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Hausdorff–Young Inequalities for Fourier Transforms over Cayley–Dickson Algebras

Shihao Fan and Guangbin Ren*

Abstract. In this study, we extend Beckner's seminal work on the Fourier transform to the domain of Cayley–Dickson algebras, establishing a precise form of the Hausdorff–Young inequality for functions that take values in these algebras. Our extension faces significant hurdles due to the unique characteristics of the Cayley–Dickson Fourier transform. This transformation diverges from the classical Fourier transform in several key aspects: it does not conform to the Plancherel theorem, alters the interplay between derivatives and multiplication, and the product of algebra elements does not necessarily maintain the magnitude relationships found in classical settings. To overcome these challenges, our approach involves constructing the Cayley–Dickson Fourier transform by sequentially applying classical Fourier transforms. A pivotal part of our strategy is the utilization of a theorem that facilitates the norm-preserving extension of linear operators between spaces L^p and L^q. Furthermore, our investigation brings new insights into the complexities surrounding the Beckner–Hirschman Entropic inequality in the context of non-associative algebras.

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Keywords. Cayley–Dickson algebras, Fourier transform, Hausdorff–Young inequality, Beckner–Hirschman entropic inequality.

1. Introduction

The Hausdorff–Young inequality is a cornerstone in Fourier analysis, stating the boundedness of the Fourier transform

$$\mathscr{F} f(\zeta) = \int_{\mathbb{R}^n} f(x) e^{-i2\pi\langle x,\zeta\rangle} dx,$$

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*Corresponding author.

between $L^p(\mathbb{R}^n)$ and $L^q(\mathbb{R}^n)$ spaces, with q being the Hölder conjugate of p for $1 < p \leqslant 2 \leqslant q < \infty$. The optimal constant in this inequality, known as the Babenko–Beckner constant, has been established for $q \geqslant 2$ and is achieved for Gaussian functions, as evidenced by analytic methods and maximization strategies [2,4,16].

Extensions of the Quaternionic Fourier transform [15] signify the evolution of multichannel signal processing. The quaternion approach is shown to better maintain signal integrity than traditional component-wise methods. This transformation offers detailed understanding of signal space structures, enhancing areas such as color imaging, vector field visualization, and speech signal processing [9].

However, extending the sharp Hausdorff–Young inequality to the non-associative octonion algebra poses significant challenges. This paper explores this area, also examining the broader Cayley–Dickson algebras, which include nested algebras developed through the Cayley–Dickson process. This sequence, represented by

$$\mathscr{C}_0 \subset \mathscr{C}_1 \subset \mathscr{C}_2 \subset \mathscr{C}_3 \subset \cdots,$$

begins with the algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$, and progresses into more intricate structures, often exhibiting more challenging properties [7,8].

We aim to further the comprehensive Fourier transform theory for $\mathscr{C}_m$ with $m \geqslant 4$, building upon the foundational work on real-valued functions by Snopek [22,23]. Our extension includes functions valued in $\mathscr{C}_m$ [10].

The octonion Fourier transform was initially introduced by Snopek [23] for real-valued functions, and later expanded by Błaszczyk [5,6] for octonion-valued functions as:

$$\mathscr{F}_{\mathbb{O}} f(\mathbf{y}) = \int_{\mathbb{R}^3} f(\mathbf{x}) e^{-2\pi e_1 x_1 y_1} e^{-2\pi e_2 x_2 y_2} e^{-2\pi e_4 x_3 y_3} d\mathbf{x}. \tag{1.1}$$

The properties of this transform have been systematically studied by Błaszczyk and Lian [5,6,15], with applications spanning various fields [13–15,22].

In the context of Cayley–Dickson algebras, Fourier transforms encounter several challenges

- The product of two algebra elements may have a magnitude that does not match the product of their individual magnitudes.
- The non-applicability of interpolation theory in this algebraic setting limits conventional proof techniques.
- The Plancherel theorem is not valid within these algebras.
- The classical relationship between derivatives and multiplications is altered in this non-associative setting.

To address these complexities, we use an innovative method, constructing the Cayley–Dickson Fourier transform through sequential classical Fourier transforms, following the complex structures within the algebra.

Specifically, the Cayley–Dickson Fourier transform $\mathscr{F}_m f$, defined for functions in $L^1(\mathbb{R}^m, \mathscr{C}m)$, is expressed as

$$\mathscr{F}_m f(\mathbf{y}) = \int_{\mathbb{R}^m} f(\mathbf{x}) e^{-2\pi e_1 x_1 y_1} e^{-2\pi e_2 x_2 y_2} \cdots e^{-2\pi e_{2m-1} x_m y_m} d\mathbf{x},$$

using a left-to-right multiplication rule for the integrand. This transformation is constructed through a series of classical Fourier transforms $\mathscr{F}_{\mathbb{C}_{e_{2t-1}}}$, aligned with the complex structure indicated by $e_{2^{t-1}}$. This composite method is represented as a successive application of $\mathscr{F}_{\mathbb{C}_{e_{2i-1}}}$ for each i from 1 to m.

Additionally, our approach utilizes a theorem about the norm-preserving extension of linear operators. Specifically, for any σ-finite measurable spaces (X, Γ_X, μ) and (Y, Γ_Y, ν), a bounded linear operator

$$T : L^p(X, \mathbb{C}) \longrightarrow L^q(Y, \mathbb{C})$$

where q is the conjugate exponent of p and $p \leqslant q$, can be extended to

$$T : L^p(X, \ell^2(\mathbb{C})) \longrightarrow L^q(Y, \ell^2(\mathbb{C}))$$

with an unchanged norm, utilizing the natural inclusion $\mathbb{C} \subset \ell^2(\mathbb{C})$.

In $\mathscr{C}_m$, we establish the Hausdorff–Young inequality for functions in $L^p(\mathbb{R}^m, \mathscr{C}_m)$ with $1 < p < 2$, expressed as

$$\|\mathscr{F} f\|_q \leqslant \left(p^{\frac{1}{p}} q^{-\frac{1}{q}}\right)^{m/2} \|f\|_p.$$

Notably, for Gaussian functions, this inequality is exact and achievable, showing a distinct jump between the cases of $m = 1$ and $m > 1$.

Our study also highlights practical implications, especially concerning the Beckner–Hirschman entropic inequality

$$S(|f|^2) + S(|\mathscr{F}_m f|^2) \geqslant m(1 - \ln 2), \tag{1.2}$$

for functions $f \in L^2(\mathbb{R}^m, \mathscr{C}_m)$ with $\|f\|_2 = 1$. Here, S denotes the von Neumann entropy.

2. Preliminaries

In this section, we will provide an overview of the Cayley–Dickson algebras and the Fourier transform in the setting of the Cayley–Dickson algebras. For the recent development related to the Cayley–Dickson algebras, we refer to [1,12,13,18–21].

Convention: In this paper, we will adopt the convention of left-to-right multiplication, due to the non-associativity of $\mathscr{C}_m$ for $m \geqslant 3$, unless explicitly stated otherwise. Thus, for any $x_j \in \mathscr{C}_m,\ j = 1, 2, \ldots, n$, we have

$$x_1 x_2 x_3 x_4 x_5 \cdots x_{n-1} x_n = (\cdots((((x_1 x_2) x_3) x_4) x_5) \cdots x_{n-1}) x_n.$$

2.1. Cayley–Dickson Process

The Cayley–Dickson algebras consist of a set of 2^m-dimensional real algebras. These algebras include all finite-dimensional division algebras over the reals that are alternative, such as the real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$, quaternions $\mathbb{H}$, and octonions $\mathbb{O}$. However, it is worth noting that Cayley–Dickson algebras with dimensions greater than 8 are not alternative and have zero divisors.

The definition of the Cayley–Dickson algebra $\mathscr{C}_m$, where $m \geqslant 1$ and $\mathscr{C}_0 = \mathbb{R}$, can be found in [3,7,8]. As a real linear space, $\mathscr{C}_{m-1}$ has a dimension of 2^{m-1}. Its standard orthogonal basis consists of 2^{m-1} elements, denoted as

$$e_0, e_1, \ldots, e_{2^{m-1}-1}, \tag{2.1}$$

where $e_0 = 1$. The multiplication rule for the basis is given by

$$e_k e_l = -\delta_{kl} + \gamma_{kls} e_s \tag{2.2}$$

for all $k, l, s = 1, 2, \ldots, 2^{m-1} - 1$, where the coefficients γ_{kls} are totally anti-symmetric with respect to the interchange of k and l, and any two of k, l, s uniquely determine the third, provided that k, l, s are not equal to each other.

We can represent any arbitrary element x in $\mathscr{C}_{m-1}$ using an orthonormal basis as:

$$x = x_0 + \sum_{k=1}^{2^{m-1}-1} x_k e_k, \tag{2.3}$$

where x_k belongs to the set of real numbers. The conjugation of x is defined as:

$$\overline{x} = x_0 - \sum_{k=1}^{2^{m-1}-1} x_k e_k. \tag{2.4}$$

Using the anti-symmetry property of the coefficients, i.e., $\gamma_{kls} = -\gamma_{lks}$, it can be shown that conjugation is an involution. Therefore,

$$\overline{\overline{x}} = x \quad \text{and} \quad \overline{xy} = \overline{y}\,\overline{x}. \tag{2.5}$$

The expressions for the real part $\mathfrak{Re}(x)$ and imaginary part $\Im(x)$ of x are given by:

$$\mathfrak{Re}(x) = \frac{x + \overline{x}}{2}, \quad \Im(x) = x - \mathfrak{Re}(x). \tag{2.6}$$

We can define the Cayley–Dickson inner product $\langle x, y \rangle$, the real inner product $\langle x, y \rangle_{\mathbb{R}}$, and the norm $|x|$ for any $x, y \in \mathscr{C}_{m-1}$ as follows:

$$\langle x, y \rangle = x\overline{y}, \tag{2.7}$$

$$\langle x, y \rangle_{\mathbb{R}} = \mathfrak{Re}\langle x, y \rangle = \sum_{k=0}^{2^{m-1}-1} x_k y_k, \tag{2.8}$$

$$|x| = \langle x, x \rangle^{1/2} = \left(\sum_{k=0}^{2^{m-1}-1} x_k^2 \right)^{1/2}. \tag{2.9}$$

Now we consider the relation between $\mathscr{C}_m$ and $\mathscr{C}_{m-1}$. We take the orthonormal basis of $\mathscr{C}_{m-1}$ in (2.1) and let $e_{2^{m-1}}$ is an imaginary unit in $\mathscr{C}_m$ that anti-commutes with the basis elements of $e_{2^{m-1}}$, i.e.,

$$e_k e_{2^{m-1}} = -e_{2^{m-1}} e_k \tag{2.10}$$

for all $k = 1, 2, \dots, 2^{m-1} - 1$. We define

$$e_{k+2^{m-1}} = e_k e_{2^{m-1}}.$$

Then we have a standard orthonormal basis of $\mathscr{C}_m$, given by

$$e_0, e_1, \dots, e_{2^{m-1}-1}, e_{2^{m-1}}, \dots, e_{2^m-1}.$$

Using the Cayley–Dickson construction, we can define the multiplication law for $\mathscr{C}_m$ in terms of $\mathscr{C}_{m-1}$ and an additional imaginary unit $e_{2^{m-1}}$. Specifically, for any $x, y, z, w \in \mathscr{C}_{m-1}$, we have:

$$(x + ye_{2^{m-1}})(z + we_{2^{m-1}}) = (xz - \overline{w}y) + (wx + y\overline{z})e_{2^{m-1}}. \tag{2.11}$$

Alternatively, we can express this as:

$$\begin{aligned} x(we_{2^{m-1}}) &= (wx)e_{2^{m-1}}, \quad (ye_{2^{m-1}})z = (y\overline{z})e_{2^{m-1}}, \quad (ye_{2^{m-1}})(we_{2^{m-1}}) \\ &= -\overline{w}y. \end{aligned}$$

The multiplication table provided by the Cayley–Dickson construction is shown below:

$$(e_{2^{m-1}} e_k)e_l = e_{2^{m-1}}(e_l e_k), \tag{2.12}$$

$$e_k(e_l e_{2^{m-1}}) = (e_l e_k)e_{2^{m-1}}, \tag{2.13}$$

$$e_{2^{m-1}}(e_{2^{m-1}} e_k) = -e_k, \tag{2.14}$$

$$(e_l e_{2^{m-1}})e_{2^{m-1}} = -e_l, \tag{2.15}$$

$$(e_{2^{m-1}} e_k)(e_l e_{2^{m-1}}) = -e_l e_k. \tag{2.16}$$

The Cayley–Dickson construction allows us to express $\mathscr{C}_m$ in terms of $\mathscr{C}_{m-1}$ and $e_{2^{m-1}}$. With this decomposition, we can compute the inner product, norm, and real inner product in $\mathscr{C}_m$. More precisely, let $u, v \in \mathscr{C}_m$ be written as

$$u = x + ye_{2^{m-1}}, \quad v = z + we_{2^{m-1}},$$

where $x, y, z, w \in \mathscr{C}_{m-1}$. We can define

$$\overline{u} := \overline{x} - ye_{2^{m-1}} \tag{2.17}$$

and obtain

$$\langle u, v\rangle = u\overline{v} = x\overline{z} + \overline{w}y + (yz - wx)e_{2^{m-1}}, \tag{2.18}$$

$$|u| = \langle u, u\rangle^{\frac{1}{2}} = (|x|^2 + |y|^2)^{\frac{1}{2}}, \tag{2.19}$$

$$\langle u, v\rangle_{\mathbb{R}} = \langle x, z\rangle_{\mathbb{R}} + \langle y, w\rangle_{\mathbb{R}} = \sum_{k=0}^{2^{m-1}-1} (x_k z_k + y_k w_k). \tag{2.20}$$

We can observe that the norm of any $u, v \in \mathscr{C}_m$ satisfies the triangle inequality

$$|u + v| \leqslant |u| + |v|. \tag{2.21}$$

Additionally, for every non-zero $u \in \mathscr{C}_m$, there exists an inverse given by:

$$u^{-1} := \frac{\overline{u}}{|u|}. \tag{2.22}$$

2.2. Multiplicativity of Absolute Value

In this subsection, we explore the relationship between multiplication and absolute value in Cayley–Dickson algebras.

If $m \geqslant 4$, for any $x, y \in \mathscr{C}_m$, the value of $|xy|$ can be greater than, equal to, or less than $|x||y|$.

To illustrate this, consider the example where $x = e_1 + e_{10}$ and $y = e_5 + e_{14}$. In this case, $xy = 0$, and so $|xy| = 0$ which is less than $|x||y| = 2$.

On the other hand, suppose $x = e_1 - e_{10}$ and $y = e_0 + e_1 + e_4 - e_{15}$. Then $|xy| = 2\sqrt{3}$, which is greater than $|x||y| = 2\sqrt{2}$.

Finally, if both x and y are real, then $|xy| = |x||y|$.

We define the set Γ_m as the collection of all complex planes generated by the imaginary unit $e_{2^{j-1}}$ for j ranging from 1 to m. In other words,

$$\Gamma_m = \bigcup_{j=1}^{m} \mathbb{C}_{e_{2^{j-1}}},$$

where

$$\mathbb{C}_{e_{2^{j-1}}} = \mathbb{R} + \mathbb{R}e_{2^{j-1}}.$$

Lemma 2.1. *Let m be a positive integer. For any $x \in \mathscr{C}_m$ and $y \in \Gamma_m$, we have*

$$|xy| = |x||y|.$$

Proof. This result can be derived through the Cayley–Dickson construction (2.11) and the process of induction. More details can be found in [10, Lemma 3.2.]. □

2.3. Vector-Valued Function Spaces

When considering function spaces, we can treat the Cayley–Dickson algebras $\mathscr{C}_m$ as $\mathbb{R}^{2^m}$, which leads to all function spaces being vector-valued. In this context, we will focus on two function spaces:

- $L^p(\mathbb{R}^m, \mathscr{C}_m)$ for $1 \leqslant p < \infty$,
- $\mathcal{S}(\mathbb{R}^m, \mathscr{C}_m)$, also known as the Schwartz space.

For any function $f : \mathbb{R}^m \to \mathscr{C}_m$, there exists a standard basis of $\mathscr{C}_m$ which allows us to express f as

$$\sum_{j=0}^{2^m-1} f_j(\mathbf{x}) e_j,$$

where $f_j : \mathbb{R}^m \to \mathbb{R}$ are real-valued functions. It is important to note that $f \in L^p(\mathbb{R}^m, \mathscr{C}_m)$ if and only if $f_j \in L^p(\mathbb{R}^m, \mathbb{R})$ for all $j = 0, 1, \ldots, 2^m - 1$. Similar results hold for other vector-valued spaces.

2.4. Cayley–Dickson Fourier Transforms

The Cayley–Dickson Fourier Transform is briefly introduced along with its properties in this subsection. For more detailed information, we refer to the paper [10].

To define the Cayley–Dickson Fourier Transform of $f \in L^1(\mathbb{R}^m, \mathscr{C}_m)$, we denote the function as $\mathscr{F}f$ and express it as follows:

$$\mathscr{F}f(\mathbf{y}) = \mathscr{F}_m f(\mathbf{y}) := \int_{\mathbb{R}^m} f(\mathbf{x}) e^{-2\pi e_1 x_1 y_1} e^{-2\pi e_2 x_2 y_2} \cdots e^{-2\pi e_{2^{m-1}} x_m y_m} d\mathbf{x}. \tag{2.23}$$

Here, $\mathbf{x}$ and $\mathbf{y}$ are m-dimensional vectors, $\mathscr{C}_m$ is identified with the 2^m-dimensional real space, and e_i is the i-th unit vector.

Proposition 2.2. *For any $f \in L^1(\mathbb{R}^m, \mathscr{C}_m)$, $\mathscr{F}f$ is uniformly continuous on $\mathbb{R}^m$, and*

$$\|\mathscr{F}f\|_{L^\infty(\mathbb{R}^m, \mathscr{C}_m)} \leqslant \|f\|_{L^1(\mathbb{R}^m, \mathscr{C}_m)}. \tag{2.24}$$

Proposition 2.3. (Parseval) *For any $f \in L^2(\mathbb{R}^m, \mathscr{C}_m)$, we have*

$$\|\mathscr{F}f\|_{L^2(\mathbb{R}^m, \mathscr{C}_m)} = \|f\|_{L^2(\mathbb{R}^m, \mathscr{C}_m)}. \tag{2.25}$$

We will now prove that the Cayley–Dickson-Fourier transform preserves the Schwartz space $\mathcal{S}(\mathbb{R}^m, \mathscr{C}_m)$. To do this, we need to introduce a critical involution given by

$$\tau_{\alpha,\beta}(\mathbf{y}) = \big(y_1, (-1)^{\alpha_1+\beta_1} y_2, \ldots, (-1)^{\sum_{l=1}^{m-1}(\alpha_l+\beta_l)} y_m\big),$$

where α and β are multi-indices in $\mathbb{N}^m$.

Proposition 2.4. [10] *Let α and β be multi-indices in $\mathbb{N}^m$. Then, the Cayley–Dickson Fourier Transform of $\partial^\alpha(\mathbf{x}^\beta f)$, evaluated at $\mathbf{y}$, can be expressed as follows*:

$$\mathscr{F}\{\partial^\alpha(\mathbf{x}^\beta f)\}(\mathbf{y}) = C\mathbf{y}^\alpha \partial^\beta \mathscr{F}f(\tau_{\alpha,\beta}(\mathbf{y})) e_{2^{m-1}}^{-\beta_m} \cdots e_1^{-\beta_1} e_{2^{m-1}}^{\alpha_m} \cdots e_1^{\alpha_1}, \tag{2.26}$$

where

$$C = (-1)^{|\beta|}(2\pi)^{|\alpha|-|\beta|}.$$

3. The Proof of Sharp Hausdorff–Young Inequalities

In order to establish our main theorem, it is necessary to extend an operator that is associated with vector-valued functions, as presented in [17, Theorem 5.5.1.].

Definition 3.1. Let $1 \leqslant p < \infty$, and (X, Γ_X, μ) be a σ-finite measure space. Let f_j be complex-valued functions in $L^p(X, \mathbb{C})$. We define f as the sequence $\{f_j\}_{j=1}^\infty$, where $f_j \in L^p(X, \mathbb{C})$ for all j. We say that $f \in L^p(X, \ell^2(\mathbb{C}))$ if

$$|f|_{\ell^2(\mathbb{C})} = \left(\sum_{j=1}^\infty |f_j|^2\right)^{1/2}$$

belongs to $L^p(X,\mathbb{C})$. We denote the norm of f in $L^p(X,\ell^2(\mathbb{C}))$

$$\|f\|_{L^p(X,\boldsymbol{\ell}^2(\mathbb{C}))} = \Big\| \, |f|_{\ell^2(\mathbb{C})} \, \Big\|_{L^p(X,\mathbb{C})} := \left\| \left(\sum_{j=1}^{\infty} |f_j|^2 \right)^{1/2} \right\|_{L^p(X,\mathbb{C})}.$$

The space $L^p(X,\ell^2(\mathbb{C}))$ is a Banach space.

Theorem 3.2. [17, Theorem 5.5.1.] *Suppose* $1 \leqslant p \leqslant q < \infty$ *and let* (X,Γ_X,μ) *and* (Y,Γ_Y,ν) *be two* σ*-finite measure spaces. Let* T *be a bounded linear operator from* $L^p(X,\mathbb{C})$ *to* $L^q(Y,\mathbb{C})$ *with norm* N*. Then* T *has a norm-preserving extension, also denoted by* T*, from* $L^p(X,\boldsymbol{\ell}^2(\mathbb{C}))$ *to* $L^q(Y,\boldsymbol{\ell}^2(\mathbb{C}))$*, where we use the canonical embedding* $\mathbb{C} \subset \boldsymbol{\ell}^2(\mathbb{C})$*. In other words, for any* $f \in L^p(X,\boldsymbol{\ell}^2(\mathbb{C}))$*, we have*

$$\|Tf\|_{L^q(X,\boldsymbol{\ell}^2(\mathbb{C}))} \leqslant N\|f\|_{L^p(X,\boldsymbol{\ell}^2(\mathbb{C}))}. \tag{3.1}$$

The Cayley–Dickson algebra $\mathscr{C}_m$ can be viewed as a complex linear space consisting of the direct sum of

$$\bigoplus_{j=0}^{2^{m-1}-1} e_j\mathbb{C}_{e_{2^{m-1}}},$$

which in turn can be seen as $\mathbb{C}^{2^{m-1}}$ in a certain way.

Lemma 3.3. *The Cayley–Dickson algebras, as a real linear space, can be represented by a direct sum of orthogonal complex planes. Specifically, there exists an isometric isomorphism*

$$I : \mathscr{C}_m \longrightarrow \bigoplus_{j=0}^{2^{m-1}-1} e_j\mathbb{C}_{e_{2^{m-1}}}$$

given by

$$I(x) = I\left(\sum_{j=0}^{2^m-1} x_j e_j\right) = \sum_{j=0}^{2^{m-1}-1} e_j(x_j + x_{j+2^{m-1}}e_{2^{m-1}}). \tag{3.2}$$

Proof. To prove this, it is sufficient to show that I is an isomorphism and isometric. It is clear that I is an isomorphism. To see that I is isometric, we have

$$|I(x)|^2 = \textstyle\sum_{j=0}^{2^{m-1}-1}(x_j^2 + x_{j+2^{m-1}}^2) = \displaystyle\sum_{j=0}^{2^m-1} x_j^2 = |x|^2.$$

The proof is complete. □

We present a generalization of Beckner's result [4] to the case of Cayley–Dickson algebras $\mathscr{C}_m$. Our main theorem is as follows.

Theorem 3.4. (Hausdorff–Young) *Let* $f \in L^p(\mathbb{R}^m,\mathscr{C}_m)$ *with* $1 < p < 2$*. Then, we have*

$$\|\mathscr{F}f\|_q \leqslant A_p^m\|f\|_p, \tag{3.3}$$

where

$$A_p = \left(p^{\frac{1}{p}} q^{-\frac{1}{q}}\right)^{1/2}$$

is sharp and can be attained if f is a Gaussian function

$$f(\mathbf{x}) = ae^{-\pi(\sum_{j=1}^{m} b_j x_j^2 + 2c_j x_j)}, \tag{3.4}$$

where b_j is positive for all j, $(c_1, c_2, \ldots, c_m) \in \mathbb{C}_{e_1} \times \mathbb{R}^{m-1}$, and

$$a \in \begin{cases} \mathbb{C}_{e_1}, & m = 1, \\ \mathbb{R}, & m \geqslant 2. \end{cases}$$

Remark 3.5. The inequality of Hausdorff–Young holds for $p = 1$ and $p = 2$, where (3.3) reduces to (2.24) and (2.25), respectively. However, the extremizers differ from the $1 < p < 2$ case, where there exist only a few functions that satisfy equality in (3.3). In the cases of $p = 1$ and $p = 2$, there are numerous functions that satisfy equality, especially in $L^2(\mathbb{R}^m, \mathscr{C}_m)$, where all functions satisfy equality as demonstrated in Theorem 2.3.

Let us return to the proof of Theorem 3.4.

Proof. Let us assume $f \in \mathcal{S}(\mathbb{R}^m, \mathscr{C}_m)$ and use an approximation process and Theorem 2.4 to show that it suffices to prove (3.3) only for functions in the Schwartz space.

To begin, we note that the constant A_p^m is sharp. We represent $\mathscr{F}_{m-1}$ as the composition of the classic Fourier transform over the complex plane $\mathbb{C}_{e_{2^t-1}}$. More precisely, we have

$$\mathscr{F}_{m-1} = \mathscr{F}_{\mathbb{C}_{e_{2^{m-2}}}} \circ \cdots \circ \mathscr{F}_{\mathbb{C}_{e_1}},$$

where $\mathscr{F}_{\mathbb{C}_{e_{2^{i-1}}}}$ for $i = 1, \ldots, m-1$ represents the classic Fourier transform over the complex plane $\mathbb{C}_{e_{2^{i-1}}}$. We then express $\mathscr{F}_{m-1} f$ in real-valued measurable components, as given by

$$\mathscr{F}_{m-1} f = \sum_{j=0}^{2^m-1} g_j e_j = \sum_{j=0}^{2^{m-1}-1} e_j (g_j + g_{j+2^{m-1}} e_{2^{m-1}}), \tag{3.5}$$

where g_j are the Fourier coefficients of f. We define auxiliary functions

$$h_j := g_j + g_{j+2^{m-1}} e_{2^{m-1}}$$

and observe that the associator

$$[e_j, h_j, e^{-2\pi e_{2^{m-1}} x_m y_m}]$$

is zero, which implies that

$$\mathscr{F} f = \mathscr{F}_{\mathbb{C}_{e_{2^{m-1}}}} \circ \mathscr{F}_{m-1} f = \sum_{j=0}^{2^{m-1}-1} e_j \mathscr{F}_{\mathbb{C}_{e_{2^{m-1}}}} h_j. \tag{3.6}$$

Lemma 3.3 implies that the complex planes $e_j \mathbb{C}_{e_{2^{m-1}}}$ are mutually orthogonal. Using this, we obtain the expressions

$$|\mathscr{F}_{m-1} f|^2 = \sum_{j=0}^{2^{m-1}-1} |h_j|^2, \tag{3.7}$$

and

$$|\mathscr{F}f|^2 = \sum_{j=0}^{2^{m-1}-1} |\mathscr{F}_{\mathbb{C}_{e_{2m-1}}} h_j|^2, \tag{3.8}$$

where h_j denotes the j-th component of f under the isometric isomorphism I from Lemma 3.3.

Using induction, we will prove our theorem by assuming that inequality (3.9) holds for $t = 1$ and $t = m - 1$ for any $\varphi \in \mathcal{S}(\mathbb{R}^m, \mathscr{C}_m)$. It states that

$$\|\mathscr{F}_t \varphi(\cdot, x_{t+1}, \ldots, x_m)\|_q \leqslant A_p^t \|\varphi(\cdot, x_{t+1}, \ldots, x_m)\|_p. \tag{3.9}$$

We should note that the $t = 1$ case corresponds to the classical result, while the $t = m - 1$ case serves as our induction hypothesis.

Then we prove that (3.9) also holds for $t = m$. We denote $\mathbf{x}' = (x_1, \ldots, x_{m-1})$. By (3.7), $f(\mathbf{x})$ is L^p integrable for almost every $x_m \in \mathbb{R}$ and induction (3.9), we have

$$\begin{aligned}\left(\int_{\mathbb{R}^{m-1}} \left(\sum_{j=0}^{2^m-1} |h_j(\mathbf{y}', x_m)|^2\right)^{q/2} d\mathbf{y}'\right)^{1/q} &= \left(\int_{\mathbb{R}^{m-1}} |\mathscr{F}_{m-1} f(\mathbf{y}', x_m)|^q d\mathbf{y}'\right)^{1/q} \\ &\leqslant A_p^{m-1} \left(\int_{\mathbb{R}^{m-1}} |f(\mathbf{x})|^p d\mathbf{x}'\right)^{1/p}.\end{aligned} \tag{3.10}$$

Integrate on both sides of (3.10) with respect to x_m, we obtain

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}^{m-1}} \left(\sum_{j=0}^{2^m-1} |h_j(\mathbf{y}', x_m)|^2\right)^{q/2} d\mathbf{y}'\right)^{p/q} dx_m \leqslant A_p^{pm-p} \|f\|_p^p.$$

Then we invoke the Minkowski inequality to the left side of above integral inequality to get

$$\begin{aligned}&\left(\int_{\mathbb{R}^{m-1}} \left(\int_{\mathbb{R}} \left(\sum_{j=0}^{2^m-1} |h_j(\mathbf{y}', x_m)|^2\right)^{p/2} dx_m\right)^{q/p} d\mathbf{y}'\right)^{p/q} \\ &\quad \leqslant \int_{\mathbb{R}} \left(\int_{\mathbb{R}^{m-1}} \left(\sum_{j=0}^{2^m-1} |h_j(\mathbf{y}', x_m)|^2\right)^{q/2} d\mathbf{y}'\right)^{p/q} dx_m < \infty.\end{aligned} \tag{3.11}$$

This show that each h_j is L^p integrable with respect to x_m for almost every $\mathbf{y}' = (y_1, \ldots, y_{m-1}) \in \mathbb{R}^{m-1}$.

Next, we claim that the quantity on the left side of (3.11) is greater than or equal to

$$A_p^{-p} \left(\int_{\mathbb{R}^m} \left(\sum_{j=0}^{2^m-1} |\mathscr{F}_{\mathbb{C}_{e_{2m-1}}} h_j(\mathbf{y})|^2\right)^{\frac{q}{2}} d\mathbf{y}\right)^{\frac{p}{q}}. \tag{3.12}$$

Indeed, we note that the functions h_j, which are $\mathbb{C}_{2^{m-1}}$-valued and operated on by the classic Fourier Transform $\mathscr{F}_{\mathbb{C}_{e_{2m-1}}}$, satisfy the conditions of Lemma 3.2 with respect to x_m. By the induction step (3.9), the norm N in

Lemma 3.2 can be taken to be A_p. Also, since $1 < p < 2 < q$, we can apply Theorem 2.4 and Tonelli's theorem to complete the proof of our claim.

Furthermore, we can see that (3.12) is exactly equal to $A_p^{-p}\|\mathscr{F}\{f\}\|_q^p$ by using (3.8).

Now we come to show that the sharp constant A_p^m can be attained by the Gaussian function given in (3.4). Additionally, we can verify that this condition indeed leads to equality in (3.3).

To do so, we refer to two well-known classical results:

$$\|g\|_{L^p(\mathbb{R},\mathbb{C})} = |a|e^{\pi b^{-1}(\mathfrak{Re}(c))^2}(pb)^{-\frac{1}{2p}}, \tag{3.13}$$

$$\begin{aligned}\|\mathscr{F}_{\mathbb{C}}g\|_{L^q(\mathbb{R},\mathbb{C})} &= \|ab^{-\frac{1}{2}}e^{-\pi b^{-1}(y+e_1c)^2}\|_{L^q(\mathbb{R},\mathbb{C})}\\ &= |a|b^{-\frac{1}{2}}e^{\pi b^{-1}(\mathfrak{Re}(c))^2}(q^{-1}b)^{\frac{1}{2q}},\end{aligned} \tag{3.14}$$

where

$$g(x) = ae^{-\pi bx^2+2\pi cx}$$

with $a \in \mathbb{C}, b > 0$, and $c \in \mathbb{C}$.

Using (3.13)–(3.14), (3.4), and Lemma 2.1, we can compute the following expressions:

$$\|f\|_{L^p(\mathbb{R}^m,\mathscr{C}_m)} = |a|e^{\pi\sum_{j=1}^m b_j^{-1}(\mathfrak{Re}(c_j))^2}\prod_{j=1}^m(pb_j)^{-\frac{1}{2p}}, \tag{3.15}$$

$$\begin{aligned}&\|\mathscr{F}f\|_{L^q(\mathbb{R},\mathscr{C}_m)}\\ &\quad= \left\|a\left(\prod_{j=1}^m b_j^{-\frac{1}{2}}\right)e^{-\pi b_1^{-1}(y_1+e_1c_1)^2}\cdots e^{-\pi b_m^{-1}(y_m+e_{2^{m-1}}c_m)^2}\right\|_{L^q(\mathbb{R},\mathscr{C}_m)}\\ &\quad= |a|\left(\prod_{j=1}^m b_j^{-\frac{1}{2}}e^{\pi b_j^{-1}(\mathfrak{Re}(c_j))^2}(q^{-1}b_j)^{\frac{1}{2q}}\right).\end{aligned} \tag{3.16}$$

It is clear that (3.15)–(3.16) make (3.3) become an equality. This completes the proof of our theorem. □

Remark 3.6. We do not know if all L^p functions that satisfy (3.4) are those that attain the optimal constant A_p^m. If $m \geqslant 2$, we can prove by induction that (3.10) is an equality if and only if f is a Gaussian function of the form

$$f(\mathbf{x}) = a(x_m)e^{-\pi\left(\sum_{j=1}^{m-1}b_j(x_m)x_j^2+2c_j(x_m)x_j\right)} \tag{3.17}$$

where $a(x_m) \in \mathbb{C}e_1$, $b_j(x_m) > 0$, $c_1(x_m) \in \mathbb{C}e_1$, and $c_j(x_m) \in \mathbb{R}$ for $j \geqslant 2$. However, it is challenging to demonstrate that $c_j(x_m)$ is independent of x_m.

Using the Hausdorff–Young inequality in Theorem 3.4, we can derive the following direct implication, known as the sharp Beckner–Hirschman entropic inequality.

Theorem 3.7. (Beckner–Hirschman) *Let $f \in L^2(\mathbb{R}^m,\mathscr{C}_m)$ and $\|f\|_2 = 1$. Then we have*

$$S(|f|^2) + S(|\mathscr{F}f|^2) \geqslant m(1-\ln 2) \tag{3.18}$$

whenever the left hand side has meaning, where

$$S(|f|) = -\int_{\mathbb{R}^m} |f(\mathbf{x})| \ln |f(\mathbf{x})| d\mathbf{x}$$

is the Shannon entropy of $|f|$.

Proof. The result follows by differentiating (3.3) with respect to p at $p = 2$. □

This theorem is the generalization of Hirschman's result [11] in the setting of Cayley–Dickson algebras.

4. Concluding Remarks

We have proved the sharp Hausdorff–Young inequality for Fourier transforms over the Cayley–Dickson algebra $\mathscr{C}_m$ for any positive integer m. This result is attained by the Gaussian function given by

$$f(\mathbf{x}) = ae^{-\pi(\sum_{j=1}^{m} b_j x_j^2 + 2c_j x_j)},$$

where b_j is positive for all j, $(c_1, c_2, \ldots, c_m) \in \mathbb{C}_{e_1} \times \mathbb{R}^{m-1}$, and

$$a \in \begin{cases} \mathbb{C}_{e_1}, & m = 1, \\ \mathbb{R}, & m \geqslant 2. \end{cases}$$

In [16], Lieb showed that when $m = 1$, the aforementioned functions are the sole extremizers of the inequality. This result remains valid for any $m \in \mathbb{N}$, as long as the extremizers are even functions. Nevertheless, it is currently unknown whether these functions remain the only extremizers even when $m = 2$ in the context of quaternions.

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Shihao Fan and Guangbin Ren
Department of Mathematics
University of Science and Technology of China
Hefei 230026
China
e-mail: rengb@ustc.edu.cn

Shihao Fan
e-mail: fsh1720@mail.ustc.edu.cn

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