^i_{\wedge_1}(t_1, \xi_1) = \frac{1}{\sqrt{2\pi}} e^{-i\left(A_1 t_1^2 + B_1 t_1 \xi_1 + C_1 \xi_1^2 + D_1 t_1 + E_1 \xi_1\right)} \tag{12}$$

and

$$\mathcal{K}^i_{\wedge_2}(t_2, \xi_2) = \frac{1}{\sqrt{2\pi}} e^{-j\left(A_2 t_2^2 + B_2 t_2 \xi_2 + C_2 \xi_2^2 + D_2 t_2 + E_2 \xi_2\right)}. \tag{13}$$

The corresponding inversion formula is given by

$$f(\boldsymbol{t}) = |B_1 B_2| \int_{\mathbb{R}^2} \overline{\mathcal{K}^i_{\wedge_1}(t_1, \xi_1)} (\mathcal{Q}^{\wedge_1,\wedge_2}_{\mathbb{H}} f)(\boldsymbol{\xi}) \overline{\mathcal{K}^i_{\wedge_2}(t_2, \xi_2)} d\boldsymbol{\xi} \tag{14}$$

3.1. Relation between QQPFT and QFT

We now see an important relation between the QQPFT and the QFT, which plays a vital role in obtaining the sharp Hausdorff-Young inequality for the QQPFT.

$$
\begin{aligned}
(\mathcal{Q}_{\mathbb{H}}^{\wedge_1,\wedge_2} f)(\boldsymbol{\xi}) &= \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{-i\left(A_1t_1^2+B_1t_1\xi_1+C_1\xi_1^2+D_1t_1+E_1\xi_1\right)} f(\boldsymbol{t})\\
&\quad\times e^{-j\left(A_2t_2^2+B_2t_2\xi_2+C_2\xi_2^2+D_2t_2+E_2\xi_2\right)} d\boldsymbol{t}\\
&= e^{-i\left(C_1\xi^2+E_1\xi_1\right)}\left\{\frac{1}{2\pi}\int_{\mathbb{R}^2} e^{-iB_1t_1\xi_1}\tilde{f}(\boldsymbol{t})e^{-jB_2t_2\xi_2} d\boldsymbol{t}\right\} e^{-j\left(C_2\xi^2+E_2\xi_2\right)},
\end{aligned}
$$

where

$$\tilde{f}(\boldsymbol{t}) = e^{-i\left(A_1t_i^2+D_1t_1\right)} f(\boldsymbol{t}) e^{-j\left(A_2t_i^2+D_2t_2\right)}. \tag{15}$$

Thus,

$$(\mathcal{Q}_{\mathbb{H}}^{\wedge_1,\wedge_2} f)(\boldsymbol{\xi}) = e^{-i\left(C_1\xi^2+E_1\xi_1\right)}\left(\mathcal{F}_{\mathbb{H}}\tilde{f}\right)(B_1\xi_1, B_2\xi_2)e^{-j\left(C_2\xi^2+E_2\xi_2\right)} \tag{16}$$

where

$$\left(\mathcal{F}_{\mathbb{H}}\tilde{f}\right)(\boldsymbol{\xi}) = \int_{\mathbb{R}^2}\frac{1}{\sqrt{2\pi}}e^{-it_1\xi_1}\tilde{f}(\boldsymbol{t})\frac{1}{\sqrt{2\pi}}e^{-jt_2\xi_2} d\boldsymbol{t}. \tag{17}$$

Based on this relation between QQPFT and the QFT, we obtain the following important inequality.

Theorem 3.1. (Sharp Hausdorff–Young Inequality) *Let* $1 \le p \le 2$, $\frac{1}{p}+\frac{1}{q}=1$ *and* $f \in L^p_{\mathbb{H}}(\mathbb{R}^2)$, *then*

$$\|\mathcal{Q}_{\mathbb{H}}^{\wedge_1,\wedge_2} f\|_{L^q_{\mathbb{H}}(\mathbb{R}^2)} \le \frac{(2\pi)^{\frac{1}{q}-\frac{1}{p}}A_p^2}{|B_1B_2|^{\frac{1}{q}}}\|f\|_{L^p_{\mathbb{H}}(\mathbb{R}^2)}, \tag{18}$$

where $A_p = \left(\frac{p^{\frac{1}{p}}}{q^{\frac{1}{q}}}\right)^{\frac{1}{2}}$.

Proof. Using the relation between the QQPFT and the QFT, we get

$$
\begin{aligned}
\|\mathcal{Q}_{\mathbb{H}}^{\wedge_1,\wedge_2} f\|_{L^q_{\mathbb{H}}(\mathbb{R}^2)} &= \left(\int_{\mathbb{R}^2}\left|\left(\mathcal{F}_{\mathbb{H}}\tilde{f}\right)(B_1\xi_1, B_2\xi_2)\right|^q d\boldsymbol{\xi}\right)^{\frac{1}{q}}\\
&= \frac{1}{|B_1B_2|^{\frac{1}{q}}}\|\mathcal{F}_{\mathbb{H}}\tilde{f}\|_{L^q_{\mathbb{H}}(\mathbb{R}^2)}.
\end{aligned}
$$

Using the sharp Hausdorff–Young inequality ([37]) for the QFT, we get

$$\|\mathcal{Q}_{\mathbb{H}}^{\wedge_1,\wedge_2} f\|_{L^q_{\mathbb{H}}(\mathbb{R}^2)} \le \frac{(2\pi)^{\frac{1}{q}-\frac{1}{p}}A_p^2}{|B_1B_2|^{\frac{1}{q}}}\|\tilde{f}\|_{L^p_{\mathbb{H}}(\mathbb{R}^2)}.$$

Substituting $\tilde{f}$, from (15), we get (18). This completes the proof. □

Theorem 3.2. (Parseval's formula) *Let* $f, g \in L^2_{\mathbb{H}}(\mathbb{R}^2)$, *then*

$$\langle f, g\rangle = |B_1B_2|\langle \mathcal{Q}_{\mathbb{H}}^{\wedge_1,\wedge_2} f, \mathcal{Q}_{\mathbb{H}}^{\wedge_1,\wedge_2} g\rangle. \tag{19}$$

In particular,

$$\|f\|^2_{L^2_{\mathbb{H}}(\mathbb{R}^2)} = |B_1B_2|\|\mathcal{Q}_{\mathbb{H}}^{\wedge_1,\wedge_2} f\|^2_{L^2_{\mathbb{H}}(\mathbb{R}^2\times\mathbb{R}^2)}. \tag{20}$$

Proof. By the Parseval's formula for the QFT of the function $\tilde{f}$ and $\tilde{g}$, we have

$$\begin{aligned}\langle \tilde{f}, \tilde{g}\rangle &= \langle \mathcal{F}_{\mathbb{H}}\tilde{f}, \mathcal{F}_{\mathbb{H}}\tilde{g}\rangle \\ &= Sc\int_{\mathbb{R}^2} |B_1B_2| \left(\mathcal{F}_{\mathbb{H}}\tilde{f}\right)(B_1\xi_1, B_2\xi_2)\overline{(\mathcal{F}_{\mathbb{H}}\tilde{g})(B_1\xi_1, B_2\xi_2)}d\boldsymbol{\xi}.\end{aligned}$$

Using the relation between the QQPFT and the QFT, we get

$$\begin{aligned}\langle \tilde{f}, \tilde{g}\rangle &= |B_1B_2|\int_{\mathbb{R}^2} Sc\left[e^{i\left(C_1\xi_1^2+E_1\xi_2\right)}\left(\mathcal{Q}_{\mathbb{H}}^{\wedge_1,\wedge_2}g\right)(\boldsymbol{\xi})\overline{\left(\mathcal{Q}_{\mathbb{H}}^{\wedge_1,\wedge_2}g\right)(\boldsymbol{\xi})}e^{-i\left(C_1\xi_1^2+E_1\xi_2\right)}\right]d\boldsymbol{\xi} \\ &= |B_1B_2|\int_{\mathbb{R}^2} Sc\left[\left(\mathcal{Q}_{\mathbb{H}}^{\wedge_1,\wedge_2}g\right)(\boldsymbol{\xi})\overline{\left(\mathcal{Q}_{\mathbb{H}}^{\wedge_1,\wedge_2}g\right)(\boldsymbol{\xi})}\right]d\boldsymbol{\xi} \\ &= |B_1B_2|\langle \mathcal{Q}_{\mathbb{H}}^{\wedge_1,\wedge_2}f, \mathcal{Q}_{\mathbb{H}}^{\wedge_1,\wedge_2}g\rangle.\end{aligned}$$

This proves Eq. (19). In particular, if we take $f = g$, in Eq. (19), we get Eq. (20).

This completes the proof. □

3.2. Rènyi and Shannon entropy uncertainty principle

In this subsection we obtain the Rènyi and Shannon entropy UPs for the proposed QQPFT. Analogous results for the FrFT of complex valued function can be found in [26]. Recently, Shannon entropy UP for the QPFT and the two-sided QLCT are studied in [35,44] respectively. Below we prove Rènyi UP for the QQPFT and obtain the Shannon UP in limiting case. We start with the following definition.

Definition 3.2. [15,26] The Rènyi entropy of a probability density function P on $\mathbb{R}^n$ is defined by

$$H_\alpha(P) = \frac{1}{1-\alpha}\log\left(\int_{\mathbb{R}^n}[P(\boldsymbol{t})]^\alpha d\boldsymbol{t}\right),\ \alpha > 0, \alpha \neq 1. \tag{21}$$

If $\alpha \to 1$, then (21) leads to the following Shannon entropy

$$E(P) = -\int_{\mathbb{R}^n} P(\boldsymbol{t})\log[P(\boldsymbol{t})]d\boldsymbol{t} \tag{22}$$

Theorem 3.3. *If* $f \in L^2_{\mathbb{H}}(\mathbb{R}^2)$, $\frac{1}{2} < \alpha < 1$ *and* $\frac{1}{\alpha} + \frac{1}{\beta} = 2$, *then*

$$\begin{aligned}&H_\alpha(|f|^2) + H_\beta\left(\left|\sqrt{|B_1B_2|}\left(\mathcal{Q}_{\mathbb{H}}^{\wedge_1,\wedge_2}f\right)(\boldsymbol{\xi})\right|^2\right) \\ &\quad\geq -\log(|B_1B_2|) - 2\log(2\pi) - \left(\frac{1}{1-\alpha}\log(2\alpha) + \frac{1}{1-\beta}\log(2\beta)\right).\end{aligned}$$

Proof. By Hausdorff–Young inequality (18), we have

$$\left(\int_{\mathbb{R}^2}\left|\left(\mathcal{Q}_{\mathbb{H}}^{\wedge_1,\wedge_2}f\right)(\boldsymbol{\xi})\right|^q d\boldsymbol{\xi}\right)^{\frac{1}{q}} \leq \frac{(2\pi)^{\frac{1}{q}-\frac{1}{p}}A_p^2}{|B_1B_2|^{\frac{1}{q}}}\left(\int_{\mathbb{R}^2}|f(\boldsymbol{t})|^p d\boldsymbol{t}\right)^{\frac{1}{p}}. \tag{23}$$

Putting $p = 2\alpha$ and $q = 2\beta$, in Eq. (23), we have

$$\frac{1}{\sqrt{|B_1B_2|}}\left(\int_{\mathbb{R}^2}\left|\sqrt{|B_1B_2|}\left(\mathcal{Q}_{\mathbb{H}}^{\wedge_1,\wedge_2}f\right)(\boldsymbol{\xi})\right|^{2\beta}d\boldsymbol{\xi}\right)^{\frac{1}{2\beta}} \leq \frac{(2\pi)^{\frac{1}{2\beta}-\frac{1}{2\alpha}}A_{2\alpha}^2}{|B_1B_2|^{\frac{1}{2\beta}}}\left(\int_{\mathbb{R}^2}|f(\boldsymbol{t})|^{2\alpha}d\boldsymbol{t}\right)^{\frac{1}{2\alpha}}.$$

This implies

$$\frac{|B_1B_2|^{\frac{1}{\beta}-1}}{(2\pi)^{\frac{1}{\alpha}-\frac{1}{\beta}}A_{2\alpha}^4} \leq \left(\int_{\mathbb{R}^2}|f(\boldsymbol{t})|^{2\alpha}d\boldsymbol{t}\right)^{\frac{1}{\alpha}}\left(\int_{\mathbb{R}^2}\left|\sqrt{|B_1B_2|}\left(\mathcal{Q}_{\mathbb{H}}^{\wedge_1,\wedge_2}f\right)(\boldsymbol{\xi})\right|^{2\beta}d\boldsymbol{\xi}\right)^{-\frac{1}{\beta}} \tag{24}$$

Since $\frac{1}{\alpha}+\frac{1}{\beta}=2$, we have

$$\frac{\alpha}{1-\alpha} = -\frac{\beta}{1-\beta}. \tag{25}$$

Raising to the power $\frac{\alpha}{1-\alpha}$ in (24) and using (25), we get

$$\frac{|B_1B_2|^{-1}}{(2\pi)^{\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)\left(\frac{\alpha}{1-\alpha}\right)}A_{2\alpha}^{\frac{4\alpha}{1-\alpha}}} \leq \left(\int_{\mathbb{R}^2}|f(\boldsymbol{t})|^{2\alpha}d\boldsymbol{t}\right)^{\frac{1}{1-\alpha}}\left(\int_{\mathbb{R}^2}\left|\sqrt{|B_1B_2|}\left(\mathcal{Q}_{\mathbb{H}}^{\wedge_1,\wedge_2}f\right)(\boldsymbol{\xi})\right|^{2\beta}d\boldsymbol{\xi}\right)^{\frac{1}{1-\beta}}.$$

Taking log on both sides, we get

$$\begin{aligned}-\log(|B_1B_2|) - \log\left((2\pi)^{\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)\left(\frac{\alpha}{1-\alpha}\right)}A_{2\alpha}^{\frac{4\alpha}{1-\alpha}}\right) &\leq \frac{1}{1-\alpha}\log\left(\int_{\mathbb{R}^2}|f(\boldsymbol{t})|^{2\alpha}d\boldsymbol{t}\right)\\ &+\frac{1}{1-\beta}\log\left(\int_{\mathbb{R}^2}\left|\sqrt{|B_1B_2|}\left(\mathcal{Q}_{\mathbb{H}}^{\wedge_1,\wedge_2}\right)(\boldsymbol{\xi})\right|^{2\alpha}d\boldsymbol{\xi}\right).\end{aligned} \tag{26}$$

Thus, it follows that

$$\begin{aligned}&H_\alpha(|f|^2) + H_\beta\left(\left|\sqrt{|B_1B_2|}\left(\mathcal{Q}_{\mathbb{H}}^{\wedge_1,\wedge_2}f\right)(\boldsymbol{\xi})\right|^2\right)\\ &\geq -\log(|B_1B_2|) - 2\log(2\pi) - \left(\frac{1}{1-\alpha}\log(2\alpha) + \frac{1}{1-\beta}\log(2\beta)\right).\end{aligned} \tag{27}$$

This is the Rènyi entropy UP for QQPFT. □

Remark 1. If $\alpha \to 1$, then $\beta \to 1$ and in this case Eq. (27) can be written as

$$E(|f|^2) + E\left(\left|\sqrt{|B_1B_2|}\left(\mathcal{Q}_{\mathbb{H}}^{\wedge_1,\wedge_2}f\right)(\boldsymbol{\xi})\right|^2\right) \geq -\log(|B_1B_2|) - 2\log(2\pi) + 2 - \log 4,$$

its right hand side is obtained using the relation $\frac{1}{\alpha}+\frac{1}{\beta}=2$ and taking the limit as $\alpha \to 1$. Thus, we have

$$E(|f|^2) + E\left(\left|\sqrt{|B_1B_2|}\left(\mathcal{Q}_{\mathbb{H}}^{\wedge_1,\wedge_2}f\right)(\boldsymbol{\xi})\right|^2\right) \geq \log\left(\frac{e^2}{16\pi^2|B_1B_2|}\right). \tag{28}$$

This is the Shannon entropy UP for QQPFT.

4. Short time quaternion quadratic phase Fourier transform

In this section we give the definition of the STQQPFT and study its properties. We obtain its relation with that of the quaternion AF and the quaternion WVD associated with the QQPFT.

Definition 4.1. Let $\wedge_l = (A_l, B_l, C_l, D_l, E_l), A_l, B_l, C_l, D_l, E_l \in \mathbb{R}$ and $B_l \neq 0$ for $l = 1, 2$. The short time quaternion quadratic phase Fourier transform (STQQPFT) of a function $f \in L^2_{\mathbb{H}}(\mathbb{R}^2)$ with respect to a quaternion window function (QWF) $g \in L^2_{\mathbb{H}}(\mathbb{R}^2) \cap L^{\infty}_{\mathbb{H}}(\mathbb{R}^2)$ is defined by

$$\left(\mathcal{S}^{\wedge_1,\wedge_2}_{\mathbb{H},g} f\right)(\boldsymbol{x}, \boldsymbol{\xi}) = \int_{\mathbb{R}^2} \mathcal{K}^i_{\wedge_1}(t_1, \xi_1) f(\boldsymbol{t}) \overline{g(\boldsymbol{t} - \boldsymbol{x})} \mathcal{K}^j_{\wedge_2}(t_2, \xi_2) d\boldsymbol{t}, (\boldsymbol{x}, \boldsymbol{\xi}) \in \mathbb{R}^2 \times \mathbb{R}^2,$$

where $\mathcal{K}^i_{\wedge_1}(t_1, \xi_1)$ and $\mathcal{K}^j_{\wedge_2}(t_2, \xi_2)$ are given by Eqs. (12) and (13), respectively.

We now derive some of the basic properties of the STQQPFT. But before that we state the following lemma:

Lemma 4.1. *Let $\boldsymbol{t} = (t_1, t_2), \boldsymbol{\xi} = (\xi_1, \xi_2), \boldsymbol{k} = (k_1, k_2) \in \mathbb{R}^2, r \in \mathbb{R}$. Then the kernel $\mathcal{K}^i_{\wedge_1}(t_1, \xi_1)$ and $\mathcal{K}^j_{\wedge_2}(t_2, \xi_2)$ satisfy the following*

$$\mathcal{K}^i_{\wedge_1}(t_1 + rk_1, \xi_1) = \mathcal{K}^i_{\wedge_1}\left(t_1, \xi_1 + \frac{2rk_1A_1}{B_1}\right) \phi^i_{\wedge_1,r}(k_1, \xi_1), \tag{29}$$

where

$$\phi^i_{\wedge_1,r}(k_1, \xi_1) = e^{-i\left(A_1 r^2 k_1^2 + D_1 rk_1 + B_1 rk_1\xi_1 - \frac{4r^2 A_1^2 C_1 k_1^2}{B_1^2} - \frac{4rA_1C_1k_1\xi_1}{B_1} - \frac{2rA_1k_1}{B_1}\right)} \tag{30}$$

and

$$\mathcal{K}^j_{\wedge_2}(t_2 + rk_2, \xi_2) = \mathcal{K}^j_{\wedge_2}\left(t_2, \xi_2 + \frac{2rk_2A_2}{B_2}\right) \phi^j_{\wedge_2,r}(k_2, \xi_2), \tag{31}$$

where

$$\phi^j_{\wedge_2,r}(k_2, \xi_2) = e^{-j\left(A_2 r^2 k_2^2 + D_2 rk_2 + B_2 rk_2\xi_2 - \frac{4r^2 A_2^2 C_2 k_2^2}{B_2^2} - \frac{4rA_2C_2k_2\xi_2}{B_2} - \frac{2rA_2k_2}{B_2}\right)}. \tag{32}$$

Proof. From the definition of $\mathcal{K}^i_{\wedge_1}$, we have

$$\begin{aligned}
&\mathcal{K}^i_{\wedge_1}(t_1+rk_1,\xi_1)\\
&\quad=\frac{1}{\sqrt{2\pi}}e^{-i\left\{A_1(t_1+rk_1)^2+B_1(t_1+rk_1)\xi_1+C_1\xi_1^2+D_1(t_1+rk_1)+E_1\xi_1\right\}}\\
&\quad=\frac{1}{\sqrt{2\pi}}e^{-i\left\{A_1t_1^2+B_1t_1\left(\xi_1+\frac{2rA_1k_1}{B_1}\right)+D_1t_1+C_1\xi_1^2+E_1\xi_1+B_1rk_1\xi_1\right\}}\\
&\qquad\times e^{-i(A_1r^2k_1^2+D_1rk_1)}\\
&\quad=\frac{1}{\sqrt{2\pi}}e^{-i\left\{A_1t_1^2+B_1t_1\left(\xi_1+\frac{2rA_1k_1}{B_1}\right)+D_1t_1+C_1\left(\xi_1+\frac{2rA_1k_1}{B_1}\right)^2+E_1\left(\xi_1+\frac{2rA_1k_1}{B_1}\right)\right\}}\\
&\qquad\phi^i_{\wedge_1,r}(k_1,\xi_1),\\
&\text{i.e., } \mathcal{K}^i_{\wedge_1}(t_1+rk_1,\xi_1)=\mathcal{K}^i_{\wedge_1}\left(t_1,\xi_1+\frac{2rk_1A_1}{B_1}\right)\phi^i_{\wedge_1,r}(k_1,\xi_1).
\end{aligned}$$

This proves Eq. (29). Similarly, Eq. (31) can be proved. □

The theorem below gives the basic properties of the proposed STQQPFT.

Theorem 4.1. *Let* $g,g_1,g_2\in L^2_{\mathbb{H}}(\mathbb{R}^2)\cap L^\infty_{\mathbb{H}}(\mathbb{R}^2)$ *be QWFs and* $f,f_1,f_2\in L^2_{\mathbb{H}}(\mathbb{R}^2)$. *Also let* $\lambda\neq 0$, $\boldsymbol{k}=(k_1,k_2)\in\mathbb{R}^2$, $p,q\in\{x+iy:x,y\in\mathbb{R}\}$, $r,s\in\{x+jy:x,y\in\mathbb{R}\}$, *then*

- (i) *Boundedness:* $\left\|\mathcal{S}^{\wedge_1,\wedge_2}_{\mathbb{H},g}f\right\|_{L^\infty_{\mathbb{H}}(\mathbb{R}^2)}\leq\frac{1}{2\pi}\|g\|_{L^2_{\mathbb{H}}(\mathbb{R}^2)}\|f\|_{L^2_{\mathbb{H}}(\mathbb{R}^2)}$.
- (ii) *Linearity:* $\mathcal{S}^{\wedge_1,\wedge_2}_{\mathbb{H},g}(pf_1+qf_2)=p\left[\mathcal{S}^{\wedge_1,\wedge_2}_{\mathbb{H},g}f_1\right]+q\left[\mathcal{S}^{\wedge_1,\wedge_2}_{\mathbb{H},g}f_2\right]$
- (iii) *Anti-linearity:* $\mathcal{S}^{\wedge_1,\wedge_2}_{\mathbb{H},rg_1+sg_2}f=\left[\mathcal{S}^{\wedge_1,\wedge_2}_{\mathbb{H},g_1}f\right]\bar{r}+\left[\mathcal{S}^{\wedge_1,\wedge_2}_{\mathbb{H},g_2}f\right]\bar{s}$.
- (iv) *Translation:* $\left(\mathcal{S}^{\wedge_1,\wedge_2}_{\mathbb{H},g}(\tau_{\boldsymbol{k}}f)\right)(\boldsymbol{x},\boldsymbol{\xi})=\phi^i_{\wedge_1,1}(k_1,\xi_1)\left(\mathcal{S}^{\wedge_1,\wedge_2}_{\mathbb{H},g}f\right)(\boldsymbol{x}-\boldsymbol{k},\boldsymbol{\xi}'_{\boldsymbol{x}})\phi^j_{\wedge_2,1}(k_2,\xi_2)$, *where* $(\tau_{\boldsymbol{k}}f)(\boldsymbol{t})=f(\boldsymbol{t}-\boldsymbol{k})$, $\xi'_{\boldsymbol{x}}=\left(\xi_1+\frac{2A_1x_1}{B_1},\xi_2+\frac{2A_2x_2}{B_2}\right)$, $\phi^i_{\wedge_1,1}(k_1,\xi_1,)$ *and* $\phi^j_{\wedge_2,1}(k_2,\xi_2)$, *are obtained from* (30) *and* (32) *by replacing* $r=1$.
- (v) *Scaling:* $\left(\mathcal{S}^{\wedge_1,\wedge_2}_{\mathbb{H},g_\lambda}f_\lambda\right)(\boldsymbol{x},\boldsymbol{\xi})=\left(\mathcal{S}^{\wedge_1',\wedge_2'}_{\mathbb{H},g}f\right)\left(\frac{1}{\lambda}\boldsymbol{x},\boldsymbol{\xi}\right)$, *where* $(f_\lambda)(\boldsymbol{t})=\frac{1}{\lambda}f\left(\frac{1}{\lambda}\boldsymbol{t}\right)$, $\wedge_l'=\left(\lambda^2A_l,\lambda B_l,C_l,\lambda D_l,E_l\right)$, $l=1,2$.

Proof. The proof of (i), (ii) and (iii) are straight forward so we omit their proof.

(iv) We have from the Definition 4.1

$$\left(\mathcal{S}^{\wedge_1,\wedge_2}_{\mathbb{H},g}(\tau_{\boldsymbol{k}}f)\right)(\boldsymbol{x},\boldsymbol{\xi})=\int_{\mathbb{R}^2}\mathcal{K}^i_{\wedge_1}(t_1+k_1,\xi_1)f(\boldsymbol{t})\overline{g(\boldsymbol{t}-(\boldsymbol{x}-\boldsymbol{k}))}\mathcal{K}^j_{\wedge_2}(t_2+k_1,\xi_2)d\boldsymbol{t}.$$

Using lemma (4.1), we get

$$
\begin{aligned}
&\left(\mathcal{S}_{\mathbb{H},g}^{\wedge_1,\wedge_2}(\tau_{\boldsymbol{k}} f)\right)(\boldsymbol{x},\boldsymbol{\xi})\\
&= \int_{\mathbb{R}^2} \mathcal{K}^i_{\wedge_1}\left(t_1,\xi_1+\frac{2A_1k_1}{B_1}\right)\phi^i_{\wedge_1,1}(k_1,\xi_1) f(\boldsymbol{t})\overline{g(\boldsymbol{t}-(\boldsymbol{x}-\boldsymbol{k}))}\mathcal{K}^j_{\wedge_2}\\
&\quad\times\left(t_2,\xi_2+\frac{2A_2k_2}{B_2}\right)\phi^j_{\wedge_2,1}(k_2,\xi_2)d\boldsymbol{t}\\
&= \phi^i_{\wedge_1,1}(k_1,\xi_1)\left\{\int_{\mathbb{R}^2}\mathcal{K}^i_{\wedge_1}\left(t_1,\xi_1+\frac{2A_1k_1}{B_1}\right) f(\boldsymbol{t})\overline{g(\boldsymbol{t}-(\boldsymbol{x}-\boldsymbol{k}))}\mathcal{K}^j_{\wedge_2}\right.\\
&\quad\left.\times\left(t_2,\xi_2+\frac{2A_2k_2}{B_2}\right)d\boldsymbol{t}\right\}\phi^j_{\wedge_2,1}(k_2,\xi_2).
\end{aligned}
$$

Thus, we have

$$
\left(\mathcal{S}_{\mathbb{H},g}^{\wedge_1,\wedge_2}(\tau_{\boldsymbol{k}} f)\right)(\boldsymbol{x},\boldsymbol{\xi}) = \phi^i_{\wedge_1,1}(k_1,\xi_1)\left(\mathcal{S}_{\mathbb{H},g}^{\wedge_1,\wedge_2} f\right)(\boldsymbol{x}-\boldsymbol{k},\boldsymbol{\xi}'_{\boldsymbol{x}})\phi^j_{\wedge_2,1}(k_2,\xi_2).
$$

This proves (iv).

(v) We have

$$
\left(\mathcal{S}_{\mathbb{H},g_\lambda}^{\wedge_1,\wedge_2} f_\lambda\right)(\boldsymbol{x},\boldsymbol{\xi}) = \textstyle\int_{\mathbb{R}^2}\mathcal{K}^i_{\wedge_1}(\lambda t_1,\xi_1) f(\boldsymbol{t})\overline{g\left(\boldsymbol{t}-\frac{1}{\lambda}\boldsymbol{x}\right)}\mathcal{K}^j_{\wedge_1}(\lambda t_2,\xi_2)d\boldsymbol{t}. \tag{33}
$$

Now,

$$
\begin{aligned}
\mathcal{K}^i_{\wedge_1}(\lambda t_1,\xi_1) &= \frac{1}{\sqrt{2\pi}}e^{-i\left((\lambda^2A_1)t_1^2+(\lambda B_1)t_1\xi_1+C_1\xi_1^2+D_1t_1+E_1\xi_1\right)}\\
&= \mathcal{K}^i_{\wedge_1'}(t_1,\xi_1).
\end{aligned} \tag{34}
$$

Similarly,

$$
\mathcal{K}^j_{\wedge_2}(\lambda t_2,\xi_2) = \mathcal{K}^j_{\wedge_2'}(t_2,\xi_2). \tag{35}
$$

Using Eqs. (34) and (35) in Eq. (33), we get

$$
\begin{aligned}
&\left(\mathcal{S}_{\mathbb{H},g_\lambda}^{\wedge_1,\wedge_2} f_\lambda\right)(\boldsymbol{x},\boldsymbol{\xi}) = \int_{\mathbb{R}^2}\mathcal{K}^i_{\wedge_1'}(t_1,\xi_1) f(\boldsymbol{t})\overline{g\left(\boldsymbol{t}-\frac{1}{\lambda}\boldsymbol{x}\right)}\mathcal{K}^j_{\wedge_2'}(t_2,\xi_2)d\boldsymbol{t},\\
&i.e., \left(\mathcal{S}_{\mathbb{H},g_\lambda}^{\wedge_1,\wedge_2} f_\lambda\right)(\boldsymbol{x},\boldsymbol{\xi}) = \left(\mathcal{S}_{\mathbb{H},g}^{\wedge_1',\wedge_2'} f\right)\left(\frac{1}{\lambda}\boldsymbol{x},\boldsymbol{\xi}\right).
\end{aligned}
$$

This completes the proof. □

Theorem 4.2. (Inner product relation) *If g_1, g_2 be two QWFs and $f_1, f_2 \in L^2_{\mathbb{H}}(\mathbb{R}^2)$, then $\mathcal{S}_{\mathbb{H},g_1}^{\wedge_1,\wedge_2} f_1,\ \mathcal{S}_{\mathbb{H},g_2}^{\wedge_1,\wedge_2} f_2 \in L^2_{\mathbb{H}}(\mathbb{R}^2\times\mathbb{R}^2)$ and*

$$
\left\langle \mathcal{S}_{\mathbb{H},g_1}^{\wedge_1,\wedge_2} f_1, \mathcal{S}_{\mathbb{H},g_2}^{\wedge_1,\wedge_2} f_2\right\rangle = \frac{1}{|B_1B_2|}\langle f_1(\overline{g_1},\overline{g_2}), f_2\rangle. \tag{36}
$$

Proof. We have

$$\begin{aligned}
&\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left|\left(\mathcal{S}_{\mathbb{H},g_1}^{\wedge_1,\wedge_2} f_1\right)(\boldsymbol{x},\boldsymbol{\xi})\right|^2 d\boldsymbol{x}d\boldsymbol{\xi}\\
&\quad=\int_{\mathbb{R}^2}\left\{\int_{\mathbb{R}^2}\left|\left(\mathcal{Q}_{\mathbb{H}}^{\wedge_1,\wedge_2}\{f_1(\cdot)\overline{g_1(\cdot-\boldsymbol{x})}\}\right)(\boldsymbol{\xi})\right|^2 d\boldsymbol{\xi}\right\}d\boldsymbol{x}\\
&\quad=\frac{1}{|B_1B_2|}\int_{\mathbb{R}^2}\left\{\int_{\mathbb{R}^2}|f_1(\boldsymbol{t})\overline{g_1(\boldsymbol{t}-\boldsymbol{x})}|^2 d\boldsymbol{t}\right\}d\boldsymbol{x},\ \text{using Parseval's Identity}\\
&\quad=\frac{1}{|B_1B_2|}\|f_1\|^2_{L^2_{\mathbb{H}}(\mathbb{R}^2)}\|g_1\|^2_{L^2_{\mathbb{H}}(\mathbb{R}^2)}.
\end{aligned}$$

Thus, $\mathcal{S}_{\mathbb{H},g_1}^{\wedge_1,\wedge_2} f_1\in L^2_{\mathbb{H}}(\mathbb{R}^2\times\mathbb{R}^2)$. Similarly, $\mathcal{S}_{\mathbb{H},g_2}^{\wedge_1,\wedge_2} f_2\in L^2_{\mathbb{H}}(\mathbb{R}^2\times\mathbb{R}^2)$.

Now,

$$\begin{aligned}
&\left\langle\mathcal{S}_{\mathbb{H},g_1}^{\wedge_1,\wedge_2} f_1,\mathcal{S}_{\mathbb{H},g_2}^{\wedge_1,\wedge_2} f_2\right\rangle\\
&\quad=Sc\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left(\mathcal{Q}_{\mathbb{H}}^{\wedge_1,\wedge_2}\{f_1(\cdot)\overline{g_1(\cdot-\boldsymbol{x})}\}\right)(\boldsymbol{\xi})\overline{\left(\mathcal{Q}_{\mathbb{H}}^{\wedge_1,\wedge_2}\{f_2(\cdot)\overline{g_2(\cdot-\boldsymbol{x})}\}\right)(\boldsymbol{\xi})}d\boldsymbol{x}d\boldsymbol{\xi}\\
&\quad=\frac{1}{|B_1B_2|}Sc\int_{\mathbb{R}^2}\left\{\int_{\mathbb{R}^2}f_1(\boldsymbol{t})\overline{g_1(\boldsymbol{t}-\boldsymbol{x})}\ \overline{f_2(\boldsymbol{t})\overline{g_2(\boldsymbol{t}-\boldsymbol{x})}}d\boldsymbol{t}\right\}d\boldsymbol{x}\\
&\quad=\frac{1}{|B_1B_2|}Sc\int_{\mathbb{R}^2}f_1(\boldsymbol{t})\left(\overline{g_1},\overline{g_2}\right)\overline{f_2(\boldsymbol{t})}d\boldsymbol{t}.
\end{aligned}$$

Thus, it follows that

$$\left\langle\mathcal{S}_{\mathbb{H},g_1}^{\wedge_1,\wedge_2} f_1,\mathcal{S}_{\mathbb{H},g_2}^{\wedge_1,\wedge_2} f_2\right\rangle=\frac{1}{|B_1B_2|}\langle f_1\left(\overline{g_1},\overline{g_2}\right),f_2\rangle.$$

This finishes the proof. □

Remark 2. From theorem 4.2, we have the following results:

1. If $g_1=g_2=g$ in Eq. (36), then
$$\left\langle\mathcal{S}_{\mathbb{H},g_1}^{\wedge_1,\wedge_2} f_1,\mathcal{S}_{\mathbb{H},g_2}^{\wedge_1,\wedge_2} f_2\right\rangle=\frac{1}{|B_1B_2|}\|g\|^2_{L^2_{\mathbb{H}}(\mathbb{R}^2)}\langle f_1,f_2\rangle.$$
2. If $f_1=f_2=f$ in Eq. (36), then
$$\left\langle\mathcal{S}_{\mathbb{H},g_1}^{\wedge_1,\wedge_2} f_1,\mathcal{S}_{\mathbb{H},g_2}^{\wedge_1,\wedge_2} f_2\right\rangle=\frac{1}{|B_1B_2|}\|f\|^2_{L^2_{\mathbb{H}}(\mathbb{R}^2)}\langle g_1,g_2\rangle.$$
3. If $f_1=f=f_2$ and $g_1=g=g_2$ in Eq. (36), then
$$\|\mathcal{S}_{\mathbb{H},g}^{\wedge_1,\wedge_2} f\|^2_{L^2_{\mathbb{H}}(\mathbb{R}^2\times\mathbb{R}^2)}=\frac{1}{|B_1B_2|}\|f\|^2_{L^2_{\mathbb{H}}(\mathbb{R}^2)}\|g\|^2_{L^2_{\mathbb{H}}(\mathbb{R}^2)}.\tag{37}$$

The theorem below gives the reconstruction formula for the STQQPFT.

Theorem 4.3. (Inversion formula) *Let g be a QWF and $f\in L^2_{\mathbb{H}}(\mathbb{R}^2)$, then*

$$f(\boldsymbol{t})=\frac{|B_1B_2|}{\|g\|^2_{L^2_{\mathbb{H}}(\mathbb{R}^2)}}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\overline{\mathcal{K}^i_{\wedge_1}(t_1,\xi_1)}\left(\mathcal{S}_{\mathbb{H},g}^{\wedge_1,\wedge_2} f\right)(\boldsymbol{x},\boldsymbol{\xi})\overline{\mathcal{K}^j_{\wedge_2}(t_2,\xi_2)}g(\boldsymbol{t}-\boldsymbol{x})d\boldsymbol{x}d\boldsymbol{\xi}.$$

Proof. We have

$$\begin{aligned}\langle f,h\rangle &= \frac{|B_1B_2|}{\|g\|^2_{L^2_{\mathbb{H}}(\mathbb{R}^2)}} Sc\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left(\mathcal{S}^{\wedge_1,\wedge_2}_{\mathbb{H},g}f\right)(\boldsymbol{x},\boldsymbol{\xi})\\ &\quad \overline{\left\{\int_{\mathbb{R}^2}\mathcal{K}^i_{\wedge_1}(t_1,\xi_1)h(\boldsymbol{t})\overline{g(\boldsymbol{t}-\boldsymbol{x})}\mathcal{K}^j_{\wedge_2}(t_2,\xi_2)d\boldsymbol{t}\right\}}d\boldsymbol{x}d\boldsymbol{\xi}\\ &= \frac{|B_1B_2|}{\|g\|^2_{L^2_{\mathbb{H}}(\mathbb{R}^2)}}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}Sc\\ &\quad \left\{\left(\mathcal{S}^{\wedge_1,\wedge_2}_{\mathbb{H},g}f\right)(\boldsymbol{x},\boldsymbol{\xi})\overline{\mathcal{K}^j_{\wedge_2}(t_2,\xi_2)}g(\boldsymbol{t}-\boldsymbol{x})\overline{h(\boldsymbol{t})}\overline{\mathcal{K}^i_{\wedge_1}(t_1,\xi_1)}\right\}d\boldsymbol{t}d\boldsymbol{x}d\boldsymbol{\xi}\\ &= \frac{|B_1B_2|}{\|g\|^2_{L^2_{\mathbb{H}}(\mathbb{R}^2)}}Sc\int_{\mathbb{R}^2}\\ &\quad \left\{\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\overline{\mathcal{K}^i_{\wedge_1}(t_1,\xi_1)}\left(\mathcal{S}^{\wedge_1,\wedge_2}_{\mathbb{H},g}f\right)(\boldsymbol{x},\boldsymbol{\xi})\overline{\mathcal{K}^j_{\wedge_2}(t_2,\xi_2)}g(\boldsymbol{t}-\boldsymbol{x})d\boldsymbol{x}d\boldsymbol{\xi}\right\}\overline{h(\boldsymbol{t})}d\boldsymbol{t}\\ &= \frac{|B_1B_2|}{\|g\|^2_{L^2_{\mathbb{H}}(\mathbb{R}^2)}}\\ &\quad \left\langle\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\overline{\mathcal{K}^i_{\wedge_1}(t_1,\xi_1)}\left(\mathcal{S}^{\wedge_1,\wedge_2}_{\mathbb{H},g}f\right)(\boldsymbol{x},\boldsymbol{\xi})\overline{\mathcal{K}^j_{\wedge_2}(t_2,\xi_2)}g(\cdot-\boldsymbol{x})d\boldsymbol{x}d\boldsymbol{\xi},h(\cdot)\right\rangle.\end{aligned}$$

Since $h\in L^2_{\mathbb{H}}(\mathbb{R}^2)$ is arbitrary, it follows that

$$f(\boldsymbol{t}) = \frac{|B_1B_2|}{\|g\|^2_{L^2_{\mathbb{H}}(\mathbb{R}^2)}}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\overline{\mathcal{K}^i_{\wedge_1}(t_1,\xi_1)}\left(\mathcal{S}^{\wedge_1,\wedge_2}_{\mathbb{H},g}f\right)(\boldsymbol{x},\boldsymbol{\xi})\overline{\mathcal{K}^j_{\wedge_2}(t_2,\xi_2)}g(\boldsymbol{t}-\boldsymbol{x})d\boldsymbol{x}d\boldsymbol{\xi}.$$

This completes the proof. □

4.1. Quaternion ambiguity function and Wigner-Ville distribution associated to the QQPFT

In this subsection, we give the definitions of two-sided QQPAF and QQP-WVD and obtain their relation with that of the proposed STQQPFT.

Definition 4.2. The two-sided quaternion quadratic phase ambiguity function (QQPAF) of $f,g\in L^2_{\mathbb{H}}(\mathbb{R}^2)$ is defined by

$$\left(\mathcal{A}^{\wedge_1,\wedge_2}_{\mathbb{H}}(f,g)\right)(\boldsymbol{x},\boldsymbol{\xi}) = \int_{\mathbb{R}^2}\mathcal{K}^i_{\wedge_1}(t_1,\xi_1)f\left(\boldsymbol{t}+\frac{1}{2}\boldsymbol{x}\right)\overline{g\left(\boldsymbol{t}-\frac{1}{2}\boldsymbol{x}\right)}\mathcal{K}^j_{\wedge_2}(t_2,\xi_2)\,d\boldsymbol{t},$$

where $\mathcal{K}^i_{\wedge_1}(t_1,\xi_1)$ and $\mathcal{K}^j_{\wedge_2}(t_2,\xi_2)$ are given by Eqs. (12) and (13) respectively.

The following theorem gives the relation between the QQPAF and the STQQPFT.

Theorem 4.4. *If g is a QWF and $f\in L^2_{\mathbb{H}}(\mathbb{R}^2)$, then*

$$\left(\mathcal{A}^{\wedge_1,\wedge_2}_{\mathbb{H}}(f,g)\right)(\boldsymbol{x},\boldsymbol{\xi}) = \phi^i_{\wedge_1,-\frac{1}{2}}(x_1,\xi_1)\left(\mathcal{S}^{\wedge_1,\wedge_2}_{\mathbb{H},g}f\right)(\boldsymbol{x},\boldsymbol{\xi}'_{\boldsymbol{x}})\phi^j_{\wedge_2,-\frac{1}{2}}(x_2,\xi_2),$$

$$\boldsymbol{\xi}'_{\boldsymbol{x}} = \left(\xi_1-\frac{A_1x_1}{B_1},\xi_2-\frac{A_2x_2}{B_2}\right)$$

where $\phi^i_{\wedge_1,-\frac{1}{2}}(x_1,\xi_1)$ *and* $\phi^j_{\wedge_2,-\frac{1}{2}}(x_2,\xi_2)$ *are obtained from Eqs.* (30) *and* (32) *by replacing* $r=-\frac{1}{2}$.

Proof. From the definition of $\mathcal{A}_{\mathbb{H}}^{\wedge_1,\wedge_2}(f,g)$, it follows that

$$\left(\mathcal{A}_{\mathbb{H}}^{\wedge_1,\wedge_2}(f,g)\right)(\boldsymbol{x},\boldsymbol{\xi})=\int_{\mathbb{R}^2}\mathcal{K}^i_{\wedge_1}\left(t_1-\frac{x_1}{2},\xi_2\right)f(\boldsymbol{t})\overline{g(\boldsymbol{t}-\boldsymbol{x})}\mathcal{K}^j_{\wedge_2}\left(t_2-\frac{x_2}{2},\xi_2\right)d\boldsymbol{t}.$$

Using Eqs. (29) and (31) for $r=-\frac{1}{2}$, we get

$$\begin{aligned}&\left(\mathcal{A}_{\mathbb{H}}^{\wedge_1,\wedge_2}(f,g)\right)(\boldsymbol{x},\boldsymbol{\xi})\\&=\int_{\mathbb{R}^2}\mathcal{K}^i_{\wedge_1}\left(t_1,\xi_1-\frac{A_1x_1}{B_1}\right)\phi^i_{\wedge_1,-\frac{1}{2}}(x_1,\xi_1)f(\boldsymbol{t})\overline{g(\boldsymbol{t}-\boldsymbol{x})}\\&\quad\mathcal{K}^j_{\wedge_2}\left(t_2,\xi_2-\frac{A_2x_2}{B_2}\right)\phi^j_{\wedge_2,-\frac{1}{2}}(x_2,\xi_2)d\boldsymbol{t}\\&=\phi^i_{\wedge_1,-\frac{1}{2}}(x_1,\xi_1)\\&\quad\left\{\int_{\mathbb{R}^2}\mathcal{K}^i_{\wedge_1}\left(t_1,\xi_1-\frac{A_1x_1}{B_1}\right)f(\boldsymbol{t})\overline{g(\boldsymbol{t}-\boldsymbol{x})}\mathcal{K}^j_{\wedge_2}\left(t_2,\xi_2-\frac{A_2x_2}{B_2}\right)d\boldsymbol{t}\right\}\\&\quad\phi^j_{\wedge_2,-\frac{1}{2}}(x_2,\xi_2).\end{aligned}$$

This gives

$$\left(\mathcal{A}_{\mathbb{H}}^{\wedge_1,\wedge_2}(f,g)\right)(\boldsymbol{x},\boldsymbol{\xi})=\phi^i_{\wedge_1,-\frac{1}{2}}(x_1,\xi_1)\left(\mathcal{S}_{\mathbb{H},g}^{\wedge_1,\wedge_2}f\right)(\boldsymbol{x},\boldsymbol{\xi}'_{\boldsymbol{x}})\phi^j_{\wedge_2,-\frac{1}{2}}(x_2,\xi_2).$$

This completes the proof. □

Definition 4.3. The two-sided quaternion quadratic phase Wigner–Ville distribution (QQPWVD) of $f,g\in L^2_{\mathbb{H}}(\mathbb{R}^2)$, is defined by

$$\left(\mathcal{W}_{\mathbb{H}}^{\wedge_1,\wedge_2}(f,g)\right)(\boldsymbol{x},\boldsymbol{\xi})=\int_{\mathbb{R}^2}\mathcal{K}^i_{\wedge_1}(t_1,\xi_1)f\left(\boldsymbol{x}+\frac{1}{2}\boldsymbol{t}\right)\overline{g\left(\boldsymbol{x}-\frac{1}{2}\boldsymbol{t}\right)}\mathcal{K}^j_{\wedge_2}(t_2,\xi_2)\,d\boldsymbol{t},$$

where $\mathcal{K}^i_{\wedge_1}(t_1,\xi_1)$ and $\mathcal{K}^j_{\wedge_2}(t_2,\xi_2)$ are given by Eqs. (12) and (13) respectively.

The following theorem gives the relation between the QQPWVD and the STQQPFT.

Theorem 4.5. *If g is a QWF and $f\in L^2_{\mathbb{H}}(\mathbb{R}^2)$, then*

$$\left(\mathcal{W}_{\mathbb{H}}^{\wedge_1,\wedge_2}(f,g)\right)(\boldsymbol{x},\boldsymbol{\xi})=4\psi^i_{\wedge_1}(x_1,\xi_1)\left(\mathcal{S}_{\mathbb{H},\tilde{g}}^{\wedge'_1,\wedge'_2}f\right)(2\boldsymbol{x},\boldsymbol{\xi}'_{\boldsymbol{x}})\psi^j_{\wedge_2}(x_2,\xi_2),$$

$$\boldsymbol{\xi}'_{\boldsymbol{x}}=\left(\xi_1-\frac{4A_1x_1}{B_1},\xi_2-\frac{4A_2x_2}{B_2}\right)$$

where $\wedge'_l=(4A_l,2B_l,C_l,2D_l,E_l),\ l=1,2,\ \tilde{g}(\boldsymbol{t})=g(-\boldsymbol{t})$,

$$\psi^i_{\wedge_1}(x_1,\xi_1)=e^{-i\left(4A_1x_1^2-2B_1x_1\xi_1-2D_1x_1-\frac{16A_1^2C_1x_1^2}{B_1^2}+\frac{8A_1C_1x_1\xi_1}{B_1}+\frac{4A_1E_1x_1}{B_1}\right)}$$

and

$$\psi^j_{\wedge_2}(x_2,\xi_2)=e^{-j\left(4A_2x_2^2-2B_2x_2\xi_2-2D_2x_2-\frac{16A_2^2C_2x_2^2}{B_2^2}+\frac{8A_2C_2x_2\xi_2}{B_2}+\frac{4A_2E_2x_2}{B_2}\right)}.$$

Proof. From the definition of $\mathcal{W}_{\mathbb{H}}^{\wedge_1,\wedge_2}(f,g)$, we have

$$\begin{aligned}
&\left(\mathcal{W}_{\mathbb{H}}^{\wedge_1,\wedge_2}(f,g)\right)(\boldsymbol{x},\boldsymbol{\xi})\\
&\quad= 4\int_{\mathbb{R}^2}\mathcal{K}^i_{\wedge_1}(2(t_1-x_1),\xi_1)f(\boldsymbol{t})\overline{g(2\boldsymbol{x}-\boldsymbol{t})}\mathcal{K}^j_{\wedge_2}(2(t_2-x_2),\xi_2)d\boldsymbol{t}\\
&\quad= 4\int_{\mathbb{R}^2}\mathcal{K}^i_{\wedge_1}(2(t_1-x_1),\xi_1)f(\boldsymbol{t})\overline{\tilde{g}(\boldsymbol{t}-2\boldsymbol{x})}\mathcal{K}^j_{\wedge_2}(2(t_2-x_2),\xi_2)d\boldsymbol{t}. \quad (38)
\end{aligned}$$

Now from the definition of $\mathcal{K}^i_{\wedge_1}$, in Eq. (12), we have

$$\begin{aligned}
&\mathcal{K}^i_{\wedge_1}(2(t_1-x_1),\xi_1)\\
&\quad=\frac{1}{\sqrt{2\pi}}e^{-i(4A_1t_1^2-8A_1x_1t_1+2B_1t_1\xi_1+2D_1t_1+E_1\xi_1+C_1\xi_2)}e^{-i(4A_1x_1^2-2B_1x_1\xi_1-2D_1x_1)}\\
&\quad=\frac{1}{\sqrt{2\pi}}e^{-i\left\{(4A_1)t_1^2+2B_1\left(\xi_1-\frac{4A_1x_1}{B_1}\right)+C_1\left(\xi_1-\frac{4A_1x_1}{B_1}\right)^2+(2D_1)t_1+E_1\left(\xi_1-\frac{4A_1x_1}{B_1}\right)\left(\xi_1-\frac{4A_1x_1}{B_1}\right)\right\}}\\
&\quad\quad\psi^i_{\wedge_1}(x_1,\xi_1)
\end{aligned}$$

i.e.,

$$\mathcal{K}^i_{\wedge_1}(2(t_1-x_1),\xi_1)=\mathcal{K}^i_{\wedge_1'}\left(t_1,\xi_1-\frac{4A_1x_1}{B_1}\right)\psi^i_{\wedge_1}(x_1,\xi_1). \quad (39)$$

Similarly, we have

$$\mathcal{K}^j_{\wedge_2}(2(t_2-x_2),\xi_2)=\mathcal{K}^j_{\wedge_2'}\left(t_2,\xi_2-\frac{4A_2x_2}{B_2}\right)\psi^j_{\wedge_2}(x_2,\xi_2). \quad (40)$$

Using Eqs. (39) and (40) in (38), we have

$$\begin{aligned}
&\left(\mathcal{W}_{\mathbb{H}}^{\wedge_1,\wedge_2}(f,g)\right)(\boldsymbol{x},\boldsymbol{\xi})\\
&\quad= 4\int_{\mathbb{R}^2}\mathcal{K}^i_{\wedge_1'}\left(t_1,\xi_1-\frac{4A_1x_1}{B_1}\right)\psi^i_{\wedge_1}(x_1,\xi_1)f(\boldsymbol{t})\overline{\tilde{g}(\boldsymbol{t}-2\boldsymbol{x})}\mathcal{K}^j_{\wedge_2'}\\
&\quad\quad\left(t_2,\xi_2-\frac{4A_2x_2}{B_2}\right)\psi^j_{\wedge_2}(x_2,\xi_2)d\boldsymbol{t}\\
&\quad= 4\psi^i_{\wedge_1}(x_1,\xi_1)\\
&\quad\quad\left\{\int_{\mathbb{R}^2}\mathcal{K}^i_{\wedge_1'}\left(t_1,\xi_1-\frac{4A_1x_1}{B_1}\right)f(\boldsymbol{t})\overline{\tilde{g}(\boldsymbol{t}-2\boldsymbol{x})}\mathcal{K}^j_{\wedge_2'}\left(t_2,\xi_2-\frac{4A_2x_2}{B_2}\right)d\boldsymbol{t}\right\}\\
&\quad\quad\psi^j_{\wedge_2}(x_2,\xi_2).
\end{aligned}$$

This gives

$$\left(\mathcal{W}_{\mathbb{H}}^{\wedge_1,\wedge_2}(f,g)\right)(\boldsymbol{x},\boldsymbol{\xi})=4\psi^i_{\wedge_1}(x_1,\xi_1)\left(\mathcal{S}^{\wedge_1',\wedge_2'}_{\mathbb{H},\tilde{g}}f\right)(2\boldsymbol{x},\boldsymbol{\xi}'_{\boldsymbol{x}})\psi^j_{\wedge_2}(x_2,\xi_2).$$

This completes the proof. □

4.2. Uncertainty principle for STQQPFT

The Heisenberg's UP gives the information about a function and its FT, it says that the function cannot be highly localized in both time and frequency domain. Wilczok [49] introduced a new class of UP that compares the localization of a functions with the localization of its wavelet transform, analogous to the Heisenberg UP governing the localization of the complex valued function and the corresponding FT. Gupta et al. [27] obtained the Lieb's and

Donoho-Stark's UP for the linear canonical wavelet transform and obtained the lower bound of the measure of its essential support.

Here, we prove the Lieb's UP for the STQQPFT, QQPWVD and QQ-PAF. Analogous result for the classical STFT and the windowed linear canonical transform can be found in [25,34] respectively. Before we move forward, let us first prove the following lemma.

Lemma 4.2. (Lieb's inequality) *Let g be a QWF, $f \in L^2_{\mathbb{H}}(\mathbb{R}^2)$ and $2 \leq q < \infty$. Then*

$$\left\|\mathcal{S}_{\mathbb{H},g}^{\wedge_1,\wedge_2} f\right\|_{L^q_{\mathbb{H}}(\mathbb{R}^2\times\mathbb{R}^2)} \leq \frac{(2\pi)^{\frac{1}{q}-\frac{1}{p}}}{|B_1B_2|^{\frac{1}{q}}}\left(\frac{2}{q}\right)^{\frac{2}{q}} \|g\|_{L^2_{\mathbb{H}}(\mathbb{R}^2)}\|f\|_{L^2_{\mathbb{H}}(\mathbb{R}^2)}. \tag{41}$$

Proof.

$$\left(\int_{\mathbb{R}^2}\left|\left(\mathcal{S}_{\mathbb{H},g}^{\wedge_1,\wedge_2} f\right)(\boldsymbol{x},\boldsymbol{\xi})\right|^q d\boldsymbol{\xi}\right)^{\frac{1}{q}} = \left(\int_{\mathbb{R}^2}\left|\left(\mathcal{Q}_{\mathbb{H}}^{\wedge_1,\wedge_2}\{f(\cdot)\overline{g(\cdot-\boldsymbol{x})}\}\right)(\boldsymbol{\xi})\right|^q d\boldsymbol{\xi}\right)^{\frac{1}{q}}. \tag{42}$$

Using Hausdorff–Young inequality, we get

$$\begin{aligned}
&\left(\int_{\mathbb{R}^2}\left|\left(\mathcal{S}_{\mathbb{H},g}^{\wedge_1,\wedge_2} f\right)(\boldsymbol{x},\boldsymbol{\xi})\right|^q d\boldsymbol{\xi}\right)^{\frac{1}{q}}\\
&\leq \frac{A_p^2(2\pi)^{\frac{1}{q}-\frac{1}{p}}}{|B_1B_2|^{\frac{1}{q}}}\left(\int_{\mathbb{R}^2}\left|f(\boldsymbol{t})\overline{g(\boldsymbol{t}-\boldsymbol{x})}\right|^p d\boldsymbol{t}\right)^{\frac{1}{p}}\\
&= \frac{A_p^2(2\pi)^{\frac{1}{q}-\frac{1}{p}}}{|B_1B_2|^{\frac{1}{q}}}\left(\int_{\mathbb{R}^2}|f(\boldsymbol{t})|^p|\tilde{g}(\boldsymbol{x}-\boldsymbol{t})|^p d\boldsymbol{t}\right)^{\frac{1}{p}},\ \tilde{g}(\boldsymbol{t}) = g(-\boldsymbol{t})\\
&= \frac{A_p^2(2\pi)^{\frac{1}{q}-\frac{1}{p}}}{|B_1B_2|^{\frac{1}{q}}}\left\{\left(|f|^p\star|\tilde{g}|^p\right)(\boldsymbol{x})\right\}^{\frac{1}{p}},
\end{aligned}$$

where $\star$ is the convolution defined as $(u\star v)(\boldsymbol{x}) = \int_{\mathbb{R}^2} u(\boldsymbol{t})v(\boldsymbol{x}-\boldsymbol{t})d\boldsymbol{t}$. This implies that

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left|\left(\mathcal{S}_{\mathbb{H},g}^{\wedge_1,\wedge_2} f\right)(\boldsymbol{x},\boldsymbol{\xi})\right|^q d\boldsymbol{x}d\boldsymbol{\xi} \leq \frac{A_p^{2q}(2\pi)^{q\left(\frac{1}{q}-\frac{1}{p}\right)}}{|B_1B_2|}\int_{\mathbb{R}^2}\left\{\left(|f|^p\star|\tilde{g}|^p\right)(\boldsymbol{x})\right\}^{\frac{q}{p}} d\boldsymbol{x}.$$

This gives

$$\begin{aligned}
&\left\{\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left|\left(\mathcal{S}_{\mathbb{H},g}^{\wedge_1,\wedge_2} f\right)(\boldsymbol{x},\boldsymbol{\xi})\right|^q d\boldsymbol{x}d\boldsymbol{\xi}\right\}^{\frac{1}{q}}\\
&\leq \frac{A_p^2(2\pi)^{\frac{1}{q}-\frac{1}{p}}}{|B_1B_2|^{\frac{1}{q}}}\left[\int_{\mathbb{R}^2}\left\{\left(|f|^p\star|\tilde{g}|^p\right)(\boldsymbol{x})\right\}^{\frac{q}{p}} d\boldsymbol{x}\right]^{\frac{q}{p}\cdot\frac{1}{q}}\\
&= \frac{A_p^2(2\pi)^{\frac{1}{q}-\frac{1}{p}}}{|B_1B_2|^{\frac{1}{q}}}\left\||f|^p\star|\tilde{g}|^p\right\|^{\frac{1}{p}}_{L^{\frac{q}{p}}_{\mathbb{H}}(\mathbb{R}^2)}.
\end{aligned} \tag{43}$$

Now we see that, if $k=\frac{2}{p}$, $l=\frac{q}{p}$, then $k\geq 1$ and $\frac{1}{k}+\frac{1}{k}=1+\frac{1}{l}$. Since $|f|^p,\ |\tilde{g}|^p\in L^k_{\mathbb{H}}(\mathbb{R}^2)$, we get, by Young's inequality

$$\||f|^p\star|\tilde{g}|^p\|^{\frac{1}{p}}_{L^{\frac{q}{p}}_{\mathbb{H}}(\mathbb{R}^2)}\leq A^4_k A^2_{l'}\|f\|^p_{L^2_{\mathbb{H}}(\mathbb{R}^2)}\|\tilde{g}\|^p_{L^2_{\mathbb{H}}(\mathbb{R}^2)},\tag{44}$$

where l' is such that $\frac{1}{l}+\frac{1}{l'}=1$. Therefore, from Eqs. (43) and (44), it follows that

$$\left\{\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left|\left(\mathcal{S}^{\wedge_1,\wedge_2}_{\mathbb{H},g}f\right)(\boldsymbol{x},\boldsymbol{\xi})\right|^q d\boldsymbol{x}d\boldsymbol{\xi}\right\}^{\frac{1}{q}}$$
$$\leq\frac{(2\pi)^{\frac{1}{q}-\frac{1}{p}}}{|B_1B_2|^{\frac{1}{q}}}A^2_pA^{\frac{4}{p}}_kA^{\frac{2}{p}}_{l'}\|g\|_{L^2_{\mathbb{H}}(\mathbb{R}^2)}\|f\|_{L^2_{\mathbb{H}}(\mathbb{R}^2)},\tag{45}$$

where $A_r=\left(\frac{r^{\frac{1}{r}}}{r'^{\frac{1}{r'}}}\right)^{\frac{1}{2}}$, $\frac{1}{r}+\frac{1}{r'}=1$. Now, we have

$$\begin{aligned}A^2_pA^{\frac{4}{p}}_kA^{\frac{2}{p}}_{l'}&=\frac{p^{\frac{1}{p}}}{q^{\frac{1}{q}}}\cdot\frac{k}{k'^{\frac{2}{k'p}}}\cdot\frac{l'^{\frac{1}{pl'}}}{\left(\frac{q}{p}\right)^{\frac{1}{q}}},\ \text{since } k=\frac{2}{q},\ l=\frac{q}{p}\\&=\frac{p}{q^{\frac{2}{q}}}\cdot\frac{l'^{\frac{1}{pl'}}}{k'^{\frac{2}{k'p}}}\\&=\frac{2}{q^{\frac{2}{q}}}\cdot\left(\frac{1}{2}\right)^{\frac{q-p}{pq}},\ \text{since } k'=2l'\\&=\left(\frac{2}{q}\right)^{\frac{2}{q}}.\end{aligned}\tag{46}$$

Thus using Eqs. (46) in (45), we get

$$\left\{\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left|\left(\mathcal{S}^{\wedge_1,\wedge_2}_{\mathbb{H},g}f\right)(\boldsymbol{x},\boldsymbol{\xi})\right|^q d\boldsymbol{x}d\boldsymbol{\xi}\right\}^{\frac{1}{q}}\leq\frac{(2\pi)^{\frac{1}{q}-\frac{1}{p}}}{|B_1B_2|^{\frac{1}{q}}}\left(\frac{2}{q}\right)^{\frac{2}{q}}\|g\|_{L^2_{\mathbb{H}}(\mathbb{R}^2)}\|f\|_{L^2_{\mathbb{H}}(\mathbb{R}^2)}.$$

This finishes the proof. □

4.3. Lieb's uncertainty principle

Definition 4.4. Let $\epsilon\geq 0$ and $\Omega\subset\mathbb{R}^n$ be measurable. A function $F\in L^2_{\mathbb{H}}(\mathbb{R}^n)$ is said to be ϵ-concentrated on Ω if

$$\|\chi_{\Omega^c}F\|_{L^2_{\mathbb{H}}(\mathbb{R}^n)}\leq\epsilon\|F\|_{L^2_{\mathbb{H}}(\mathbb{R}^n)},$$

where χ_{Ω^c} denotes the indicator function on $\Omega^c=\mathbb{R}^n\setminus\Omega$.

If $0\leq\epsilon\leq\frac{1}{2}$, then majority of the energy is concentrated on Ω and Ω is said to be the essential support of F. Support of F is contained in Ω, if $\epsilon=0$.

Theorem 4.6. *Let g be a QWF and $f\in L^2_{\mathbb{H}}(\mathbb{R}^2)$, such that $f\neq 0$. Let $\epsilon\geq 0$ and $\Omega\subset\mathbb{R}^2\times\mathbb{R}^2$ is a measurable set. If $\mathcal{S}^{\wedge_1,\wedge_2}_{\mathbb{H},g}f$, on Ω, is ϵ-concentrated, then for every $q>2$*

$$|\Omega|\geq\frac{(2\pi)^2}{|B_1B_2|}(1-\epsilon^2)^{\frac{q}{q-2}}\left(\frac{q}{2}\right)^{\frac{4}{q-2}}.$$

Proof. Since $\mathcal{S}_{\mathbb{H},g}^{\wedge_1,\wedge_2} f$ is ϵ-concentrated on Ω, we have

$$\left\|\chi_{\Omega^c}\mathcal{S}_{\mathbb{H},g}^{\wedge_1,\wedge_2} f\right\|^2_{L^2_{\mathbb{H}}(\mathbb{R}^2\times\mathbb{R}^2)} \leq \frac{\epsilon^2}{|B_1B_2|}\|f\|^2_{L^2_{\mathbb{H}}(\mathbb{R}^2)}\|g\|^2_{L^2_{\mathbb{H}}(\mathbb{R}^2)}.$$

This implies

$$\left\|\chi_{\Omega}\mathcal{S}_{\mathbb{H},g}^{\wedge_1,\wedge_2} f\right\|^2_{L^2_{\mathbb{H}}(\mathbb{R}^2\times\mathbb{R}^2)} \geq \frac{1}{|B_1B_2|}(1-\epsilon^2)\|f\|^2_{L^2_{\mathbb{H}}(\mathbb{R}^2)}\|g\|^2_{L^2_{\mathbb{H}}(\mathbb{R}^2)}. \tag{47}$$

Now, using Holder's inequality with exponents $\frac{q}{q-2}$ and $\frac{q}{2}$, we have

$$\begin{aligned}\left\|\chi_{\Omega}\mathcal{S}_{\mathbb{H},g}^{\wedge_1,\wedge_2} f\right\|^2_{L^2_{\mathbb{H}}(\mathbb{R}^2\times\mathbb{R}^2)} &\leq \left\{\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left(\chi_{\Omega}(\boldsymbol{x},\boldsymbol{\xi})\right)^{\frac{q}{q-2}}d\boldsymbol{x}d\boldsymbol{\xi}\right\}^{\frac{q-2}{q}}\\ &\quad\left\{\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left(\left|\left(\mathcal{S}_{\mathbb{H},g}^{\wedge_1,\wedge_2} f\right)(\boldsymbol{x},\boldsymbol{\xi})\right|^2\right)^{\frac{q}{2}}d\boldsymbol{x}d\boldsymbol{\xi}\right\}^{\frac{2}{q}}\\ &= |\Omega|^{\frac{q-2}{q}}\left\|\mathcal{S}_{\mathbb{H},g}^{\wedge_1,\wedge_2} f\right\|^2_{L^q_{\mathbb{H}}(\mathbb{R}^2)}.\end{aligned}$$

Using, the Lieb's inequality (41), we get

$$\left\|\chi_{\Omega}\mathcal{S}_{\mathbb{H},g}^{\wedge_1,\wedge_2} f\right\|^2_{L^2_{\mathbb{H}}(\mathbb{R}^2\times\mathbb{R}^2)} \leq |\Omega|^{\frac{q-2}{q}}\frac{(2\pi)^{\frac{2}{q}-\frac{2}{p}}}{|B_1B_2|^{\frac{2}{q}}}\left(\frac{4}{q}\right)^{\frac{4}{q}}\|f\|^2_{L^2_{\mathbb{H}}(\mathbb{R}^2)}\|g\|^2_{L^2_{\mathbb{H}}(\mathbb{R}^2)}. \tag{48}$$

From Eqs. (47) and (48), we get

$$|\Omega|^{\frac{q-2}{q}}\frac{(2\pi)^{\frac{2}{q}-\frac{2}{p}}}{|B_1B_2|^{\frac{2}{q}}}\left(\frac{2}{q}\right)^{\frac{4}{q}} \geq \frac{1}{|B_1B_2|}(1-\epsilon^2).$$

This gives

$$\begin{aligned}&|\Omega| \geq \frac{1}{|B_1B_2|}(2\pi)^{2\left(1-\frac{2}{q}\right)\frac{q}{q-2}}(1-\epsilon^2)^{\frac{q}{q-2}}\left(\frac{q}{2}\right)^{\frac{4}{q-2}}, \text{ since } \frac{1}{p}+\frac{1}{q}=1\\ &\text{i.e., } |\Omega| \geq \frac{1}{|B_1B_2|}(2\pi)^2(1-\epsilon^2)^{\frac{q}{q-2}}\left(\frac{q}{2}\right)^{\frac{4}{q-2}}.\end{aligned}$$

This completes the proof. □

Remark 3. Taking $\epsilon = 0$, in the above theorem, we get the following lower bound for the support of $\mathcal{S}_{\mathbb{H},g}^{\wedge_1,\wedge_2} f$

$$\begin{aligned}\left|\text{supp}\left(\mathcal{S}_{\mathbb{H},g}^{\wedge_1,\wedge_2} f\right)\right| &\geq \frac{(2\pi)^2}{|B_1B_2|}\lim_{q\to 2+}\left(\frac{q}{2}\right)^{\frac{4}{q-2}}\\ \text{i.e., } \left|\text{supp}\left(\mathcal{S}_{\mathbb{H},g}^{\wedge_1,\wedge_2} f\right)\right| &\geq \frac{(2\pi e)^2}{|B_1B_2|}.\end{aligned} \tag{49}$$

i.e., measure of the support of $\mathcal{S}_{\mathbb{H},g}^{\wedge_1,\wedge_2} f \geq \frac{(2\pi e)^2}{|B_1B_2|}$.

Corollary 4.8.1. *Let g be a QWF and $f \in L^2_{\mathbb{H}}(\mathbb{R}^2)$, such that $f \neq 0$. Let $\epsilon \geq 0$ and $\Omega \subset \mathbb{R}^2\times\mathbb{R}^2$ is measurable. If $\mathcal{A}_{\mathbb{H}}^{\wedge_1,\wedge_2}(f,g)$, on Ω, is ϵ-concentrated, then for every $q > 2$*

$$|\Omega| \geq \frac{(2\pi)^2}{|B_1B_2|}(1-\epsilon^2)^{\frac{q}{q-2}}\left(\frac{q}{2}\right)^{\frac{4}{q-2}}. \tag{50}$$

In particular, if $\epsilon = 0$*, then*

$$\left|\text{supp}\left(\mathcal{A}_{\mathbb{H}}^{\wedge_1,\wedge_2}(f,g)\right)\right| \geq \frac{(2\pi e)^2}{|B_1 B_2|}. \tag{51}$$

Proof. From Theorem 4.4, it follows that

$$\left|\left(\mathcal{A}_{\mathbb{H}}^{\wedge_1,\wedge_2}(f,g)\right)(\boldsymbol{x},\boldsymbol{\xi})\right| = \left|\left(\mathcal{S}_{\mathbb{H},g}^{\wedge_1,\wedge_2} f\right)(\boldsymbol{x},\boldsymbol{\xi}'_{\boldsymbol{x}})\right|, \boldsymbol{\xi}'_{\boldsymbol{x}} = \left(\xi_1 - \frac{A_1 x_1}{B_1}, \xi_2 - \frac{A_2 x_2}{B_2}\right).$$

Since $\mathcal{A}_{\mathbb{H}}^{\wedge_1,\wedge_2}(f,g)$ is ϵ-concentrated on Ω, it can be shown that $\mathcal{S}_{\mathbb{H},g}^{\wedge_1,\wedge_2} f$ is ϵ-concentrated on $P^{-1}\Omega$, where P is the non-singular matrix given by $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{A_1}{B_1} & 0 & 1 & 0 \\ 0 & \frac{A_2}{B_2} & 0 & 1 \end{bmatrix}$ and $P^{-1}\Omega = \{P^{-1}\boldsymbol{x} : \boldsymbol{x} \in \Omega\}$. So, by Theorem 4.6, we have

$$|P^{-1}\Omega| \geq \frac{(2\pi)^2}{|B_1 B_2|}(1-\epsilon^2)^{\frac{q}{q-2}} \left(\frac{q}{2}\right)^{\left(\frac{4}{q-2}\right)}.$$

This gives

$$|\Omega| \geq \frac{(2\pi)^2}{|B_1 B_2|}(1-\epsilon^2)^{\frac{q}{q-2}} \left(\frac{q}{2}\right)^{\left(\frac{4}{q-2}\right)}, \text{ since } det(P^{-1}) = 1.$$

This proves Eq. (50). □

Corollary 4.8.2. *Let* g *be a QWF and* $f \in L^2_{\mathbb{H}}(\mathbb{R}^2)$*, such that* $f \neq 0$*. Let* $\epsilon \geq 0$ *and* $\Omega \subset \mathbb{R}^2 \times \mathbb{R}^2$ *is measurable. If* $\mathcal{W}_{\mathbb{H}}^{\wedge_1,\wedge_2}(f,g)$*, on* Ω*, is* ϵ*-concentrated, then for every* $q > 2$

$$|\Omega| \geq \frac{(2\pi)^2}{16|B_1 B_2|}(1-\epsilon^2)^{\frac{q}{q-2}} \left(\frac{q}{2}\right)^{\frac{4}{q-2}}. \tag{52}$$

In particular, if $\epsilon = 0$*, then*

$$\left|\text{supp}\left(\mathcal{W}_{\mathbb{H}}^{\wedge_1,\wedge_2}(f,g)\right)\right| \geq \frac{(\pi e)^2}{4|B_1 B_2|}. \tag{53}$$

Proof. From Theorem 4.5, it follows that

$$\begin{aligned} \left|\left(\mathcal{W}_{\mathbb{H}}^{\wedge_1,\wedge_2}(f,g)\right)(\boldsymbol{x},\boldsymbol{\xi})\right| &= 4\left|\left(\mathcal{S}_{\mathbb{H},\tilde{g}}^{\wedge'_1,\wedge'_2} f\right)(2\boldsymbol{x},\boldsymbol{\xi}'_{\boldsymbol{x}})\right|, \\ \boldsymbol{\xi}'_{\boldsymbol{x}} &= \left(\xi_1 - \frac{4A_1 x_1}{B_1}, \xi_2 - \frac{4A_2 x_2}{B_2}\right). \end{aligned}$$

Since $\mathcal{W}_{\mathbb{H}}^{\wedge_1,\wedge_2}(f,g)$ is ϵ-concentrated on Ω, it can be shown that $\mathcal{S}_{\mathbb{H},g}^{\wedge'_1,\wedge'_2} f$ is ϵ-concentrated on $P^{-1}\Omega$, where P is the non-singular matrix given by $\begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ \frac{4A_1}{B_1} & 0 & 1 & 0 \\ 0 & \frac{4A_2}{B_2} & 0 & 1 \end{bmatrix}$. So, by Theorem 4.6, we have

$$|P^{-1}\Omega| \geq \frac{(2\pi)^2}{4|B_1 B_2|}(1-\epsilon^2)^{\frac{q}{q-2}} \left(\frac{q}{2}\right)^{\left(\frac{4}{q-2}\right)}.$$

This gives

$$|\Omega| \geq \frac{(2\pi)^2}{16|B_1B_2|}(1-\epsilon^2)^{\frac{q}{q-2}}\left(\frac{q}{2}\right)^{\left(\frac{4}{q-2}\right)}, \text{ since } det(P^{-1})=4.$$

This proves Eq. (52). ☐

4.4. Entropy uncertainty principle

As a consequence of the inner product relation and the Lieb's inequality we have the following theorem, the proof of which is motivated from the [40].

Theorem 4.7. *Let $f \in L^2_{\mathbb{H}}(\mathbb{R}^2)$ and g be a QWF such that $\|g\|_{L^2_{\mathbb{H}}(\mathbb{R}^2)}\|f\|_{L^2_{\mathbb{H}}(\mathbb{R}^2)} = 1$, then*

$$\mathcal{E}_S(f,g,\wedge_1,\wedge_2) \geq \frac{2}{|B_1B_2|}, \tag{54}$$

where

$$\begin{aligned}\mathcal{E}_S(f,g,\wedge_1,\wedge_2) \ &= -\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left|\left(\mathcal{S}^{\wedge_1,\wedge_2}_{\mathbb{H},g}f\right)(\boldsymbol{x},\boldsymbol{\xi})\right|^2 \\ &\qquad \log\left(\left|\left(\mathcal{S}^{\wedge_1,\wedge_2}_{\mathbb{H},g}f\right)(\boldsymbol{x},\boldsymbol{\xi})\right|^2\right) d\boldsymbol{x}\boldsymbol{d}\xi.\end{aligned}$$

Proof. Define

$$I(f,g,\wedge_1,\wedge_2,q) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left|\left(\mathcal{S}^{\wedge_1,\wedge_2}_{\mathbb{H},g}f\right)(\boldsymbol{x},\boldsymbol{\xi})\right|^q d\boldsymbol{x}d\boldsymbol{\xi}. \tag{55}$$

Then using (55) in (37), we get

$$I(f,g,\wedge_1,\wedge_2,2) = \frac{1}{|B_1B_2|}. \tag{56}$$

Also, from (41) and (56), it can be shown that

$$I(f,g,\wedge_1,\wedge_2,q) \leq \frac{(2\pi)^{2-q}}{|B_1B_2|}\left(\frac{2}{q}\right)^2. \tag{57}$$

Define, for $\lambda > 0$,

$$R(\lambda) = \frac{I(f,g,\wedge_1,\wedge_2,2) - I(f,g,\wedge_1,\wedge_2,2+2\lambda).}{\lambda}$$

Then

$$\begin{aligned} R(\lambda) &\geq \frac{1}{\lambda}\left\{\frac{1}{|B_1B_2|} - \frac{(2\pi)^{-2\lambda}}{|B_1B_2|}\left(\frac{1}{1+\lambda}\right)^2\right\} \\ &> \frac{1}{\lambda|B_1B_2|}\left\{1 - \frac{1}{(1+\lambda^2)}\right\}\end{aligned}$$

i.e.,

$$R(\lambda) > \frac{2+\lambda}{|B_1B_2|(1+\lambda)^2}. \tag{58}$$

Assume that $\mathcal{E}_S(f,g,\wedge_1,\wedge_2) < \infty$, otherwise (54) is obvious.

Now using the inequality $1+\lambda\log a\leq a^{\lambda},\ \lambda>0$, we have

$$\begin{aligned} 0 &\leq \frac{1}{\lambda}\left|\left(\mathcal{S}_{\mathbb{H},g}^{\wedge_1,\wedge_2}f\right)(\boldsymbol{x},\boldsymbol{\xi})\right|^2\left(1-\left|\left(\mathcal{S}_{\mathbb{H},g}^{\wedge_1,\wedge_2}f\right)(\boldsymbol{x},\boldsymbol{\xi})\right|^{2\lambda}\right)\\ &\leq -\left|\left(\mathcal{S}_{\mathbb{H},g}^{\wedge_1,\wedge_2}f\right)(\boldsymbol{x},\boldsymbol{\xi})\right|^2\log\left(\left|\left(\mathcal{S}_{\mathbb{H},g}^{\wedge_1,\wedge_2}f\right)(\boldsymbol{x},\boldsymbol{\xi})\right|^2\right). \end{aligned} \tag{59}$$

Since, $-\left|\left(\mathcal{S}_{\mathbb{H},g}^{\wedge_1,\wedge_2}f\right)(\boldsymbol{x},\boldsymbol{\xi})\right|^2\log\left(\left|\left(\mathcal{S}_{\mathbb{H},g}^{\wedge_1,\wedge_2}f\right)(\boldsymbol{x},\boldsymbol{\xi})\right|^2\right)$ is integrable, in view of Eq. (59), using Lebesgue dominated convergence theorem, we have

$$\begin{aligned} \lim_{\lambda\to 0+}R(\lambda) &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\lim_{\lambda\to 0+}\left\{\frac{1}{\lambda}\left|\left(\mathcal{S}_{\mathbb{H},g}^{\wedge_1,\wedge_2}f\right)(\boldsymbol{x},\boldsymbol{\xi})\right|^2\left(1-\left|\left(\mathcal{S}_{\mathbb{H},g}^{\wedge_1,\wedge_2}f\right)(\boldsymbol{x},\boldsymbol{\xi})\right|^{2\lambda}\right)\right\}d\boldsymbol{x}\,d\boldsymbol{\xi}\\ &= \mathcal{E}_S(f,g,\wedge_1,\wedge_2). \end{aligned} \tag{60}$$

Again from (58), we get

$$\lim_{\lambda\to 0+}R(\lambda)\geq\frac{2}{|B_1B_2|}. \tag{61}$$

Thus from (60) and (61), we have Eq. (54). This completes the proof. □

Corollary 4.9.1. *Let $f\in L^2_{\mathbb{H}}(\mathbb{R}^2)$ and g be a QWF such that $\|g\|_{L^2_{\mathbb{H}}(\mathbb{R}^2)}\|f\|_{L^2_{\mathbb{H}}(\mathbb{R}^2)}=1$, then*

$$\mathcal{E}_A(f,g,\wedge_1,\wedge_2)\geq\frac{2}{|B_1B_2|}, \tag{62}$$

where

$$\begin{aligned} \mathcal{E}_A(f,g,\wedge_1,\wedge_2) &= -\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left|\left(\mathcal{A}_{\mathbb{H}}^{\wedge_1,\wedge_2}(f,g)\right)(\boldsymbol{x},\boldsymbol{\xi})\right|^2\\ &\qquad\log\left(\left|\left(\mathcal{A}_{\mathbb{H}}^{\wedge_1,\wedge_2}(f,g)\right)(\boldsymbol{x},\boldsymbol{\xi})\right|^2\right)d\boldsymbol{x}\boldsymbol{d}\xi. \end{aligned}$$

Proof. From Theorem 4.4, it follows that

$$\left|\left(\mathcal{A}_{\mathbb{H}}^{\wedge_1,\wedge_2}(f,g)\right)(\boldsymbol{x},\boldsymbol{\xi})\right|=\left|\left(\mathcal{S}_{\mathbb{H},g}^{\wedge_1,\wedge_2}f\right)(\boldsymbol{x},\boldsymbol{\xi}'_{\boldsymbol{x}})\right|,\boldsymbol{\xi}'_{\boldsymbol{x}}=\left(\xi_1-\frac{A_1x_1}{B_1},\xi_2-\frac{A_2x_2}{B_2}\right).$$

So, we have

$$\begin{aligned} \mathcal{E}_A(f,g,\wedge_1,\wedge_2) &= -\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left|\left(\mathcal{S}_{\mathbb{H},g}^{\wedge_1,\wedge_2}f\right)(\boldsymbol{x},\boldsymbol{\xi}'_{\boldsymbol{x}})\right|^2\\ &\qquad\log\left(\left|\left(\mathcal{S}_{\mathbb{H},g}^{\wedge_1,\wedge_2}f\right)(\boldsymbol{x},\boldsymbol{\xi}'_{\boldsymbol{x}})\right|^2\right)d\boldsymbol{x}\boldsymbol{d}\xi\\ &= \mathcal{E}_S(f,g,\wedge_1,\wedge_2). \end{aligned}$$

Thus using Theorem 4.7, we have Eq. (62). □

Corollary 4.9.2. *Let $f\in L^2_{\mathbb{H}}(\mathbb{R}^2)$ and g be a QWF such that $\|g\|_{L^2_{\mathbb{H}}(\mathbb{R}^2)}\|f\|_{L^2_{\mathbb{H}}(\mathbb{R}^2)}=1$. Then*

$$\mathcal{E}_W(f,g,\wedge_1,\wedge_2)\geq\frac{2-\log 16}{|B_1B_2|},$$

where

$$\mathcal{E}_W(f,g,\wedge_1,\wedge_2) = -\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left|\left(\mathcal{W}_{\mathbb{H}}^{\wedge_1,\wedge_2}(f,g)\right)(\boldsymbol{x},\boldsymbol{\xi})\right|^2 \log\left(\left|\left(\mathcal{W}_{\mathbb{H}}^{\wedge_1,\wedge_2}(f,g)\right)(\boldsymbol{x},\boldsymbol{\xi})\right|^2\right) d\boldsymbol{x}\boldsymbol{d}\xi.$$

Proof. From Theorem 4.5, it follows that

$$\left|\left(\mathcal{W}_{\mathbb{H}}^{\wedge_1,\wedge_2}(f,g)\right)(\boldsymbol{x},\boldsymbol{\xi})\right| = 4\left|\left(\mathcal{S}_{\mathbb{H},\tilde{g}}^{\wedge_1',\wedge_2'}f\right)(2\boldsymbol{x},\boldsymbol{\xi}'_{\boldsymbol{x}})\right|, \boldsymbol{\xi}'_{\boldsymbol{x}} = \left(\xi_1 - \frac{4A_1x_1}{B_1}, \xi_2 - \frac{4A_2x_2}{B_2}\right).$$

So, we have

$$\begin{aligned}
&\mathcal{E}_W(f,g,\wedge_1,\wedge_2)\\
&= -16\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left|\left(\mathcal{S}_{\mathbb{H},\tilde{g}}^{\wedge_1',\wedge_2'}f\right)(2\boldsymbol{x},\boldsymbol{\xi}'_{\boldsymbol{x}})\right|^2 \log\left(16\left|\left(\mathcal{S}_{\mathbb{H},\tilde{g}}^{\wedge_1',\wedge_2'}f\right)(2\boldsymbol{x},\boldsymbol{\xi}'_{\boldsymbol{x}})\right|^2\right) d\boldsymbol{x}\boldsymbol{d}\xi\\
&= -4\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left|\left(\mathcal{S}_{\mathbb{H},\tilde{g}}^{\wedge_1',\wedge_2'}f\right)(\boldsymbol{x},\boldsymbol{\xi})\right|^2 \log\left(16\left|\left(\mathcal{S}_{\mathbb{H},\tilde{g}}^{\wedge_1',\wedge_2'}f\right)(\boldsymbol{x},\boldsymbol{\xi})\right|^2\right) d\boldsymbol{x}\boldsymbol{d}\xi\\
&= -\frac{4\log 16}{|4B_1B_2|} - 4\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left|\left(\mathcal{S}_{\mathbb{H},\tilde{g}}^{\wedge_1',\wedge_2'}f\right)(\boldsymbol{x},\boldsymbol{\xi})\right|^2 \log\left(\left|\left(\mathcal{S}_{\mathbb{H},\tilde{g}}^{\wedge_1',\wedge_2'}f\right)(\boldsymbol{x},\boldsymbol{\xi})\right|^2\right) d\boldsymbol{x}\boldsymbol{d}\xi\\
&= -\frac{\log 16}{|B_1B_2|} + 4\mathcal{E}_S(f,\tilde{g},\wedge_1',\wedge_2').
\end{aligned}$$

Therefore, using Theorem 4.7, we have

$$\mathcal{E}_W(f,g,\wedge_1,\wedge_2) \geq \frac{2-\log 16}{|B_1B_2|}.$$

This finishes the proof. □

Example of STQQPFT: Consider the functions $f(\boldsymbol{t}) = e^{-\left(t_1^2+t_2^2\right)}$ and $g(\boldsymbol{t}) = \begin{cases} 1, & 0 \leq t_1 < \frac{1}{2}, 0 \leq t_2 < \frac{1}{2} \\ -1, & \frac{1}{2} \leq t_1 < 1, \frac{1}{2} \leq t_2 < 1 \\ 0, & otherwise \end{cases}$, $\boldsymbol{t} = (t_1,t_2) \in \mathbb{R}^2$. Using the Definition 4.1, the STQQPFT of f with respect to the window function g is given by

$$\left(\mathcal{S}_{\mathbb{H},g}^{\wedge_1,\wedge_2}f\right)(\boldsymbol{x},\boldsymbol{\xi}) = \int_{\mathbb{R}^2}\mathcal{K}_{\wedge_1}^{i}(t_1,\xi_1)f(\boldsymbol{t})\overline{g(\boldsymbol{t}-\boldsymbol{x})}\mathcal{K}_{\wedge_2}^{j}(t_2,\xi_2)d\boldsymbol{t}, \quad (\boldsymbol{x},\boldsymbol{\xi}) \in \mathbb{R}^2\times\mathbb{R}^2, \tag{63}$$

where $\boldsymbol{x} = (x_1, x_2)$, $\boldsymbol{\xi} = (\xi_1, \xi_2)$, $\boldsymbol{t} = (t_1, t_2) \in \mathbb{R}^2$. Thus for the chosen function f and the window function g, we get from (63)

$$\begin{aligned}
&\left(\mathcal{S}_{\mathbb{H},g}^{\wedge_1,\wedge_2} f\right)(\boldsymbol{x},\boldsymbol{\xi}) \\
&= \left\{\int_{x_1}^{x_1+\frac{1}{2}} \frac{1}{\sqrt{2\pi}} e^{-i\left(A_1 t_1^2 + B_1 t_1 \xi_1 + C_1 \xi_1^2 + D_1 t_1 + E_1 \xi_1\right)} e^{-t_1^2} dt_1\right\} \\
&\quad \left\{\int_{x_2}^{x_2+\frac{1}{2}} \frac{1}{\sqrt{2\pi}} e^{-j\left(A_2 t_2^2 + B_2 t_2 \xi_2 + C_2 \xi_2^2 + D_2 t_1 + E_2 \xi_1\right)} e^{-t_2^2} dt_2\right\} \\
&\quad - \left\{\int_{x_1+\frac{1}{2}}^{x_1+1} \frac{1}{\sqrt{2\pi}} e^{-i\left(A_1 t_1^2 + B_1 t_1 \xi_1 + C_1 \xi_1^2 + D_1 t_1 + E_1 \xi_1\right)} e^{-t_1^2} dt_1\right\} \\
&\quad \left\{\int_{x_2+\frac{1}{2}}^{x_2+1} \frac{1}{\sqrt{2\pi}} e^{-j\left(A_2 t_2^2 + B_2 t_2 \xi_2 + C_2 \xi_2^2 + D_2 t_1 + E_2 \xi_1\right)} e^{-t_2^2} dt_2\right\} \qquad (64)
\end{aligned}$$

We first consider the integral

$$\begin{aligned}
&\int_{x_1}^{x_1+\frac{1}{2}} \frac{1}{\sqrt{2\pi}} e^{-i\left(A_1 t_1^2 + B_1 t_1 \xi_1 + C_1 \xi_1^2 + D_1 t_1 + E_1 \xi_1\right)} e^{-t_1^2} dt_1 \\
&= \frac{1}{\sqrt{2\pi}} e^{-i(C_1 \xi_1^2 + E_1 \xi_1)} \int_{x_1}^{x_1+\frac{1}{2}} e^{-(1+iA_1)t_1^2 - i(B_1 \xi_1 + D_1) t_1} dt_1 \\
&= \frac{1}{\sqrt{2\pi}} e^{-i(C_1 \xi_1^2 + E_1 \xi_1) - \left(\frac{B_1 \xi_1 + D_1}{2\sqrt{1+iA_1}}\right)^2} \int_{x_1}^{x_1+\frac{1}{2}} e^{-\left(\sqrt{1+iA_1} t_1 + \frac{B_1 \xi_1 + D_1}{2\sqrt{1+iA_1}} i\right)^2} dt_1,
\end{aligned}$$

i.e.,

$$\begin{aligned}
&\int_{x_1}^{x_1+\frac{1}{2}} \frac{1}{\sqrt{2\pi}} e^{-i\left(A_1 t_1^2 + B_1 t_1 \xi_1 + C_1 \xi_1^2 + D_1 t_1 + E_1 \xi_1\right)} e^{-t_1^2} dt_1 \\
&= J(\wedge_1, \xi_1, i) \\
&\quad \left[erf\left(\left(x_1 + \frac{1}{2}\right) X(A_1, i) + Y(\wedge_1, \xi_1, i)\right) - erf\left(x_1 X(A_1, i) + Y(\wedge_1, \xi_1, i)\right)\right],
\end{aligned} \qquad (65)$$

where

$$\begin{aligned}
J(\wedge_1, \xi_1, i) &= \frac{1}{2\sqrt{2}} \frac{e^{-i(C_1 \xi_1^2 + E_1 \xi_1) - \left(\frac{B_1 \xi_1 + D_1}{2\sqrt{1+iA_1}}\right)^2}}{\sqrt{1+iA_1}}, \\
X(A_1, i) &= \sqrt{1+iA_1}, Y(\wedge_1, \xi_1, i) = \frac{B_1 \xi_1 + D_1}{2\sqrt{1+iA_1}} i
\end{aligned}$$

and $erf(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-x^2} dx$. Similarly, we have

$$\begin{aligned} &\int_{x_2}^{x_2+\frac{1}{2}} \frac{1}{\sqrt{2\pi}} e^{-j\left(A_2 t_2^2 + B_2 t_2 \xi_2 + C_2 \xi_2^2 + D_2 t_2 + E_2 \xi_2\right)} e^{-t_2^2} dt_2 \\ &\quad = J(\wedge_2, \xi_2, j) \\ &\qquad \left[erf\left(\left(x_2 + \frac{1}{2} \right) X(A_2, j) + Y(\wedge_2, \xi_2, j) \right) \right. \\ &\qquad\quad \left. - erf\left(x_2 X(A_2, j) + Y(\wedge_2, \xi_2, j) \right) \right]. \end{aligned} \tag{66}$$

Also, we have

$$\begin{aligned} &\int_{x_1+\frac{1}{2}}^{x_1+1} \frac{1}{\sqrt{2\pi}} e^{-i\left(A_1 t_1^2 + B_1 t_1 \xi_1 + C_1 \xi_1^2 + D_1 t_1 + E_1 \xi_1\right)} e^{-t_1^2} dt_1 \\ &\quad = J(\wedge_1, \xi_1, i) \\ &\qquad \left[erf\left((x_1 + 1) X(A_1, i) + Y(\wedge_1, \xi_1, i) \right) \right. \\ &\qquad\quad \left. - erf\left(\left(x_1 + \frac{1}{2} \right) X(A_1, i) + Y(\wedge_1, \xi_1, i) \right) \right] \end{aligned} \tag{67}$$

and

$$\begin{aligned} &\int_{x_2+\frac{1}{2}}^{x_2+1} \frac{1}{\sqrt{2\pi}} e^{-j\left(A_2 t_2^2 + B_2 t_2 \xi_2 + C_2 \xi_2^2 + D_2 t_2 + E_2 \xi_2\right)} e^{-t_2^2} dt_2 \\ &\quad = J(\wedge_2, \xi_2, j) \\ &\qquad \left[erf\left((x_2 + 1) X(A_2, j) + Y(\wedge_2, \xi_2, j) \right) \right. \\ &\qquad\quad \left. - erf\left(\left(x_2 + \frac{1}{2} \right) X(A_2, j) + Y(\wedge_2, \xi_2, j) \right) \right]. \end{aligned} \tag{68}$$

Using Eqs. (65), (66), (67) and (68) in Eq. (64), we have

$$\begin{aligned} &\left(\mathcal{S}_{\mathbb{H},g}^{\wedge_1,\wedge_2} f \right) (\boldsymbol{x}, \boldsymbol{\xi}) \\ &\quad = J(\wedge_1, \xi_1, i) \left[erf\left(\left(x_1 + \frac{1}{2} \right) X(A_1, i) + Y(\wedge_1, \xi_1, i) \right) \right. \\ &\qquad \left. - erf\left(x_1 X(A_1, i) + Y(\wedge_1, \xi_1, i) \right) \right] \\ &\qquad \times J(\wedge_2, \xi_2, j) \left[erf\left(\left(x_2 + \frac{1}{2} \right) X(A_2, j) + Y(\wedge_2, \xi_2, j) \right) \right. \\ &\qquad \left. - erf\left(x_2 X(A_2, j) + Y(\wedge_2, \xi_2, j) \right) \right] \\ &\qquad - J(\wedge_1, \xi_1, i) \left[erf\left((x_1 + 1) X(A_1, i) + Y(\wedge_1, \xi_1, i) \right) \right. \\ &\qquad \left. - erf\left(\left(x_1 + \frac{1}{2} \right) X(A_1, i) + Y(\wedge_1, \xi_1, i) \right) \right] \\ &\qquad \times J(\wedge_2, \xi_2, j) \left[erf\left((x_2 + 1) X(A_2, j) + Y(\wedge_2, \xi_2, j) \right) \right. \\ &\qquad \left. - erf\left(\left(x_2 + \frac{1}{2} \right) X(A_2, j) + Y(\wedge_2, \xi_2, j) \right) \right]. \end{aligned}$$

5. Conclusions

In this article, we have studied Parseval's identity and sharp Hausdorff-Young inequality for the two-sided QQPFT of quaternion-valued functions. Based on the sharp Hausdorff–Young inequality, we have obtained the sharper Rènyi entropy UP for the proposed QPFT of quaternion-valued functions. We have extended the STQPFT of complex-valued functions to the functions of quaternion-valued and studied the properties like boundedness, linearity, translation, and scaling. We have also obtained the inner product relation and inversion formula for the proposed two-sided STQQPFT. We have also obtained the relations of STQQPFT with that of the QQPAF and the QQP-WVD of the quaternion-valued function associated with the QQPFT. We have obtained the sharper version of the Lieb's and entropy UPs for all these three transforms based on the sharp Hausdorff-Young inequality for the QQPFT.

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Declarations

Conflict of interest No potential conflict of interest was reported by the authors.

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Bivek Gupta and Amit K. Verma
Department of Mathematics
Indian Institute of Technology Patna
Bihta
Patna 801103
India
e-mail: akverma@iitp.ac.in

Bivek Gupta
e-mail: bivekgupta040792@gmail.com

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