Adv. Appl. Clifford Algebras (2021) 31:43

0188-7009/030001-12
published online June 1, 2021
https://doi.org/10.1007/s00006-021-01149-0

Advances in Applied Clifford Algebras

Generalizing Classical Clifford Algebras, Graded Clifford Algebras and their Associated Geometry

Michaela Vancliff*

Abstract. This article is based on a talk given by the author at the *12th International Conference on Clifford Algebras and their Applications in Mathematical Physics*. A generalization, introduced by Cassidy and the author, of a classical Clifford algebra is discussed together with connections between that generalization and a generalization of a graded Clifford algebra. A geometric approach to studying the algebras, viewed through the lens of Artin, Tate and Van den Bergh's noncommutative algebraic geometry, is also presented.

Mathematics Subject Classification. Primary 15A66; Secondary 16W55, 16W50, 14A22.

Keywords. Base point, Quadratic form, Quadric, Regular algebra, Skew Clifford algebra .

1. Introduction

This article is based on a talk given by the author at the *12th International Conference on Clifford Algebras and their Applications in Mathematical Physics*. That talk's focus was a generalization of a classical Clifford algebra and a generalization of a graded Clifford algebra that were introduced by Cassidy and the author in [3,4]. In particular, the talk describes behavior of the algebras determined by certain geometric data, viewed through the lens of Artin et al. noncommutative algebraic geometry [1,2].

Some results of the author herein are based upon work that was supported in part by grants DMS-0900239 and DMS-1302050 from the National Science Foundation.

This article is part of the Topical Collection on Proceedings ICCA 12, Hefei, 2020, edited by Guangbin Ren, Uwe Kähler, Rafał Abłamowicz, Fabrizio Colombo, Pierre Dechant, Jacques Helmstetter, G. Stacey Staples, Wei Wang.

*Corresponding author.

In the second section of the article, the focus is graded skew Clifford algebras as presented in [3]. To that end, the setting of graded Clifford algebras is recalled, including a result (Theorem 2.4) that uses the existence of base points of a certain quadric system to determine algebraic properties of the algebra. In order to develop a "quantized" analogue of a graded Clifford algebra, a generalization of the notion of symmetric matrix is given in Definition 2.6. The notion of graded skew Clifford algebra is provided in Definition 2.8. Section 2 also provides many examples, some of which are drawn from the area of quantum groups.

Section 3 is concerned with describing a quantized version of the geometry used in the theory of graded Clifford algebras. To this end, quadratic forms, quadric systems and base points of a quadric system are generalized to the noncommutative setting in the third section. This new terminology allows Theorem 2.4 to be modified for the context of graded skew Clifford algebras in Theorem 3.6. This result (Theorem 3.6) uses the existence of base points of a certain quadric system (but, now, in the setting of noncommutative algebraic geometry) to determine algebraic properties of the algebra, like its commutative counterpart, Theorem 2.4. The results are applied to the examples from Sect. 2, and, additionally, an example with Weyl algebras is presented.

The fourth, and final, section of the article considers a "quantized" version of the map that sends a graded Clifford algebra onto a classical Clifford algebra. A "quantized" analogue, called a skew Clifford algebra, of a classical Clifford algebra was introduced in [4], and is provided in Definition 4.2. This definition is followed by some examples of skew Clifford algebras of various finite dimensions. Theorem 4.9 shows that a skew Clifford algebra is the quotient of a graded skew Clifford algebra that is generated by degree-1 elements. It also shows that, under certain conditions, the graded skew Clifford algebra can be taken to be quadratic with many of the properties that are satisfied by polynomial rings.

Throughout this article, unless otherwise stated, $\Bbbk$ denotes a field where $\operatorname{char}(\Bbbk) \neq 2$, and $M(n, \Bbbk)$ denotes the ring of $n \times n$ matrices with entries in $\Bbbk$. Whenever any geometry is discussed, the field $\Bbbk$ is assumed to be algebraically closed.

2. Graded Skew Clifford Algebras

In this section, we recall the notion of a graded skew Clifford algebra which was introduced in [3] and which generalizes the notion of a graded Clifford algebra.

We first recall the notion of a graded Clifford algebra, as presented in [6].

Definition 2.1. [6] Let $M_1, \ldots, M_n \in M(n, \Bbbk)$ denote symmetric matrices. The graded Clifford algebra $C(M_1, \ldots, M_n)$, associated to $M_1, \ldots, M_n$, is defined to be the associative $\mathbb{Z}$-graded $\Bbbk$-algebra on degree-1 generators

$x_1, \ldots, x_n$ and on degree-2 generators $y_1, \ldots, y_n$ with defining relations given by:

(a) $x_i x_j + x_j x_i = \sum_{k=1}^{n} (M_k)_{ij}\, y_k$ for all $i, j = 1, \ldots, n$, and

(b) the requirement that the subalgebra generated by $y_1, \ldots, y_n$ is a polynomial ring contained in the center of $C(M_1, \ldots, M_n)$.

Example 2.2. Suppose $n = 2$ and let $M_1 = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$ and $M_2 = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$. With these data, the degree-2 defining relations of the associated graded Clifford algebra C are:

$$2x_1^2 = 2y_1, \quad 2x_2^2 = 2y_2, \quad x_1x_2 + x_2x_1 = 1y_1 + 0y_2 \ (= x_1^2),$$

from which it follows that the subalgebra generated by y_1 and y_2 is a polynomial ring contained in the center of C. Hence,

$$C = \Bbbk\langle x_1,\, x_2\rangle / \langle x_1x_2 + x_2x_1 - x_1^2\rangle,$$

where $\Bbbk\langle x_1,\, x_2\rangle$ denotes the free algebra generated by x_1 and x_2.

Since the degree-2 elements $y_1, \ldots, y_n$ are central of positive degree and since the algebra $C(M_1, \ldots, M_n)/\langle y_1, \ldots, y_n\rangle$ is noetherian, [1, Lemma 8.2] implies that graded Clifford algebras are noetherian. One can associate geometry to graded Clifford algebras $C(M_1, \ldots, M_n)$ via the symmetric matrices $M_1, \ldots, M_n$, which is illustrated as follows using the data in our previous example.

Example 2.3. To the matrices M_1 and M_2 in Example 2.2, we associate the quadratic forms $2(t_1^2 + t_1t_2)$ and $2t_2^2$, respectively. The points on which both quadratic forms vanish in $\mathbb{P}^1$ are called the base points of the quadric system given by the two quadratic forms, and, since there are no such points in $\mathbb{P}^1$, we say that the quadric system associated to $2(t_1^2 + t_1t_2)$ and $2t_2^2$ is base-point free. This fact is of note since the next result implies that this geometric observation is sufficient to conclude that $C = \Bbbk\langle x_1,\, x_2\rangle / \langle x_1x_2 + x_2x_1 - x_1^2\rangle$ without having to check that y_1 and y_2 generate a polynomial ring in $\Bbbk\langle x_1,\, x_2\rangle / \langle x_1x_2 + x_2x_1 - x_1^2\rangle$.

In fact, the next result implies that the property of being base-point free is intimately tied to the graded Clifford algebra satisfying many properties shared by polynomial rings, including some homological properties.

Theorem 2.4. [6] *The graded Clifford algebra $C(M_1, \ldots, M_n)$ is quadratic, Auslander-regular of global dimension n and satisfies the Cohen–Macaulay property with Hilbert series equal to that of the polynomial ring on n variables if and only if the quadric system in $\mathbb{P}^{n-1}$ associated to $M_1, \ldots, M_n$ is base-point free; in this case, $C(M_1, \ldots, M_n)$ is Artin-Schelter regular and a noetherian domain.*

Example 2.5. Suppose $n = 2$ and let $M_1 = \begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix}$ and $M_2 = \begin{bmatrix} 0 & -1 \\ -1 & 2 \end{bmatrix}$. The graded Clifford algebra C associated to these data has defining relations:

$$2x_1^2 = 2y_1, \quad 2x_2^2 = 2y_2, \quad x_1x_2 + x_2x_1 + x_1^2 + x_2^2 = 0, \quad x_1^2x_2 = x_2x_1^2,$$

since the ambiguities given by Bergman's Diamond Lemma from these relations are resolvable (so the subalgebra generated by x_1^2 and x_2^2 is a polynomial ring). The quadric system associated to M_1 and M_2 is given by $t_1^2 - t_1t_2$ and $t_2^2 - t_1t_2$, yielding the base point $(1, 1) \in \mathbb{P}^1$ on which both quadratic forms vanish. This example illustrates that graded Clifford algebras need not be quadratic, and, clearly, C is not a domain as $(x_1 + x_2)^2 = 0$.

The objective in [3] was to find a larger class of algebras to which Theorem 2.4 (or an analogue thereof) applies. That work led to producing a class of algebras in [3, Section 5.1] that solved a certain open problem in the area of noncommutative algebraic geometry. In order to "skew" the algebras and Theorem 2.4, the notions of symmetric matrix, graded Clifford algebra, quadratic form, quadric system and base-point were generalized. We next address the first of those constructions.

Definition 2.6. [3] For this definition, we temporarily allow $\Bbbk$ to denote an arbitrary field. Let $\mu = (\mu_{ij}) \in M(n, \Bbbk)$ be a matrix with the property that $\mu_{ij}\mu_{ji} = 1$ for all i, j such that $i \neq j$. A matrix $M = (M_{ij}) \in M(n, \Bbbk)$ is called μ-symmetric if $M_{ij} = \mu_{ij}M_{ji}$ for all $i, j = 1, \ldots, n$.

Clearly, if $\mu_{ij} = 1$ for all i, j, then any μ-symmetric matrix is symmetric. On the other hand, if $\mathrm{char}(\Bbbk) \neq 2$ and $\mu_{ij} = -1$ for all i, j, then any μ-symmetric matrix is skew symmetric. Hence, the notion of μ-symmetry generalizes the notions of symmetry and skew symmetry.

Example 2.7. If $n = 3$, and if $\mu_{kk} = 1$ for all k, then the matrix

$$\begin{bmatrix} a & b & c \\ \mu_{21}b & d & e \\ \mu_{31}c & \mu_{32}e & f \end{bmatrix} \in M(3, \Bbbk)$$

is μ-symmetric.

For the next definition, the reader should note that a normalizing sequence of homogeneous elements in a $\mathbb{Z}$-graded $\Bbbk$-algebra A consists of homogeneous elements $f_1, \ldots, f_m \in A \setminus \Bbbk$ such that f_1 is normal in A (that is, $Af_1 = f_1A$) and, for each $k \in \{2, \ldots, m\}$, the image of f_k is nonzero and normal in $A/\langle f_1, \ldots, f_{k-1}\rangle$.

Definition 2.8. [3] Recall that $\mathrm{char}(\Bbbk) \neq 2$. Let $\mu = (\mu_{ij}) \in M(n, \Bbbk)$ satisfy $\mu_{kk} = 1 = \mu_{ij}\mu_{ji}$ for all i, j, k, and suppose $M_1, \ldots, M_n \in M(n, \Bbbk)$ are μ-symmetric matrices. A graded skew Clifford algebra $A(\mu, M_1, \ldots, M_n)$, associated to $M_1, \ldots, M_n$ and μ, is an associative $\mathbb{Z}$-graded $\Bbbk$-algebra on degree-1 generators $x_1, \ldots, x_n$ and on degree-2 generators $y_1, \ldots, y_n$ with defining relations given by:

(a) $x_i x_j + \mu_{ij} x_j x_i = \sum_{k=1}^{n} (M_k)_{ij} y_k$ for all $i, j = 1, \ldots, n$, and

(b) the existence of a normalizing sequence $\{y'_1, \ldots, y'_n\}$ consisting of homogeneous degree-2 elements of $A(\mu, M_1, \ldots, M_n)$ that span $\Bbbk y_1 + \cdots + \Bbbk y_n$.

Clearly, graded Clifford algebras are graded skew Clifford algebras.

Example 2.9. Let $n = 2$, $M_1 = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ and $M_2 = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$. Both M_1 and M_2 are μ-symmetric matrices for any matrix μ that satisfies the properties given in Definition 2.8. The degree-2 defining relations of a graded skew Clifford algebra associated to these data are:

$$2x_1^2 = 2y_1, \quad 2x_2^2 = 2y_2, \quad x_1 x_2 + \mu_{12} x_2 x_1 = 0,$$

from which it follows that $\{y_1, y_2\}$ is a normalizing sequence. Hence, up to isomorphism, there is only one graded skew Clifford algebra associated to these data, and it is the quadratic $\Bbbk$-algebra on generators x_1 and x_2 with one defining relation, namely $x_1 x_2 + \mu_{12} x_2 x_1 = 0$. It follows that this graded skew Clifford algebra is the coordinate ring of the quantum affine plane [5].

Example 2.10. Generalizing the previous example, any skew polynomial ring with generators $x_1, \ldots, x_n$, with defining relations $x_i x_j + \mu_{ij} x_j x_i = 0$ (where $\mu_{ij}\mu_{ji} = 1$), for all $i \neq j$, is a graded skew Clifford algebra. In particular, any polynomial ring on a finite number of generators is a graded skew Clifford algebra.

Example 2.11. Let $n = 2$, $\mu = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$, $M_1 = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$ and $M_2 = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$. The degree-2 defining relations of a graded skew Clifford algebra associated to these data are:

$$2x_1^2 = 2y_1, \quad 2x_2^2 = 2y_2, \quad x_1 x_2 - x_2 x_1 = y_1 = x_1^2.$$

Since x_1^2 is normal in the algebra $A = \Bbbk\langle x_1,\, x_2\rangle / \langle x_1 x_2 - x_2 x_1 - x_1^2\rangle$ and since the image of x_2^2 is normal in the algebra $A/\langle x_1^2\rangle$, it follows that $\{x_1^2,\, x_2^2\}$ is a normalizing sequence in A. Hence, up to isomorphism, there is only one graded skew Clifford algebra associated to μ, M_1 and M_2, and it is the quadratic algebra A. The algebra A is often called the Jordan plane.

It is perhaps worthwhile to make some observations before we continue.

Remarks 2.12. 1. For any graded skew Clifford algebra on $2n$ generators as in Definition 2.8, there are $n + \binom{n}{2}$ defining relations that are homogeneous of degree two, since

$$\begin{aligned} x_j x_i + \mu_{ji} x_i x_j &= \sum_{k=1}^{n} (M_k)_{ji}\, y_k = \sum_{k=1}^{n} \mu_{ji} (M_k)_{ij}\, y_k \\ &= \mu_{ji}(x_i x_j + \mu_{ij} x_j x_i), \end{aligned}$$

for all $i, j = 1, \ldots, n$. Moreover, if the normalizing sequence that spans $\Bbbk y_1 + \cdots + \Bbbk y_n$ is not determined by the degree-2 defining relations, then the algebra has degree-3 defining relations and possibly also degree-4 defining relations. Thus, in spite of the above examples, a graded skew Clifford algebra is not, in general, a quadratic algebra, and this situation mirrors that of graded Clifford algebras (cf. Example 2.5).

2. By [3, Lemma 1.13], a graded skew Clifford algebra is generated by degree-1 elements if and only if the matrices $M_1, \ldots, M_n$ are linearly independent.
3. Since the elements $y'_1, \ldots, y'_n$ in Definition 2.8 are normalizing of positive degree in the graded skew Clifford algebra A, and since $A/\langle y'_1, \ldots, y'_n\rangle = A/\langle y_1, \ldots, y_n\rangle$ is a noetherian algebra, [1, Lemma 8.2] implies that graded skew Clifford algebras are noetherian.
4. Given $\mu, M_1, \ldots, M_n$ as in Definition 2.8, the definition of graded skew Clifford algebra does not, in general, determine the graded skew Clifford algebra uniquely, even up to isomorphism; there could, conceivably, be different ways of obtaining a normalizing sequence satisfying (b) of Definition 2.8. However, if a graded skew Clifford algebra is quadratic, then the data $\mu, M_1, \ldots, M_n$ determine the algebra up to isomorphism.

3. Associating Geometry to Graded Skew Clifford Algebras

In this section, our goal is the main theorem of [3] which is a generalization of Theorem 2.4. To this end, we associate geometric data to a μ-symmetric matrix and describe analogues of the notions of quadratic form, quadric system and base point in the noncommutative setting.

Definition 3.1. [3] To a μ-symmetric matrix $M \in M(n, \Bbbk)$, as given in Definition 2.6, we associate

(a) the skew polynomial ring S on generators $z_1, \ldots, z_n$ with defining relations $z_j z_i = \mu_{ij} z_i z_j$ for all $i \neq j$ (where we write S_d for the span of the homogeneous elements of S of degree d), and

(b) the element $[z_1 \ \cdots \ z_n] M \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} \in S_2$.

We call the nonzero elements of S_2 quadratic forms.

Example 3.2. To the matrices in Example 2.11, we associate the quadratic forms

$$[z_1 \ z_2] M_1 \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = 2z_1^2 + z_1 z_2 - z_2 z_1 = 2z_1^2 + 2z_1 z_2 \in S_2, \tag{3.1}$$

$$[z_1 \ z_2] M_2 \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = 2z_2^2 \in S_2, \tag{3.2}$$

using $\mu_{12} = -1$ in (3.1).

One can associate geometry to quadratic forms and to defining relations of S by applying them to elements of $\mathbb{P}(S_1{}^*) \times \mathbb{P}(S_1{}^*)$. In particular,

$$(z_j z_i - \mu_{ij} z_i z_j)(\ (a_1, \ldots, a_n),\ (b_1, \ldots, b_n)\ ) = a_j b_i - \mu_{ij} a_i b_j \in \{0,1\} \subset \Bbbk,$$

and the element $z_i z_j + z_k^2 \in S_2$ would be evaluated as

$$(z_i z_j + z_k^2)(\ (a_1, \ldots, a_n),\ (b_1, \ldots, b_n)\ ) = a_i b_j + a_k b_k \in \{0,1\} \subset \Bbbk.$$

Definition 3.3. [3] Define the quadric, $\mathcal{Z}(q)$, determined by any quadratic form q to be the set of points in $\mathbb{P}(S_1{}^*) \times \mathbb{P}(S_1{}^*)$ on which q and all the defining relations of S vanish.

In the commutative setting, that is $\mu_{ij} = 1$ for all i, j, we have that the algebra S is the polynomial ring on n generators. In this case, the zero locus in $\mathbb{P}(S_1{}^*) \times \mathbb{P}(S_1{}^*)$ of the defining relations of S is the graph of the identity map on $\mathbb{P}(S_1{}^*)$. This implies that evaluation of quadratic forms on points of $\mathbb{P}(S_1{}^*) \times \mathbb{P}(S_1{}^*)$ reduces to the traditional evaluation from the commutative case. For instance, evaluation of $z_i z_j + z_k^2$ on a point $(\,(a_1, \ldots, a_n),\ (b_1, \ldots, b_n)\,)$ in the zero locus of the defining relations of S yields, in this case, $a_i a_j + a_k^2 \in \{0,1\}$. Moreover, in this case, the base points of a quadric system Q in $\mathbb{P}^{n-1} = \mathbb{P}(S_1{}^*)$ are parametrized by graded modules $N = \bigoplus_{i=0}^{\infty} N_i$ over $S/\langle q : \mathcal{Z}(q) \in Q\rangle$ that are cyclic (generated by N_0) and satisfy $\dim_{\Bbbk}(N_i) = 1$ for all i. Hence, Q is base-point free if and only if there are no such graded modules. We will use certain modules analogous to these graded modules to extend the notion of base point to the noncommutative setting.

Definition 3.4. [3] Let μ and S be as in Definition 3.1.

(a) If $q_1, \ldots, q_m \in S_2 \setminus \{0\}$, we call their span a quadric system.

(b) A quadric system Q is said to be normalizing if it is given by a normalizing sequence of S; that is, Q is spanned by elements $q_1, \ldots, q_m \in S_2 \setminus \{0\}$ such that q_1 is normal in S, and, for each $k \in \{2, \ldots, m\}$, the image of q_k is nonzero and normal in $S/\langle q_1, \ldots, q_{k-1}\rangle$.

(c) A right base point of a quadric system Q is a graded right $\frac{S}{\langle Q\rangle}$-module $N = \bigoplus_{i=0}^{\infty} N_i$ that satisfies
  - (i) $N = N_0 \frac{S}{\langle Q\rangle}$, and
  - (ii) there exists $c \in \mathbb{N} \setminus \{0\}$ such that $\dim_{\Bbbk}(N_i) = c$ for all i, and
  - (iii) $\dim_{\Bbbk}(N/N') < \infty$ for all nonzero (graded) submodules N' of N.

  A left base point of a quadric system is defined analogously.

(d) We say a quadric system is right (respectively, left) base-point free if it has no right (respectively, left) base points.

It should be noted that, by [3, Proposition 10], a normalizing quadric system is right base-point free if and only if it is left base-point free. Moreover, in condition (c)(ii) of Definition 3.4, if $c = 1$ for any right base point, then condition (c)(iii) is redundant for that base point.

Assuming $\Bbbk$ is algebraically closed, if S is commutative, then any right base point corresponds to a base point in the traditional sense, which can be seen as follows. The definition of N implies that there exists $i \in \{1, \ldots, n\}$ such that $z_i \in S_1$ does not annihilate N. Let R denote the subring of degree-0 elements of the localization of S formed by inverting the powers of z_i. The definition of N guarantees the existence of a c-dimensional simple R-module, but the Nullstellensatz implies that $c = 1$. The statement now follows from the previous paragraph and the comments immediately preceding Definition 3.4.

Definition 3.5. (a) [3] If a normalizing quadric system Q is right base-point free, then we say it is base-point free.

(b) [1] If $c = 1$ in Definition 3.4(c), then the module N is called a point module.

With this new terminology, we can now say that if Q is a normalizing quadric system, then the isomorphism classes of point modules over $S/\langle Q\rangle$ are parametrized by the points of $\bigcap_{q\in Q} \mathcal{Z}(q)$. This mirrors precisely the situation in the commutative case. We are now ready to state the promised generalization of Theorem 2.4.

Theorem 3.6. [3] *Let μ and $M_1, \ldots, M_n \in M(n, \Bbbk)$ be as in Definition 2.8. A graded skew Clifford algebra $A(\mu, M_1, \ldots, M_n)$ is quadratic, Auslander regular of global dimension n and satisfies the Cohen-Macaulay property with Hilbert series equal to that of the polynomial ring on n variables if and only if the quadric system associated to $M_1, \ldots, M_n$ is normalizing and base-point free; in this case, $A(\mu, M_1, \ldots, M_n)$ is Artin-Schelter regular and a noetherian domain.*

Example 3.7. Referring to Example 2.9, the quadric system Q associated to the matrices M_1 and M_2 in that case is $\Bbbk z_1^2 \oplus \Bbbk z_2^2$, and $S = \Bbbk\langle z_1,\, z_2\rangle/\langle z_2z_1 - \mu_{12}z_1z_2\rangle$. Since each of z_1^2 and z_2^2 is normal in S, the quadric system Q is normalizing in this case. Suppose $N = \bigoplus_{i=0}^{\infty} N_i = N_0S/\langle Q\rangle$ is a right base point. Since $\dim_{\Bbbk}(N) = \infty$ and $\dim_{\Bbbk}(N_0) < \infty$, it follows that $\dim_{\Bbbk}(S/\langle Q\rangle) = \infty$. However,

$$\frac{S}{\langle Q\rangle} = \frac{\Bbbk\langle z_1,\, z_2\rangle}{\langle z_2z_1 - \mu_{12}z_1z_2,\; z_1^2,\; z_2^2\rangle},$$

which has dimension $4 \neq \infty$, so such a module N does not exist. Hence, Q is normalizing and base-point free, and, by Theorem 3.6, the graded skew Clifford algebra associated to these data is quadratic, Auslander regular etc.

Example 3.8. Referring to Example 2.11, the quadric system Q associated to the matrices M_1 and M_2 in that case is $\Bbbk(z_1^2 + z_1z_2) \oplus \Bbbk z_2^2$, and $S = \Bbbk\langle z_1,\, z_2\rangle/\langle z_2z_1 + z_1z_2\rangle$. In this case, $\{z_2^2,\; z_1^2 + z_1z_2\}$ is a normalizing sequence in S since z_2^2 is normal in S and

$$\begin{aligned}(z_1^2 + z_1z_2)z_1 &= (z_1 - 2z_2)(z_1^2 + z_1z_2),\\ z_1(z_1^2 + z_1z_2) &= (z_1^2 + z_1z_2)(z_1 + 2z_2),\\ (z_1^2 + z_1z_2)z_2 &= z_2(z_1^2 + z_1z_2)\end{aligned}$$

in $S/\langle z_2^2\rangle$. Hence, Q is a normalizing quadric system. Since $S/\langle Q\rangle$ has dimension four, an argument similar to that used in the preceding example implies that Q has no base points. Thus, Q is normalizing and base-point free, and, by Theorem 3.6, the graded skew Clifford algebra associated to these data is quadratic, Auslander regular etc.

Example 3.9. The author is grateful to Michael Reed, of Crucial Flow Research, for asking a question during the talk on which this article is based as to whether or not Weyl algebras are related to graded skew Clifford algebras. In this example, we will show that the Nth Weyl algebra is a quotient of a quadratic graded skew Clifford algebra on $2N+1$ generators that satisfies the properties stated in Theorem 3.6. Let N be a positive integer and write $n = 2N+1$. Let $\mu \in M(n, \Bbbk)$, where $\mu_{kk} = 1$ for all k and $\mu_{ij} = -1$ for all $i \neq j$. For each $k \in \{1, \ldots, n-1\}$, let $M_k \in M(n, \Bbbk)$ denote the matrix with kk-entry equal to 2, and 0 in all other entries. Moreover, we define $M_n \in M(n, \Bbbk)$ by taking its ij-entry to be:

$$(M_n)_{ij} = \begin{cases} 2 & \text{if } j = i = n, \\ (-1)^i & \text{if } j = i + (-1)^{i+1},\ i \neq n \neq j, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, for instance, if $n = 5$, then

$$M_5 = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}.$$

The matrices $M_1, \ldots, M_n$ are μ-symmetric, and the degree-2 defining relations of the graded skew Clifford algebra $A(\mu, M_1, \ldots, M_n)$ are:

(i) $x_k^2 = y_k$ for all k, and
(ii) $x_i x_j - x_j x_i = 0$ for all $i \neq j$ where either $j \neq i + (-1)^{i+1}$ or $n \in \{i, j\}$, and
(iii) $x_i x_j - x_j x_i = (-1)^i y_n = (-1)^i x_n^2$ for all i, j where $j = i + (-1)^{i+1}$ and $i \neq n \neq j$.

The quadric system Q determined by $M_1, \ldots, M_n$ is

$$Q = \Bbbk z_1^2 \oplus \cdots \oplus \Bbbk z_{n-1}^2 \oplus \Bbbk(z_n^2 + \sum_{i \in I} z_{i+1} z_i),$$

where $I = \{1, 3, 5, \ldots, n-2\}$. The elements $z_1^2, \ldots, z_{n-1}^2, z_n^2 + \sum_{i \in I} z_{i+1} z_i$ form a normalizing sequence in S, since the image of each element is central in the factor ring given by the preceding elements in the sequence. Moreover, the ring $S/\langle Q \rangle$ has finite dimension over $\Bbbk$, since $S/\langle Q \rangle$ has a basis that is a subset of $\{z_1^{i_1} \cdots z_n^{i_n} : i_k \in \{0, 1\} \text{ for all } k\}$. It follows that Q is base-point free (by an argument similar to that used for the previous two examples). Hence, by Theorem 3.6, $A(\mu, M_1, \ldots, M_n)$ is a graded skew Clifford algebra that satisfies the properties stated in that theorem. In particular, the algebra is generated by $x_1, \ldots, x_n$ with defining relations given by (ii) and (iii) above. In order to see that $A(\mu, M_1, \ldots, M_n)$ maps onto the Nth Weyl algebra, we define a homomorphism χ from $A(\mu, M_1, \ldots, M_n)$ to a $\Bbbk$-algebra on $2N$ generators by

$$\begin{aligned} &x_i \mapsto X_{(i+1)/2} \text{ for all } i \in 2\mathbb{Z}+1,\ i \neq n, \\ &x_i \mapsto d_{i/2} \text{ for all } i \in 2\mathbb{Z}, \\ &x_n \mapsto 1. \end{aligned}$$

The image of χ is the $\Bbbk$-algebra on generators $X_1, \ldots, X_N, d_1, \ldots, d_N$ with defining relations

$$\begin{aligned} &d_k X_k - X_k d_k = 1 \text{ for all } k \in \{1, \ldots, N\}, \\ &X_i X_j - X_j X_i = 0 = d_i d_j - d_j d_i \text{ for all } i,\, j, \\ &X_i d_j - d_j X_i = 0 \text{ for all } i \neq j \end{aligned}$$

(where $\chi(0) = \chi(x_i x_n - x_n x_i) = 0$ for all i). It follows that

$$\frac{A(\mu, M_1, \ldots, M_n)}{\langle x_n - 1 \rangle}$$

is isomorphic to the Nth Weyl algebra.

We close this section by noting that if a graded skew Clifford algebra $A(\mu, M_1, \ldots, M_n)$ is quadratic, then its Koszul dual is isomorphic to $S/\langle Q \rangle$, where Q is the quadric system given by $M_1, \ldots, M_n$.

4. Skew Clifford Algebras

Until now, our discussion has centered on graded skew Clifford algebras, from [3], which are $\mathbb{Z}$-graded algebras that can be viewed as quantized analogues of graded Clifford algebras. Since a graded Clifford algebra maps onto a classical Clifford algebra, our goal in this section is to replicate an analogue of this mapping in the setting of graded skew Clifford algebras. To this end, a quantized analogue of a classical Clifford algebra will be presented as in [4].

Remark 4.1. The reader should note that the algebra defined in Definition 4.2 below can, sometimes, depending on the data, be the zero algebra. This can be seen by comparing

$$\frac{\Bbbk\langle x_1,\, x_2 \rangle}{\langle x_1^2,\, x_2^2,\, x_1 x_2 + x_2 x_1 - 1 \rangle} \quad \text{and} \quad \frac{\Bbbk\langle x_1,\, x_2 \rangle}{\langle x_1^2,\, x_2^2,\, x_1 x_2 - x_2 x_1 - 1 \rangle};$$

the first algebra is a Clifford algebra of dimension four, whereas the second algebra is the zero algebra and not a Clifford algebra.

Definition 4.2. [4]

(a) Let V be a vector space with ordered basis $\mathcal{B} = \{x_1, \ldots, x_n\}$ and let μ be as in Definition 2.8. We call a bilinear form $\phi : V \times V \to \Bbbk$ μ-symmetric (relative to $\mathcal{B}$) if $\phi(x_i,\, x_j) = \mu_{ij} \phi(x_j,\, x_i)$ for all $i,\, j$.

(b) Let V be a vector space with ordered basis $\mathcal{B} = \{x_1, \ldots, x_n\}$, and let ϕ be a μ-symmetric bilinear form (relative to $\mathcal{B}$). The skew Clifford algebra $\mathsf{sCl}(V,\, \mu,\, \phi)$ associated with ϕ is the quotient of the tensor algebra on V by the ideal generated by all elements of the form $x_i \otimes x_j + \mu_{ij} x_j \otimes x_i - 2\phi(x_i,\, x_j) \cdot 1$ for all i, j.

Example 4.3. The skew Clifford algebra associated to the data $V = \Bbbk^2$ with ordered basis $\mathcal{B} = \{x_1, x_2\}$,

$$\mu = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad \text{and} \quad (\phi(x_i,\, x_j)) = \begin{bmatrix} 0 & 1/2 \\ -1/2 & 0 \end{bmatrix}$$

is the second algebra in Remark 4.1, which is the zero algebra.

Example 4.4. Any Clifford algebra is a skew Clifford algebra.

Example 4.5. If we take V and μ as in Definition 4.2, and let $\phi = 0$, then the skew Clifford algebra $\mathsf{sCl}(V, \mu, 0)$ is the quantum exterior algebra $\Lambda_\mu(V)$. In this case, we have $\dim_{\Bbbk}(\mathsf{sCl}(V, \mu, 0)) = 2^{\dim(V)}$. Moreover, the Koszul dual of $\Lambda_\mu(V)$ is the algebra S, where $S_1^* = V$ and $\{z_1, \dots, z_n\}$ is the dual basis to $\mathcal{B}$.

Example 4.6. Suppose $\dim_{\Bbbk}(V) = 3$ and let

$$\mu = \begin{bmatrix} 1 & a & 1 \\ 1/a & 1 & a \\ 1 & 1/a & 1 \end{bmatrix} \quad \text{and} \quad (\phi(x_i, x_j)) = \begin{bmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ b & 0 & 0 \end{bmatrix},$$

where $a, b \in \Bbbk$ with $a \neq 0$. In this case, $\dim_{\Bbbk}(\mathsf{sCl}(V, \mu, \phi)) = 8$.

Example 4.7. Suppose $\dim_{\Bbbk}(V) = 4$ and let

$$\mu = \begin{bmatrix} 1 & \mu_{12} & \mu_{13} & 1 \\ \mu_{21} & 1 & \mu_{23} & 1 \\ \mu_{31} & \mu_{32} & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad (\phi(x_i, x_j)) = \begin{bmatrix} 0 & 0 & 0 & a \\ 0 & 0 & 0 & b \\ 0 & 0 & 0 & c \\ a & b & c & 1 \end{bmatrix}$$

where $a, b, c \in \Bbbk$, such that $abc \neq 0$. If $\mu_{23} = \mu_{13} = 1 \neq \mu_{12}$, then we have that $\dim_{\Bbbk}(\mathsf{sCl}(V, \mu, \phi)) = 8$. On the other hand, if $\mu_{23} \neq 1$ or $\mu_{13} \neq 1$, then $\dim_{\Bbbk}(\mathsf{sCl}(V, \mu, \phi)) = 4$.

Let $g : V \to \mathsf{sCl}(V, \mu, \phi)$ denote the composition

$$V \hookrightarrow T(V) \twoheadrightarrow \mathsf{sCl}(V, \mu, \phi),$$

where $T(V)$ denotes the tensor algebra on V. For Clifford algebras, g is always injective, but, as seen in Example 4.3, for arbitrary $\mathsf{sCl}(V, \mu, \phi)$, g need not be injective. Nevertheless, using g, there is a universal mapping property analogous to that for Clifford algebras ([4, Theorem 2.6]). The next result determines when g is injective.

Theorem 4.8. [4] *For V, μ and ϕ as in Definition 4.2, the following are equivalent:*

(a) *the map $g : V \to \mathsf{sCl}(V, \mu, \phi)$ is injective*;
(b) $\dim_{\Bbbk}(\mathsf{sCl}(V, \mu, \phi)) = 2^{\dim(V)}$;
(c) *the quadratic form* $\begin{bmatrix} z_1 & \cdots & z_n \end{bmatrix} (\phi(x_i, x_j)) \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} \in S_2,$

determined by ϕ, is central in S.

In contrast, if the map g is not injective for some nonzero skew Clifford algebra $\mathsf{sCl}(V, \mu, \phi)$, then, by [4, Corollary 3.13], $\dim_{\Bbbk}(\mathsf{sCl}(V, \mu, \phi)) = 2^j$ for some $j \in \{1, \dots, \dim_{\Bbbk}(V) - 1\}$.

We conclude by relating skew Clifford algebras to graded skew Clifford algebras.

Theorem 4.9. [4] *Let V, μ and ϕ be as in Definition 4.2. Write $n = \dim_{\Bbbk}(V)$ and suppose $\phi \neq 0$.*

(a) *The skew Clifford algebra* $\mathsf{sCl}(V, \mu, \phi)$ *is a quotient of a graded skew Clifford algebra* $A(\mu, M_1, \ldots, M_n)$*, for some* μ*-symmetric matrices* $M_1, \ldots, M_n \in M(n, \Bbbk)$*, where* $A(\mu, M_1, \ldots, M_n)$ *is generated by degree-1 elements.*
(b) *If* $\dim_{\Bbbk}(\mathsf{sCl}(V, \mu, \phi)) = 2^n$*, then there exists a quadratic graded skew Clifford algebra* $A(\mu, M_1, \ldots, M_n)$*, for some* μ*-symmetric matrices* $M_1, \ldots, M_n \in M(n, \Bbbk)$*, that satisfies the properties of Theorem* 3.6*, where* $A(\mu, M_1, \ldots, M_n)$ *maps onto* $\mathsf{sCl}(V, \mu, \phi)$ *if and only if* $\mu_{ij}^2 = 1$ *for all* i, j*.*

Acknowledgements

The author warmly thanks the organizers of the *12th International Conference on Clifford Algebras and their Applications in Mathematical Physics* for their invitation to give a plenary lecture at the conference.

References

[1] Artin, M., Tate, J., Van den Bergh, M.: Some Algebras Associated to Automorphisms of Elliptic Curves. In: P. Cartier et al. (eds.) The Grothendieck Festschrift **1**, pp 33–85. Birkhäuser, Boston, MA (1990). https://doi.org/10.1007/978-0-8176-4574-8_3

[2] Artin, M., Tate, J., Van den Bergh, M.: Modules over regular algebras of dimension 3. Invent. Math. **106**, 335–388 (1991). https://doi.org/10.1007/BF01243916

[3] Cassidy, T., Vancliff, M.: Generalizations of graded clifford algebras and of complete intersections. J. Lond. Math. Soc. (2) **81**(1), 91–112 (2010). (Corrigendum: **90** No. 2, 631-636 (2014).) https://doi.org/10.1112/jlms/jdp057

[4] Cassidy, T., Vancliff, M.: Skew clifford algebras. J. Pure Appl. Algebra **223**, 5091–5105 (2019). https://doi.org/10.1016/j.jpaa.2019.03.012

[5] Faddeev, L.D., Reshetikhin, NYu., Takhtadzhyan, L.A.: Quantization of lie groups and lie algebras. Leningrad Math. J. **1**(1), 193–225 (1990). https://doi.org/10.1016/B978-0-12-400465-8.50019-5

[6] Le Bruyn, L.: Central singularities of quantum spaces. J. Algebra **177**(1), 142–153 (1995). https://doi.org/10.1006/jabr.1995.1290

Michaela Vancliff
Department of Mathematics
University of Texas at Arlington
Box 19408Arlington TX76019-0408
USA
e-mail: vancliff@uta.edu

Received: February 26, 2021.
Accepted: May 18, 2021.