Adv. Appl. Clifford Algebras (2023) 33:2

0188-7009/010001-12
published online November 12, 2022
https://doi.org/10.1007/s00006-022-01244-w

Advances in Applied Clifford Algebras

Pitt's Inequality and Logarithmic Uncertainty Principle for the Clifford-Fourier Transform

Shanshan Li and Minggang Fei*

Communicated by Uwe Kaehler

Abstract. In this paper, we prove the sharp Pitt's inequality for a generalized Clifford-Fourier transform which is given by a similar operator exponential as the classical Fourier transform but containing generators of Lie superalgebra. As an application, the Beckner's logarithmic uncertainty principle for the Clifford-Fourier transform is established.

Mathematics Subject Classification. 42B10, 30G30.

Keywords. Pitt's inequality, Logarithmic uncertainty principle, Clifford-Fourier transform, Clifford analysis.

1. Introduction

In harmonic analysis the uncertainty principles play an important role, which states that a function f and its Fourier transform $\mathcal{F}f$ cannot be at the same time simultaneously and sharply localized [19, 25]. One of the most important of these uncertainty principles is the well-known Beckner's logarithmic inequality [5], which is closely related to the logarithmic Sobolev inequality and which implies, in particular, to the well-known Heisenberg-Pauli-Weyl uncertainty principle [26].

W. Beckner in [5] showed that for every $f \in \mathcal{S}(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d} \ln(|x|)|f(x)|^2 dx + \int_{\mathbb{R}^d} \ln(|y|)|\mathcal{F}f(y)|^2 dy \geq \left(\psi\left(\frac{d}{4}\right) + \ln 2\right) \int_{\mathbb{R}^d} |f(x)|^2 dx, \tag{1.1}$$

The authors were partially supported by "the Fundamental Research Funds for the Central Universities", Southwest Minzu University with 2021NYB07.

*Corresponding author.

where $\mathcal{F}f$ is the classical Fourier transform of f defined through

$$\mathcal{F}f(y) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x) e^{-i\langle x,y\rangle} dx,$$

with $\langle x, y\rangle := \sum_{j=1}^d x_j y_j$, ψ is the logarithmic derivative of the Gamma function Γ, and $\mathcal{S}(\mathbb{R}^d)$ denotes the Schwartz space.

The key ingredient to prove Beckner's logarithmic inequality (1.1) is the following Pitt's inequality for the Fourier transform [5]

$$\left\| |\cdot|^{-\beta} \mathcal{F}f \right\|_2 \leq c(\beta) \left\| |\cdot|^{\beta} f \right\|_2 \tag{1.2}$$

for $f \in \mathcal{S}(\mathbb{R}^d)$, $0 \leq \beta < d/2$, with sharp constant

$$c(\beta) = 2^{-\beta} \frac{\Gamma\left(\frac{1}{2}\left(\frac{d}{2} - \beta\right)\right)}{\Gamma\left(\frac{1}{2}\left(\frac{d}{2} + \beta\right)\right)}.$$

It it noted that by Parseval's identity, Pitt's inequality can be viewed as a Hardy-Rellich inequality

$$\left\| |\cdot|^{-\beta} f \right\|_2 \leq c(\beta) \left\| |\cdot|^{\beta} \mathcal{F}f \right\|_2 = c(\beta) \left\| (-\Delta)^{\beta/2} f \right\|_2,$$

whose proofs and extensions can be found in [18,33].

The original proof of (1.2) by Beckner in [5] is based on an equivalent integral realization as a Stein-Weiss fractional integral on $\mathbb{R}^d$. In [33], Yafaev used the following decomposition of $L^2(\mathbb{R}^d)$ [32] to study inequality (1.2) on the subsets of $L^2(\mathbb{R}^d)$ which are invariant under the Fourier transform:

$$L^2(\mathbb{R}^d) = \sum_{n=0}^{\infty} \oplus \mathcal{R}_n, \tag{1.3}$$

where $\mathcal{R}_0$ denotes the space of radial functions, and $\mathcal{R}_n := \mathcal{R}_0 \otimes \mathcal{H}_n$ denotes the space of functions on $\mathbb{R}^d$ which are products of radial functions and spherical harmonics of degree n.

Following Yafaev's idea, Gorbachev et al. in [22,23] recently proved the sharp Pitt's inequalities for the Hankel transform [11,13,17], Dunkl transform [14,16] and (k, a)-generalized Fourier transform [6].

In this paper, following the idea in [22,23], and using the decomposition (1.3) of the space $L^2(\mathbb{R}^d)$, we prove the sharp Pitt's inequality for a generalized Clifford-Fourier transform which is given by a similar operator exponential as the classical Fourier transform but containing generators of Lie superalgebra.

In signal processing, multiplexing which is originated in telegraphy and now widely applied in the areas of electronic, telecommunications, digital video and computer net works motivates us to develop the function theory for multivector-valued functions $f = (f_1, \ldots, f_n)$. From a mathematical point of view, the idea of multiplexing is to encode n independent functions $f_j \in L^2(\mathbb{R}^d; \mathbb{C})$, $j = 1, \ldots, n$, as a single function f that captures the information of each component f_j([1,3,4,24]). It is well-known that Clifford algebra $\mathcal{C}l_{0,d}$, a noncommutative complex 2^d-dimensional universal algebra generated by the orhthonormal basis $\{e_1, \ldots, e_d\}$, provides an explicit way to present the multivector-valued functions. And Clifford analysis [7,15] is a refinement of harmonic analysis in $\mathbb{R}^d$, in the sense that Lie algebra $\mathfrak{sl}_2$ generated by the Laplace operator Δ and the norm squared of a vector $|x|^2$, $x \in \mathbb{R}^d$, in harmonic analysis (see, e.g. [28]) is refined to the Lie superalgebra $\mathfrak{osp}(1|2)$ (containing $\mathfrak{sl}_2$ as its even subalgebra).

Several attempts have been considered to introduce the generalizations of the classical Fourier transform to the setting of Clifford analysis (see [2,9,10,20,21,27,29] and the references therein). In this paper we consider the so-called Clifford-Fourier transform in literature first introduced in [8] and further developed in [12], because it is given by a similar operator exponential as the classical Fourier transform but now containing generators of $\mathfrak{osp}(1|2)$. More precisely, denote $\partial_y := \sum_{j=1}^d e_j \partial_{y_j}$ to be the Dirac operator in Clifford analysis. The Clifford-Fourier transform can be written as the following integral from

$$\mathcal{F}_{\pm}(f)(y) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} K_{\pm}(x,y) f(x) dx, \tag{1.4}$$

here the kernel function $K_{\pm}(x,y)$ is given by

$$K_{\pm}(x,y) = e^{\mp i \frac{\pi}{2} \Gamma_y} e^{-i\langle x,y\rangle},$$

with $\Gamma_y := (\partial_y y - y\partial_y)/2 + d/2$. In [12], the authors give the algebra background and full discussion about the kernel and the corresponding Clifford-Fourier transform. They obtain a completely explicit description of the kernel in terms of a finite sum of Bessel functions when d is even, and for the odd case they show that it is enough to identify the kernel in dimension 3, form which kernels in higher odd dimensions can be deduced by taking suitable derivatives. Moreover, they also express the kernel in dimension 3 as a single integral of a combination of Bessel functions.

Our main goal is to study the Pitt's inequality for the Clifford-Fourier transform $\mathcal{F}_- f$ of function f on Clifford-valued Schwartz space $\mathcal{S}(\mathbb{R}^d; \mathcal{C}l_{0,d})$

$$\Big\| |\cdot|^{-\beta} \mathcal{F}_- f \Big\|_2 \leq c(\beta) \Big\| |\cdot|^{\beta} f \Big\|_2 \tag{1.5}$$

with sharp constant

$$c(\beta) = 2^{-\beta} \frac{\Gamma\Big(\frac{1}{2}\Big(\frac{d}{2} - \beta\Big)\Big)}{\Gamma\Big(\frac{1}{2}\Big(\frac{d}{2} + \beta\Big)\Big)},$$

and the Beckner's logarithmic uncertainty principle

$$\int_{\mathbb{R}^d} \ln(|x|)|f(x)|^2 dx + \int_{\mathbb{R}^d} \ln(|y|)|\mathcal{F}_- f(y)|^2 dy \geq \Big(\psi\Big(\frac{d}{4}\Big) + \ln 2\Big) \int_{\mathbb{R}^d} |f(x)|^2 dx, \tag{1.6}$$

provided that

$$0 \leq \beta < d/2.$$

This paper is organized as follows. The next section is devoted to recalling some definitions and basic properties of the Clifford analysis and the Clifford-Fourier transform. In Sect. 3, based on the direct sum decomposition (1.3) whose subspaces are also invariant under the Clifford-Fourier transform, we prove the sharp Pitt's inequality (1.5) and the Beckner's logarithmic uncertainty principle (1.6) for the Clifford-Fourier transform.

2. Preliminaries

Let $\{e_1, e_2, \ldots, e_d\}$ be an orthonormal basis of $\mathbb{R}^d$ satisfying the anti-commutation relationship

$$e_j e_k + e_k e_j = -2\delta_{jk}, \tag{2.1}$$

where δ_{jk} is the Kronecker symbol. The complex universal Clifford algebra $\mathcal{C}l_{0,d}$ is defined as the 2^d-dimensional associative algebra with basis given by $e_0 = 1$ and $e_A = e_{h_1} e_{h_2} \ldots e_{h_n}$, where $A = \{h_1, h_2, \ldots, h_n\} \subset \{1, 2, \ldots, d\}$, for $1 \leq h_1 < h_2 < \cdots < h_n \leq d$. Hence, each element $x \in \mathcal{C}l_{0,d}$ will be represented by $x = \sum_A x_A e_A$, $x_A \in \mathbb{C}$, here $\mathbb{C}$ denotes the complex plane. The complex Clifford algebra $\mathcal{C}l_{0,d}$ is a complex linear, associate, but non-commutative algebra.

The typical element of $\mathbb{R}^d$ is denoted by vector $x = x_1 e_1 + x_2 e_2 + \cdots + x_d e_d$, $x_j \in \mathbb{R}$, $j = 1, 2, \ldots, d$. The inner product and the wedge product of two vectors x and y in $\mathbb{R}^d$ are defined as follows:

$$\langle x, y \rangle := \sum_{j=1}^{d} x_j y_j = -\frac{1}{2}(xy + yx),$$

$$x \wedge y := \sum_{j<k} e_j e_k (x_j y_k - x_k y_j) = \frac{1}{2}(xy - yx).$$

Here the multiplication of vectors x and y is defined using the relation (2.1) of the Clifford algebra. If we define the Clifford conjugate of any number $a \in \mathcal{C}l_{0,d}$ as

$$\bar{a} := \sum_{|A|=n} \bar{a}_A \bar{e}_A, \ \bar{e}_A = (-1)^{\frac{n(n+1)}{2}} e_A, \ a_A \in \mathbb{C},$$

where $\bar{a}_A$ denotes the conjugate of a complex number, then $\overline{ab} = \bar{b}\bar{a}$ for any $a, b \in \mathcal{C}l_{0,d}$ and $\overline{x \wedge y} = y \wedge x$ for any vectors x and y in $\mathbb{R}^d$.

The above conjugate leads to the scalar part of the product $f\bar{g}$ for $f, g \in \mathcal{C}l_{0,d}$ given by

$$[f\bar{g}]_0 := \sum_A f_A \bar{g}_A, \ f_A, g_A \in \mathbb{C}. \tag{2.2}$$

For $f = g$ in (2.2), we have the modulus $|f|$ of any $f \in \mathcal{C}l_{0,d}$ defined as

$$|f| := \sqrt{[f\bar{f}]_0} = \sqrt{\sum_A |f_A|^2} = |\bar{f}|. \tag{2.3}$$

We note that the square of a vector $x \in \mathbb{R}^d$ is scalar-valued and equals to the norm squared up to a minus sign, i.e., $x^2 = -|x|^2$. For any $a, b \in \mathcal{C}l_{0,d}$, there has $|ab| \leq 2^d|a||b|$ and $|a+b| \leq |a| + |b|$. But if $a \in \mathbb{R}^d$ and $b \in \mathcal{C}l_{0,d}$, it holds([34]):

$$|ab| = |a||b|. \tag{2.4}$$

Denote the space $L^p(\mathbb{R}^d; \mathcal{C}l_{0,d})$ as the module of all Clifford-valued functions $f: \mathbb{R}^d \to \mathcal{C}l_{0,d}$ with finite norm

$$\|f\|_p = \begin{cases} \left(\int_{\mathbb{R}^d} |f(x)|^p dx \right)^{\frac{1}{p}}, \ 1 \leq p < \infty, \\ \\ \text{ess sup}_{x\in\mathbb{R}^d} |f(x)|, \ p = \infty, \end{cases} \tag{2.5}$$

where $dx = dx_1 \ldots dx_d$ represents the usual Lebesgue measure in $\mathbb{R}^d$. In particular, the L^2-norm for $L^2(\mathbb{R}^d; \mathcal{C}l_{0,d})$ is introduced by the scalar inner product

$$\langle f, g \rangle := \int_{\mathbb{R}^d} [f(x)\overline{g(x)}]_0 dx. \tag{2.6}$$

We remark here that the definition of the scalar inner product, defined by (2.6), is reduced to the standard one in $L^2(\mathbb{R}^d; \mathbb{C})$ which is a subset of $L^2(\mathbb{R}^d; \mathcal{C}l_{0,d})$, where $f, g \in L^2(\mathbb{R}^d; \mathbb{C})$ leads to $[f(x)\overline{g(x)}]_0 = f(x)\overline{g(x)}$.

Finally, we recall some definitions and results of the Clifford-Fourier transform from [12]:

Definition 2.1. On the Schwartz class of Clifford-valued functions $\mathcal{S}(\mathbb{R}^d; \mathcal{C}l_{0,d})$, we define the Clifford-Fourier transform as

$$\mathcal{F}_{\pm} f(y) := (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} K_{\pm}(x, y) f(x) dx, \tag{2.7}$$

and their inverses as

$$\mathcal{F}_{\pm}^{-1} f(y) := (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} \widetilde{K}_{\pm}(x, y) f(x) dx, \tag{2.8}$$

where

$$K_{\pm}(x, y) := e^{\mp i \frac{\pi}{2} \Gamma_y} e^{-i\langle x,y \rangle}, \tag{2.9}$$

and

$$\widetilde{K}_{\pm}(x, y) := e^{\pm i \frac{\pi}{2} \Gamma_y} e^{i\langle x,y \rangle}, \tag{2.10}$$

are the corresponding kernel functions, $\Gamma_y := (\partial_y y - y\partial_y)/2 + d/2 = -\sum_{j<k} e_j e_k (y_j \partial_{y_k} - y_k \partial_{y_j})$ is the so-called Gamma operator([15]).

We note that the above kernel functions are not symmetric, in the sense that for example $K_-(x, y) \neq K_-(y, x)$ (see [12]). Hence, we adopt the convention that we always integrate over the first variable in the kernels. Throughout this paper, we only focus on the derivation of the kernel $K_-(x, y) = e^{i \frac{\pi}{2} \Gamma_y} e^{-i\langle x,y \rangle}$. The other ones can be derived similarly.

From [12], we have the following explicit form of the kernel by using Gegenbauer polynomials $C_k^\lambda(\omega)$ and Bessel functions $J_\alpha(t)$:

$$K_-(x, y) = A_\lambda + B_\lambda + (x \wedge y) C_\lambda, \tag{2.11}$$

with

$$A_\lambda = 2^{\lambda-1}\Gamma(\lambda+1)\sum_{k=0}^{\infty}(i^d+(-1)^k)(|x||y|)^{-\lambda}J_{k+\lambda}(|x||y|)C_k^\lambda(\langle x',y'\rangle),$$

$$B_\lambda = -2^{\lambda-1}\Gamma(\lambda)\sum_{k=0}^{\infty}(i^d-(-1)^k)(|x||y|)^{-\lambda}J_{k+\lambda}(|x||y|)C_k^\lambda(\langle x',y'\rangle),$$

$$C_\lambda = -(2\lambda)2^{\lambda-1}\Gamma(\lambda)\sum_{k=0}^{\infty}(i^d+(-1)^k)(|x||y|)^{-\lambda-1}J_{k+\lambda}(|x||y|)C_{k-1}^{\lambda+1}(\langle x',y'\rangle),$$

where $x' = x/|x|$, $y' = y/|y|$, and $\lambda = (d-2)/2$.

Note that when the dimension d is even, A_λ, B_λ and C_λ are real-valued. In particular, when $d = 2$, the Clifford-Fourier kernel is given by

$$K_-(x,y) = e^{i\frac{\pi}{2}\Gamma_y}e^{-i\langle x,y\rangle} = \cos(x_1y_2 - x_2y_1) + e_1e_2\sin(x_1y_2 - x_2y_1).$$

We note that, as $(e_1e_2)^2 = -1$, the above formula implies that, upon substituting e_1e_2 by the imaginary unit i, the kernel is equal to the kernel of classical Fourier transform. But this is clearly not the case for higher dimensions. For instance, the authors in [12] show that $K_-(x,z)K_-(y,z) \neq K_-(x+y,z)$ if the dimension d is even and $d > 2$.

Furthermore, the explicit representation (2.11) of the kernel $K_-(x,y)$ allows the authors in [12] to study the following Bochner-type identities for the Clifford-Fourier transform:

Proposition 2.2. *Let $\mathcal{M}_n := \ker \partial_x \cap \mathcal{P}_n$ denote the space of spherical monogenics of degree n, here $\mathcal{P}_n$ is the space of homogeneous polynomials of degree n. Then*

(1). for functions of type $f(x) = M_n(x')f_0(r)$ with $x = rx'$, $M_n \in \mathcal{M}_n$ and $f_0(r) \in \mathcal{S}(\mathbb{R}^d)$ being real-valued radial function, there has

$$\mathcal{F}_-(f)(y) = (-1)^n M_n(y')\rho^n H_{n+\lambda}(f_0(r)r^{-n})(\rho),\ \ y = \rho y'; \tag{2.12}$$

(2). for functions of type $f(x) = x'M_{n-1}(x')f_0(r)$ with $M_{n-1} \in \mathcal{M}_{n-1}$ and $f_0(r) \in \mathcal{S}(\mathbb{R}^d)$, there has

$$\mathcal{F}_-(f)(y) = -i^d y' M_{n-1}(y')\rho^n H_{n+\lambda}(f_0(r)r^{-n})(\rho), \tag{2.13}$$

where H_λ denotes the Hankel transform (see (3.1) in the next section).

3. Pitt's inequality and logarithmic uncertainty principle for the CFT

Before we prove the Pitt's inequality for the Clifford-Fourier transform, let us recall some known results for Hankel transform. The Hankel transform is defined through

$$H_\lambda(f)(\rho) = \int_0^\infty f(r)j_\lambda(\rho r)d\nu_\lambda(r) \tag{3.1}$$

where $j_\lambda(t) := 2^\lambda\Gamma(\lambda+1)t^{-\lambda}J_\lambda(t)$ denotes the normalized Bessel function with $\lambda \geq -1/2$, the normalized Lebesgue measure $d\nu_\lambda(r) := b_\lambda r^{2\lambda+1}dr$ with constant $b_\lambda = (2^\lambda \Gamma(\lambda+1))^{-1}$. From [22,31,33], the Pitt's inequality for the Hankel transform is given as

$$\|(\cdot)^{-\beta} H_\lambda f\|_{2,d\nu_\lambda} \leq c(\beta,\lambda)\|(\cdot)^\beta f\|_{2,d\nu_\lambda} \tag{3.2}$$

for $f \in \mathcal{S}(\mathbb{R}^d)$, $0 < \beta < \lambda + 1$ and $\lambda > -1$, with sharp constant

$$c(\beta,\lambda) = 2^{-\beta} \frac{\Gamma\Big(\frac{1}{2}\Big(\lambda + 1 - \beta\Big)\Big)}{\Gamma\Big(\frac{1}{2}\Big(\lambda + 1 + \beta\Big)\Big)}, \tag{3.3}$$

where the above weight L^2-norm is defined through

$$\|f\|_{2,d\nu_\lambda} := \Big(\int_0^\infty |f(r)|^2 d\nu_\lambda(r)\Big)^{1/2}.$$

We are now in a position to prove the Pitt's inequality (1.5) for the CFT.

Theorem 3.1. *Let $0 \leq \beta < d/2$. For any $f \in \mathcal{S}(\mathbb{R}^d; \mathcal{C}l_{0,d})$, the following Pitt's inequality*

$$\Big\| |\cdot|^{-\beta} \mathcal{F}_- f \Big\|_2 \leq c(\beta) \Big\| |\cdot|^\beta f \Big\|_2 \tag{3.4}$$

holds with the sharp constant

$$c(\beta) = 2^{-\beta} \frac{\Gamma\Big(\frac{1}{2}\Big(\frac{d}{2} - \beta\Big)\Big)}{\Gamma\Big(\frac{1}{2}\Big(\frac{d}{2} + \beta\Big)\Big)}. \tag{3.5}$$

Proof. For $\beta = 0$ we have $c(\beta) = 1$ and the Pitt's inequality (3.4) becomes the Parseval's identity (see [30]). In the rest of the proof, we assume that $0 < \beta < d/2$. From the direct decomposition (1.3), we let l_n be the dimension of $\mathcal{H}_n$, and denote by $\{Y_n^j : j = 1, \ldots, l_n\}$ the real-valued orthonormal basis $\mathcal{H}_n$. Then for $f \in \mathcal{S}(\mathbb{R}^d)$, we have

$$f(x) = \sum_{n=0}^\infty \sum_{j=1}^{l_n} f_{nj}(r) Y_n^j(x'), \;\; x = rx', \tag{3.6}$$

where

$$f_{nj}(r) = \int_{S^{d-1}} f(rx') Y_n^j(x') d\sigma(x').$$

Furthermore, there has

$$\int_{S^{d-1}} |f(rx')|^2 d\sigma(x') = \sum_{n=0}^\infty \sum_{j=1}^{l_n} |f_{nj}(r)|^2,$$

and

$$\begin{aligned}\int_{\mathbb{R}^d}|x|^{2\beta}|f(x)|^2dx &= \int_0^\infty r^{2\beta+d-1}\int_{S^{d-1}}|f(rx')|^2d\sigma(x')dr\\ &= \int_0^\infty r^{2\beta+d-1}\sum_{n=0}^{\infty}\sum_{j=1}^{l_n}|f_{nj}(r)|^2dr\\ &= \sum_{n=0}^{\infty}\sum_{j=1}^{l_n}\int_0^\infty |f_{nj}(r)|^2r^{2\beta}b_\lambda^{-1}d\nu_\lambda(r). \qquad (3.7)\end{aligned}$$

Now, due to the following Fisher decomposition:

$$\mathcal{H}_n = \mathcal{M}_n \oplus x'\mathcal{M}_{n-1}, \qquad (3.8)$$

the expression (3.6) of f and Bochner-type identities (2.12) and (2.13), there holds for $y = \rho y'$

$$\mathcal{F}_-(f)(y) = \sum_{n=0}^{\infty}\sum_{j=1}^{l_n}\left((-1)^nZ_n^j(y') + (-i^d)y'W_{n-1}^j(y')\right)\rho^nH_{n+\lambda}(f_{nj}(r)r^{-n})(\rho),$$

where $Z_n^j \in \mathcal{M}_n$ and $W_{n-1}^j \in \mathcal{M}_{n-1}$.

Thus, using spherical coordinates, the direct sum decomposition of $L^2(\mathbb{R}^d)$ (1.3) and the Fisher decomposition (3.8), we have

$$\begin{aligned}&\int_{\mathbb{R}^d}|y|^{-2\beta}|\mathcal{F}_-f(y)|^2dy\\ &= \int_0^\infty \rho^{-2\beta+(d-1)}d\rho\int_{S^{d-1}}\left|\sum_{n=1}^{n}\sum_{j=1}^{l_n}\rho^nH_{n+\lambda}(f_{nj}(r)r^{-n})(\rho)\tilde{Y}_n^j(y')\right|^2dy'\\ &\le \sum_{n=0}^{\infty}\sum_{j=1}^{l_n}\int_0^\infty |H_{n+\lambda}(f_{nj}(r)r^{-n})(\rho)|^2\rho^{-2(\beta-n)}b_\lambda^{-1}d\nu_\lambda(\rho), \qquad (3.9)\end{aligned}$$

where $\tilde{Y}_n^j(y') = (-1)^nZ_n^j(y') + (-i^d)y'W_{n-1}^j(y')$.

Furthermore, by using the Pitt's inequality (3.2) for the Hankel transform, we have

$$\begin{aligned}&\int_0^\infty |H_{n+\lambda}(f_{nj}(r)r^{-n})(\rho)|^2\rho^{-2(\beta-n)}b_\lambda^{-1}d\nu_\lambda(\rho)\\ &\quad= \int_0^\infty |H_{n+\lambda}(f_{nj}(r)r^{-n})(\rho)|^2\rho^{-2(\beta-n)}\frac{b_\lambda}{b_{n+\lambda}}d\nu_{n+\lambda}(\rho)\\ &\quad\le c^2(\beta,\lambda)\int_0^\infty |f_{nj}(r)r^{-n}|^2r^{2\beta}\frac{b_\lambda}{b_{n+\lambda}}d\nu_{n+\lambda}(r)\\ &\quad= c^2(\beta,\lambda)\int_0^\infty |f_{nj}(r)r^{-n}|^2r^{2\beta}d\nu_\lambda(r). \qquad (3.10)\end{aligned}$$

Since $\lambda = (d-2)/2$, then using (3.7), (3.9) and (3.10), we arrive at

$$\int_{\mathbb{R}^d} |y|^{-2\beta} |\mathcal{F}_- f(y)|^2 dy \le \sum_{n=0}^{\infty} \sum_{j=1}^{l_n} c^2(\beta,\lambda) \int_0^\infty |f_{nj}(r) r^{-n}|^2 r^{2\beta} d\nu_\lambda(r)$$
$$= c^2(\beta) \int_{\mathbb{R}^d} |x|^{2\beta} |f(x)|^2 dx. \tag{3.11}$$

□

Using the Pitt's inequality (3.4) we obtain the Beckner's logarithmic uncertainty principle for the CFT.

Theorem 3.2. *Suppose that* $0 \le \beta < d/2$. *Then the inequality*

$$\int_{\mathbb{R}^d} \ln(|x|) |f(x)|^2 dx + \int_{\mathbb{R}^d} \ln(|y|) |\mathcal{F}_- f(y)|^2 dy \ge \left(\psi\left(\frac{d}{4}\right) + \ln 2\right) \int_{\mathbb{R}^d} |f(x)|^2 dx, \tag{3.12}$$

holds for any $f \in \mathcal{S}(\mathbb{R}^d)$, *here* $\psi(t) = \Gamma'(t)/\Gamma(t)$ *being the psi function.*

Proof. To simplify our proof, we rewrite the Pitt's inequality (1.5) in the following form

$$\int_{\mathbb{R}^d} |y|^{-\beta} |\mathcal{F}_- f(y)|^2 dy \le c^2(\beta/2) \int_{\mathbb{R}^d} |x|^{\beta} |f(x)|^2 dx,$$

here $0 \le \beta < d$. Now for $\beta \in (-d, d)$, we define the function

$$\varphi(\beta) := \int_{\mathbb{R}^d} |y|^{-\beta} |\mathcal{F}_- f(y)|^2 dy - c^2(\beta/2) \int_{\mathbb{R}^d} |x|^{\beta} |f(x)|^2 dx.$$

The Pitt's inequality (1.5) and Parseval's identity for the CFT imply that $\varphi(\beta) \le 0$ for $\beta > 0$ and $\varphi(0) = 0$, respectively. Hence,

$$\varphi'_+(0) = \lim_{\beta \to 0^+} \frac{\varphi(\beta) - \varphi(0)}{\beta} \le 0. \tag{3.13}$$

Since $f, \mathcal{F}_- f \in \mathcal{S}(\mathbb{R}^d)$, then for any $|\beta| < d$,

$$\int_{|x|>1} |x|^{\beta} \ln(|x|) |f(x)|^2 dx$$

and

$$\int_{|y|>1} |y|^{\beta} \ln(|y|) |\mathcal{F}_- f(y)|^2 dy$$

are well-defined. Furthermore, by spherical coordinates,

$$\int_{|x|\le 1} |x|^{\beta} |\ln(|x|)| dx = \int_0^1 r^{\beta+d-1} |\ln r| dr \int_{S^{d-1}} dx' < \infty,$$

which gives

$$|x|^{\beta} \ln(|x|) |f(x)|^2 \in L^1(\mathbb{R}^d) \qquad \text{and} \qquad |y|^{\beta} \ln(|y|) |\mathcal{F}_- f(y)|^2 \in L^1(\mathbb{R}^d).$$

Thus,

$$\varphi'(\beta) = -\int_{\mathbb{R}^d} |y|^{-\beta} \ln(|y|)|\mathcal{F}_- f(y)|^2 dy$$
$$-c^2(\beta/2)\int_{\mathbb{R}^d} |x|^{\beta} \ln(|x|)|f(x)|^2 dx$$
$$-\frac{dc^2(\beta/2)}{d\beta}\int_{\mathbb{R}^d} |x|^{\beta}|f(x)|^2 dx. \tag{3.14}$$

In addition, from (3.5) we have

$$-\frac{dc^2(\beta/2)}{d\beta} = \psi\left(\frac{d}{4}\right) + \ln 2. \tag{3.15}$$

Combining (3.14), (3.14) and (3.15), we conclude the proof of (3.12). □

Data Availibility Statement No datasets were generated or analysed during the current study.

Declarations

Conflict of interest This work does not have any conflicts of interest.

References

[1] Abreu, L.D.: Sampling and interpolation in Bargmann–Fock spaces of polyanalytic functions. Appl. Comput. Harmon. Anal. **29**, 287–302 (2010)

[2] Bahri, M., Hitzer, E.: Clifford Fourier transformation and uncertainty principle for the Clifford geometric algebra $Cl_{3,0}$. Adv. Appl. Clifford Algebra **16**(1), 41–61 (2006)

[3] Balan, R.: Density and redundancy of the noncoherent Weyl-Heisenberg superframes, The functional and harmonic analysis of wavelets and frames (San Antonio, TX, 1999), vol. 247 of Contemp. Math. 29–41 (1999)

[4] Balan, R.: Multiplexing of signals using superframes. SPIE Wavelets Applications, vol. 4119 of Signal and Image Processing VIII, 118–129 (2000)

[5] Beckner, W.: Pitt's inequality and the uncertainty principle. Proc. Am. Math. Soc. **123**, 1897–1905 (1995)

[6] Ben Saïd, S., Kobayashi, T., Ørsted, B.: Laguerre semigroup and Dunkl operators. Compos. Math. **148**(4), 1265–1336 (2012)

[7] Brackx, F, Delanghe, R., Sommen, F.: Clifford Analysis, Research Notes in Mathematics 76. Boston, MA: Pitman (Advanced Publishing Program) (1982)

[8] Brackx, F., De Schepper, N., Sommen, F.: The Clifford–Fourier transform. J. Fourier Anal. Appl. **11**(6), 669–681 (2005)

[9] Brackx, F., De Schepper, N., Sommen, F.: The two-dimensional Clifford-Fourier transform. J. Math. Imaging Vision **26**(1–2), 5–18 (2006)

[10] Brackx, F., De Schepper, N., Sommen, F.: The Fourier transform in Clifford analysis. Adv. Imaging Electron Phys. **156**, 55–201 (2009)

[11] Colzani, L., Crespi, A., Travaglini, L., Vignati, M.: Equiconvergence theorems for Fourier-Bessel expansions with applications to the harmonic analysis of radial functions in euclidean and noneuclidean spaces. Trans. Am. Math. Soc. **338**(1), 43–55 (1975)

[12] De Bie, H., Xu, Y.: On the Clifford-Fourier transform. Int. Math. Res. Not. IMRN **22**, 5123–5163 (2011)

[13] De Carli, L.: On the $L^p - L^q$ norm of the Hankel transform and related operators. J. Math. Anal. Appl. **348**(1), 366–382 (2008)

[14] de Jeu, M.: The Dunkl transform. Invent. Math. **113**, 147–162 (1993)

[15] Delanghe, R., Sommen, F., Soucek, V.: Clifford Algebra and Spinor Valued Functions. A Function Theory for Dirac Operator. Kluwer, Dordrecht (1992)

[16] Dunkl, C.F., Xu, Y.: Orthogonal Polynomials of Several Variables. Cambridge University Press, Cambridge (2001)

[17] Erdélyi, A., Magnus, W., Oberhettinger, F., Tricomi, F.G.: Tables of Integral Transforms, vol. 2. McGraw-Hill Book Company, New York (1954)

[18] Eilertsen, S.: On weighted fractional integral inequalities. J. Funct. Anal. **185**(1), 342–366 (2001)

[19] Folland, G.B., Sitaram, A.: The uncertainty principle: a mathematical survey. J. Fourier Anal. Appl. **3**, 207–238 (1997)

[20] Fu, Y.X., Li, L.Q.: Real Paley-Wiener theorems for the Clifford Fourier transform. Sci. Chin. Math. **57**(11), 2381–2392 (2014)

[21] Fu, Y.X., Li, L.Q.: Uncertainty principle for multivector-valued functions. Int. J. Wavelets Multiresolut. Inf. Process **13**(1), 1550005 (2015)

[22] Gorbachev, D.V., Ivanov, V.I., Tikhonov, SYu.: Sharp Pitt inequality and logarithmic uncertainty principle for Dunkl transform in L^2. J. Approx. Theor. **202**, 109–118 (2016)

[23] Gorbachev, D.V., Ivanov, V.I., Yu, S., Tikhonov, A.N.: Pitt's inequalities and uncertainty principle for generalized Fourier transform. Int. Math. Res. Not. IMRN **23**, 7179–7200 (2016)

[24] Gröchenig, K., Lyubarskii, Y.: Gabor (super)frames with Hermite functions. Math. Ann. **345**(2), 267–286 (2009)

[25] Havin, V., Jöricke, B.: The Uncertainty Principle in Harmonic Analysis. Springer, Berlin (1994)

[26] Heisenberg, W.: Über den anschaulichen Inhalt der quantentheoretischen Kimeatik und Mechanik. Z. Physik **43**, 172–198 (1927)

[27] Hitzer, E., Bahri, M.: Clifford Fourier transform on multivector fields and uncertainty principles for dimensions $n = 2(mod 4)$ and $n = 3(mod 4)$. Adv. Appl. Clifford Algebra **18**, 715–736 (2008)

[28] Howe, R., Tan, E.-C.: Nonabelian Harmonic Analysis. Universitext, Springer, New York (1992)

[29] Kou, K.I., Qian, T.: The Paley-Wiener theorem in $\mathbb{R}^n$ with the Clifford analysis setting. J. Funct. Anal. **189**, 227–241 (2002)

[30] Li, S., Leng, J., Fei, M.: Paley-Wiener-type theorems for the Clifford-Fourier transform. Math. Methods Appl. Sci. **42**(18), 6101–6113 (2019)

[31] Omri, S.: Logarithmic uncertainty principle for the Hankel transform. Integral Transforms Spec. Funct. **22**(9), 655–670 (2011)

[32] Stein, E.M., Weiss, G.: Introduction to Fourier Analysis on Euclidean Spaces. Princeton University Press, Princeton (1971)

[33] Yafaev, D.: Sharp constants in the Hardy-Rellich inequalities. J. Funct. Anal. **168**(1), 121–144 (1999)

[34] Yang, Y., Qian, T.: Schwarz lemma in Euclidean spaces. Complex Var. Elliptic Eq. **51**(7), 653–659 (2006)

Shanshan Li
School of Mathematics
Southwest Minzu University
Chengdu 610041
People's Republic of China

Minggang Fei
School of Mathematical Sciences
University of Electronic Science and Technology of China
Chengdu 611731
People's Republic of China
e-mail: fei@uestc.edu.cn

Received: December 28, 2021.

Accepted: September 13, 2022.