Adv. Appl. Clifford Algebras (2024) 34:14
0188-7009/030001-15
published online April 8, 2024
https://doi.org/10.1007/s00006-024-01321-2

Advances in Applied Clifford Algebras

Lipschitz Norm Estimate for a Higher Order Singular Integral Operator

Tania Rosa Gómez Santiesteban*, Ricardo Abreu Blaya, Juan Carlos Hernández Gómez and José Luis Sánchez Santiesteban

Communicated by Swanhild Bernstein

Abstract. Let Γ be a d-summable surface in $\mathbb{R}^m$, i.e., the boundary of a Jordan domain in $\mathbb{R}^m$, such that $\int_0^1 N_\Gamma(\tau)\tau^{d-1}\mathrm{d}\tau < +\infty$, where $N_\Gamma(\tau)$ is the number of balls of radius τ needed to cover Γ and $m-1 < d < m$. In this paper, we consider a singular integral operator S_Γ^* associated with the iterated equation $\mathcal{D}_{\underline{x}}^k f = 0$, where $\mathcal{D}_{\underline{x}}$ stands for the Dirac operator constructed with the orthonormal basis of $\mathbb{R}^m$. The fundamental result obtained establishes that if $\alpha > \frac{d}{m}$, the operator S_Γ^* transforms functions of the higher order Lipschitz class $\mathrm{Lip}(\Gamma, k+\alpha)$ into functions of the class $\mathrm{Lip}(\Gamma, k+\beta)$, for $\beta = \frac{m\alpha-d}{m-d}$. In addition, an estimate for its norm is obtained.

Mathematics Subject Classification. Primary 30G35, 47A30; Secondary 47G10, 28A80 .

Keywords. Dirac operator, D-summable surface, Higher order Lipschitz class, Norm estimate, Singular integral operator.

1. Introduction

The problem about the existence of the limit boundary values of the complex Cauchy Transform

$$\mathcal{C}_\Gamma f(z) := \frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)}{\zeta - z}d\zeta,$$

*Corresponding author.

with $f \in C^{0,\alpha}(\Gamma)$, is widely treated in [8], which leads to the Plemelj-Sojotski formulas:

$$\mathcal{C}_\Gamma^+ f(t) = \lim_{\substack{z\to t\\ z\in\Omega_+}} \mathcal{C}_\Gamma f(z) = \frac{1}{2}\left(f(t) + \mathcal{S}_\Gamma f(t)\right)$$

$$\mathcal{C}_\Gamma^- f(t) = \lim_{\substack{z\to t\\ z\in\Omega_-}} \mathcal{C}_\Gamma f(z) = \frac{1}{2}(-f(t) + \mathcal{S}_\Gamma f(t)), \tag{1.1}$$

where

$$\mathcal{S}_\Gamma f(t) := \frac{1}{\pi i}\int_\Gamma \frac{f(\zeta)}{\zeta - t} d\zeta,\ t\in\Gamma$$
$$= \lim_{\epsilon\to 0}\frac{1}{\pi i}\int_{\Gamma\setminus\Gamma\cap B(t,\epsilon)} \frac{f(\zeta)}{\zeta - t} d\zeta.$$

is a singular integral operator understood in the sense of the Cauchy principal value.

From the Plemelj-Sojotski formulas (1.1), we obtain an alternative definition of this singular operator, given by the expression

$$\mathcal{S}_\Gamma f(t) = \mathcal{C}_\Gamma^+ f(t) + \mathcal{C}_\Gamma^- f(t). \tag{1.2}$$

In addition, we have

$$\mathcal{C}_\Gamma^+ f(t) - \mathcal{C}_\Gamma^- f(t) = f(t),$$

from where

$$\mathcal{C}_\Gamma^- f(t) = \mathcal{C}_\Gamma^+ f(t) - f(t). \tag{1.3}$$

Also, note that, by substituting (1.3) into (1.2),

$$\mathcal{S}_\Gamma f(t) = 2\mathcal{C}_\Gamma^+ f(t) - f(t). \tag{1.4}$$

The expression just obtained for $\mathcal{S}_\Gamma f$ will be the starting point in the present research.

The Plemelj–Privalov theorem, named in honor of the Slovenian mathematician Josip Plemelj (1873–1967) and the Russian mathematician Ivan Ivanovich Privalov (1891–1941), is of great importance in the theory of singular integral equations and complex variable function theory. This theorem affirms the invariance of the Hölder classes under the action of the Cauchy singular integral operator, i.e., $\mathcal{S}_\Gamma : C^{0,\alpha}(\Gamma) \to C^{0,\alpha}(\Gamma)$, $0 < \alpha < 1$.

The result was obtained by Plemelj [13] for the case of smooth curves. It was rediscovered independently by Privalov [14] for the circle and, later on, for any smooth piecewise curve without cusps [15]. The reader is referred to the work of [11] for further historical details.

A higher order version of the above mentioned result was established in [5,6], where the higher order Lipschitz class $\text{Lip}(\Gamma, k+\alpha)$ plays the role of the more traditional Hölder classes. More precisely, in these works the invariance of a generalized singular integral operator $S_\Gamma^{(k)}$ when acting on $\text{Lip}(\Gamma, k+\alpha)$, is proved in complex and Clifford settings.

In the more general context of non-smooth boundaries, such invariance is no longer true. This time, the corresponding singular integral operator acts between differents Hölder classes (see [1]). For the case where the boundary Γ is d-summable, an estimation of its norm is also established in the above mentioned paper.

This brief introduction leads to the aim of the present paper: to characterize the behaviour of a singular integral operator related to iterated Dirac equations in domains with fractal boundaries. We will prove a sort of invariance of this operator between higher order Lipschitz classes. Moreover, an estimation for its norm is obtained.

2. Preliminaries

2.1. Polymonogenic Functions

Denote by $e_1, e_2, \ldots, e_m$ an orthonormal basis of $\mathbb{R}^m$, subjected to the multiplication relations

$$e_i^2 = -1,\, e_i e_j = -e_j e_i,\, i, j = 1, 2, \ldots m,\, i < j.$$

Thus the Euclidean space

$$\mathbb{R}^m = \{\underline{x} = x_1 e_1 + x_2 e_2 + \ldots x_m e_m,\, x_i \in \mathbb{R}, i = 1, 2, \ldots m\}$$

is embedded in the real Clifford algebra $\mathbb{R}_{0,m}$ generated by $e_1, e_2, \ldots e_m$ over the field of real numbers $\mathbb{R}$. An element $a \in \mathbb{R}_{0,m}$ may be written as $a = \sum_A a_A e_A$, where a_A are real constants and A runs over all the possible ordered sets

$$A = \{0 \leq i_1 < \cdots < i_k \leq m\}, \text{ or } A = \emptyset,$$

and

$$e_A := e_{i_1} e_{i_2} \cdots e_{i_k},\; e_0 = e_\emptyset = 1.$$

In particular, $Sc\, a := a_0$ is referred as the scalar part of a. Conjugation in $\mathbb{R}_{0,m}$ is defined as the anti-involution $a \mapsto \overline{a}$ for which $\overline{e_i} = -e_i$. A norm $\|.\|$ on $\mathbb{R}_{0,m}$ is defined by $\|a\|^2 = Sc[a\overline{a}]$ for $a \in \mathbb{R}_{0,m}$. We remark that for $\underline{x} \in \mathbb{R}^m$ we have $\|\underline{x}\| = |\underline{x}|$, the symbol $|.|$ denotes the usual Euclidean norm.

We will consider functions defined on subsets of $\mathbb{R}^m$ and taking values in $\mathbb{R}_{0,m}$. Those functions might be written as $f = \sum_A f_A e_A$, where f_A are $\mathbb{R}$-valued functions. The notions of continuity, differentiability and integrability of a $\mathbb{R}_{0,m}$-valued function f have the usual component-wise meaning. In particular, the spaces of all k-time continuous differentiable and p-integrable functions are denoted by $C^k(\mathbf{E})$ and $L^p(\mathbf{E})$ respectively, where $\mathbf{E}$ is a suitable subset of $\mathbb{R}^m$.

The Dirac operator $\mathcal{D}_{\underline{x}}$ for C^1-functions on $\mathbb{R}^m$ is defined by

$$\mathcal{D}_{\underline{x}} = \partial_{x_1} e_1 + \partial_{x_2} e_2 + \cdots + \partial_{x_m} e_m.$$

It is worth pointing out that $\mathcal{D}_{\underline{x}}$ factorizes the Laplace operator in $\mathbb{R}^m$ in the sense that

$$\mathcal{D}_{\underline{x}} \mathcal{D}_{\underline{x}} = -\triangle.$$

The fundamental solution of $\mathcal{D}_{\underline{x}}$ is thus given by

$$E_0(\underline{x}) = \partial_{\underline{x}} E_1(\underline{x}),$$

where

$$E_1(\underline{x}) = \frac{1}{(m-2)\sigma_m |\underline{x}|^{m-2}},\ \underline{x} \neq 0$$

is the fundamental solution of $\triangle$ and σ_m stands for the surface area of the unit sphere in $\mathbb{R}^m$.

The function

$$E_0(\underline{x}) = -\frac{1}{\sigma_m}\frac{\underline{x}}{|\underline{x}|^m},$$

called Clifford–Cauchy kernel, satisfy in $\mathbb{R}^m \setminus \{0\}$ the equations

$$\mathcal{D}_{\underline{x}} E_0 = E_0 \mathcal{D}_{\underline{x}} = 0.$$

An $\mathbb{R}_{0,m}$-valued function f, defined and differentiable in an open region Ω of $\mathbb{R}^m$, is called left monogenic (right monogenic) if $\mathcal{D}_{\underline{x}} f = 0$ ($f\mathcal{D}_{\underline{x}} = 0$) in Ω. Functions that are both left and right monogenic are called two-sided monogenic. We refer the reader to [4,10] for the more classical setting of Clifford analysis.

More generally, an $\mathbb{R}_{0,m}$-valued function f in $C^k(\Omega)$ is called polymonogenic (left) of order k, or simply k-monogenic (left) if $\mathcal{D}^k_{\underline{x}} f = 0$ in Ω. In particular, bimonogenic functions are nothing more than $\mathbb{R}_{0,m}$-valued harmonic functions. See papers such as [2,3,16] for further general information concerning polymonogenic functions.

2.2. Higher Order Lipschitz Classes and Whitney Extension Theorem

Here and subsequently, $\mathbf{j} := (j_1, j_2, ..., j_m)$ and $\mathbf{l} := (l_1, l_2, ..., l_m)$ denote multi-indexes, with $\mathbf{j}! := j_1! j_2! ... j_m!$, $|\mathbf{j}| := j_1 + j_2 + ... + j_m$; $\underline{x}^{\mathbf{l}} = x_1^{l_1} x_2^{l_2} ... x_m^{l_m}$ and $\partial^{\mathbf{j}} := \frac{\partial^{|\mathbf{j}|}}{\partial_{x_1}^{j_1} ... \partial_{x_m}^{j_m}}$.

Definition 2.1. [17] Let $\mathbf{E}$ be a closed subset of $\mathbb{R}^m$, k a non-negative integer and $0 < \alpha \leq 1$. We shall say that a real valued function f, defined in $\mathbf{E}$, belongs to $\mathrm{Lip}(\mathbf{E}, k+\alpha)$ if there exist real valued bounded functions $f^{\mathbf{j}}$, $0 < |\mathbf{j}| \leq k$, defined on $\mathbf{E}$, with $f^{\mathbf{0}} = f$, and such that if

$$f^{\mathbf{j}}(\underline{x}) = \sum_{|\mathbf{j}+\mathbf{l}| \leq k} \frac{f^{\mathbf{j}+\mathbf{l}}(\underline{y})}{\mathbf{l}!}(\underline{x}-\underline{y})^{\mathbf{l}} + R_{\mathbf{j}}(\underline{x}, \underline{y}),$$

then

$$|f^{\mathbf{j}}(\underline{x})| \leq M, |R_{\mathbf{j}}(\underline{x}, \underline{y})| \leq M|\underline{x}-\underline{y}|^{k+\alpha-|\mathbf{j}|},\ \forall \underline{x}, \underline{y} \in \mathbf{E}, |\mathbf{j}| \leq k, \quad (2.1)$$

where M is a positive constant.

The above compatibility conditions (2.1) are equivalent to the fact that the field of polynomials

$$\sum_{|\mathbf{j}| \leq k} f^{\mathbf{j}}(\underline{y})(\underline{x}-\underline{y})^{\mathbf{j}},\ \underline{y} \in \Gamma$$

is the field of Taylor polynomials of a $C^{k,\alpha}$-function. In 1934 H. Whitney proves that given a function $f \in \mathrm{Lip}(\mathbf{E}, k+\alpha)$ there exists $\tilde{f} \in \mathrm{Lip}(\mathbb{R}^m, k+\alpha)$ such that $\tilde{f} \in C^\infty(\mathbb{R}^m \setminus \mathbf{E})$ [19], a result which we state here without proof.

Theorem 2.2. [17] *Let $f \in Lip(\mathbf{E}, k+\alpha)$. Then, there exists a function $\tilde{f} \in Lip(\mathbb{R}^m, k+\alpha)$ satisfying*

(i) $\tilde{f}|_{\mathbf{E}} = f^{\mathbf{0}}$, $\partial^{\mathbf{j}}\tilde{f}|_{\mathbf{E}} = f^{\mathbf{j}}$,

(ii) $\tilde{f} \in C^{k+1}(\mathbb{R}^m \setminus \mathbf{E})$,

(iii) $|\partial^{\mathbf{j}}\tilde{f}(\underline{x})| \leq c \;\; dist(\underline{x}, \mathbf{E})^{\alpha-1}$, *for* $|\mathbf{j}| = k+1$, $\underline{x} \in \mathbb{R}^m \setminus \mathbf{E}$.

Here and subsequently, c denotes a generic constant, not necessarily the same at each occurrence.

The proof of Theorem 2.2 uses the so-called Whitney decomposition, which involves a collection of disjoint cubes Q whose lengths are proportional to their distance from $\mathbf{E}$. This decomposition, usually denoted by $\mathcal{W}$, is so that

$$\mathbb{R}^m \setminus \mathbf{E} = \bigcup_{Q \in \mathcal{W}} Q.$$

In what follows we use the symbol $|Q|$ to denote the diameter of the cube $Q \in \mathcal{W}$. For details we refer the reader to [17].

In our context we will say that an $\mathbb{R}_{0,m}$-valued function f belongs to $\mathrm{Lip}(\mathbf{E}, k+\alpha)$ if each of its real components does so. It is easy to see that this component-wise definition is equivalent to the assumption that there exist $\mathbb{R}_{0,m}$-valued functions $f^{\mathbf{j}}$ such that if

$$f^{\mathbf{j}}(\underline{x}) = \sum_{|\mathbf{j}+\mathbf{l}| \leq k} \frac{f^{\mathbf{j}+\mathbf{l}}(\underline{y})}{\mathbf{l}!}(\underline{x}-\underline{y})^{\mathbf{l}} + R_{\mathbf{j}}(\underline{x}, \underline{y}), \tag{2.2}$$

then

$$\|f^{\mathbf{j}}(\underline{x})\| \leq M, \|R_{\mathbf{j}}(\underline{x}, \underline{y})\| \leq M|\underline{x}-\underline{y}|^{k+\alpha-|\mathbf{j}|}, \; \forall \underline{x}, \underline{y} \in \mathbf{E}, |\mathbf{j}| \leq k. \tag{2.3}$$

The following norm in $\mathrm{Lip}(\mathbf{E}, k+\alpha)$ was introduced in [18]:

$$\|f\|_{k+\alpha,\mathbf{E}} = \sup_{0 \leq |\mathbf{j}| \leq k} \left\{ \sup_{\underline{x} \in \mathbf{E}} \|f^{\mathbf{j}}(\underline{x})\|, \sup_{\underline{x},\underline{y} \in \mathbf{E}} \frac{\|R_{\mathbf{j}}(\underline{x}, \underline{y})\|}{|\underline{x}-\underline{y}|^{k+\alpha-|\mathbf{j}|}} \right\}, \tag{2.4}$$

where $\|.\|$ denotes the Clifford norm on $\mathbb{R}_{0,m}$.

Before going further, it is necessary to consider the notion of d-summable sets. This concept was introduced by Harrison and Norton in [12]. We say that $\mathbf{E}$ is d-summable for some $m-1 < d < m$ if the improper integral

$$\int_0^1 N_{\mathbf{E}}(\tau)\tau^{d-1}\mathrm{d}\tau \quad \text{converges},$$

where $N_{\mathbf{E}}(\tau)$ is the number of balls of radius τ needed to cover $\mathbf{E}$.

The following lemma establishes a relationship between the Whitney decomposition $\mathcal{W}$ and the concept of d-summability. The reader is invited to review [12] for details of the proof.

Lemma 2.3. *If Ω is a Jordan domain of $\mathbb{R}^m$ with d-summable boundary Γ, then the d-sum $\sum_{Q\in\mathcal{W}}|Q|$ of the Whitney decomposition $\mathcal{W}$ of Ω is finite.*

For notational convenience, we will use the symbol $\mathbf{s}(d)$ to denote the d-sum of a Jordan domain Ω with d-summable boundary.

For $f \in \text{Lip}(\Gamma, k+\alpha)$, the polymonogenic Cauchy Transform

$$\mathcal{C}^{(k)}_{\Gamma^*}f(\underline{x}) = \tilde{f}(\underline{x})\chi_\Omega - (-1)^{k+1}\int_\Omega E_k(\underline{y}-\underline{x})\mathcal{D}^{k+1}_{\underline{y}}\tilde{f}(\underline{y})\mathrm{d}\underline{y}, \tag{2.5}$$

with Γ being d-summable, was introduced in [9].

Here and below, E_s $(s \geq 1)$ are given by:

$$E_{s-1}(\underline{x}) = \begin{cases} c(m,s)\dfrac{\underline{x}}{|\underline{x}|^{m-s+1}} & \text{if m and s are odd,} \\ & \text{or if m is even, s is odd and } s<m, \\ c(m,s)\dfrac{1}{|\underline{x}|^{m-s}} & \text{if m is odd and s is even} \\ & \text{or if m is even, s is even and } s<m, \\ c(m,s)\underline{x}^{s-m}\left(\log|\underline{x}| + d(m,s)\right) & \text{if m is even and } s\geq m, \end{cases}$$

where $c(m,s)$ is a constant that depends on m and s; and $d(m,s)$ is a real constant that depends on m and s [16].

For the kernels E_s, the following estimation:

$$\left|\partial^{\mathbf{j}}E_s(\underline{x})\right| \leq \frac{c}{|\underline{x}|^{m-s-1+|\mathbf{j}|}}, \tag{2.6}$$

is obtained in [6].

Let us define the singular integral operator:

$$\mathcal{S}^{(k)}_{\Gamma^*}f(\underline{t}) = 2(\mathcal{C}^{(k)}_{\Gamma^*})^+f(\underline{t}) - f(\underline{t}), \tag{2.7}$$

which may be written as

$$\mathcal{S}^{(k)}_{\Gamma^*}f(\underline{t}) = \tilde{f}(\underline{t}) - 2\mathcal{T}_{k+1}\mathcal{D}^{k+1}_{\underline{x}}\tilde{f}(\underline{t}), \tag{2.8}$$

where

$$\mathcal{T}_{k+1}g(\underline{x}) := (-1)^{k+1}\int_\Omega E_k(\underline{y}-\underline{x})g(\underline{y})\mathrm{d}\underline{y},$$

denotes the higher-order Teodorescu operator.

We will adopt (2.8) as the definition of our singular integral operator, whose properties are studied bellow.

3. Main Results

The following result is a generalization of [7, Lemma 1].

Lemma 3.1. *Let be $F \in Lip(\mathbb{R}^m, k+\alpha)$ and $\mathbf{E}\subset\mathbb{R}^m$ a compact set. Then $f = F|_{\mathbf{E}}$ belongs to $Lip(\mathbf{E}, k+\alpha)$. Moreover, we have*

$$\|f\|_{k+\alpha,\mathbf{E}} \leq \sup_{|\mathbf{j}|=k}\left\{\frac{2^{\frac{m}{2}}m!}{(m-k)!k!}\|\partial^{\mathbf{j}}F\|_{\alpha,\mathbb{R}^m}, \sup_{\underline{x}\in\mathbf{E}}\|f(\underline{x})\|\right\}. \tag{3.1}$$

Proof. Let f be defined as the trace of F in $\mathbf{E}$, i. e., $f = F|_{\mathbf{E}}$. Taking $f^{\mathbf{j}} = F^{\mathbf{j}}|_{\mathbf{E}} = \partial^{\mathbf{j}} F|_{\mathbf{E}}$ obviously yields $f \in \mathrm{Lip}(\mathbf{E}, k+\alpha)$.

On the other hand, if $|\mathbf{j}| = 0$, it follows that

$$\sup_{\underline{x}\in\mathbf{E}} \|f^{\mathbf{j}}(\underline{x})\| = \sup_{\underline{x}\in\mathbf{E}} \|f(\underline{x})\|, \tag{3.2}$$

since by definition $f^{\mathbf{0}} = f$.

Let us introduce the following notations

$$f^{\mathbf{j}} = \sum_A f_A^{\mathbf{j}} e_A,$$

$$\partial^{\mathbf{j}} F = \sum_A \partial^{\mathbf{j}} F_A e_A,$$

and

$$R_{\mathbf{j}} = \sum_A R_{\mathbf{j}_A} e_A,$$

where $f_A^{\mathbf{j}}$, $\partial^{\mathbf{j}} F_A$, and $R_{\mathbf{j}_A}$ are $\mathbb{R}$-valued functions.

For $|\mathbf{j}| = 0$, we have

$$R_{\mathbf{j}}(\underline{x},\underline{y}) = f(\underline{x}) - \sum_{|\mathbf{l}|\le k} \frac{f^{\mathbf{l}}(\underline{y})}{\mathbf{l}!}(\underline{x}-\underline{y})^{\mathbf{l}}, \ \ \underline{x},\underline{y} \in \mathbf{E},$$

or equivalently,

$$\begin{aligned} R_{\mathbf{j}}(\underline{x},\underline{y}) =& F(\underline{x}) - \sum_{|\mathbf{l}|\le k} \frac{\partial^{\mathbf{l}} F(\underline{y})}{\mathbf{l}!}(\underline{x}-\underline{y})^{\mathbf{l}} \\ =& F(\underline{x}) - \sum_{|\mathbf{l}|\le k-1} \frac{\partial^{\mathbf{l}} F(\underline{y})}{\mathbf{l}!}(\underline{x}-\underline{y})^{\mathbf{l}} - \sum_{|\mathbf{l}|= k} \frac{\partial^{\mathbf{l}} F(\underline{y})}{\mathbf{l}!}(\underline{x}-\underline{y})^{\mathbf{l}}. \end{aligned}$$

Thus, for each $R_{\mathbf{j}_A}(\underline{x},\underline{y})$ it turns out that

$$R_{\mathbf{j}_A}(\underline{x},\underline{y}) = F_A(\underline{x}) - \sum_{|\mathbf{l}|\le k-1} \frac{\partial^{\mathbf{l}} F_A(\underline{y})}{\mathbf{l}!}(\underline{x}-\underline{y})^{\mathbf{l}} - \sum_{|\mathbf{l}|= k} \frac{\partial^{\mathbf{l}} F_A(\underline{y})}{\mathbf{l}!}(\underline{x}-\underline{y})^{\mathbf{l}}. \tag{3.3}$$

At this stage we make use of the mean value theorem, which leads to

$$F_A(\underline{x}) - \sum_{|\mathbf{l}|\le k-1} \frac{\partial^{\mathbf{l}} F_A(\underline{y})}{\mathbf{l}!}(\underline{x}-\underline{y})^{\mathbf{l}} = \sum_{|\mathbf{l}|= k} \frac{\partial^{\mathbf{l}} F_A(\underline{y}^*)}{\mathbf{l}!}(\underline{x}-\underline{y})^{\mathbf{l}}. \tag{3.4}$$

where $\underline{y}^*$ belongs to the segment joining $\underline{x}$ and $\underline{y}$.

By substituting (3.4) into (3.3), one has

$$\begin{aligned} R_{\mathbf{j}_A}(\underline{x},\underline{y}) &= \sum_{|\mathbf{l}|= k} \frac{\partial^{\mathbf{l}} F_A(\underline{y}^*)}{\mathbf{l}!}(\underline{x}-\underline{y})^{\mathbf{l}} - \sum_{|\mathbf{l}|= k} \frac{\partial^{\mathbf{l}} F_A(\underline{y})}{\mathbf{l}!}(\underline{x}-\underline{y})^{\mathbf{l}} \\ &= \sum_{|\mathbf{l}|= k} \left[\frac{\partial^{\mathbf{l}} F_A(\underline{y}^*) - \partial^{\mathbf{l}} F_A(\underline{y})}{\mathbf{l}!}\right](\underline{x}-\underline{y})^{\mathbf{l}}. \end{aligned}$$

Hence,

$$\begin{aligned}|R_{\mathbf{j}_A}(\underline{x},\underline{y})| &\leq \sum_{|\mathbf{l}|=k} \frac{1}{\mathbf{l}!}|\partial^{\mathbf{l}} F_A(\underline{y}^*) - \partial^{\mathbf{l}} F_A(\underline{y})||(\underline{x}-\underline{y})^{\mathbf{l}}| \\ &\leq \sum_{|\mathbf{l}|=k} \frac{1}{\mathbf{l}!}|\partial^{\mathbf{l}} F_A(\underline{y}^*) - \partial^{\mathbf{l}} F_A(\underline{y})||\underline{x}-\underline{y}|^{|\mathbf{l}|} \\ &\leq \sum_{|\mathbf{l}|=k} \frac{1}{\mathbf{l}!}|\partial^{\mathbf{l}} F_A(\underline{y}^*) - \partial^{\mathbf{l}} F_A(\underline{y})||\underline{x}-\underline{y}|^{k} \\ &\leq \sum_{|\mathbf{l}|=k} \frac{1}{\mathbf{l}!}\frac{|\partial^{\mathbf{l}} F_A(\underline{y}^*) - \partial^{\mathbf{l}} F_A(\underline{y})|}{|\underline{x}-\underline{y}|^{\alpha}}|\underline{x}-\underline{y}|^{k+\alpha} \\ &\leq \frac{m!}{(m-k)!k!} \sup_{|\mathbf{l}|=k} \frac{|\partial^{\mathbf{l}} F_A(\underline{y}^*) - \partial^{\mathbf{l}} F_A(\underline{y})|}{|\underline{y}^*-\underline{y}|^{\alpha}}|\underline{x}-\underline{y}|^{k+\alpha} \\ &\leq \frac{m!}{(m-k)!k!}|\partial^{\mathbf{l}} F_A|_{\alpha,\mathbb{R}^m}|\underline{x}-\underline{y}|^{k+\alpha}.\end{aligned}$$

Consequently,

$$\frac{|R_{\mathbf{j}_A}(\underline{x},\underline{y})|}{|\underline{x}-\underline{y}|^{k+\alpha}} \leq \frac{m!}{(m-k)!k!}|\partial^{\mathbf{l}} F_A|_{\alpha,\mathbb{R}^m}.$$

Thus,

$$\begin{aligned}\|R_{\mathbf{j}}(\underline{x},\underline{y})\| &= \|\sum_A R_{\mathbf{j}_A}(\underline{x},\underline{y})e_A\| \\ &= \sqrt{\sum_A R^2_{\mathbf{j}_A}(\underline{x},\underline{y})} \\ &\leq \sqrt{\sum_A \max_A R^2_{\mathbf{j}_A}(\underline{x},\underline{y})} \\ &\leq \sqrt{2^m \max_A R^2_{\mathbf{j}_A}(\underline{x},\underline{y})} \\ &\leq 2^{\frac{m}{2}} \max_A |R_{\mathbf{j}_A}(\underline{x},\underline{y})|.\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{\|R_{\mathbf{j}}(\underline{x},\underline{y})\|}{|\underline{x}-\underline{y}|^{k+\alpha}} &\leq \frac{2^{\frac{m}{2}} m!}{(m-k)!k!} \max_A \left\{|\partial^{\mathbf{l}} F_A|_{\alpha,\mathbb{R}^m}\right\} \\ &\leq \frac{2^{\frac{m}{2}} m!}{(m-k)!k!}\|\partial^{\mathbf{l}} F\|_{\alpha,\mathbb{R}^m}.\end{aligned} \tag{3.5}$$

The proof of the general case $1 \leq |\mathbf{j}| \leq k$, follows a completely analogous procedure and for the sake of brevity will be omitted. □

The following Lemma will be useful.

Lemma 3.2. *Let $h \in L^p(\Omega)$, with $p > m$. Then*

$$\mathcal{H}(\underline{x}) = \int_{\Omega} \frac{(\underline{y}-\underline{x})^{\mathbf{j}}}{|\underline{y}-\underline{x}|^{m+|\mathbf{j}|-1}} h(\underline{y}) \mathrm{d}\underline{y} \tag{3.6}$$

is a function belonging to $C^{0,\beta}(\mathbb{R}^m)$ with $\beta = \frac{p-m}{p}$.

Proof. We have

$$\begin{aligned}
&\left| \frac{(\underline{y}-\underline{x})^{\mathbf{j}}}{|\underline{y}-\underline{x}|^{m+|\mathbf{j}|-1}} - \frac{(\underline{y}-\underline{z})^{\mathbf{j}}}{|\underline{y}-\underline{z}|^{m+|\mathbf{j}|-1}} \right| \\
&\quad = \left| \frac{(\underline{y}-\underline{x})^{\mathbf{j}}}{|\underline{y}-\underline{x}|^{m+|\mathbf{j}|-1}} - \frac{(\underline{y}-\underline{x})^{\mathbf{j}}}{|\underline{y}-\underline{z}|^{m+|\mathbf{j}|-1}} + \frac{(\underline{y}-\underline{x})^{\mathbf{j}}}{|\underline{y}-\underline{z}|^{m+|\mathbf{j}|-1}} - \frac{(\underline{y}-\underline{z})^{\mathbf{j}}}{|\underline{y}-\underline{z}|^{m+|\mathbf{j}|-1}} \right| \\
&\quad \le \left| (\underline{y}-\underline{x})^{\mathbf{j}} \left[\frac{1}{|\underline{y}-\underline{x}|^{m+|\mathbf{j}|-1}} - \frac{1}{|\underline{y}-\underline{z}|^{m+|\mathbf{j}|-1}} \right] \right| + \left| \frac{(\underline{y}-\underline{x})^{\mathbf{j}} - (\underline{y}-\underline{z})^{\mathbf{j}}}{|\underline{y}-\underline{z}|^{m+|\mathbf{j}|-1}} \right| \\
&\quad := I_1 + I_2.
\end{aligned}$$

First, we estimate I_1.

$$I_1 \le \left|\underline{y}-\underline{x}\right|^{|\mathbf{j}|} \frac{\left| |\underline{y}-\underline{z}|^{m+|\mathbf{j}|-1} - |\underline{y}-\underline{x}|^{m+|\mathbf{j}|-1} \right|}{|\underline{y}-\underline{x}|^{m+|\mathbf{j}|-1} |\underline{y}-\underline{z}|^{m+|\mathbf{j}|-1}}. \tag{3.7}$$

Developing the power difference in (3.7), we obtain:

$$I_1 \le c\,|\underline{z}-\underline{x}| \frac{\left| |\underline{y}-\underline{z}|^{m+|\mathbf{j}|-2} + |\underline{y}-\underline{z}|^{m+|\mathbf{j}|-3}|\underline{y}-\underline{x}| + \cdots + |\underline{y}-\underline{x}|^{m+|\mathbf{j}|-2} \right|}{|\underline{y}-\underline{x}|^{m-1} |\underline{y}-\underline{z}|^{m+|\mathbf{j}|-1}} \tag{3.8}$$

Since $|\underline{y}-\underline{z}| \ge \frac{|\underline{y}-\underline{x}|}{2}$, it follows

$$\begin{aligned}
I_1 &\le \frac{c\,|\underline{z}-\underline{x}|\,|\underline{y}-\underline{z}|^{m+|\mathbf{j}|-2}}{|\underline{y}-\underline{x}|^{m-1}|\underline{y}-\underline{z}|^{m+|\mathbf{j}|-1}} \\
&= \frac{c\,|\underline{z}-\underline{x}|}{|\underline{y}-\underline{x}|^{m-1}|\underline{y}-\underline{z}|},
\end{aligned} \tag{3.9}$$

and hence

$$I_1 \le \frac{c\,|\underline{z}-\underline{x}|}{|\underline{y}-\underline{x}|^{m}}. \tag{3.10}$$

Next, we will estimate I_2. By the multi-binomial Theorem, we have

$$(\underline{a}+\underline{b})^{\mathbf{j}} = \sum_{\mathbf{l} \le \mathbf{j}} \binom{\mathbf{j}}{\mathbf{l}} \underline{a}^{\mathbf{l}} \underline{b}^{\mathbf{j}-\mathbf{l}}, \tag{3.11}$$

and so

$$(\underline{y}-\underline{x})^{\mathbf{j}} = (\underline{y}-\underline{z}+\underline{z}-\underline{x})^{\mathbf{j}} = \sum_{\mathbf{l} \le \mathbf{j}} \binom{\mathbf{j}}{\mathbf{l}} (\underline{y}-\underline{z})^{\mathbf{l}} (\underline{z}-\underline{x})^{\mathbf{j}-\mathbf{l}}.$$

Therefore,

$$(\underline{y}-\underline{x})^{\mathbf{j}}-(\underline{y}-\underline{z})^{\mathbf{j}}=\sum_{\mathbf{l}\leq\mathbf{j}}\binom{\mathbf{j}}{\mathbf{l}}(\underline{y}-\underline{z})^{\mathbf{l}}(\underline{z}-\underline{x})^{\mathbf{j}-\mathbf{l}}-(\underline{y}-\underline{z})^{\mathbf{j}}$$
$$=\sum_{|\mathbf{l}|\leq|\mathbf{j}|-1}\binom{\mathbf{j}}{\mathbf{l}}(\underline{y}-\underline{z})^{\mathbf{l}}(\underline{z}-\underline{x})^{\mathbf{j}-\mathbf{l}}.$$

Thus,

$$I_2\leq\left|\frac{\sum_{|\mathbf{l}|\leq|\mathbf{j}|-1}\binom{\mathbf{j}}{\mathbf{l}}(\underline{y}-\underline{z})^{\mathbf{l}}(\underline{z}-\underline{x})^{\mathbf{j}-\mathbf{l}}}{|\underline{y}-\underline{z}|^{m+|\mathbf{j}|-1}}\right|$$
$$\leq\frac{\sum_{|\mathbf{l}|\leq|\mathbf{j}|-1}\binom{\mathbf{j}}{\mathbf{l}}\left|\underline{y}-\underline{z}\right|^{|\mathbf{l}|}\left|\underline{z}-\underline{x}\right|^{|\mathbf{j}|-|\mathbf{l}|}}{|\underline{y}-\underline{z}|^{m+|\mathbf{j}|-1}}$$
$$=\frac{c\left|\underline{y}-\underline{z}\right|^{|\mathbf{l}|}\left|\underline{z}-\underline{x}\right|^{|\mathbf{j}|-|\mathbf{l}|-1}\left|\underline{z}-\underline{x}\right|}{|\underline{y}-\underline{z}|^{m+|\mathbf{j}|-1}}$$
$$\leq\frac{c\left|\underline{y}-\underline{z}\right|^{|\mathbf{l}|}\left|\underline{y}-\underline{z}\right|^{|\mathbf{j}|-|\mathbf{l}|-1}\left|\underline{z}-\underline{x}\right|}{|\underline{y}-\underline{z}|^{m+|\mathbf{j}|-1}}$$
$$\leq\frac{c\,|\underline{z}-\underline{x}|}{|\underline{y}-\underline{z}|^{m}}\leq\frac{c\,|\underline{z}-\underline{x}|}{|\underline{y}-\underline{x}|^{m}}.$$

Sumarizing,

$$\left|\frac{(\underline{y}-\underline{z})^{\mathbf{j}}}{|\underline{y}-\underline{z}|^{m+|\mathbf{j}|-1}}-\frac{(\underline{y}-\underline{x})^{\mathbf{j}}}{|\underline{y}-\underline{x}|^{m+|\mathbf{j}|-1}}\right|\leq\frac{c\,|\underline{z}-\underline{x}|}{|\underline{y}-\underline{x}|^{m}}.$$

The rest of the proof is completely analogous to those used in [10, Proposition 8.1]. □

A rather simple consecuence of the above Lemma is the following

Proposition 3.3. *Let $g\in L^p(\Omega)$, with $p>m$. Then,*

$$\mathcal{T}_{k+1}g\in C^{k,\beta}(\mathbb{R}^m),$$

for $\beta=\frac{p-m}{p}$. □

We are now in a position to formulate our main result.

Theorem 3.4. *Let Γ be d-summable and $\alpha>\dfrac{d}{m}$. Then*

$$\mathcal{S}^{(k)}_{\Gamma^*}:Lip(\Gamma,k+\alpha)\to Lip(\Gamma,k+\beta),$$

for $\beta = \dfrac{m\alpha - d}{m-d}$. *In addition,*

$$\|\mathcal{S}_{\Gamma^*}^{(k)}\|_{k+\beta,\Gamma} \leq 1 + c_1 \mathbf{s}(d)^{\frac{1-\beta}{m}} |\Gamma|^{k+\beta} + \frac{2^{\frac{m}{2}} m!}{(m-k)!k!} \left(|\Gamma|^{\alpha-\beta} + c_2 \mathbf{s}(d)^{\frac{1-\beta}{m}} |\Gamma|^{\beta} + c_3 \mathbf{s}(d)^{\frac{1-\beta}{m}} \right),$$

where c_1, c_2, *and* c_3 *are constants depending on* α, d, *and* m.

Proof. Since $\alpha > \dfrac{d}{m}$, it follows that

$$m < \frac{m-d}{1-\alpha}.$$

Let $p = \frac{m-d}{1-\alpha}$. Then, from [9, Proposition 2] we have that $\mathcal{D}_{\underline{y}}^{k+1} \tilde{f} \in L^p(\Omega)$. On the other hand, Proposition 3.3 yields $T_{k+1}\mathcal{D}_{\underline{y}}^{k+1} \tilde{f} \in \mathrm{Lip}(\mathbb{R}^m, k + \frac{p-m}{p})$, i. e., $T_{k+1}\mathcal{D}_{\underline{y}}^{k+1} \tilde{f} \in \mathrm{Lip}(\mathbb{R}^m, k+\beta)$, where use has been made of the obvious equality

$$\frac{p-m}{p} = \frac{\dfrac{m-d}{1-\alpha} - m}{\dfrac{m-d}{1-\alpha}} = \frac{m\alpha - d}{m-d}.$$

Thus, $\mathcal{C}_{\Gamma^*}^{(k)}$ has continuous extensions up to $\overline{\Omega}$ and $\mathcal{S}_{\Gamma^*}^{(k)} \in \mathrm{Lip}(\Gamma, k+\beta)$.

Now, we are able to examine $\|\mathcal{S}_{\Gamma^*}^{(k)}\|_{k+\beta,\Gamma}$. It easily follows from the Hölder inequality that

$$\begin{aligned} \|\mathcal{S}_{\Gamma^*}^{(k)} f(\underline{t})\| &\leq \|\tilde{f}\|_{k+\alpha,\Gamma} + 2\|T_{k+1}\mathcal{D}_{\underline{y}}^{k+1}\tilde{f}\| \\ &\leq \|f\|_{k+\alpha,\Gamma} + 2\|\mathcal{D}_{\underline{y}}^{k+1}\tilde{f}\|_p \|E_k\|_q. \end{aligned} \tag{3.12}$$

On the other hand, for $\underline{t} \in \Gamma$ we have

$$\int_{\Omega} \left|E_k(\underline{y} - \underline{t})\right|^q \mathrm{d}\underline{y} = c \int_{\Omega} \left| \frac{\underline{y} - \underline{t}}{|\underline{y} - \underline{t}|^{m-k}} \right|^q \mathrm{d}\underline{y}.$$

Since $|\underline{y} - \underline{t}| \leq |\Gamma|$,

$$\int_{\Omega} \left|E_k(\underline{y} - \underline{t})\right|^q \mathrm{d}\underline{y} \leq c|\Gamma|^{m-(m-k-1)q}.$$

Accordingly,

$$\|E_k\|_q \leq c|\Gamma|^{\frac{m}{q}-(m-k-1)} = c|\Gamma|^{k+1+\frac{1-q}{q}m} = c|\Gamma|^{k+1-\frac{m}{p}}. \tag{3.13}$$

On the other hand, by [9, Proposition 2], it follows that

$$\int_{\Omega} \|\mathcal{D}_{\underline{y}}^{k+1}\tilde{f}(\underline{y})\|^p \mathrm{d}\underline{y} \leq c\|\mathcal{D}_{\underline{y}}^{k}\tilde{f}\|_{\alpha,\Gamma}^p \mathbf{s}(d).$$

From the above, we deduce

$$\|\mathcal{D}_{\underline{y}}^{k+1}\tilde{f}\|_p \leq c\mathbf{s}(d)^{\frac{1}{p}} \|f\|_{k+\alpha,\Gamma}. \tag{3.14}$$

Combining (3.12), (3.13) and (3.14) leads to

$$\|\mathcal{S}^{(k)}_{\Gamma^*} f(\underline{t})\| \leq \|f\|_{k+\alpha,\Gamma} + c_1 \mathbf{s}(d)^{\frac{1}{p}} \|f\|_{k+\alpha,\Gamma} |\Gamma|^{k+1-\frac{m}{p}},$$

or equivalently,

$$\|\mathcal{S}^{(k)}_{\Gamma^*} f(\underline{t})\| \leq \left(1 + c_1 \mathbf{s}(d)^{\frac{1}{p}} |\Gamma|^{k+1-\frac{m}{p}}\right) \|f\|_{k+\alpha,\Gamma}. \tag{3.15}$$

Let us now proceed to estimate $\left\| \left[\mathcal{S}^{(k)}_{\Gamma^*}\right]^{\mathbf{j}} \right\|_{\beta,\mathbb{R}^m}$.

A repeated use of the Hölder inequality yields

$$\|\mathcal{T}^{\mathbf{j}}_{k+1} \mathcal{D}^{k+1}_{\underline{y}} \tilde{f}\| \leq c \|\mathcal{D}^{k+1}_{\underline{y}} \tilde{f}\|_p \left\| \frac{(\underline{y}-\underline{t})^{\mathbf{j}}}{|\underline{y}-\underline{t}|^{m+|\mathbf{j}|-1}} \right\|_q.$$

Next, after some rather direct computations we have

$$\begin{aligned}
\left\| \frac{(\underline{y}-\underline{t})^{\mathbf{j}}}{|\underline{y}-\underline{t}|^{m+|\mathbf{j}|-1}} \right\|_q &= \left(\int_\Omega \left| \frac{(\underline{y}-\underline{t})^{\mathbf{j}}}{|\underline{y}-\underline{t}|^{m+|\mathbf{j}|-1}} \right|^q \mathrm{d}\underline{y} \right)^{\frac{1}{q}} \\
&= \left(\int_\Omega \frac{1}{|\underline{y}-\underline{t}|^{(m-1)q}} \mathrm{d}\underline{y} \right)^{\frac{1}{q}} \\
&\leq \left(|\Gamma|^{m-(m-1)q} \right)^{\frac{1}{q}} = |\Gamma|^{\frac{m}{q}-(m-1)} \\
&\leq |\Gamma|^{m\frac{p-1}{p}-(m-1)} = |\Gamma|^{\frac{p-m}{p}} = |\Gamma|^{\beta}.
\end{aligned} \tag{3.16}$$

Consequently, we obtain

$$\|\mathcal{T}^{\mathbf{j}}_{k+1} \mathcal{D}^{k+1}_{\underline{y}} \tilde{f}\| \leq c_2 \mathbf{s}(d)^{\frac{1}{p}} |\Gamma|^{\beta} \|f\|_{k+\alpha,\Gamma}, \tag{3.17}$$

which easily follows from (3.14) and (3.16).

As stated by Lemma 3.1, it is required to estimate $\|\mathcal{T}^{\mathbf{j}}_{k+1} \mathcal{D}^{k+1}_{\underline{y}} \tilde{f}\|_{\beta,\mathbb{R}^m}$. On applying Lemma 3.2, we obtain

$$\begin{aligned}
\left\| \mathcal{T}^{\mathbf{j}}_{k+1} \mathcal{D}^{k+1}_{\underline{y}} \tilde{f}(\underline{y}) - \mathcal{T}^{\mathbf{j}}_{k+1} \mathcal{D}^{k+1}_{\underline{y}} \tilde{f}(\underline{x}) \right\| \leq & c \|\mathcal{D}^{k+1}_{\underline{y}} \tilde{f}\|_p |\underline{y}-\underline{x}|^{\frac{p-m}{p}} \\
\leq & c_3 \mathbf{s}(d)^{\frac{1}{p}} \|f\|_{k+\alpha,\Gamma} |\underline{y}-\underline{x}|^{\beta}.
\end{aligned}$$

Accordingly,

$$\frac{\left\| \mathcal{T}^{\mathbf{j}}_{k+1} \mathcal{D}^{k+1}_{\underline{y}} \tilde{f}(\underline{y}) - \mathcal{T}^{\mathbf{j}}_{k+1} \mathcal{D}^{k+1}_{\underline{y}} \tilde{f}(\underline{x}) \right\|}{|\underline{y}-\underline{x}|^{\beta}} \leq c_3 \mathbf{s}(d)^{\frac{1}{p}} \|f\|_{k+\alpha,\Gamma}. \tag{3.18}$$

Then, combining (3.17) and (3.18) we obtain

$$\|\mathcal{T}^{\mathbf{j}}_{k+1} \mathcal{D}^{k+1}_{\underline{y}} \tilde{f}\|_{\beta,\mathbb{R}^m} \leq \left(c_2 |\Gamma|^{\beta} + c_3\right) \mathbf{s}(d)^{\frac{1}{p}} \|f\|_{k+\alpha,\Gamma}. \tag{3.19}$$

It remains to examine $\|\partial^{\mathbf{j}} \tilde{f}\|_{\beta,\mathbb{R}^m}$ for $|\mathbf{j}| = k$. By properties of the Whitney extension, $\|\partial^{\mathbf{j}} \tilde{f}\|_{\beta,\mathbb{R}^m} = \|f^{\mathbf{j}}\|_{\beta,\Gamma}$ holds. Actually, since $\alpha > \beta$, $\|f^{\mathbf{j}}\|_{\beta,\Gamma} \leq |\Gamma|^{\alpha-\beta} \|f^{\mathbf{j}}\|_{\alpha,\Gamma}$, it turns out that

$$\|\partial^{\mathbf{j}} \tilde{f}\|_{\beta,\mathbb{R}^m} \leq |\Gamma|^{\alpha-\beta} \|f\|_{k+\alpha,\Gamma}. \tag{3.20}$$

On substituting (3.15), (3.19), and (3.20) into (3.1) it yields

$$\|\mathcal{S}_{\Gamma^*}^{(k)} f\|_{k+\beta,\Gamma} \leq \Big\{1 + c_1\mathbf{s}(d)^{\frac{1-\beta}{m}}|\Gamma|^{k+\beta} + \frac{2^{\frac{m}{2}}m!}{(m-k)!k!}\left(|\Gamma|^{\alpha-\beta} + c_2\mathbf{s}(d)^{\frac{1-\beta}{m}}|\Gamma|^{\beta} + c_3\mathbf{s}(d)^{\frac{1-\beta}{m}}\right)\Big\}\|f\|_{k+\alpha,\Gamma}.$$

Finally,

$$\|\mathcal{S}_{\Gamma^*}^{(k)}\|_{k+\beta,\Gamma} \leq 1 + c_1\mathbf{s}(d)^{\frac{1-\beta}{m}}|\Gamma|^{k+\beta} + \frac{2^{\frac{m}{2}}m!}{(m-k)!k!}\left(|\Gamma|^{\alpha-\beta} + c_2\mathbf{s}(d)^{\frac{1-\beta}{m}}|\Gamma|^{\beta} + c_3\mathbf{s}(d)^{\frac{1-\beta}{m}}\right),$$

which completes the proof. □

Funding The main author was supported by a doctoral Postgraduate Study Fellowship from the Consejo Nacional de Humanidades, Ciencia y Tecnología (Grant Number 895783). She would like to express her gratitude for the support.

Declarations

Conflicts of interest The authors declare that they have no Conflict of interest regarding the publication of this paper.

References

[1] Abreu Blaya, R., Bory Reyes, J.: Hölder norm estimate for the Hilbert transform in Clifford analysis. Bull. Braz. Math. Soc. New Ser. **41**, 389–398 (2010). https://doi.org/10.1007/s00574-010-0017-9

[2] Brackx, F.: Non-(k)-monogenic points of functions of a quaternion variable. Funct. Theor. Methods Partial Differ. Equ. **561**, 138–149 (1976). https://doi.org/10.1007/BFb0087632

[3] Brackx, F.: On-(k)-monogenic functions of a quaternion variable. Funct. Theor. Methods Differ. Equ. **8**, 22–44 (1976)

[4] Brackx, F., Delanghe, R., Sommen, F.: Clifford analysis. Chapman & Hall/CRC Research Notes in Mathematics Series. Pitman Advanced Publishing Program, Boston (1982)

[5] De la Cruz Toranzo, L., Abreu Blaya, R., Bory Reyes, J.: The Plemelj–Privalov theorem in polyanalytic function theory. J. Math. Anal. Appl. **463**(2), 517–533 (2018). https://doi.org/10.1016/j.jmaa.2018.03.023

[6] De la Cruz Toranzo, L., Abreu Blaya, R., Bory Reyes, J.: On the Plemelj–Privalov theorem in Clifford analysis involving higher order Lipschitz classes. J. Math. Anal. Appl. **480**(2), 1–13 (2019). https://doi.org/10.1016/j.jmaa.2019.123411

[7] De la Cruz Toranzo, L., Moreno García, A., Moreno García, T., Abreu Blaya, R., Bory Reyes, J.: A bimonogenic Cauchy transform on higher order Lipschitz classes. Mediterr. J. Math. **16**, 1–14 (2019). https://doi.org/10.1007/s00009-018-1280-z

[8] Gakhov, F.D.: Boundary value problems. In: International Series of Monographs on Pure and Applied Mathematics, vol. 85. Pergamon Press (1966). https://doi.org/10.1016/C2013-0-01739-2

[9] Gómez Santiesteban, T.R., Abreu Blaya, R., Hernández Gómez, J.C., Sigarreta Almira, J.M.: A Cauchy transform for polymonogenic functions on fractal domains. Complex Anal. Oper. Theory **16**(3), 42 (2022). https://doi.org/10.1007/s11785-022-01228-5

[10] Gürlebeck, K., Habetha, K., Sprößig, W.: Holomorphic Functions in the Plane and n-Dimensional Space. Birkhäuser, Basel (2007). https://doi.org/10.1007/978-3-7643-8272-8

[11] Guseynov, E.G.: The Plemelj–Privalov theorem for generalized Hölder classes. Russ. Acad. Sci. Sb. Math. **75**(1), 165–182 (1993). https://doi.org/10.1070/SM1993v075n01ABEH003378

[12] Harrison, J., Norton, A.: The Gauss–Green theorem for fractal boundaries. Duke Math. J. (1992). https://doi.org/10.1215/S0012-7094-92-06724-X

[13] Plemelj, J.: Ein ergänzungssatz zur Cauchyschen integraldarstellung analytischer funktionen, randwerte betreffend. Monatsh. Math. Phys. **19**(1), 205–210 (1908). https://doi.org/10.1007/BF01736696

[14] Privalov, I.: Sur les fonctions conjuguées. Bull. Soc. Math. Fr. **44**(2), 100–103 (1916). https://doi.org/10.24033/bsmf.965

[15] Privalov, I.: Sur les intégrales du type de Cauchy. C. R. (Dokl.) Acad. Sci. URSS **23**, 859–863 (1939)

[16] Ryan, J.: Basic Clifford analysis. Cubo Math. Educ. **2**, 226–256 (2000)

[17] Stein, E.M.: Singular integrals and differentiability properties of functions. In: Princeton Mathematical Series, vol. 30. Princeton University Press, Princeton (1970). https://www.jstor.org/stable/j.ctt1bpmb07

[18] Tamayo Castro, C., Abreu Blaya, R., Bory Reyes, J.: Compactness of embedding of generalized higher order Lipschitz classes. Anal. Math. Phys. **9**, 1719–1727 (2019). https://doi.org/10.1007/s13324-018-0268-y

[19] Whitney, H.: Analytic extensions of differentiable functions defined in closed sets. Trans. Am. Math. Soc. **36**(1), 63–89 (1934). https://doi.org/10.1090/S0002-9947-1934-1501735-3

Tania Rosa Gómez Santiesteban, Ricardo Abreu Blaya, Juan Carlos Hernández Gómez and José Luis Sánchez Santiesteban
Facultad de Matemáticas
Universidad Autónoma de Guerrero
Ave. Lázaro Cárdenas
Chilpancingo 39087 Guerrero
México
e-mail: taniasantiesteban1991@gmail.com

Ricardo Abreu Blaya
e-mail: rabreublaya@yahoo.es

Juan Carlos Hernández Gómez
e-mail: jcarloshg@gmail.com

José Luis Sánchez Santiesteban
e-mail: jlsanchezsantiesteban@gmail.com

Ricardo Abreu Blaya
Investigador Invitado
Universidad UTE
Quito
Ecuador

Received: September 13, 2023.
Accepted: February 27, 2024.