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**Advances in Applied Clifford Algebras**

# A New Way to Construct the Riemann Curvature Tensor Using Geometric Algebra and Division Algebraic Structure

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**Abstract.** The Riemann curvature tensor is constructed using the Clifford-Dirac geometric algebra and division-algebraic operator structure.

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## 1. Introduction

In this paper the Clifford-Dirac geometric algebra (also known as the Space Time Algebra [1]) and the operator structure of the four unique division algebras are used in a relatively straightforward way to construct the Riemann curvature tensor. Conceptually why such a mathematical technique would ever be of any interest for use in theoretical physics becomes evident in the method's development.

## 2. The Division-Algebraic Operator Structure

Composition algebras are algebras $\mathbb{A}$ such that for any two elements the algebraic norm of their product equals the product of their norms [5,6,10,16,19]: $\|xy\| = \|x\|\,\|y\|\ \forall x, y \in \mathbb{A}$. These composition algebras exist in 1, 2, 4 and 8 dimensions, corresponding to $\mathbb{K} = \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$ [6,11,16]. Only the $\mathbb{K}$ are division algebras, that is, composition algebras without zero divisors [6,16,19].

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The division algebras $\mathbb{K}$ have an interesting and unique binary coupling operation $\circ$ which maps pairs of such $\mathbb{K}$-structures into Euclidean structure, given by [1,4–6,8,12,16][1]

$$(A, \boldsymbol{A}) \circ (B, \boldsymbol{B}) \longmapsto (AB - \boldsymbol{A} \cdot \boldsymbol{B},\ A\boldsymbol{B} + \boldsymbol{A}B + \boldsymbol{A} \times \boldsymbol{B}) \tag{1}$$

when acting within a space with local Euclidean metric $\delta_{ab}$ permitting the standard $\mathbb{E}^3$ or $\mathbb{E}^7$ dot and cross products on the RHS, and where the LHS $(A, \boldsymbol{A})$ and $(B, \boldsymbol{B})$ are $\mathbb{K}$-structures - with specific use in this paper of $\mathbb{H}$ (quaternion) structures along with the $\mathbb{E}^3$ dot and cross products. Since the quaternions are subsumed by geometric algebra [1], when using the quaternions (1) can be seen as more fundamentally geometric-algebraic operator structure.

Of notable interest in (1) is that LHS $\mathbb{H}$-structure generates the RHS mapping $AB - \boldsymbol{A} \cdot \boldsymbol{B} = A_\mu B^\mu$. This mapping thus signals a Lorentz metric signature $(1, -1, -1, -1)$. Therefore (1) with LHS $\mathbb{H}$-structure is taken to generate a mapping from a minimally $\mathbb{H}$-structured space not simply into Euclidean space $\mathbb{E}^3$, but instead into the Minkowski space-time continuum $\mathbb{M}$.

## 3. The Clifford-Dirac Algebra

The symbols $\gamma_\beta$ will refer to the four Clifford-Dirac algebra elements having the following pertinent properties [15,17,22,28][2]

$$\begin{aligned} &(\gamma_n)^2 = -1; \quad (\gamma_0)^2 = 1\ (n: 1 \rightarrow 3) \\ &\gamma^n = -\gamma_n; \quad \gamma_\alpha \gamma_\beta = -\gamma_\beta \gamma_\alpha\ (\alpha \neq \beta). \end{aligned} \tag{2}$$

Clifford algebras are structures naturally associated with quadratic forms on vector spaces [1,15], such as a metric on a manifold with vector space structure. The $\gamma_\beta$ are specifically "Lorentzian" in structure [9,17], being associated with the vector space and metric structure of the Minkowski space-time manifold $(\mathbb{M}, \boldsymbol{\eta})$, with Lorentz metric $\boldsymbol{\eta} \equiv \eta_{\mu\nu}$. This Lorentzian Clifford-Dirac algebra structure is determined by the anti-commutator relation [13,15,17]

$$\{\gamma_\alpha, \gamma_\beta\} = 2\eta_{\alpha\beta}. \tag{3}$$

Since according to the equivalence principle a locally flat space-time inertial frame $(\mathbb{M}, \boldsymbol{\eta})$ is recoverable in a sufficiently small neighborhood $N$ of any space-time point $p \in (\mathbb{B}, \boldsymbol{g})$ of a general space-time manifold $\mathbb{B}$ with general metric $g_{\mu\nu}$ [7,21], the $\gamma_\beta$ so locally exist via (3) in the tangent space $\mathbb{M} = T_p\mathbb{B}$.

[1]See, e.g., Ref. [16], p. 2; Ref. [1], p. 9.; Ref. [8], Eq. (3); Ref. [12], Eq. (5).
[2]Ref. [28], p. 89.

## 4. Combining Sections 2 and 3

We would like to combine the Clifford-Dirac algebra elements and the quaternions into specific structures, and then make use of (1) on these combinations. In that light we define the following structures:

$$\eta = \mathbf{1}\gamma_{0} + \boldsymbol{i}\gamma_{1} + \boldsymbol{j}\gamma_{2} + \boldsymbol{k}\gamma_{3} \qquad \partial = \mathbf{1}\partial_{0} + \boldsymbol{i}\partial_{x} + \boldsymbol{j}\partial_{y} + \boldsymbol{k}\partial_{z} \tag{4}$$

with $\mathbf{1}$ the identity element, the $\partial_\mu$ being the standard derivative operators, and $\tau_n = \{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ the imaginary quaternion elements with $\tau_n^2 = -1$ [1,5]. An over-bar, e.g., $\bar{\eta}$, is the quaternion conjugate, and with $\boldsymbol{\gamma} = \boldsymbol{i}\gamma_{1} + \boldsymbol{j}\gamma_{2} + \boldsymbol{k}\gamma_{3}$ and $\nabla = \boldsymbol{i}\partial_{x} + \boldsymbol{j}\partial_{y} + \boldsymbol{k}\partial_{z}$.

Looking at (4), it is clear that $\partial$ is a local operator. It is in fact nothing other than the generalization of the Cauchy-Riemann operator $\partial_x + \mathrm{i}\partial_y$ of complex analysis to $(1,3)$ dimensions used throughout Clifford algebraic analysis [1].[3] Since the Clifford-Dirac algebra elements are conceived as local operators [17,28],[4] which exist in $\mathbb{M} = T_p\mathbb{B}$, $\eta$ is also a local operator.

Thus a combination of $\eta$ and $\partial$ using (1) will also yield new, coupled operators - but now they will be coupled local space-time operators. In this paper we will focus on the development of two coupled operators, $\eta \circ \partial$ and $\bar{\eta} \circ \eta$.

### 4.1. The Operator $\boldsymbol{\eta} \circ \boldsymbol{\partial}$

Applying (1) to $\eta$ and $\partial$ and defining the result as $\phi_{\eta\partial}$, we have

$$\begin{aligned} \phi_{\eta\partial} &\equiv \eta \circ \partial = (\gamma_{0}, \boldsymbol{\gamma}) \circ (\partial_{0}, \nabla) \\ &= (\gamma_0 \partial/\partial t - \boldsymbol{\gamma}\cdot\nabla, \gamma_0\nabla + \boldsymbol{\gamma}\partial/\partial t + \boldsymbol{\gamma}\times\nabla) = (\phi^0_{\eta\partial}, \phi^1_{\eta\partial}). \end{aligned} \tag{5}$$

There are two operators in (5), which are coupled together. The first is

$$\phi^0_{\eta\partial} \equiv \gamma_0 \partial/\partial t - \boldsymbol{\gamma}\cdot\nabla. \tag{6}$$

This happens to be the well-known Dirac operator $\not{\partial}$, that is: $\phi^0_{\eta\partial} \equiv \not{\partial}$, which is known to operate on a spinor field $\psi$ representing an elementary fermion such as an electron [3,9,13,17,18,28].

The second operator

$$\phi^1_{\eta\partial} \equiv \gamma_0\nabla + \boldsymbol{\gamma}\partial/\partial t + \boldsymbol{\gamma}\times\nabla \tag{7}$$

is not nearly as transparent.

Nevertheless, it has been shown that $\phi^1_{\eta\partial}$ is a Clifford field operator $\not{\Delta}$ which operates on a Clifford field $\not{\Phi}$ [23]. [5] A Clifford field (using for our purposes the Clifford-Dirac algebra) is a field given by: $\not{\Phi} = \Phi^{ab\cdots}_{\mu}\gamma^\mu$, with each $\Phi^{ab\cdots}_{\mu}$ being a function or batch of functions attached to their assigned Clifford element $\gamma^\mu$.

Further, it happens that when $\phi^1_{\eta\partial} \equiv \not{\Delta}$ operates on a Clifford field identified with the gauge field of a Standard Model interaction, such as the electromagnetic potential $A_\mu \to \not{A} = A_\mu\gamma^\mu$, the operation generates the

[3] Ref. [1], Ch. 3 & Sec. 3.3.
[4] Ref. [17], Secs. 11.5 & 24.6.
[5] See Ref. [23], Sec. III, for the step-by-step derivation, which results in Eq. (8) herein.

linear portion of the field tensor associated with that gauge field interaction [23].

Relabeling the Clifford field operator $\phi^1_{\eta\partial}$ as $\not{\Delta}$, we have the operation on a general Clifford field $\not{\Phi}$ given by the intrinsic expression [23]

$$\not{\Delta}\not{\Phi} = \mathbf{H} \tag{8}$$

where $\mathbf{H}$ is an anti-symmetric field tensor [23]. If $\not{\Phi}$ is then set as the electromagnetic potential $\not{\Phi} \rightarrow \not{A} = A_\mu \gamma^\mu$, $\mathbf{H}$ becomes the electromagnetic field tensor $\mathbf{F}$ with components $F_{\mu\nu}$ given in the local basis of vector fields $\{\partial_\mu\}$ as [2,7,13,14]

$$\begin{aligned} &\mathbf{F}(\partial_\mu, \partial_\nu) = F_{\mu\nu}, \\ &F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \end{aligned} \tag{9}$$

### 4.2. The Operator $\bar{\eta} \circ \eta$

For (1) applied to $\bar{\eta}$ and $\eta$ we have

$$\begin{aligned} \phi_{\bar{\eta}\eta} \equiv \bar{\eta} \circ \eta &= (\gamma_0, -\boldsymbol{\gamma}) \circ (\gamma_0, \boldsymbol{\gamma}) \\ &= (C, \gamma_0 \boldsymbol{\gamma} - \boldsymbol{\gamma}\gamma_0 + \boldsymbol{\gamma} \times \boldsymbol{\gamma}) \end{aligned} \tag{10}$$

where $C$ is a constant irrelevant for the purposes of this paper. Thus focusing on the second operator on the RHS and using $[r, s] = r \times s$ [9], we can write

$$\phi_{\bar{\eta}\eta} = [\gamma_0, \boldsymbol{\gamma}] + [\boldsymbol{\gamma}, \boldsymbol{\gamma}]. \tag{11}$$

Consider the commutator $[\gamma_\alpha, \gamma_\beta]$, with $\alpha, \beta : 1 \rightarrow 4$. Comparing $[\gamma_\alpha, \gamma_\beta]$ with $[\boldsymbol{\gamma}, \boldsymbol{\gamma}]$, the difference between the two resides within the commutator $[\gamma_0, \gamma_a]$, with $a : 1 \rightarrow 3$. The expression $[\gamma_0, \gamma_a]$ is identical to the first term $[\gamma_0, \boldsymbol{\gamma}]$ of (11), and therefore we may rewrite (11) as

$$\phi_{\bar{\eta}\eta} = \left[\gamma_\alpha, \gamma_\beta\right]. \tag{12}$$

Let us first focus on the operator $[\boldsymbol{\gamma}, \boldsymbol{\gamma}]$ of (11). Interestingly enough, this operator has been shown to generate the *non-linear* portions of the field strength tensor for a Standard Model interaction [24].

For example, consider the components $W^i_\mu$, with $i : 1 \rightarrow 3$, of the weak interaction gauge field $\boldsymbol{W}_\mu$. Taken as a Clifford field we have the isomorphism

$$\boldsymbol{W}_\mu \cong \not{W}_\mu = \gamma_1 W^1_\mu + \gamma_2 W^2_\mu + \gamma_3 W^3_\mu. \tag{13}$$

These will be the entities that $[\boldsymbol{\gamma}, \boldsymbol{\gamma}]$ operates on. Consider the '3' component of $[\boldsymbol{\gamma}, \boldsymbol{\gamma}]$:

$$[\gamma_1, \gamma_2] = \gamma_1\gamma_2 - \gamma_2\gamma_1. \tag{14}$$

This is a bilinear operator which will act on the $(\gamma_1 W^1_\mu, \gamma_2 W^2_\nu)$ components of (13). We have

$$\begin{aligned} [\gamma_1, \gamma_2]\,(\gamma_1 W^1_\mu, \gamma_2 W^2_\nu) &= \gamma_1\gamma_2\gamma_1 W^1_\mu \gamma_2 W^2_\nu - \gamma_2\gamma_1\gamma_2 W^2_\mu \gamma_1 W^1_\nu \\ &= -\left(W^1_\mu W^2_\nu - W^2_\mu W^1_\nu\right) = -\sum_{i,j} \epsilon_{ij3} W^i_\mu W^j_\nu \\ &= -\left(\boldsymbol{W}_\mu \times \boldsymbol{W}_\nu\right)_{k=3} = -\left[W^1_\mu, W^2_\nu\right] \end{aligned} \tag{15}$$

where the negative sign arises from multiplying the Clifford elements out and using (2), $\epsilon_{ijk}$ is the Levi-Civita symbol, and where the $\mu\nu$ indices are understood as not operated on (switched) by the commutator in the last expression.

We can write this more generally as

$$[\gamma_i, \gamma_j]\,(\gamma_i W^i_\mu, \gamma_j W^j_\nu) = \gamma_i\gamma_j\gamma_i W^i_\mu \gamma_j W^j_\nu - \gamma_j\gamma_i\gamma_j W^j_\mu \gamma_i W^i_\nu$$
$$= -\sum_{i,j} \epsilon_{ijk} W^i_\mu W^j_\nu = -\,(\boldsymbol{W}_\mu \times \boldsymbol{W}_\nu)_k = -\left[W^i_\mu, W^j_\nu\right]. \tag{16}$$

Thus we can write

$$\left[\boldsymbol{\gamma}_i, \boldsymbol{\gamma}_j\right](\not{W}^i_\mu, \not{W}^j_\nu)_k = -\,[\boldsymbol{W}_\mu, \boldsymbol{W}_\nu]_k\,. \tag{17}$$

This is the $k$-th component of the non-linear portion of the weak interaction's field strength tensor [3,15,18][6]

$$G_{\mu\nu} = \partial_\mu \boldsymbol{W}_\nu - \partial_\nu \boldsymbol{W}_\mu + i\,[\boldsymbol{W}_\mu, \boldsymbol{W}_\nu] \tag{18}$$

where for purposes of analysis we have set any coupling constant to 1.

Combining (8)–(9) and (17) we can thus write the intrinsic expression for the weak interaction field strength tensor $\mathbf{G}$ in Clifford field/field operator formalism

$$\not{\Delta}\not{W} - [\boldsymbol{\gamma}, \boldsymbol{\gamma}]\,(\not{W}, \not{W}) = \mathbf{G},$$
$$\mathbf{G}(\partial_\mu, \partial_\nu) = G_{\mu\nu}. \tag{19}$$

It has previously been shown that all three field strength tensors of the Standard Model interactions as well as that for the electroweak interaction can be generated using this technique [23–26].

## 5. The Riemann Curvature Tensor

Given the above it is quite natural to inquire as to if the Riemann curvature tensor, which is the analogous field strength tensor for the gravitational interaction as modeled in General Relativity theory [2,13,14,21], can be generated using the selfsame method as used for the Standard Model and electroweak interactions.

### 5.1. The Christoffel Connection Coefficients

We must first ask what fields are to be used in attempting to construct the Riemann curvature tensor, such fields which would be mathematically analogous to the gauge fields of the Standard Model.

Consider a vector bundle $E$ over a manifold $M$ with connection $D$ and associated covariant derivative $D_\nu$ [2,15,21]. The covariant derivative $D_\nu s$ takes the derivative of a section $s$ of $E$ in the direction of a vector field $\nu$ on an open subset $U \subset M$ [2]. With basis vector fields $\partial_\mu$ on $M$ and basis $e_i$ of sections of $E$ and writing $D_\mu \equiv D_{\partial_\mu}$ one can express $D_\mu e_i$ uniquely as a linear combination of the $e_i$

$$D_\mu e_i = A^j_{\mu i} e_j \tag{20}$$

with the $A^j_{\mu i}$ being a batch of functions on $U$ [2,7,13].

[6]See, e.g., Ref. [3], Eqs. (9.29)–(9.30).

In abelian and non-abelian gauge theory for particle physics the connection coefficients $A^j_{\mu i}$ become the coefficients of the gauge fields for a Standard Model interaction, namely the gauge fields (vector potentials) $\Delta_\mu$ having $\Delta^j_{\mu i}$ in (20) as connection coefficients for the principal $G$-bundles with structure Lie groups $G = \{U(1), SU(2), SU(3)\}$ [2,7,13].

A separate and special case of (20) arises when considering the torsion-free, canonical Levi-Civita metric connection $\nabla$ on the tangent bundle $TM$ of the semi-Riemannian space-time manifold $M \equiv (\mathbb{B}, \mathbf{g})$ and associated covariant derivative $\nabla_\mu$ [2,7,13]. In such a case the connection coefficients are none other than the symmetric Christoffel connection coefficients $\Gamma^\mu_{\alpha\beta} = \Gamma^\mu_{\beta\alpha}$ given through the metric as [2,7,13]

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2} g^{\mu\omega} \left( \partial_\alpha g_{\omega\beta} - \partial_\omega g_{\beta\alpha} + \partial_\beta g_{\alpha\omega} \right). \tag{21}$$

Using the basis of coordinate vector fields $e_\mu \equiv \partial_\mu$ we can thus write (20) in this special case as [13,15]

$$\nabla_\beta e_\alpha = \Gamma^\mu_{\alpha\beta} e_\mu. \tag{22}$$

The Christoffel connection coefficients of (22) thus function analogously to the Standard Model vector potentials, but specifically concern action in the tangent space fiber $T_p\mathbb{B}$ and how the basis $\{e_\alpha\}$ changes from point to point on $(\mathbb{B}, \mathbf{g})$.

Next consider the curvature tensor $\mathbf{F}(u, v)$ for some gauge theoretic bundle, with vector fields $u$ and $v$, which defines the curvature of a connection on the bundle, and is given in component form by [2,7,13][7]

$$F^j_{\alpha\beta i} = \partial_\alpha A^j_{i\beta} - \partial_\beta A^j_{i\alpha} + A^j_{k\alpha} A^k_{i\beta} - A^j_{k\beta} A^k_{i\alpha}. \tag{23}$$

written with suppressed internal indices $i$, $j$ and $k$ as [2,7,13,15]

$$\mathbf{F}(\partial_\alpha, \partial_\beta) = F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha + [A_\alpha, A_\beta]. \tag{24}$$

For the special case of (21)-(22) $\mathbf{F}$ is the Riemann curvature tensor $R^\mu_{\lambda\beta\alpha}$ [13,21], which is given in component form as [2,7,13,14,21][8]

$$R^\mu_{\lambda\beta\alpha} = \partial_\alpha \Gamma^\mu_{\lambda\beta} - \partial_\beta \Gamma^\mu_{\lambda\alpha} + \Gamma^\mu_{\lambda\alpha}\Gamma^\nu_{\lambda\beta} - \Gamma^\mu_{\lambda\beta}\Gamma^\nu_{\lambda\alpha}. \tag{25}$$

Given these well-known, deep relations between $\Gamma^\mu_{\alpha\beta}$ and $\Delta_\mu$ we are naturally led to attempt construction of the Riemann tensor within the new formalism using the $\Gamma^\mu_{\alpha\beta}$.

### 5.2. The Christoffel-Clifford Field

We construct a Clifford field containing the $\Gamma^\mu_{\alpha\beta}$ as components. As alluded to above the $\Gamma^\mu_{\alpha\beta}$ are functions on some open set $U \subset \mathbb{B}$. Once again appealing to the the equivalence principle where $(\mathbb{M}, \boldsymbol{\eta})$ is recoverable in a sufficiently small neighborhood $N$ of any space-time point $x \in (\mathbb{B}, \boldsymbol{g})$, we have for $x \in N \subset U$ that $(\mathbb{M}, \boldsymbol{\eta}) \subset U$ and thus can work with the $\Gamma^\mu_{\alpha\beta}$ in the tangent space $T_p\mathbb{B} \equiv \mathbb{M}$.

[7] Ref. [2], p. 246.
[8] See, e.g., Ref. [7], p. 404.

Since Clifford fields may contain scalar field components such as the connection coefficients of (20) and (22), we may define the Christoffel-Clifford field $\gamma^{\beta}\Gamma^{\mu}_{\alpha\beta}$ as a local field in $T_p\mathbb{B}$ flowing over $T\mathbb{B}$ for all applicable space-time events $x$. We thus define the four component Clifford field

$$\gamma^{\beta}\Gamma^{\mu}_{\alpha\beta} \equiv \not{\Gamma} \tag{26}$$

with each $\Gamma^{\mu}_{\alpha\beta}$ being a *"batch* of functions" on its $\beta$-term.[9]

In (26) we have the Clifford $\beta$-index, which is typically a flat (Lorentz) space-time index, coupled to the Christoffel $\beta$-index, which is a curved (or world) index. Thus for this field conception of $\not{\Gamma}$ flowing over $U$ in $T\mathbb{B}$ to be well-defined we must construct a frame field for the $\gamma^{\beta}$ over $U$ [2,7]. This frame field is also known as a trivialization of the tangent bundle $TU \subset T\mathbb{B}$ of the open subset $U$ of $\mathbb{B}$,[10] and for our 4-dimensional space-time continuum is called a *tetrad* or a *vierbein* [2,7,13].

For every space-time point $x \in U$ we have the trivialization [2]

$$e : U \times \mathbb{R}^{1,3} \longrightarrow TU \tag{27}$$

which is a vector bundle isomorphism sending each fiber $\{x\} \times \mathbb{R}^{1,3}$ of the trivial bundle $U \times \mathbb{R}^{1,3}$ to the tangent space $T_xU$. This permits the mapping of the standard basis vector fields $\{\partial_\mu\}$ and dual basis $\{dx^\mu\}$ covector fields of $\mathbb{R}^{1,3}$ at any $x \in U$ to a basis of tangent vectors and dual covectors - for our purposes the Clifford-Dirac basis at some space-time event $x : \gamma^{\mu}(x) \;\; \forall x \in U$ and their dual relations $\gamma_\mu(x)$ at $x$.

We have the non-singular linear combinations [2,7]

$$\begin{aligned} \gamma_\mu(x) &= e^{\nu}_{\mu}(x)\partial_\nu, \\ \gamma^\mu(x) &= e^{\mu}_{\nu}(x)dx^\nu \end{aligned} \tag{28}$$

with $e^{\nu}_{\mu}(x)$ and $e^{\mu}_{\nu}(x)$ being smooth, non-singular matrices dependent on $x$ such that we have the following duality relations for all $x$ [2,7]:

$$\begin{aligned} e^{\mu}_{\nu}(x)e^{\nu}_{\kappa}(x) &= A^{\mu}_{\kappa} \rightarrow \eta^{\mu}_{\kappa} \\ e^{\nu}_{\mu}(x)e^{\mu}_{\lambda}(x) &= A^{\nu}_{\lambda} \rightarrow \eta^{\nu}_{\lambda}, \end{aligned} \tag{29}$$

where $A^{\mu}_{\kappa}$ and $A^{\nu}_{\lambda}$ are symmetric constant matrices at any point $x$, thus allowing reproduction of (2) in $T_xU$ for all $x \in U$, with $A^{\mu}_{\epsilon} \rightarrow \eta^{u}_{\epsilon}$ at $x$ being a general result of matrix theory.[11]

### 5.3. The Linear Portion of the Riemann Tensor

Inspection of (23)-(25) reveals what indices are to be used in constructing the Riemann tensor, namely the exchange $(\alpha\beta) \rightarrow (\beta\alpha)$ in (23)-(25). The $\not{\partial}$ action on $\not{\Gamma}$ is then identical in form to its operation on the Standard Model gauge fields as given in (8) above, with $\not{\partial}\not{\Gamma}$ generating

$$\not{\partial}\not{\Gamma} \longrightarrow \partial_\alpha \Gamma^{\mu}_{\lambda\beta} - \partial_\beta \Gamma^{\mu}_{\lambda\alpha} \tag{30}$$

which comprises the first portion of the Riemann curvature tensor.

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[9] Ref. [13], p. 224.

[10] Ref. [2], pp. 403–405.

[11] Ref. [20], Sec. 6.2, p. 155: Any symmetric matrix can be transformed into a diagonal matrix with main diagonal ±1 entries.

### 5.4. The Non-Linear Portion of the Riemann Tensor

For construction of the non-linear portion of the Riemann tensor the full space-time operator $[\gamma_\alpha, \gamma_\beta]$ of (12) must be used vice the operator $[\boldsymbol{\gamma}, \boldsymbol{\gamma}]$ of (11) which was used above for generating the non-linear field strength tensor terms for the weak interaction. Why is this?

The structural reason that $[\boldsymbol{\gamma}, \boldsymbol{\gamma}]$ is not sufficient for generating gravitational interaction terms is that $\not{\Gamma}$ is a 4-component field in the indices. This distinguishes it from the 3-component weak gauge field $W^i_\mu$ for which the 3-component operator $[\boldsymbol{\gamma}, \boldsymbol{\gamma}]$ is prescribed for generating the non-linear tensor terms. Thus the $\not{W}$-field requires only $[\boldsymbol{\gamma}, \boldsymbol{\gamma}]$ to generate the weak interaction's non-linear field strength tensor terms. In contrast $\not{\Gamma}$ with four components in the indices requires the full 4-component space-time operator $[\gamma_\alpha, \gamma_\beta]$ to generate its non-linear field strength tensor terms.[12]

Now the same procedure above for generating the non-linear portions of the Standard Model field strength tensors is used, with $[\gamma_\alpha, \gamma_\beta] = [\gamma, \gamma]$ replacing $[\boldsymbol{\gamma}, \boldsymbol{\gamma}]$. For some examples, on the $(\alpha\beta) = (12)$ and $(03)$ indices we have

$$\begin{aligned} [\gamma_1, \gamma_2](\gamma^1\Gamma^\mu_{\nu 1}, \gamma^2\Gamma^\nu_{\lambda 2}) &= \gamma_1\gamma_2\gamma^1\Gamma^\mu_{\nu 1}\gamma^2\Gamma^\nu_{\lambda 2} - \gamma_2\gamma_1\gamma^2\Gamma^\mu_{\nu 2}\gamma^1\Gamma^\nu_{\lambda 1} \\ &= -\Gamma^\mu_{\nu 1}\Gamma^\nu_{\lambda 2} + \Gamma^\mu_{\nu 2}\Gamma^\nu_{\lambda 1}, \\ [\gamma_0, \gamma_3](\gamma^0\Gamma^\mu_{\nu 0}, \gamma^3\Gamma^\nu_{\lambda 3}) &= \gamma_0\gamma_3\gamma^0\Gamma^\mu_{\nu 0}\gamma^3\Gamma^\nu_{\lambda 3} - \gamma_3\gamma_0\gamma^3\Gamma^\mu_{\nu 3}\gamma^0\Gamma^\nu_{\lambda 0} \\ &= -\Gamma^\mu_{\nu 0}\Gamma^\nu_{\lambda 3} + \Gamma^\mu_{\nu 3}\Gamma^\nu_{\lambda 0}, \end{aligned} \tag{31}$$

and in general we can write

$$\left[\gamma_\alpha, \gamma_\beta\right]\left(\gamma^\alpha\Gamma^\mu_{\nu\alpha}, \gamma^\beta\Gamma^\nu_{\lambda\beta}\right) = -\left(\Gamma^\mu_{\nu\alpha}\Gamma^\nu_{\lambda\beta} - \Gamma^\mu_{\nu\beta}\Gamma^\nu_{\lambda\alpha}\right). \tag{32}$$

Combining (30) and (32) we arrive at the intrinsic expression for the Riemann curvature tensor $\mathbf{R}$ derived from Clifford-Christoffel fields acted on by Clifford field operators [7][13]:

$$\begin{aligned} \not{\Delta}\not{\Gamma} - [\gamma, \gamma]\,(\not{\Gamma}, \not{\Gamma}) &= \mathbf{R}, \\ \mathbf{R}(\partial_\alpha, \partial_\beta)\partial_\lambda &= R^\mu_{\alpha\beta\lambda}\partial_\mu. \end{aligned} \tag{33}$$

## 6. Conclusion

This is the seminal result herein: *General Relativity's field strength tensor used for modeling the gravitational interaction—the Riemann curvature tensor—is generated using the same formalism as has been used to generate the field strength tensors for the Standard Model interactions.*

The result is promising, as it allows one to see all four interactions of nature arising from the same mathematical formalism, and thus permits visualizing them as emanating, presumably, from some underlying structure

[12] In like manner the 8-component strong interaction gauge field: $G^a_\mu \mid a : 1 \to 8$, requires an analogous 8-component operator, which has been constructed (Ref. [26], Sec. 3.3).

[13] See, e.g., Ref. [7], p. 404, for the component expression of $\mathbf{R}$ in the $\{\partial_\mu\}$ basis.

which generates the subject formalism. Indeed, such an underlying structure has recently been postulated [25–27].

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