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Fractional Powers of the Quaternionic d-Bar Derivative

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Abstract. This work introduces fractional d-bar derivatives in the setting of quaternionic analysis, by giving meaning to fractional powers of the quaternionic d-bar derivative. The definition is motivated by starting from nth-order d-bar derivatives for $n \in \mathbb{N}$, and further justified by various natural properties such as composition laws and its action on special functions such as Fueter polynomials.

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1. Introduction

Fractional calculus is the branch of mathematical analysis dedicated to defining and studying derivative and integral operators of fractional (non-integer) orders. These operators include fractional derivatives, fractional integrals, fractional partial derivatives, fractional Laplacians, and many more. Although the concept dates back to a letter of Leibniz in 1695, and real-world applications date back to a paper of Abel in 1823, the more detailed mathematical and scientific development of fractional calculus has mostly taken place during the last half-century, and so it is often seen as a new and modern branch of analysis. For more details on the background and development of fractional calculus, we refer to some of the standard textbooks of the field [10–12].

Although complex analysis historically played a major role in the development of fractional calculus, thanks to the work of Nekrassov in 1888 [9] as well as further studies of Osler in the early 1970s, nowadays the study of fractional calculus is often done in a purely real setting. Although fractional

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derivatives to complex orders are well known, and defining fractional derivatives using complex contours is a classical concept albeit less well known nowadays, there is only one paper, by the first author, which has attempted to define a fractional power of the complex d-bar derivative [3]. This is only a first advance towards many other forthcoming developments in the intersection of fractional calculus and complex analysis, such as a concept of fractional polyanalytic functions and a d-bar formalism for solving fractional partial differential equations.

Another way of generalising the fractional complex d-bar derivative is to pose the following research question. **How can we define a fractional version of the d-bar derivative in higher-dimensional settings?** The usual higher-dimensional analogue of complex analysis is Clifford analysis, in which a higher-dimensional analogue of the complex d-bar derivative plays a very important role in constructing higher-dimensional analogues of complex differentiable functions. Therefore, what we are seeking is a fractional version of the d-bar derivative in a general Clifford algebra. To keep the problem at a manageable scale, for now, we investigate the most fundamental Clifford algebra of dimension higher than the complex numbers: namely, the space $\mathbb{H}$ of quaternions, which is a 4-dimensional real vector space and the only Clifford algebra that is also a non-commutative division algebra.

In the space of a few paragraphs, we have reached our main research question, how to define a fractional quaternionic d-bar derivative, the answering of which will require knowledge of both fractional calculus and quaternionic analysis. The remainder of this paper is organised as follows. In Sect. 2, we introduce required preliminaries from fractional calculus, complex analysis, and quaternionic analysis. Section 3 contains the main definition together with its motivation and some basic properties, and the following sections delve into more advanced properties: in Sect. 4 we conduct a detailed investigation of composition properties, and in Sect. 5 we apply the new operator to various example functions. Section 6 is devoted to a brief comparative analysis with previous research bringing together fractional calculus with Clifford analysis, and Sect. 7 concludes the work with some pointers towards future continuations.

2. Preliminaries

2.1. Fractional Calculus

We begin this section with some fundamental definitions and facts from fractional calculus.

Definition 2.1. (*Riemann–Liouville fractional calculus* [10–12]) The fractional integral to order α of a function $f(x)$ with constant of integration c is defined as

$$ {}^{RL}_{c}I^{\alpha}_{x} f(x) = \frac{1}{\Gamma(\alpha)} \int_c^x (x-t)^{\alpha-1} f(t)\,\mathrm{d}t, \tag{2.1} $$

where $\alpha > 0$ is a real number and f is a function such that this integral exists. (It is also possible to take α complex, with positive real part, but in this paper we will use real fractional orders, for reasons that will become clear later.) Usually, we consider x to be a real variable greater than c, e.g. we may assume $x \in [c, d]$ and $f \in L^1[c, d]$.

The fractional derivative to order α of a function $f(x)$ with constant c is defined by a combination of ordinary repeated derivatives and fractional integrals:

$$ {}^{RL}_{\ c}D^{\alpha}_{x} f(x) = \frac{\mathrm{d}^n}{\mathrm{d}x^n}\left({}^{RL}_{\ c}I^{n-\alpha}_{x} f(x) \right), \tag{2.2} $$

where $\alpha \geq 0$ is a real number and n can be any natural number greater than α, usually taken to be $n = \lfloor \alpha \rfloor + 1$ or $n = \lceil \alpha \rceil$ (both choices are equivalent for $\alpha \in \mathbb{N}$, and both work in general under the assumption that $\alpha \in \mathbb{R}$, although only the former is valid for non-real α with integer real part). Possible function spaces for f include $AC^n[c, d]$ if x is a real variable in $[c, d]$.

Remark 2.2. The above definition is according to the Riemann–Liouville convention of fractional calculus. There are other ways to define fractional integral and derivative operators—for example, by swapping the order of the repeated derivative and the fractional integral in (2.2), we would obtain the so-called Caputo fractional derivative [2], which is often preferred in applications of fractional calculus due to the fact that it is associated with classical-type initial conditions rather than fractional initial conditions when creating well-posed initial value problems. In the current work, we are sticking with the Riemann–Liouville definition of fractional calculus, since it is mathematically natural, easy to justify as an extension of classical calculus, and has many known useful properties as it is the most heavily studied of all fractional calculus definitions. We use the term *differintegral* to mean a fractional operator that may be either a derivative or an integral, and also use the convention [10, 12] that derivatives to negative order are integrals to positive order and vice versa:

$$ {}^{RL}_{\ c}D^{-\alpha}_{x} f(x) = {}^{RL}_{\ c}I^{\alpha}_{x} f(x), $$

enabling these operators to be defined for all $\alpha \in \mathbb{R}$, positive or negative.

Lemma 2.3. [10–12] *Choosing $c = 0$ gives natural formulae for the fractional differintegrals of power functions:*

$$ {}^{RL}_{\ 0}I^{\alpha}_{x}\left(x^{\beta}\right) = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)}\, x^{\beta+\alpha}, \qquad \beta > -1, \tag{2.3} $$

$$ {}^{RL}_{\ 0}D^{\alpha}_{x}\left(x^{\beta}\right) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}\, x^{\beta-\alpha}, \qquad \beta > -1, \tag{2.4} $$

or more generally for any finite $c \in \mathbb{R}$,

$$ {}^{RL}_{\ c}I^{\alpha}_{x}\left((x-c)^{\beta}\right) = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)}\, (x-c)^{\beta+\alpha}, \qquad \beta > -1, \tag{2.5} $$

$$ {}^{RL}_{\ c}D^{\alpha}_{x}\left((x-c)^{\beta}\right) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}\, (x-c)^{\beta-\alpha}, \qquad \beta > -1, \tag{2.6} $$

Meanwhile, choosing $c = -\infty$ *gives natural formulae for the fractional differintegrals of exponential functions:*

$$ {}_{-\infty}^{RL}I_x^{\alpha}\left(e^{ax}\right) = a^{-\alpha}e^{ax}, \qquad a > 0, \tag{2.7} $$

$$ {}_{-\infty}^{RL}D_x^{\alpha}\left(e^{ax}\right) = a^{\alpha}e^{ax}, \qquad a > 0. \tag{2.8} $$

Lemma 2.4. [10–12] *Fractional differintegrals of fractional integrals, and ordinary repeated derivatives of fractional differintegrals, have a semigroup property with respect to the fractional order:*

$$ {}_{c}^{RL}I_x^{\alpha} \circ {}_{c}^{RL}I_x^{\beta} = {}_{c}^{RL}I_x^{\alpha+\beta}, \qquad \alpha,\beta \in \mathbb{R}, \beta > 0; \tag{2.9} $$

$$ \frac{\mathrm{d}^n}{\mathrm{d}x^n} \circ {}_{c}^{RL}D_x^{\alpha} = {}_{c}^{RL}D_x^{\alpha+n}, \qquad \alpha \in \mathbb{R}, n \in \mathbb{N}. \tag{2.10} $$

Fractional differintegrals of fractional derivatives do not, in general, have a semigroup property; instead, we have the following formula:

$$ {}_{c}^{RL}D_x^{\alpha}\left({}_{c}^{RL}D_x^{\beta}f(x)\right) = {}_{c}^{RL}D_x^{\alpha+\beta}f(x) - \sum_{k=1}^{n} \frac{(x-c)^{-\alpha-k}}{\Gamma(1-\alpha-k)} \left[{}_{c}^{RL}D_x^{\beta-k}f(x)\right]_{x=c}, \tag{2.11} $$

where $\alpha,\beta \in \mathbb{R}$*,* $\beta > 0$*, and* $n = \lceil \beta \rceil$*.*

2.2. Fractional Complex d-Bar Derivatives

Following the above brief introduction to the well-known fundamentals of fractional calculus, we provide the key definitions and facts of the theory of fractional d-bar derivatives in complex analysis, as introduced in [3].

Definition 2.5. (*Fractional d-bar derivatives in* $\mathbb{C}$ [3]) The complex partial derivatives with respect to a complex variable $z = x + yi$ and its complex conjugate $\overline{z} = x - yi$, namely the operators

$$ \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \qquad \textit{and} \qquad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right), $$

have fractional powers defined as follows:

$$ {}_{c}^{1}\partial_z^{\alpha} f(z) = 2^{-\alpha} \sum_{k=0}^{\infty} \binom{\alpha}{k} {}_{c}^{RL}D_x^{\alpha-k} \left(-iD_y\right)^k f(x+yi), $$

$$ {}_{c}^{2}\partial_z^{\alpha} f(z) = 2^{-\alpha} \sum_{k=0}^{\infty} \binom{\alpha}{k} \left(-i\,{}_{c}^{RL}D_y\right)^{\alpha-k} D_x^k f(x+yi), $$

$$ {}_{c}^{1}\partial_{\overline{z}}^{\alpha} f(z) = 2^{-\alpha} \sum_{k=0}^{\infty} \binom{\alpha}{k} {}_{c}^{RL}D_x^{\alpha-k} \left(iD_y\right)^k f(x+yi), $$

$$ {}_{c}^{2}\partial_{\overline{z}}^{\alpha} f(z) = 2^{-\alpha} \sum_{k=0}^{\infty} \binom{\alpha}{k} \left(i\,{}_{c}^{RL}D_y\right)^{\alpha-k} D_x^k f(x+yi), $$

where $\alpha \in \mathbb{R}$ and $f(z)$ is a complex function in the function space $C_{\mathcal{B}}^{\infty}$ defined in [3, Definition 2.4].

Note that there are two different definitions for the fractional powers of each of ∂_z and $\partial_{\overline{z}}$. These correspond to the two possible types of fractional binomial theorem, based on natural generalisation of the finite binomial-type sum for the nth power of ∂_z and $\partial_{\overline{z}}$ when $n \in \mathbb{N}$.

Further justification for the naturality of the above definition is provided by the following two results, both identical to the corresponding results in classical (real) fractional calculus.

Lemma 2.6. [3] *Fractional d-bar derivatives have the following semigroup properties with respect to the fractional order:*

$$ {}^1_c\partial^{\alpha}_{\overline{z}} \circ {}^1_c\partial^{\beta}_{\overline{z}} = {}^1_c\partial^{\alpha+\beta}_{\overline{z}}, \qquad \alpha, \beta \in \mathbb{R}, \beta < 0; \tag{2.12} $$

$$ \frac{\partial^n}{\partial \overline{z}^n} \circ {}^1_c\partial^{\alpha}_{\overline{z}} = {}^1_c\partial^{\alpha+n}_{\overline{z}}, \qquad \alpha \in \mathbb{R}, n \in \mathbb{N}, \tag{2.13} $$

with exactly the same relations also holding for the ${}^2_c\partial^{\alpha}_{\overline{z}}$ *and* ${}^1_c\partial^{\alpha}_{z}$ *and* ${}^2_c\partial^{\alpha}_{z}$ *operators.*

Lemma 2.7. [3] *Fractional d-bar derivatives satisfy the following fractional Leibniz rule:*

$$ {}^1_c\partial^{\alpha}_{\overline{z}}\big(f(z)g(z)\big) = \sum_{k=0}^{\infty} \binom{\alpha}{k} {}^1_c\partial^{\alpha-k}_{\overline{z}} f(z) \cdot \partial^n_{\overline{z}} g(z), $$

with exactly the same relations also holding for the ${}^2_c\partial^{\alpha}_{\overline{z}}$ *and* ${}^1_c\partial^{\alpha}_{z}$ *and* ${}^2_c\partial^{\alpha}_{z}$ *operators.*

2.3. Quaternionic Analysis

A particularly important and fundamental Clifford algebra, often considered as the next simplest case after $\mathbb{R}$ and $\mathbb{C}$, is the set $\mathbb{H}$ of quaternions [8]. According to a theorem of Frobenius [5], the only finite-dimensional associative division algebras over $\mathbb{R}$ are precisely $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$, which means these are the only Clifford algebras that are also division algebras, and $\mathbb{H}$ is the only non-commutative one. As a vector space, it is generated by four basis elements $1, e_1, e_2, e_3$ satisfying $e_i^2 = -1$ for $i = 1, 2, 3$ and $e_i e_{i+1} = e_{i+2}$ where $i = 1, 2, 3$ is taken modulo 3.

Quaternionic analysis studies quaternion-valued functions f of a quaternionic variable $x = x_0 + e_1 x_1 + e_2 x_2 + e_3 x_3$, where x_i is a real independent variable for $i = 0, 1, 2, 3$. As the standard definition of differentiability from real and complex analysis does not give a useful class of functions in quaternionic analysis [6, Theorem 5.8], holomorphicity of quaternionic functions is instead defined using the d-bar operator, applied to functions from the left or right as follows:

$$ \overline{\partial} f = \frac{\partial f}{\partial x_0} + e_1 \frac{\partial f}{\partial x_1} + e_2 \frac{\partial f}{\partial x_2} + e_3 \frac{\partial f}{\partial x_3}, $$

$$ f\overline{\partial} = \frac{\partial f}{\partial x_0} + \frac{\partial f}{\partial x_1} e_1 + \frac{\partial f}{\partial x_2} e_2 + \frac{\partial f}{\partial x_3} e_3, $$

A quaternionic function f is said to be left-holomorphic if $\overline{\partial} f = 0$, or right-holomorphic if $f\overline{\partial} = 0$. (Here we use the "left" and "right" convention according to [6], although some other works [13] use the opposite convention.)

The above definitions of holomorphicity of quaternionic functions are natural generalisations of the d-bar condition for holomorphicity of complex functions. However, variables and polynomials must be defined in a non-obvious way in order to guarantee their holomorphicity. The simple identity function $f(x) = x$ is neither left-holomorphic nor right-holomorphic over the quaternions, which led to the introduction of the Fueter variables:

$$z_k(x) = x_k - x_0 e_k, \qquad k = 1, 2, 3.$$

These functions of x are both left-holomorphic and right-holomorphic [6, §5.2.2], and they can be used to construct Fueter polynomials of any order as follows.

Definition 2.8. (*Fueter polynomials* [6, Definition 6.1]) Let $\mathbf{k} = (k_1, k_2, k_3)$ be a multi-index where each k_i is a non-negative integer. The degree of this multi-index, and the corresponding Fueter term, are defined as

$$k = |\mathbf{k}| = \sum_{i=1}^{3} k_i \quad \text{and} \quad z_{\mathbf{k}} = z_1^{k_1} z_2^{k_2} z_3^{k_3}.$$

The Fueter polynomial $P_{\mathbf{k}}(x)$, for any $x \in \mathbb{H}$, is defined as follows.

- If there exist any negative k_i, then we define $P_{\mathbf{k}}(x) := 0$.
- If $\mathbf{k} = (0, 0, 0)$, shortly denoted as $\mathbf{k} = \mathbf{0}$, then we define $P_{\mathbf{0}}(x) := 1$.
- For $\mathbf{k}$ with $|\mathbf{k}| > 0$, the Fueter polynomial can be defined in the following way:

  $$P_{\mathbf{k}}(x) := \frac{1}{k!} \sum_{\sigma \in perm(k)} \sigma\left(z_{\mathbf{k}}\right),$$

  where $perm(k)$ is the permutation group on k elements.

Example. To clarify the definition above, we consider some examples of Fueter polynomials for small values of k.

- If $\mathbf{k} = (1, 0, 0)$, then $k = |\mathbf{k}| = 1$ and $z_{\mathbf{k}} = z_1$. Hence,

  $$P_{\mathbf{k}}(x) = \frac{1}{1!} \left(\sum_{\sigma \in perm(1)} \sigma(z_1) \right) = z_1.$$

- If $\mathbf{k} = (1, 1, 0)$, then $k = |\mathbf{k}| = 2$ and $z_{\mathbf{k}} = z_1 z_2$. Hence,

  $$P_{\mathbf{k}}(x) = \frac{1}{2!} \left(\sum_{\sigma \in perm(2)} \sigma(z_1 z_2) \right) = \frac{1}{2} \left(z_1 z_2 + z_2 z_1 \right).$$

- If $\mathbf{k} = (1, 2, 0)$, then $k = |\mathbf{k}| = 3$ and $z_{\mathbf{k}} = z_1 {z_2}^2$. Hence,

  $$\begin{aligned} P_{\mathbf{k}}(x) &= \frac{1}{3!} \left(\sum_{\sigma \in perm(3)} \sigma(z_1 {z_2}^2) \right) \\ &= \frac{1}{6} \Big[z_1 z_2 z_2 + z_1 z_2 z_2 + z_2 z_1 z_2 + z_2 z_1 z_2 + z_2 z_2 z_1 + z_2 z_2 z_1 \Big] \\ &= \frac{1}{3} \left(z_1 {z_2}^2 + z_2 z_1 z_2 + {z_2}^2 z_1 \right). \end{aligned}$$

The Fueter polynomials are necessary to define in order to have a quaternionic version of polynomials which are holomorphic. One of the important results concerning them [6, Theorem 6.2] is that every Fueter polynomial is both left-holomorphic and right-holomorphic: we have $\overline{\partial}P_{\mathbf{k}}(x) = P_{\mathbf{k}}(x)\overline{\partial} = 0$ for any multi-index $\mathbf{k}$.

3. A Fractional Quaternionic d-Bar Derivative

Before beginning to define fractional powers of the quaternionic d-bar derivative $\overline{\partial} = \frac{\partial}{\partial x_0} + \frac{\partial}{\partial x_1}e_1 + \frac{\partial}{\partial x_2}e_2 + \frac{\partial}{\partial x_3}e_3$, we should firstly consider the nth powers of this operator for $n \in \mathbb{N}$. A good fractional derivative definition generalises not only the derivatives to order 0 and 1, but the derivatives to all natural-number orders.

When $n = 2$, following the cancellation of mixed partial derivatives in the product of $\frac{\partial}{\partial x_0} + \frac{\partial}{\partial x_1}e_1 + \frac{\partial}{\partial x_2}e_2 + \frac{\partial}{\partial x_3}e_3$ with itself, we find:

$$\begin{aligned}\overline{\partial}^2 = &\frac{\partial^2}{\partial x_0^2} - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2}\\ &+2\left(\frac{\partial^2}{\partial x_0 \partial x_1}e_1 + \frac{\partial^2}{\partial x_0 \partial x_2}e_2 + \frac{\partial^2}{\partial x_0 \partial x_3}e_3\right).\end{aligned}$$

When $n = 3$, some more elementary calculation yields:

$$\begin{aligned}\overline{\partial}^3 = &\frac{\partial^3}{\partial x_0^3} - \frac{\partial^3}{\partial x_1^3}e_1 - \frac{\partial^3}{\partial x_2^3}e_2 - \frac{\partial^3}{\partial x_3^3}e_3\\ &+3\left(\frac{\partial^3}{\partial x_0^2 \partial x_1}e_1 + \frac{\partial^3}{\partial x_0^2 \partial x_2}e_2 + \frac{\partial^3}{\partial x_0^2 \partial x_3}e_3\right)\\ &-3\left(\frac{\partial^3}{\partial x_0 \partial x_1^2} + \frac{\partial^3}{\partial x_0 \partial x_2^2} + \frac{\partial^3}{\partial x_0 \partial x_3^2}\right)\\ &-\left(\frac{\partial^3}{\partial x_1 \partial x_2^2}e_1 + \frac{\partial^3}{\partial x_1 \partial x_3^2}e_1 + \frac{\partial^3}{\partial x_1^2 \partial x_2}e_2\right.\\ &\left.+\frac{\partial^3}{\partial x_2 \partial x_3^2}e_2 + \frac{\partial^3}{\partial x_1^2 \partial x_3}e_3 + \frac{\partial^3}{\partial x_2^2 \partial x_3}e_3\right).\end{aligned}$$

When $n = 4$, after a lot of calculation we find:

$$\begin{aligned}\overline{\partial}^4 = &\frac{\partial^4}{\partial x_0^4} + \frac{\partial^4}{\partial x_1^4} + \frac{\partial^4}{\partial x_2^4} + \frac{\partial^4}{\partial x_3^4}\\ &+4\left(\frac{\partial^4}{\partial x_0^3 \partial x_1}e_1 + \frac{\partial^4}{\partial x_0^3 \partial x_2}e_2 + \frac{\partial^4}{\partial x_0^3 \partial x_3}e_3\right)\\ &-6\left(\frac{\partial^4}{\partial x_0^2 \partial x_1^2} + \frac{\partial^4}{\partial x_0^2 \partial x_2^2} + \frac{\partial^4}{\partial x_0^2 \partial x_3^2}\right) + 2\left(\frac{\partial^4}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4}{\partial x_2^2 \partial x_3^2} + \frac{\partial^4}{\partial x_1^2 \partial x_3^2}\right)\\ &-4\left(\frac{\partial^4}{\partial x_0 \partial x_1^3}e_1 + \frac{\partial^4}{\partial x_0 \partial x_2^3}e_2 + \frac{\partial^4}{\partial x_0 \partial x_3^3}e_3\right)\\ &-4\left(\frac{\partial^4}{\partial x_0 \partial x_1 \partial x_2^2}e_1 + \frac{\partial^4}{\partial x_0 \partial x_1 \partial x_3^2}e_1 + \frac{\partial^4}{\partial x_0 \partial x_1^2 \partial x_2}e_2\right.\end{aligned}$$

$$+\frac{\partial^4}{\partial x_0 \partial x_2 \partial x_3^2}e_2+\frac{\partial^4}{\partial x_0 \partial x_1^2 \partial x_3}e_3+\frac{\partial^4}{\partial x_0 \partial x_2^2 \partial x_3}e_3\bigg).$$

In all of the above cases, we see that there is symmetry between x_1/e_1 and x_2/e_2 and x_3/e_3: these can be permuted in any way and the overall formula would remain the same. However, x_0 behaves differently, as might be expected for a quaternionic operator since 1 behaves differently from e_1 and e_2 and e_3 in the algebra. The coefficients in the $n = 4$ case indicate that these are not any sort of binomial sums, but the most natural way to order the terms in each ∂^n expression appears to be by gathering them according to the degree of the x_0 derivative.

Thinking further in this direction, we notice that the original operator $\overline{\partial}$ is a sum of a scalar part $\frac{\partial}{\partial x_0}$ and a vector part $\frac{\partial}{\partial x_1}e_1+\frac{\partial}{\partial x_2}e_2+\frac{\partial}{\partial x_3}e_3$. Therefore, recalling from [3] the idea of writing powers of an operator by formulating it as a sum of two parts and using the binomial theorem, we find that the nth-order quaternionic d-bar derivative can be written as follows:

$$\overline{\partial}_x^n=\sum_{k=0}^{n}\binom{n}{k}\left(\frac{\partial}{\partial x_0}\right)^{n-k}\left(\frac{\partial}{\partial x_1}e_1+\frac{\partial}{\partial x_2}e_2+\frac{\partial}{\partial x_3}e_3\right)^k.$$

Here, the Moisil–Teodorescu operator $\frac{\partial}{\partial x_1}e_1+\frac{\partial}{\partial x_2}e_2+\frac{\partial}{\partial x_3}e_3$ has a scalar square,

$$\left(\frac{\partial}{\partial x_1}e_1+\frac{\partial}{\partial x_2}e_2+\frac{\partial}{\partial x_3}e_3\right)^2=-\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial x_3^2}\right), \tag{3.1}$$

which makes it a Dirac operator and will make its powers easy to calculate.

Extending the idea of using the binomial theorem, in its form as an infinite sum for a fractional power rather than a finite sum for a natural-number power, we arrive naturally at the following definition for fractional powers of the quaternionic d-bar derivative.

Definition 3.1. The fractional quaternionic d-bar derivative is defined to act (from left or right) on a suitable quaternion-valued function $f(x) = f(x_0 + e_1x_1 + e_2x_2 + e_3x_3)$ as follows:

$${}^{RL}_{c}\overline{\partial}_x^\alpha f=\sum_{k=0}^{\infty}\binom{\alpha}{k}{}^{RL}_{c}D_{x_0}^{\alpha-k}\left(\frac{\partial}{\partial x_1}e_1+\frac{\partial}{\partial x_2}e_2+\frac{\partial}{\partial x_3}e_3\right)^k f, \tag{3.2}$$

$$f\,{}^{RL}_{c}\overline{\partial}_x^\alpha=\sum_{k=0}^{\infty}\binom{\alpha}{k}{}^{RL}_{c}D_{x_0}^{\alpha-k}f\left(\frac{\partial}{\partial x_1}e_1+\frac{\partial}{\partial x_2}e_2+\frac{\partial}{\partial x_3}e_3\right)^k. \tag{3.3}$$

Here, we use a constant of differintegration $c \in \mathbb{R}$ and a fractional order $\alpha \in \mathbb{R}$, and we make use of the Riemann–Liouville fractional differintegral with respect to x_0.

Because of the Dirac property of the Moisil–Teodorescu operator (3.1), these formulae can also be written equivalently as follows, separating the even and odd values of k:

$${}^{RL}_{c}\overline{\partial}_x^\alpha f=\sum_{\ell=0}^{\infty}(-1)^\ell\binom{\alpha}{2\ell}{}^{RL}_{c}D_{x_0}^{\alpha-2\ell}\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial x_3^2}\right)^\ell f$$

$$+\sum_{\ell=0}^{\infty}(-1)^{\ell}\binom{\alpha}{2\ell+1}{}^{RL}_{c}D^{\alpha-2\ell-1}_{x_0}$$
$$\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial x_3^2}\right)^{\ell}\left(e_1\frac{\partial f}{\partial x_1}+e_2\frac{\partial f}{\partial x_2}+e_3\frac{\partial f}{\partial x_3}\right), \quad (3.4)$$

$$f\,{}^{RL}_{c}\overline{\partial}^{\alpha}_{x}=\sum_{\ell=0}^{\infty}(-1)^{\ell}\binom{\alpha}{2\ell}{}^{RL}_{c}D^{\alpha-2\ell}_{x_0}\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial x_3^2}\right)^{\ell}f$$
$$+\sum_{\ell=0}^{\infty}(-1)^{\ell}\binom{\alpha}{2\ell+1}{}^{RL}_{c}D^{\alpha-2\ell-1}_{x_0}$$
$$\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial x_3^2}\right)^{\ell}\left(\frac{\partial f}{\partial x_1}e_1+\frac{\partial f}{\partial x_2}e_2+\frac{\partial f}{\partial x_3}e_3\right). \quad (3.5)$$

Using the trinomial formula for the ℓth power of a sum of three (scalar, hence commutative) terms, we can also break down these formulae into their components in each direction (scalar, e_1, e_2, e_3) by means of the following equivalent expressions:

$$\begin{aligned}{}^{RL}_{c}\overline{\partial}^{\alpha}_{x}=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}(-1)^{m+n+r}\frac{(m+n+r)!}{m!n!r!}\\
\Bigg[\binom{\alpha}{2m+2n+2r}{}^{RL}_{c}D^{\alpha-2m-2n-2r}_{x_0}\frac{\partial^{2m+2n+2r}}{\partial x_1^{2m}\partial x_2^{2n}\partial x_3^{2r}}\\
+\binom{\alpha}{2m+2n+2r+1}{}^{RL}_{c}D^{\alpha-2m-2n-2r-1}_{x_0}\frac{\partial^{2m+2n+2r+1}}{\partial x_1^{2m+1}\partial x_2^{2n}\partial x_3^{2r}}e_1\\
+\binom{\alpha}{2m+2n+2r+1}{}^{RL}_{c}D^{\alpha-2m-2n-2r-1}_{x_0}\frac{\partial^{2m+2n+2r+1}}{\partial x_1^{2m}\partial x_2^{2n+1}\partial x_3^{2r}}e_2\\
+\binom{\alpha}{2m+2n+2r+1}{}^{RL}_{c}D^{\alpha-2m-2n-2r-1}_{x_0}\frac{\partial^{2m+2n+2r+1}}{\partial x_1^{2m}\partial x_2^{2n}\partial x_3^{2r+1}}e_3\Bigg].\end{aligned}$$

Our definition makes sense from the viewpoint of the binomial theorem, formally constructing a fractional power of a sum of derivative operators. But, does it make sense as a fractional quaternionic derivative? We must find out whether standard properties of fractional derivatives are preserved as expected, and also whether standard properties of quaternionic derivatives are preserved as expected, in this new setting that combines two different mathematical fields. The remainder of this paper is essentially devoted to justifying Definition 3.1 by establishing some natural properties of the operator defined therein.

Example. How does the operator ${}^{RL}_{c}\overline{\partial}^{\alpha}_{x}$ apply to a constant function? All classical derivatives of $f(x)=A$ with respect to x_1,x_2,x_3 will be zero, so this will come down to what happens in the very first term of the binomial-type series:

$${}^{RL}_{c}\overline{\partial}^{\alpha}_{x}(A)=\sum_{k=0}^{\infty}\binom{\alpha}{k}{}^{RL}_{c}D^{\alpha-k}_{x_0}\left(\frac{\partial}{\partial x_1}e_1+\frac{\partial}{\partial x_2}e_2+\frac{\partial}{\partial x_3}e_3\right)^{k}(A)$$

$$= \binom{\alpha}{0} {}^{RL}_{\ c}D^{\alpha-0}_{x_0}(A) = \frac{(x_0 - c)^{-\alpha}}{\Gamma(1-\alpha)} A,$$

and similarly for the right-sided operator, giving the results as:

$$ {}^{RL}_{\ c}\overline{\partial}^{\alpha}_{x}(A) = \frac{A(x_0 - c)^{-\alpha}}{\Gamma(1-\alpha)} = (A)\, {}^{RL}_{\ c}\overline{\partial}^{\alpha}_{x}. \tag{3.6}$$

Note that this is consistent with the way that fractional differintegrals of Riemann–Liouville type are expected to behave. In the original fractional calculus, we know from Lemma 2.3 that the Riemann–Liouville derivative of a constant is

$$ {}^{RL}_{\ c}D^{\alpha}_{x}(A) = \frac{A(x-c)^{\beta-\alpha}}{\Gamma(1-\alpha)},$$

which looks very much like the formula (3.6) that we discovered here for the fractional quaternionic d-bar derivative of a constant.

Theorem 3.2. *The fractional quaternionic d-bar derivative* ${}^{RL}_{\ c}\overline{\partial}^{\alpha}_{x} f$ *is well defined if* f *is infinitely differentiable (with respect to* x_0, x_1, x_2, x_3*) inside a ball* $B(c, R)$ *and one of the following further conditions is satisfied.*

(a) There exists $M \in \mathbb{R}^+$ *such that, for all* $k \in \mathbb{Z}^+_0$*, the function*

$$\left(\frac{\partial}{\partial x_1}e_1 + \frac{\partial}{\partial x_2}e_2 + \frac{\partial}{\partial x_3}e_3\right)^k f$$

is bounded uniformly by M *or by* M^k*.*

(b) There exists $M \in \mathbb{R}^+$ *such that, for all* $\ell, m, n \in \mathbb{Z}^+_0$*, the function*

$$\frac{\partial^{\ell+m+n} f}{\partial x_1^{\ell}\partial x_2^{m}\partial x_2^{n}}$$

is bounded uniformly by M *or by* $M^{\ell+m+n}$*.*

Similarly for the fractional quaternionic d-bar derivative $f\, {}^{RL}_{\ c}\overline{\partial}^{\alpha}_{x}$*, with the condition in part (a) being on*

$$f\left(\frac{\partial}{\partial x_1}e_1 + \frac{\partial}{\partial x_2}e_2 + \frac{\partial}{\partial x_3}e_3\right)^k$$

instead and the condition in part (b) remaining the same.

Proof. Suppose $f(x)$ is infinitely differentiable inside a ball $B(c, R)$ with respect to x_0, x_1, x_2, x_3. Then each term of the series (3.2) and (3.3) is certainly well-defined, and we only need to show that these series converge.

Clearly,

$$\begin{aligned}\left|{}^{RL}_{\ c}\overline{\partial}^{\alpha}_{x} f(x)\right| &= \left|\sum_{k=0}^{\infty}\binom{\alpha}{k}\, {}^{RL}_{\ c}D^{\alpha-k}_{x_0}\left(\frac{\partial}{\partial x_1}e_1 + \frac{\partial}{\partial x_2}e_2 + \frac{\partial}{\partial x_3}e_3\right)^k f(x)\right| \\ &\le \sum_{k=0}^{\infty}\left|\binom{\alpha}{k}\right| \left\| {}^{RL}_{\ c}D^{\alpha-k}_{x_0}\left(\frac{\partial}{\partial x_1}e_1 + \frac{\partial}{\partial x_2}e_2 + \frac{\partial}{\partial x_3}e_3\right)^k f\right\|_{\infty}\end{aligned}$$

and

$$\begin{aligned}\left|\left(f\,{}^{RL}_{c}\overline{\partial}^{\alpha}_{x}\right)(x)\right| &= \left|\sum_{k=0}^{\infty}\binom{\alpha}{k}{}^{RL}_{c}D^{\alpha-k}_{x_0}f\left(\frac{\partial}{\partial x_1}e_1+\frac{\partial}{\partial x_2}e_2+\frac{\partial}{\partial x_3}e_3\right)^k(x)\right| \\ &\leq \sum_{k=0}^{\infty}\left|\binom{\alpha}{k}\right|\left\|{}^{RL}_{c}D^{\alpha-k}_{x_0}f\left(\frac{\partial}{\partial x_1}e_1+\frac{\partial}{\partial x_2}e_2+\frac{\partial}{\partial x_3}e_3\right)^k\right\|_{\infty}.\end{aligned}$$

For the convergence, we only care about when k is very large, so we can assume $k > \lfloor \operatorname{Re}\alpha \rfloor + 1$, in which case ${}^{RL}_{c}D^{\alpha-k}_{x_0} = {}^{RL}_{c}I^{k-\alpha}_{x_0}$ is a fractional integral. And we know [12] that fractional integral operators are bounded for bounded functions.

Under an assumption of type (a), we now have

$$\left\|{}^{RL}_{c}D^{\alpha-k}_{x_0}\left(\frac{\partial}{\partial x_1}e_1+\frac{\partial}{\partial x_2}e_2+\frac{\partial}{\partial x_3}e_3\right)^k f\right\|_{\infty} \leq \frac{R^{k-\operatorname{Re}\alpha}}{(k-\operatorname{Re}\alpha)\left|\Gamma(k-\alpha)\right|}\cdot M_k$$

and

$$\left\|{}^{RL}_{c}D^{\alpha-k}_{x_0}f\left(\frac{\partial}{\partial x_1}e_1+\frac{\partial}{\partial x_2}e_2+\frac{\partial}{\partial x_3}e_3\right)^k\right\|_{\infty} \leq \frac{R^{k-\operatorname{Re}\alpha}}{(k-\operatorname{Re}\alpha)\left|\Gamma(k-\alpha)\right|}\cdot M_k,$$

where M_k equals either M or M^k. This gives that both ${}^{RL}_{c}\overline{\partial}^{\alpha}_{x}f(x)$ and $\left(f\,{}^{RL}_{c}\overline{\partial}^{\alpha}_{x}\right)(x)$ are bounded by the series

$$\sum_{k=1}^{\infty}\left|\binom{\alpha}{k}\right|\frac{M_k R^{k-\operatorname{Re}\alpha}}{(k-\operatorname{Re}\alpha)\left|\Gamma(k-\alpha)\right|},$$

which is convergent by the ratio test, giving the result under assumption (a).

If instead we make an assumption of the form (b), then we use the expansion of $\left(\frac{\partial}{\partial x_1}e_1+\frac{\partial}{\partial x_2}e_2+\frac{\partial}{\partial x_3}e_3\right)^k$ as a trinomial sum, with all powers of e_1 and e_2 and e_3 being boundable by the binomial theorem:

$$\begin{aligned}&\left\|{}^{RL}_{c}D^{\alpha-k}_{x_0}\left(\frac{\partial}{\partial x_1}e_1+\frac{\partial}{\partial x_2}e_2+\frac{\partial}{\partial x_3}e_3\right)^k f\right\|_{\infty} \\ &\quad\leq \frac{R^{k-\operatorname{Re}\alpha}}{(k-\operatorname{Re}\alpha)\left|\Gamma(k-\alpha)\right|}\left\|\left(\frac{\partial}{\partial x_1}e_1+\frac{\partial}{\partial x_2}e_2+\frac{\partial}{\partial x_3}e_3\right)^k f\right\|_{\infty} \\ &\quad\leq \frac{R^{k-\operatorname{Re}\alpha}}{(k-\operatorname{Re}\alpha)\left|\Gamma(k-\alpha)\right|}\sum_{\ell+m+n=k}\frac{k!}{\ell!m!n!}\left\|\frac{\partial^k}{\partial x_1^{\ell}\partial x_2^{m}\partial x_3^{n}}f\right\|_{\infty} \\ &\quad\leq \frac{R^{k-\operatorname{Re}\alpha}}{(k-\operatorname{Re}\alpha)\left|\Gamma(k-\alpha)\right|}\sum_{\ell+m+n=k}\left(\frac{k!}{\ell!m!n!}\right)M_k \\ &\quad= \frac{R^{k-\operatorname{Re}\alpha}}{(k-\operatorname{Re}\alpha)\left|\Gamma(k-\alpha)\right|}3^k M_k,\end{aligned}$$

where again M_k equals either M or M^k. This gives that both ${}^{RL}_{c}\overline{\partial}^{\alpha}_{x} f(x)$ and $\left(f\,{}^{RL}_{c}\overline{\partial}^{\alpha}_{x}\right)(x)$ are bounded by the series

$$\sum_{k=1}^{\infty}\left|\binom{\alpha}{k}\right|\frac{3^k M_k R^{k-\operatorname{Re}\alpha}}{(k-\operatorname{Re}\alpha)\,|\Gamma(k-\alpha)|},$$

which is again convergent by the ratio test, giving the result under assumption (b). This completes the proof. □

4. Composition Properties

In this section, we examine what happens when we combine the newly defined fractional quaternionic d-bar operator with itself using different fractional orders. Finding natural results here will go a long way towards justifying our definition as something that makes sense as a fractional differintegral operator for quaternionic functions.

As a general note, in the results of this section, we shall understand the assumption of "suitable" quaternionic functions f to be an assumption that the relevant d-bar operators are well-defined: for example, assumptions as written in Theorem 3.2 would be sufficient.

Theorem 4.1. *For $\alpha\in\mathbb{R}$ and any suitable quaternionic function f, the following hold:*

$$\overline{\partial}\left({}^{RL}_{c}\overline{\partial}^{\alpha}_{x} f\right)={}^{RL}_{c}\overline{\partial}^{\alpha+1}_{x} f,\qquad f\,{}^{RL}_{c}\overline{\partial}^{\alpha+1}_{x}=\left(f\,{}^{RL}_{c}\overline{\partial}^{\alpha}_{x}\right)\overline{\partial}.$$

Proof. We prove only the left-sided case, since the work in the right-sided case is almost identical. Let us use the notation $\partial_{123}=\frac{\partial}{\partial x_1}e_1+\frac{\partial}{\partial x_2}e_2+\frac{\partial}{\partial x_3}e_3$ for simplicity, so that $\overline{\partial}=\frac{\partial}{\partial x_0}+\partial_{123}$ and

$$\begin{aligned}{}^{RL}_{c}\overline{\partial}^{\alpha}_{x} f&=\sum_{\ell=0}^{\infty}(-1)^{\ell}\binom{\alpha}{2\ell}{}^{RL}_{c}D^{\alpha-2\ell}_{x_0}\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial x_3^2}\right)^{\ell} f\\&\quad+\sum_{\ell=0}^{\infty}(-1)^{\ell}\binom{\alpha}{2\ell+1}{}^{RL}_{c}D^{\alpha-2\ell-1}_{x_0}\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial x_3^2}\right)^{\ell}\partial_{123} f.\end{aligned}\tag{4.1}$$

Applying $\overline{\partial}$ to the first sum in (4.1) yields, using the semigroup properties of Riemann–Liouville operators from Lemma 2.4:

$$\begin{aligned}&\sum_{\ell=0}^{\infty}(-1)^{\ell}\binom{\alpha}{2\ell}{}^{RL}_{c}D^{\alpha+1-2\ell}_{x_0}\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial x_3^2}\right)^{\ell} f\\&\quad+\sum_{\ell=0}^{\infty}(-1)^{\ell}\binom{\alpha}{2\ell}{}^{RL}_{c}D^{\alpha-2\ell}_{x_0}\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial x_3^2}\right)^{\ell}\partial_{123} f.\end{aligned}$$

Applying $\overline{\partial}$ to the second sum in (4.1) yields, again using the semigroup properties of Riemann–Liouville operators from Lemma 2.4 as well as the relation (3.1):

$$\sum_{\ell=0}^{\infty}(-1)^{\ell}\binom{\alpha}{2\ell+1}{}^{RL}_{c}D^{\alpha-2\ell}_{x_0}\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial x_3^2}\right)^{\ell}\partial_{123}f$$
$$+\sum_{\ell=0}^{\infty}(-1)^{\ell+1}\binom{\alpha}{2\ell+1}{}^{RL}_{c}D^{\alpha-2\ell-1}_{x_0}\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial x_3^2}\right)^{\ell+1}f.$$

Putting all four of the above sums together, while separating out the $\ell = 0$ term in the first one and substituting ℓ for $\ell + 1$ in the second one, we get:

$$\begin{aligned}
&\overline{\partial}\left({}^{RL}_{c}\overline{\partial}^{\alpha}_{x}f\right)\\
&={}^{RL}_{c}D^{\alpha+1}_{x_0}f+\sum_{\ell=1}^{\infty}(-1)^{\ell}\binom{\alpha}{2\ell}{}^{RL}_{c}D^{\alpha+1-2\ell}_{x_0}\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial x_3^2}\right)^{\ell}f\\
&\quad+\sum_{\ell=0}^{\infty}(-1)^{\ell}\left[\binom{\alpha}{2\ell}+\binom{\alpha}{2\ell+1}\right]{}^{RL}_{c}D^{\alpha-2\ell}_{x_0}\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial x_3^2}\right)^{\ell}\partial_{123}f\\
&\quad+\sum_{\ell=1}^{\infty}(-1)^{\ell}\binom{\alpha}{2\ell-1}{}^{RL}_{c}D^{\alpha-2\ell+1}_{x_0}\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial x_3^2}\right)^{\ell}f.
\end{aligned}$$

Using the fact that $\binom{\alpha}{2\ell}+\binom{\alpha}{2\ell+1}=\binom{\alpha+1}{2\ell+1}$, we find

$$\begin{aligned}
&\overline{\partial}\left({}^{RL}_{c}\overline{\partial}^{\alpha}_{x}f\right)\\
&={}^{RL}_{c}D^{\alpha+1}_{x_0}f+\sum_{\ell=1}^{\infty}(-1)^{\ell}\binom{\alpha+1}{2\ell}{}^{RL}_{c}D^{\alpha+1-2\ell}_{x_0}\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial x_3^2}\right)^{\ell}f\\
&\quad+\sum_{\ell=0}^{\infty}(-1)^{\ell}\binom{\alpha+1}{2\ell+1}{}^{RL}_{c}D^{\alpha+1-2\ell-1}_{x_0}\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial x_3^2}\right)^{\ell}\partial_{123}f,
\end{aligned}$$

which is exactly ${}^{RL}_{c}\overline{\partial}^{\alpha+1}_{x}f$ according to the formulation (4.1). □

Corollary 4.2. *For $\alpha \in \mathbb{R}$ and $n \in \mathbb{N}$ and any suitable quaternionic function f, the following hold:*

$$\overline{\partial}^{n}\left({}^{RL}_{c}\overline{\partial}^{\alpha}_{x}f\right)={}^{RL}_{c}\overline{\partial}^{\alpha+n}_{x}f,\qquad f\,{}^{RL}_{c}\overline{\partial}^{\alpha+n}_{x}=\left(f\,{}^{RL}_{c}\overline{\partial}^{\alpha}_{x}\right)\overline{\partial}^{n}.$$

Proof. This follows immediately from Theorem 4.1 by induction on n. □

Theorem 4.3. *For $\alpha, \beta \in \mathbb{R}$ with $\beta < 0$, and for any suitable quaternionic function f, the following hold:*

$${}^{RL}_{c}\overline{\partial}^{\alpha}_{x}\left({}^{RL}_{c}\overline{\partial}^{\beta}_{x}f\right)={}^{RL}_{c}\overline{\partial}^{\alpha+\beta}_{x}f,\qquad f\,{}^{RL}_{c}\overline{\partial}^{\alpha+\beta}_{x}=\left(f\,{}^{RL}_{c}\overline{\partial}^{\beta}_{x}\right){}^{RL}_{c}\overline{\partial}^{\alpha}_{x}.$$

Proof. Again, we prove only the left-sided case, since the work in the right-sided case is almost identical. This time using the formula (3.2), we have

$${}^{RL}_{c}\overline{\partial}^{\alpha}_{x}\left({}^{RL}_{c}\overline{\partial}^{\beta}_{x}f\right)$$

$$
\begin{aligned}
&= \sum_{k=0}^{\infty} \binom{\alpha}{k} {}^{RL}_{c}D^{\alpha-k}_{x_0} \left(\frac{\partial}{\partial x_1} e_1 + \frac{\partial}{\partial x_2} e_2 + \frac{\partial}{\partial x_3} e_3 \right)^k \\
&\sum_{n=0}^{\infty} \binom{\beta}{n} {}^{RL}_{c}D^{\beta-n}_{x_0} \left(\frac{\partial}{\partial x_1} e_1 + \frac{\partial}{\partial x_2} e_2 + \frac{\partial}{\partial x_3} e_3 \right)^n (f) \\
&= \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \binom{\alpha}{k} \binom{\beta}{n} {}^{RL}_{c}D^{\alpha-k}_{x_0} {}^{RL}_{c}D^{\beta-n}_{x_0} \left(\frac{\partial}{\partial x_1} e_1 + \frac{\partial}{\partial x_2} e_2 + \frac{\partial}{\partial x_3} e_3 \right)^{k+n} f \\
&= \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \binom{\alpha}{k} \binom{\beta}{n} {}^{RL}_{c}D^{\alpha+\beta-(k+n)}_{x_0} \left(\frac{\partial}{\partial x_1} e_1 + \frac{\partial}{\partial x_2} e_2 + \frac{\partial}{\partial x_3} e_3 \right)^{k+n} f \\
&= \sum_{p=0}^{\infty} \sum_{k=0}^{p} \binom{\alpha}{k} \binom{\beta}{p-k} {}^{RL}_{c}D^{\alpha+\beta-p}_{x_0} \left(\frac{\partial}{\partial x_1} e_1 + \frac{\partial}{\partial x_2} e_2 + \frac{\partial}{\partial x_3} e_3 \right)^{p} f, \qquad (4.2)
\end{aligned}
$$

where we have written $p = k + n$ in the last step and used the semigroup property for Riemann–Liouville operators (Lemma 2.4 applies since $\beta - n < 0$ for all $n \geq 0$). Now, by the Chu–Vandermonde identity, the finite sum over k of the two binomial coefficients becomes a single binomial coefficient $\binom{\alpha+\beta}{p}$, and the expression above becomes:

$$
\begin{aligned}
&= \sum_{p=0}^{\infty} \binom{\alpha+\beta}{p} {}^{RL}_{c}D^{\alpha+\beta-p}_{x_0} \left(\frac{\partial}{\partial x_1} e_1 + \frac{\partial}{\partial x_2} e_2 + \frac{\partial}{\partial x_3} e_3 \right)^{p} f(x) \\
&= {}^{RL}_{c}\overline{\partial}^{\alpha+\beta}_{x} f(x),
\end{aligned}
$$

thus proving the result as stated. □

The above results all show cases where a semigroup property is valid for the fractional quaternionic d-bar operator. To verify that it is not universally valid when $\beta > 0$, we also include the following result which is for $\beta = 1$.

Theorem 4.4. *For $\alpha \in \mathbb{R}$ and any suitable quaternionic function f, the following hold:*

$$
\begin{aligned}
{}^{RL}_{c}\overline{\partial}^{\alpha}_{x} (\overline{\partial} f)(x) &= {}^{RL}_{c}\overline{\partial}^{\alpha+1}_{x} f(x) - \sum_{k=0}^{\infty} \binom{\alpha}{k} \frac{(x_0-c)^{k-\alpha-1}}{\Gamma(k-\alpha)} \\
&\quad \times \left(\frac{\partial}{\partial x_1} e_1 + \frac{\partial}{\partial x_2} e_2 + \frac{\partial}{\partial x_3} e_3 \right)^k f(c + e_1x_1 + e_2x_2 + e_3x_3), \\
\left((f\overline{\partial})\, {}^{RL}_{c}\overline{\partial}^{\alpha}_{x} \right)(x) &= \left(f\, {}^{RL}_{c}\overline{\partial}^{\alpha+1}_{x} \right)(x) - \sum_{k=0}^{\infty} \binom{\alpha}{k} \frac{(x_0-c)^{k-\alpha-1}}{\Gamma(k-\alpha)} \\
&\quad \times f(c + e_1x_1 + e_2x_2 + e_3x_3) \left(\frac{\partial}{\partial x_1} e_1 + \frac{\partial}{\partial x_2} e_2 + \frac{\partial}{\partial x_3} e_3 \right)^k .
\end{aligned}
$$

Proof. Again, we prove only the left-sided case, since the work in the right-sided case is almost identical. In the previous proof, everything up to and including (4.2) does not use the assumption that $\beta < 0$, so we can start from (4.2) here in the case $\beta = 1$:

$$
{}^{RL}_{c}\overline{\partial}^{\alpha}_{x} \left({}^{RL}_{c}\overline{\partial}^{\beta}_{x} f \right)
$$

$$= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\binom{\alpha}{k}\binom{1}{n}\,{}^{RL}_{\;c}D^{\alpha-k}_{x_0}\,{}^{RL}_{\;c}D^{1-n}_{x_0}\left(\frac{\partial}{\partial x_1}e_1+\frac{\partial}{\partial x_2}e_2+\frac{\partial}{\partial x_3}e_3\right)^{k+n} f. \tag{4.3}$$

Given the appearance of the binomial coefficient $\binom{1}{n}$, the infinite sum over n is in fact a finite sum, with only the values $n = 0$ and $n = 1$ giving nonzero contributions. When $n = 1$, the inner differintegral with respect to x_0 is the identity operator and so we certainly have a semigroup property in this case. When $n = 0$, the inner differintegral with respect to x_0 is a first-order derivative, and so Lemma 2.4 tells us that

$${}^{RL}_{\;c}D^{\alpha-k}_{x_0}\left({}^{RL}_{\;c}D^{1}_{x_0}g(x_0)\right) = {}^{RL}_{\;c}D^{\alpha-k+1}_{x_0}g(x_0) - \frac{(x_0-c)^{\alpha-k-1}}{\Gamma(\alpha-k)}\big[g(x_0)\big]_{x_0=c}.$$

Thus, overall in (4.3), the only deviation from a perfect semigroup property is given by, for every k and for $n = 0$ only, the subtraction of a term $\binom{\alpha}{k}\frac{(x_0-c)^{\alpha-k-1}}{\Gamma(\alpha-k)}\big[g(x_0)\big]_{x_0=c}$ inside the sum. For the terms exhibiting a perfect semigroup property, we can use the same Chu–Vandermonde argument as in the proof of Theorem 4.3 to obtain exactly ${}^{RL}_{\;c}\overline{\partial}^{\alpha+1}_{x} f$. Then (4.3) becomes:

$${}^{RL}_{\;c}\overline{\partial}^{\alpha+\beta}_{x} f - \sum_{k=0}^{\infty}\binom{\alpha}{k}\frac{(x_0-c)^{\alpha-k-1}}{\Gamma(\alpha-k)}\left[\left(\frac{\partial}{\partial x_1}e_1+\frac{\partial}{\partial x_2}e_2+\frac{\partial}{\partial x_3}e_3\right)^{k+0} f\right]_{x_0=c},$$

which gives the result as stated. □

The above Theorem 4.4 is a special case of the following theorem, which we have stated separately because it is a particularly elegant formula (more so than the following more general result) and will be particularly useful in the next section.

Theorem 4.5. *For $\alpha, \beta \in \mathbb{R}$ with $\beta > 0$, and for any suitable quaternionic function f, the following hold:*

$$\begin{aligned}\left({}^{RL}_{\;c}\overline{\partial}^{\alpha}_{x}\,{}^{RL}_{\;c}\overline{\partial}^{\beta}_{x} f\right)(x) &= \left({}^{RL}_{\;c}\overline{\partial}^{\alpha+\beta}_{x} f\right)(x)\\ &\quad - \sum_{p=0}^{\infty}\sum_{q=1}^{\lceil\beta\rceil} A_{p,q-1}(\alpha,\beta)\frac{(x_0-c)^{p-q-\alpha}}{\Gamma(p-q-\alpha+1)}\\ &\quad \times\left[{}^{RL}_{\;c}D^{\beta-q}_{x_0}\left(\frac{\partial}{\partial x_1}e_1+\frac{\partial}{\partial x_2}e_2+\frac{\partial}{\partial x_3}e_3\right)^{p} f\right]_{x_0=c},\\ \left(f\,{}^{RL}_{\;c}\overline{\partial}^{\beta}_{x}\,{}^{RL}_{\;c}\overline{\partial}^{\alpha}_{x}\right)(x) &= \left(f\,{}^{RL}_{\;c}\overline{\partial}^{\alpha+\beta}_{x}\right)(x)\\ &\quad - \sum_{p=0}^{\infty}\sum_{q=1}^{\lceil\beta\rceil} A_{p,q-1}(\alpha,\beta)\frac{(x_0-c)^{p-q-\alpha}}{\Gamma(p-q-\alpha+1)}\\ &\quad \times\left[{}^{RL}_{\;c}D^{\beta-q}_{x_0} f\left(\frac{\partial}{\partial x_1}e_1+\frac{\partial}{\partial x_2}e_2+\frac{\partial}{\partial x_3}e_3\right)^{p}\right]_{x_0=c},\end{aligned}$$

where the coefficient $A_{p,q-1}(\alpha,\beta)$ *is defined by the following combinatorial sum:*

$$A_{p,q-1}(\alpha,\beta)=\sum_{n=0}^{\min(p,q-1)}\binom{\alpha}{p-n}\binom{\beta}{n}.$$

Proof. Again, we prove only the left-sided case, since the work in the right-sided case is almost identical. The same as in the previous proof, we can start from the expression (4.2) for ${}^{RL}_{c}\overline{\partial}^{\alpha}_{x}\,{}^{RL}_{c}\overline{\partial}^{\beta}_{x}f$. By Lemma 2.4, we have a semigroup property in (4.2) when $\beta\leq n$, and the following result when $\beta>n$ (i.e. when $n\leq\lceil\beta\rceil-1$):

$$\begin{aligned}{}^{RL}_{c}D^{\alpha-k}_{x_0}\left({}^{RL}_{c}D^{\beta-n}_{x_0}g(x_0)\right)&={}^{RL}_{c}D^{\alpha+\beta-k-n}_{x_0}g(x_0)\\&\quad-\sum_{i=1}^{\lceil\beta-n\rceil}\frac{(x_0-c)^{-\alpha+k-i}}{\Gamma(1-\alpha+k-i)}\left[{}^{RL}_{c}D^{\beta-n-i}_{x}g(x_0)\right]_{x_0=c}.\end{aligned}$$

Substituting this into (4.2), and using the fact that the terms exhibiting a perfect semigroup property will together give ${}^{RL}_{c}\overline{\partial}^{\alpha+\beta}_{x}f$ by the same Chu–Vandermonde argument as in the proof of Theorem 4.3, we find the following:

$$\begin{aligned}\left({}^{RL}_{c}\overline{\partial}^{\alpha}_{x}\,{}^{RL}_{c}\overline{\partial}^{\beta}_{x}f\right)(x)&=\left({}^{RL}_{c}\overline{\partial}^{\alpha+\beta}_{x}f\right)(x)\\&\quad-\sum_{k=0}^{\infty}\sum_{n=0}^{\lceil\beta\rceil-1}\sum_{i=1}^{\lceil\beta\rceil-n}\binom{\alpha}{k}\binom{\beta}{n}\frac{(x_0-c)^{-\alpha+k-i}}{\Gamma(1-\alpha+k-i)}\\&\quad\times\left[{}^{RL}_{c}D^{\beta-n-i}_{x}\left(\frac{\partial}{\partial x_1}e_1+\frac{\partial}{\partial x_2}e_2+\frac{\partial}{\partial x_3}e_3\right)^{k+n}f(x)\right]_{x_0=c}\end{aligned}$$

Now the result follows from rewriting the triple sum, setting $p=k+n$ and $q=n+i$ so that $k=p-n$ and $k-i=p-q$, where

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\lceil\beta\rceil-1}\sum_{i=1}^{\lceil\beta\rceil-n}=\sum_{p=0}^{\infty}\sum_{q=1}^{\lceil\beta\rceil}\sum_{n=1}^{\min(p,q-1)}.$$

This substitution is done in order to achieve a final answer where only one of the dummy-variable indices appears on each differintegral operator (the orders change from $\beta-n-i$ and $k+n$ to $\beta-q$ and p respectively), and where the innermost sum over n contains only binomial coefficients and nothing else. □

We have now obtained various semigroup and composition relations for the new fractional quaternionic d-bar derivative, which are natural in the sense that they are similar or analogous to the corresponding relations for the classical (Riemann–Liouville) fractional derivative. In the same way as in [3], such results help to justify our definition, and they will also be useful in the next section for calculating the result of actually applying our new operator to some well-known functions.

5. Applications to Functions

So far, we have considered the application of the fractional quaternionic d-bar derivative to only one type of function: the basic constant function. In the usual calculus, one might consider the second-simplest function after a constant function to be the identity function $f(x) = x$. But in quaternionic calculus, this function is neither left-holomorphic nor right-holomorphic, so we do not expect a neat result from applying the fractional d-bar derivative to this function, if even the classical (first-order) d-bar derivative does not give zero.

Instead, let us start by considering the Fueter variables: $z_1 = x_1 - x_0e_1$, $z_2 = x_2 - x_0e_2$, $z_3 = x_3 - x_0e_3$. These are all both left-holomorphic and right-holomorphic, and can be considered as partially analogous to the identity function in quaternionic calculus.

Theorem 5.1. *The Fueter variables $z_i = x_i - x_0e_i$ in $\mathbb{H}$ have fractional d-bar derivatives given by*

$$ {}^{RL}_{c}\overline{\partial}^{\alpha}_{x}(z_i) = (z_i)\,{}^{RL}_{c}\overline{\partial}^{\alpha}_{x} = \frac{(x_0-c)^{-\alpha}}{\Gamma(1-\alpha)}z_i, $$

and therefore, as a particular case,

$$ {}^{RL}_{0}\overline{\partial}^{\alpha}_{x}(z_i) = (z_i)\,{}^{RL}_{0}\overline{\partial}^{\alpha}_{x} = \frac{x_0^{-\alpha}}{\Gamma(1-\alpha)}z_i, $$

valid for $i = 1, 2, 3$ and any $\alpha \in \mathbb{R}$.

Proof. We start with the left-sided case, and use the formula (3.4) for the fractional quaternionic d-bar derivative, noting that every term with $\ell \geq 1$ will contribute zero to the sum. Firstly, for $c = 0$ and for any $i = 1, 2, 3$, we have:

$$ \begin{aligned} {}^{RL}_{0}\overline{\partial}^{\alpha}_{x}(z_i) &= \binom{\alpha}{0}\,{}^{RL}_{0}D^{\alpha}_{x_0}(x_i - x_0e_i) \\ &\quad + \binom{\alpha}{1}\,{}^{RL}_{c}D^{\alpha-1}_{x_0}\left(\frac{\partial}{\partial x_1}e_1 + \frac{\partial}{\partial x_2}e_2 + \frac{\partial}{\partial x_3}e_3\right)(x_i - x_0e_i) \\ &= {}^{RL}_{0}D^{\alpha}_{x_0}(x_i - x_0e_i) + \alpha\,{}^{RL}_{0}D^{\alpha-1}_{x_0}(e_i) \\ &= \frac{x_0^{-\alpha}}{\Gamma(1-\alpha)}x_i - \frac{x_0^{1-\alpha}}{\Gamma(2-\alpha)}e_i + \alpha\frac{x_0^{1-\alpha}}{\Gamma(2-\alpha)}e_i \\ &= \frac{x_0^{-\alpha}}{\Gamma(1-\alpha)}x_i - \frac{1-\alpha}{\Gamma(2-\alpha)}x_0^{1-\alpha}e_i \\ &= \frac{x_0^{-\alpha}}{\Gamma(1-\alpha)}x_i - \frac{x_0^{-\alpha}}{\Gamma(1-\alpha)}x_0e_i = \frac{x_0^{-\alpha}}{\Gamma(1-\alpha)}(x_i - x_0e_i), \end{aligned} $$

where we have used Lemma 2.3 for handling the fractional differintegrals of constants and of x_0.

For the general (finite) c, we need to be more careful with handling these fractional differintegrals, since $(x_0 - c)$ does not immediately appear. We have:

$$ {}^{RL}_{c}\overline{\partial}^{\alpha}_{x}(z_i) = {}^{RL}_{c}D^{\alpha}_{x_0}(x_i - x_0e_i) + \alpha\,{}^{RL}_{c}D^{\alpha-1}_{x_0}(e_i) $$

$$
\begin{aligned}
&= {}^{RL}_{\ c}D^{\alpha}_{x_0}\big(-(x_0-c)e_i - ce_i + x_i\big) + \alpha\, {}^{RL}_{\ c}D^{\alpha-1}_{x_0}(e_i) \\
&= -\frac{(x_0-c)^{1-\alpha}}{\Gamma(2-\alpha)}e_i + \frac{(x_0-c)^{-\alpha}}{\Gamma(1-\alpha)}(-ce_i+x_i) + \alpha\frac{(x_0-c)^{1-\alpha}}{\Gamma(2-\alpha)}e_i \\
&= -\frac{1-\alpha}{\Gamma(2-\alpha)}(x_0-c)^{1-\alpha}e_i + \frac{(x_0-c)^{-\alpha}}{\Gamma(1-\alpha)}(-ce_i+x_i) \\
&= -\frac{(x_0-c)^{-\alpha}(x_0-c)}{\Gamma(1-\alpha)}e_i + \frac{(x_0-c)^{-\alpha}}{\Gamma(1-\alpha)}(-ce_i+x_i) \\
&= \frac{(x_0-c)^{-\alpha}}{\Gamma(1-\alpha)}\big(-e_i(x_0-c) + (-ce_i+x_i)\big) \\
&= \frac{(x_0-c)^{-\alpha}}{\Gamma(1-\alpha)}(x_i - x_0e_i),
\end{aligned}
$$

which is the required general result in the left-sided case. The work in the right-sided case will be exactly analogous, since $\frac{(x_0-c)^{-\alpha}}{\Gamma(1-\alpha)}$ is a scalar and so multiplying by it on the left or right yields the same result. □

How can we generalise Theorem 5.1 on Fueter variables to a broader class of functions?

Fueter variables are a special case of Fueter polynomials, and we have also dealt with another special case already, namely constant functions. For the Fueter polynomial $P_{\mathbf{k}}(x)$, we now know that, if the degree of the multi-index $\mathbf{k}$ is either 0 or 1, then the result of applying the fractional d-bar derivative to $P_{\mathbf{k}}(x)$, on the left or right, is multiplication by the scalar function $\frac{(x_0-c)^{-\alpha}}{\Gamma(1-\alpha)}$. Let us now see what happens if the degree of $\mathbf{k}$ is 2.

Theorem 5.2. *The Fueter polynomials $P_{\mathbf{k}}$ with $|\mathbf{k}| = 2$ come in two types: firstly, $P_{2\varepsilon_i} = z_i^2$ for $i = 1, 2, 3$, with for any $\alpha \in \mathbb{R}$*

$$ {}^{RL}_{\ c}\overline{\partial}^{\alpha}_{x} P_{2\varepsilon_i}(x) = \Big(P_{2\varepsilon_i}\, {}^{RL}_{\ c}\overline{\partial}^{\alpha}_{x}\Big)(x) = \frac{(x_0-c)^{-\alpha}}{\Gamma(1-\alpha)}P_{2\varepsilon_i}(x), $$

and secondly, $P_{\varepsilon_i+\varepsilon_j} = \frac{1}{2}(z_iz_j + z_jz_i)$ for $i < j$, with for any $\alpha \in \mathbb{R}$

$$ {}^{RL}_{\ c}\overline{\partial}^{\alpha}_{x} P_{\varepsilon_i+\varepsilon_j}(x) = \Big(P_{\varepsilon_i+\varepsilon_j}\, {}^{RL}_{\ c}\overline{\partial}^{\alpha}_{x}\Big)(x) = \frac{(x_0-c)^{-\alpha}}{\Gamma(1-\alpha)}P_{\varepsilon_i+\varepsilon_j}(x). $$

Proof. Without loss of generality, we consider just $\mathbf{k} = (2, 0, 0)$ and $\mathbf{k} = (1, 1, 0)$, as the other cases will then follow from a symmetry argument. We again consider only the left-sided case, since the right-sided case will be exactly analogous with the same scalar multiplier.

Firstly, for $\mathbf{k} = (2, 0, 0)$ we have $P_{\mathbf{k}}(x) = x_1^2 - 2x_0x_1e_1 - x_0^2$. All partial derivatives with respect to x_2 and x_3 are zero, as are all partial derivatives of order greater than 2 with respect to x_1. So, using the formula (3.4), we only take account of $\ell = 0$ and $\ell = 1$ in the first sum, and only $\ell = 0$ in the second sum, to get:

$$
\begin{aligned}
&{}^{RL}_{\ c}\overline{\partial}^{\alpha}_{x}\big({x_1}^2 - 2x_0x_1e_1 - {x_0}^2\big) \\
&\quad = (-1)^0\binom{\alpha}{0}\, {}^{RL}_{\ c}D^{\alpha}_{x_0}\big({x_1}^2 - 2x_0x_1e_1 - {x_0}^2\big)
\end{aligned}
$$

$$+ (-1)^1 \binom{\alpha}{2} {}^{RL}_{c}D^{\alpha-2}_{x_0} \left(\frac{\partial^2}{\partial x_1^2}\right)^1 \left({x_1}^2 - 2x_0x_1e_1 - {x_0}^2\right)$$
$$+ (-1)^0 \binom{\alpha}{1} {}^{RL}_{c}D^{\alpha-1}_{x_0} \left(\frac{\partial}{\partial x_1}e_1\right) \left({x_1}^2 - 2x_0x_1e_1 - {x_0}^2\right)$$
$$= {}^{RL}_{c}D^{\alpha}_{x_0} \left({x_1}^2 - 2(x_0 - c + c)x_1e_1 - (x_0 - c + c)^2\right)$$
$$+ \alpha {}^{RL}_{c}D^{\alpha-1}_{x_0} (2x_1e_1 + 2x_0) - \binom{\alpha}{2} {}^{RL}_{c}D^{\alpha-2}_{x_0}(2).$$

As before, we simplify this expression by expanding the polynomials in terms of $(x_0 - c)$ instead of x_0:

$$= {}^{RL}_{c}D^{\alpha}_{x_0} \left[{x_1}^2 - 2x_1e_1(x_0 - c) - 2cx_1e_1 - (x_0 - c)^2 - 2x_0c + c^2\right]$$
$$+ \alpha {}^{RL}_{c}D^{\alpha-1}_{x_0} (2x_1e_1 + 2(x_0 - c) + 2c) - \frac{\alpha(\alpha-1)}{2} {}^{RL}_{c}D^{\alpha-2}_{x_0}(2)$$
$$= {}^{RL}_{c}D^{\alpha}_{x_0} \left[{x_1}^2 - 2cx_1e_1 - c^2 - 2x_1e_1(x_0 - c) - (x_0 - c)^2 - 2(x_0 - c)c\right]$$
$$+ \alpha {}^{RL}_{c}D^{\alpha-1}_{x_0} (2x_1e_1 + 2c + 2(x_0 - c)) - \frac{\alpha(\alpha-1)}{2} \cdot \frac{2(x_0 - c)^{2-\alpha}}{\Gamma(3-\alpha)}$$
$$= \left[{x_1}^2 - 2cx_1e_1 - c^2\right] \frac{(x_0 - c)^{-\alpha}}{\Gamma(1-\alpha)}$$
$$- 2(x_1e_1 + c) \frac{(x_0 - c)^{1-\alpha}}{\Gamma(2-\alpha)} - \frac{\Gamma(3)}{\Gamma(3-\alpha)}(x_0 - c)^{2-\alpha}$$
$$+ 2\alpha(x_1e_1 + c)\frac{(x_0 - c)^{1-\alpha}}{\Gamma(2-\alpha)} + 2\alpha\frac{(x_0 - c)^{2-\alpha}}{\Gamma(3-\alpha)}$$
$$- \alpha(\alpha-1)\frac{(x_0 - c)^{2-\alpha}}{\Gamma(3-\alpha)}.$$

Observing the factors of $(x_1e_1 + c)$ appearing in later terms, we also notice that the factor in square brackets in the first term is $-(x_1e_1 + c)^2$, enabling us to simplify further:

$$= -\left[{x_1}^2e_1^2 + 2cx_1e_1 + c^2\right] \frac{(x_0 - c)^{-\alpha}}{\Gamma(1-\alpha)} - 2(x_1e_1 + c)\frac{1-\alpha}{\Gamma(2-\alpha)}(x_0 - c)^{1-\alpha}$$
$$- \frac{2 - 2\alpha + \alpha^2 - \alpha}{\Gamma(3-\alpha)}(x_0 - c)^{2-\alpha}$$
$$= -(x_1e_1 + c)^2\frac{(x_0 - c)^{-\alpha}}{\Gamma(1-\alpha)} - 2(x_1e_1 + c)\frac{(x_0 - c)^{1-\alpha}}{\Gamma(1-\alpha)} - \frac{(x_0 - c)^{2-\alpha}}{\Gamma(1-\alpha)}$$
$$= \frac{(x_0 - c)^{-\alpha}}{\Gamma(1-\alpha)}\left[-(x_1e_1 + c)^2 - 2(x_1e_1 + c)(x_0 - c) - (x_0 - c)^2\right]$$
$$= \frac{(x_0 - c)^{-\alpha}}{\Gamma(1-\alpha)}\left(x_1^2 - 2x_0x_1e_1 - x_0^2\right),$$

which is the stated result.

Secondly, for $\mathbf{k} = (1, 1, 0)$ we have $P_{\mathbf{k}}(x) = x_1x_2 - x_0x_2e_1 - x_0x_1e_2$. All partial derivatives with respect to x_3 are zero, as are all partial derivatives of order greater than 1 with respect to x_1 or x_2. So, using the formula (3.4), we only take account of $\ell = 0$ in each sum, to get:

$$\begin{aligned}
&{}^{RL}_{\ c}\overline{\partial}^{\alpha}_{x}\big(x_1x_2 - x_0x_2e_1 - x_0x_1e_2\big)\\
&\quad = (-1)^0\binom{\alpha}{0}\,{}^{RL}_{\ c}D^{\alpha}_{x_0}\big(x_1x_2 - x_0x_2e_1 - x_0x_1e_2\big)\\
&\qquad + (-1)^0\binom{\alpha}{1}\,{}^{RL}_{\ c}D^{\alpha-1}_{x_0}\left(\frac{\partial}{\partial x_1}e_1 + \frac{\partial}{\partial x_2}e_2\right)\big(x_1x_2 - x_0x_2e_1 - x_0x_1e_2\big)\\
&\quad = {}^{RL}_{\ c}D^{\alpha}_{x_0}\big(x_1x_2 - x_0x_2e_1 - x_0x_1e_2\big)\\
&\qquad + \alpha\,{}^{RL}_{\ c}D^{\alpha-1}_{x_0}\big(x_2e_1 - x_0e_1e_2 + x_1e_2 - x_0e_2e_1\big)\\
&\quad = x_1x_2\frac{(x_0-c)^{-\alpha}}{\Gamma(1-\alpha)} - \big(x_1e_2 + x_2e_1\big)\frac{(x_0-c)^{1-\alpha}}{\Gamma(2-\alpha)}\\
&\qquad + \alpha\big(x_2e_1 + x_1e_2\big)\frac{(x_0-c)^{-\alpha}}{\Gamma(1-\alpha)}\\
&\quad = \frac{(x_0-c)^{1-\alpha}}{\Gamma(1-\alpha)}\left(x_1x_2 + \frac{\alpha-1}{1-\alpha}(x_je_i + x_ie_j)\right) = \frac{(x_0-c)^{-\alpha}}{\Gamma(1-\alpha)}P_{\varepsilon_i+\varepsilon_j}(x),
\end{aligned}$$

which is the stated result. □

After seeing the results of Theorems 5.1 and 5.2, as well as that of the Example in Sect. 3, we may wonder if a similar result is true for an arbitrary holomorphic quaternionic function. It is known [3] that any holomorphic complex function $g(z)$ has fractional d-bar derivatives given by multiplying $g(z)$ with $\frac{(2(x-c))^{-\alpha}}{\Gamma(1-\alpha)}$ or $\frac{(2(ci-yi))^{-\alpha}}{\Gamma(1-\alpha)}$, and therefore plausible that a similar result may be true in the quaternionic setting, at least for functions which are both left-holomorphic and right-holomorphic such as Fueter polynomials.

However, increasing the degree of the multi-index $\mathbf{k}$ to 3, we found that this is not the case: when $\mathbf{k} = (1, 2, 0)$, we have the following result.

Example. If $\mathbf{k} = (1, 2, 0)$, then the Fueter polynomial is

$$\begin{aligned}
P_{\mathbf{k}}(x) &= \frac{1}{3}\left(z_1{z_2}^2 + z_2z_1z_2 + {z_2}^2z_1\right)\\
&= x_1x_2^2 - x_0^2x_1 - 2x_0x_1e_2 - x_0x_2^2e_1 + \frac{1}{3}x_0^3e_1,
\end{aligned}$$

and its left-sided fractional d-bar derivative (with constant $c = 0$) is

$$\begin{aligned}
{}^{RL}_{\ c}\overline{\partial}^{\alpha}_{x}P_{\mathbf{k}}(x) = \frac{x_0^{-\alpha}}{\Gamma(1-\alpha)}\Bigg[&x_1x_2^2 + \frac{x_0}{1-\alpha}\big(-2x_1e_2 - x_2^2e_1(1-\alpha) - 2x_1x_2e_2\big)\\
&+\frac{x_0^2}{(1-\alpha)(2-\alpha)}\big(-2x_1 + 2\alpha e_1e_2 - 2\alpha x_2e_1e_2 - \alpha(\alpha-1)x_1\big)\\
&+\frac{x_0^3}{(1-\alpha)(2-\alpha)(3-\alpha)}\left(\frac{e_1}{3} - 2\alpha e_1 + \alpha(\alpha-1)e_1\right)\Bigg],
\end{aligned}$$

which cannot be equal to $\frac{x_0^{-\alpha}}{\Gamma(1-\alpha)}P_{\mathbf{k}}(x)$ since the latter has no component in the direction of $e_1e_2 = e_3$ while the fractional derivative formula has a nonzero component in this direction. Thus, by a symmetry argument, we find that

$$ {}^{RL}_{c}\overline{\partial}^{\alpha}_{x} P_{\mathbf{k}}(x) \neq \frac{(x_0 - c)^{-\alpha}}{\Gamma(1-\alpha)} P_{\mathbf{k}}(x) \neq \left(P_{\mathbf{k}} \, {}^{RL}_{c}\overline{\partial}^{\alpha}_{x} \right)(x), $$

and the pattern previously observed for low values of $|\mathbf{k}|$ does not hold true in general.

All of the fractional d-bar derivatives found above were done by direct calculation. Is there an easier way to find fractional d-bar derivatives of functions such as Fueter polynomials? The answer is yes: recalling that Fueter polynomials are left-holomorphic and right-holomorphic, we can make use of Theorem 4.4 together with a neat trick from fractional calculus to achieve the following result.

Theorem 5.3. *For any $\alpha \in \mathbb{R}$ and any multi-index $\mathbf{k}$ as in Definition 2.8, we have the following formula for the fractional d-bar derivatives of the Fueter polynomial $P_{\mathbf{k}}$:*

$$ {}^{RL}_{c}\overline{\partial}^{\alpha}_{x} P_{\mathbf{k}}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x_0 - c)^{n-\alpha}}{n!\Gamma(1-\alpha)} \times \left[\left(\frac{\partial}{\partial x_1} e_1 + \frac{\partial}{\partial x_2} e_2 + \frac{\partial}{\partial x_3} e_3 \right)^n P_{\mathbf{k}}(x) \right]_{x_0 = c} ; \quad (5.1) $$

$$ \left(P_{\mathbf{k}} \, {}^{RL}_{c}\overline{\partial}^{\alpha}_{x} \right)(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x_0 - c)^{n-\alpha}}{n!\Gamma(1-\alpha)} \times \left[P_{\mathbf{k}}(x) \left(\frac{\partial}{\partial x_1} e_1 + \frac{\partial}{\partial x_2} e_2 + \frac{\partial}{\partial x_3} e_3 \right)^n \right]_{x_0 = c} . \quad (5.2) $$

Proof. As usual, we prove only the left-sided case, the proof for the right-sided case being analogous.

Using Theorem 4.4, we know that the order-α fractional d-bar derivative can be expressed in terms of the order-$(\alpha - 1)$ fractional d-bar derivative of the simple d-bar derivative, which latter for all Fueter polynomials we know to be zero:

$$ \begin{aligned} {}^{RL}_{c}\overline{\partial}^{\alpha}_{x} P_{\mathbf{k}}(x) &= {}^{RL}_{c}\overline{\partial}^{\alpha-1}_{x} (\overline{\partial} P_{\mathbf{k}})(x) + \sum_{n=0}^{\infty} \binom{\alpha-1}{n} \frac{(x_0 - c)^{n-\alpha}}{\Gamma(n-\alpha+1)} \\ &\quad \times \left(\frac{\partial}{\partial x_1} e_1 + \frac{\partial}{\partial x_2} e_2 + \frac{\partial}{\partial x_3} e_3 \right)^n P_{\mathbf{k}}(c + e_1 x_1 + e_2 x_2 + e_3 x_3) \\ &= \sum_{n=0}^{\infty} \frac{\Gamma(\alpha)(x_0 - c)^{n-\alpha}}{\Gamma(\alpha - n) n! \Gamma(n-\alpha+1)} \left(\frac{\partial}{\partial x_1} e_1 + \frac{\partial}{\partial x_2} e_2 + \frac{\partial}{\partial x_3} e_3 \right)^n \left[P_{\mathbf{k}}(x) \right]_{x_0 = c} . \end{aligned} $$

Using the reflection formula for the gamma function, we have

$$ \Gamma(\alpha - n)\Gamma(n - \alpha + 1) = \frac{\pi}{\sin(\pi\alpha - \pi n)} = (-1)^n \Gamma(\alpha)\Gamma(1-\alpha), $$

which leads to the desired formula. ∎

Remark 5.4. From [6, Theorem 6.2], we know that the operator $\partial_{123} = \frac{\partial}{\partial x_1}e_1 + \frac{\partial}{\partial x_2}e_2 + \frac{\partial}{\partial x_3}e_3$ has the following effect on Fueter polynomials:

$$\partial_{123}P_{\mathbf{k}}(x) = k_1e_1P_{\mathbf{k}-\varepsilon_1}(x) + k_2e_2P_{\mathbf{k}-\varepsilon_2}(x) + k_3e_3P_{\mathbf{k}-\varepsilon_3}(x).$$

This would lead to a reformulation of (5.1) and (5.2) expressing the fractional d-bar derivatives of $P_{\mathbf{k}}(x)$ as sums of Fueter polynomials of smaller degree. However, we were unable to rewrite these expressions in a neat or concise form, so we have omitted them from this paper.

The following result, a consequence of Theorem 5.3, provides at least one infinite family of Fueter polynomials with very neat and natural formulae for their fractional d-bar derivatives.

Theorem 5.5. *For any* $i = 1, 2, 3$ *and* $k \in \mathbb{N}$, *we have* $P_{k\varepsilon_i}(x) = (x_i - x_0e_i)^k$, *and*

$${}^{RL}_{\ c}\overline{\partial}^{\alpha}_{x}P_{k\varepsilon_i}(x) = \left(P_{k\varepsilon_i}\,{}^{RL}_{\ c}\overline{\partial}^{\alpha}_{x}\right)(x) = \frac{(x_0-c)^{-\alpha}}{\Gamma(1-\alpha)}P_{k\varepsilon_i}(x).$$

Proof. We use the formulae (5.1) and (5.2). When applied to functions of the type $P_{k\varepsilon_i}(x) = (x_i - x_0e_i)^k$, the operator $\partial_{123} = \frac{\partial}{\partial x_1}e_1 + \frac{\partial}{\partial x_2}e_2 + \frac{\partial}{\partial x_3}e_3$ acts simply as $\frac{\partial}{\partial x_i}e_i$, since all partial derivatives with respect to x_j will be zero for $j \neq i$. So we have

$$\begin{aligned}
{}^{RL}_{\ c}\overline{\partial}^{\alpha}_{x}P_{k\varepsilon_i}(x) &= \sum_{n=0}^{\infty}\frac{(-1)^n(x_0-c)^{n-\alpha}}{n!\Gamma(1-\alpha)}\left[\left(\frac{\partial}{\partial x_i}e_i\right)^n P_{k\varepsilon_i}(x)\right]_{x_0=c} \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n(x_0-c)^{n-\alpha}}{n!\Gamma(1-\alpha)}\left(\frac{\partial}{\partial x_i}e_i\right)^n (x_i-ce_i)^k \\
&= \frac{(x_0-c)^{-\alpha}}{\Gamma(1-\alpha)}\sum_{n=0}^{\infty}\frac{(-1)^n(x_0-c)^n}{n!}e_i^n\frac{\partial^n}{\partial x_i^n}(x_i-ce_i)^k \\
&= \frac{(x_0-c)^{-\alpha}}{\Gamma(1-\alpha)}\sum_{n=0}^{k}\frac{(-1)^n(x_0-c)^n(e_i)^n}{n!}\cdot\frac{k!}{(k-n)!}(x_i-ce_i)^{k-n} \\
&= \frac{(x_0-c)^{-\alpha}}{\Gamma(1-\alpha)}\sum_{n=0}^{k}\binom{k}{n}(ce_i-x_0e_i)^n(x_i-ce_i)^{k-n} \\
&= \frac{(x_0-c)^{-\alpha}}{\Gamma(1-\alpha)}(x_i-x_0e_i)^k = \frac{(x_0-c)^{-\alpha}}{\Gamma(1-\alpha)}P_{k\varepsilon_i}(x),
\end{aligned}$$

where in the last step we used the classical binomial theorem, valid since $ce_i - x_0e_i$ and $x_i - ce_i$ always commute, being vectors in the plane generated by 1 and e_i. Similarly,

$$\begin{aligned}
\left(P_{k\varepsilon_i}\,{}^{RL}_{\ c}\overline{\partial}^{\alpha}_{x}\right)(x) &= \sum_{n=0}^{\infty}\frac{(-1)^n(x_0-c)^{n-\alpha}}{n!\Gamma(1-\alpha)}\left[P_{k\varepsilon_i}(x)\left(\frac{\partial}{\partial x_i}e_i\right)^n\right]_{x_0=c} \\
&= \frac{(x_0-c)^{-\alpha}}{\Gamma(1-\alpha)}\sum_{n=0}^{\infty}\frac{(-1)^n(x_0-c)^n}{n!}\left[\frac{\partial^n}{\partial x_i^n}(x_i-ce_i)^k\right]e_i^n
\end{aligned}$$

$$= \frac{(x_0-c)^{-\alpha}}{\Gamma(1-\alpha)} \sum_{n=0}^{k} \frac{1}{n!} \left[\frac{k!}{(k-n)!}(x_i - ce_i)^{k-n} \right] (-1)^n (x_0-c)^n (e_i)^n$$
$$= \frac{(x_0-c)^{-\alpha}}{\Gamma(1-\alpha)} \sum_{n=0}^{k} \binom{k}{n} (x_i - ce_i)^{k-n} (ce_i - x_0 e_i)^n$$
$$= \frac{(x_0-c)^{-\alpha}}{\Gamma(1-\alpha)} (x_i - x_0 e_i)^k = \frac{(x_0-c)^{-\alpha}}{\Gamma(1-\alpha)} P_{k\varepsilon_i}(x),$$

again using the classical binomial theorem and the fact that $x_i - ce_i$ and $ce_i - x_0 e_i$ always commute with each other. □

6. Comparative Analysis

It would be remiss of us not to mention the previous work on "fractional Clifford analysis" [1,7], and to compare our work with this to clarify its novelty.

There are two key differences between the fractional operator defined in our work and the one defined by Kähler and Vieira [7] using Caputo fractional derivatives as follows:

$$D^\alpha = e_1 {}_0^C D_{x_1}^\alpha + e_2 {}_0^C D_{x_2}^\alpha + e_3 {}_0^C D_{x_3}^\alpha,$$

an idea later used by Cerejeiras et al [1], Ferreira and Vieira [4], etc.

The first (simpler) difference is that they defined a fractional Dirac operator, i.e. a fractional version of $\partial_{123} = \frac{\partial}{\partial x_1} e_1 + \frac{\partial}{\partial x_2} e_2 + \frac{\partial}{\partial x_3} e_3$, while we defined a fractional d-bar operator, i.e. a fractional version of $\overline{\partial} = \frac{\partial}{\partial x_0} + \partial_{123}$. This means from the beginning that the goals of the two projects are different: in our work, we have never attempted to define a fractional power of the Dirac operator ∂_{123}, using only kth powers of this operator for $k \in \mathbb{Z}_0^+$.

The second (more subtle) difference is that our operator ${}_c^{RL}\overline{\partial}_x^\alpha$ is an attempt to defined a fractional *power* of the original operator $\overline{\partial}$, in such a way that putting $\alpha = k$ recovers the kth power of the d-bar derivative for every $k \in \mathbb{N}$. The fractional Dirac operator defined in [7] is not a fractional power of the original Dirac operator, in the sense that putting $\alpha = k$ does not recover the kth power of the Dirac operator except when $k = 1$. For example:

$$D^2 = e_1 \frac{\partial^2}{\partial x_1^2} + e_2 \frac{\partial^2}{\partial x_2^2} + e_3 \frac{\partial^2}{\partial x_3^2} \neq \left(\frac{\partial}{\partial x_1} e_1 + \frac{\partial}{\partial x_2} e_2 + \frac{\partial}{\partial x_3} e_3 \right)^2 .$$

The operator D^α is a fractional version of the Dirac operator, but not a fractional power thereof. By contrast, our operator is defined with the intention of being a fractional power of the d-bar operator, and this is why we were able to recover properties such as the semigroup relation which are typically associated with powers.

The fractional Dirac operator of Kähler and Vieira [7] makes a lot of sense from the Clifford-algebraic viewpoint, with properties such as Weyl relations and Lie superalgebras emerging naturally in this setting. However, from the viewpoint of fractional calculus, we believe that our fractional d-bar operator has more of the expected fractional properties, such as natural semigroup relations and action on polynomial analogues.

7. Conclusions

In this work, we have introduced a new operator acting on quaternionic functions which we call a fractional d-bar derivative. We constructed this operator by thinking about what a fractional power of the ordinary d-bar operator might look like, starting from natural-number powers and extrapolating using the binomial theorem. We also justified our definition by considering some key properties of our operator which showcase behaviours similar to those of classical fractional derivatives, such as composition formulae and application to holomorphic polynomial-type functions.

The work done so far is enough for a full investigation of the current project, introducing a new definition and justifying it mathematically, but it also inspires many new thoughts which may be investigated in future projects. For example, it may be possible to extend the work done here in the quaternionic space $\mathbb{H}$ to the more general setting of $C\ell(n)$, the Clifford algebra built on the n-dimensional vector space $\mathbb{R}^n$. It may also be possible to define fractional versions of other quaternionic or Clifford differential operators, such as the radial differential operators [6, Definition 11.28] that are used for exponential and trigonometric functions in higher-dimensional Clifford algebras. From the viewpoint of fractional calculus, everything that we have done so far has been using only the Riemann–Liouville definition of fractional derivatives; it may be worth investigating what happens when this fractional model is replaced by others, such as that of Caputo or more general classes of operators.

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