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Conjugate Harmonic Functions of Fueter Type

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Abstract. Let $\mathcal{H}$ be an oriented three-dimensional manifold and $\mathbb{H}_+ = \mathbb{R}_+ \oplus \mathcal{H}$. The author introduces non-abelian vector valued Fourier transforms on $\mathcal{H}$ and Poisson integrals on $\mathbb{H}_+$. Through the boundary behaviour of Poisson integral, the author obtains the characterization of conjugate harmonic functions of Fueter type via Riesz transforms.

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1. Introduction

The quaternion algebra is a four dimensional associative algebra over $\mathbb{R}$ with generators 1, e_1, e_2, e_3 satisfying $e_1^2 = e_2^2 = e_3^2 = -1$, $e_1e_2 = -e_2e_1 = e_3$, $e_2e_3 = -e_3e_2 = e_1$ and $e_3e_1 = -e_1e_3 = e_2$ (see [1,2,4,6,10,27] and the references given there). In what follows, we always denote the quaternion algebra by $\mathbb{H}$. For any $q \in \mathbb{H}$, it can be written as these linear combinations of 1, e_1, e_2 and e_3, namely, $q = x_0 + x_1e_1 + x_2e_2 + x_3e_3$, where x_0, x_1, x_2, $x_3 \in \mathbb{R}$.

Our purpose is to deal with the following respects:

1.1. Quaternion Upper Half Planes

(1.1.a) Suppose the involution of $\mathbb{H}$ is

$$\mathbb{H} \ni q = x_0 + x_1e_1 + x_2e_2 + x_3e_3 \to \widehat{q} = x_0 - x_1e_1 + x_2e_2 + x_3e_3 \in \mathbb{H}.$$

The quaternion upper half plane is given by

$$\widetilde{\mathbb{H}}_+ := \{q = x_0 + x_1e_1 + x_2e_2 + x_3e_3 \in \mathbb{H} : \ x_1 > 0\}. \tag{1.1}$$

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The group of linear fractional transformations on $\widetilde{\mathbb{H}}_+$ is given by

$$S := \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in M_2(\mathbb{H}) : \begin{pmatrix} \widehat{\alpha} & \widehat{\gamma} \\ \widehat{\beta} & \widehat{\delta} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}.$$

The upper half plane $\widetilde{\mathbb{H}}_+$ as in (1.1) has been studied since 1980's, revealing connections with geometric, the theory of numbers, harmonic analysis and PDE. The geometric properties of $\widetilde{\mathbb{H}}_+$ were studied by Kähler [15]. Gritsenko [12,13] made a detailed investigation about the theory of numbers in $\widetilde{\mathbb{H}}_+$, and in a more general context in [17]. For a deeper discussion of harmonic analysis and PDE on $\widetilde{\mathbb{H}}_+$ is due to Lax and Phillips [18–21].

(1.1.b) In [27, p. 201], Sudbery introduced the vector space $\mathbb{R}\oplus\mathcal{H}$, where $\mathcal{H}$ denotes an oriented three-dimensional manifold. Suppose the involution of $\mathbb{H}$ is

$$\mathbb{H} \ni q = x_0 + x_1e_1 + x_2e_2 + x_3e_3 \to q^\star = x_0 - x_1e_1 - x_2e_2 - x_3e_3 \in \mathbb{H}.$$

Narita [23, p. 641] considered the following form upper half plane

$$\mathbb{H}_+ = \mathbb{R}_+ \oplus \mathcal{H} := \{q = x_0 + x_1e_1 + x_2e_2 + x_3e_3 \in \mathbb{H} : \ x_0 > 0\}. \quad (1.2)$$

The group of linear fractional transformations on $\mathbb{H}_+$ is defined by

$$Q := \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in M_2(\mathbb{H}) : \begin{pmatrix} \alpha^\star & \gamma^\star \\ \beta^\star & \delta^\star \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}.$$

Comparing (1.1.a) with (1.1.b), we find that there's a big difference between the Lie groups S and Q. Notice that $\mathcal{H}$ is the boundary of $\mathbb{H}_+$, where $\mathbb{H}_+$ is as in (1.2). Because $\mathcal{H}$ has a simple structure, thus it is easy to establish some analysis theories. The choice of $\mathbb{H}_+$ seems to be the best adapted to our theory. In this paper, we are trying to follow the classical harmonic analysis of Stein and Weiss [26] into the oriented three-dimensional manifold $\mathcal{H}$.

1.2. Fourier Transforms

We shall consider the Fourier transforms on $\mathcal{H}$. Let N be the nilpotent subgroup of Q as in (1.1.b). Since $N \simeq \mathcal{H}$, the definitions of left and right Fourier transforms are unambiguous. The Fourier transforms are described by Kirillov's orbit method [16] and Vergne's polarizing subalgebra [28]. Motivated by [25], we also obtain the Plancherel theorem adapted to these Fourier transforms. The Plancherel theorem will turn to be important for Riesz transform.

1.3. Fueter Conjugates Harmonic Functions and Cimmino Systems

The investigation was mainly aimed at constructing Fueter conjugates harmonic functions. This construction was motivated by [24]. In order to obtain these functions, we recall the notions of two systems as follows:

(1.3.a) A $\mathbb{R}$-differentiable function $F = F_0+e_1F_1+e_2F_2+e_3F_3$ is Fueter left regular at the neighborhood U of $q \in \mathbb{H}$ if and only if F_j for $j = 0, 1, 2, 3$ are differentiable functions and

$$\frac{\partial F}{\partial x_0} + \sum_{j=1}^{3} e_j \frac{\partial F}{\partial x_j} = 0. \quad (1.3)$$

This is so-called Cauchy–Riemann–Fueter equations (see [9] or [27, Proposition 3]).

(1.3.b) For $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ and $\nu(x) \in \mathcal{H}$, we consider the function F as in (1.3)

$$F(x_0 + \nu(x)) = F_0(x_0, x) + \sum_{j=1}^{3} e_j F_j(x_0, x).$$

Function F is a left Fueter conjugate harmonic function, if (F_0, F_1, F_2, F_3) satisfy the first order linear partial differential equations:

$$\begin{cases} \frac{\partial F_0}{\partial x_0} - \frac{\partial F_1}{\partial x_1} - \frac{\partial F_2}{\partial x_2} - \frac{\partial F_3}{\partial x_3} = 0; \\ \frac{\partial F_0}{\partial x_1} + \frac{\partial F_1}{\partial x_0} - \frac{\partial F_2}{\partial x_3} + \frac{\partial F_3}{\partial x_2} = 0; \\ \frac{\partial F_0}{\partial x_2} + \frac{\partial F_1}{\partial x_3} + \frac{\partial F_2}{\partial x_0} - \frac{\partial F_3}{\partial x_1} = 0; \\ \frac{\partial F_0}{\partial x_3} - \frac{\partial F_1}{\partial x_2} + \frac{\partial F_2}{\partial x_1} + \frac{\partial F_3}{\partial x_0} = 0. \end{cases}$$

This is so-called left Cilmmino systems (see, for example, [3,5,8]).

By (1.3.a) and (1.3.b), we introduce Poisson integral (see Sect. 3). Through the boundary behaviour of Poisson integral (see Proposition 3.10 below), we obtain the characterization of Fueter conjugates harmonic functions via Riesz transforms (see Theorem 3.11 below). Then Riesz transform provides a necessary and sufficient condition for Fueter conjugate harmonic functions, where these functions satisfy left (and right, resp.) Cimmino systems.

Main results and structure of this paper The paper starts with Sect. 2, which contains the needed preliminaries. In particular, for a nilpotent Lie group N, we introduce left and right Fourier transforms. Section 3 contains a detailed study of set of characterization of the Poisson integral. Then the Fueter conjugate harmonic systems of functions are obtained via Riesz transforms. We can now formulate our main results. For any $x_0 > 0$ and any $\nu(x) \in \mathcal{H}$, let $x_0 + \nu(x) \in \mathbb{H}_+$, left and right Fueter conjugate harmonic functions $F_L(x_0+\nu(x))$ and $F_R(x_0+\nu(x))$, respectively, are given by, for any $f \in L^2(\mathbb{R}^3)$,

$$F_L(x_0 + \nu(x)) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \left(1 + \frac{\nu(\xi)}{|\nu(\xi)|} i\right) e^{-x_0|\xi|} \widehat{f}(\xi) e^{ix\cdot\xi}\, d\xi$$

and

$$F_R(x_0 + \nu(x)) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{ix\cdot\xi} e^{-x_0|\xi|} \widehat{f}(\xi) \left(1 + i\frac{\nu(\xi)}{|\nu(\xi)|}\right) d\xi,$$

where

$$\widehat{f}(\xi) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} f(x) e^{-i\xi\cdot x}\, dx.$$

2. Harmonic Analysis on $\mathcal{H}$

Let $q = x_0 + x_1e_1 + x_2e_2 + x_3e_3 \in \mathbb{H}$, where x_0, x_1, x_2 and $x_3 \in \mathbb{R}$. The involution of q is defined by $q^\star := x_0 - x_1e_1 - x_2e_2 - x_3e_3$. The Euclidean norm on $\mathbb{H}$ can be expressed in terms of this involution by $|q|^2 = qq^\star = x_0^2 + x_1^2 + x_2^2 + x_3^2$. As usual, for $q = x_0 + x_1e_1 + x_2e_2 + x_3e_3$ and $p = y_0 + y_1e_1 + y_2e_2 + y_3e_3$, we denote the product of two elements of $\mathbb{H}$ by

$$qp = \mathrm{Re}\langle q,\, p\rangle + \sum_{j=1}^{3}(x_0y_j + y_0x_j)e_j + \widetilde{q} \times \widetilde{p},$$

where

$$\mathrm{Re}\langle q,\, p\rangle := qp^\star = x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3$$

and

$$\widetilde{q} \times \widetilde{p} := \begin{vmatrix} e_1 & e_2 & e_3 \\ y_1 & y_2 & y_3 \\ x_1 & x_2 & x_3 \end{vmatrix}.$$

Define a Lie algebra $\mathfrak{g}$ by

$$\left\{ X \in M_2(\mathbb{H}) : X^\star Q + QX = \mathbf{0},\, Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\},$$

where $M_2(\mathbb{H})$ is a quaternion 2×2-matrix and $\mathbf{0}$ denotes a zero 2×2-matrix. Let τ be the Cantan involution of $\mathfrak{g}$ defined by $\tau(X) = -{}^tX^\star \in \mathfrak{g}$, where $X \in \mathfrak{g}$. Then the Cartan decomposition is given by $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k} = \{X \in \mathfrak{g} :\ \tau(X) = X\}$ and $\mathfrak{p} = \{X \in \mathfrak{g} :\ \tau(X) = -X\}$. Let $\exp(\mathfrak{g}) = Sp(1,\, 1)$ and $Sp^*(1) = \{p \in \mathbb{H} : pp^\star = 1\}$. From [23, p. 641], it follows that the symmetric space $Sp(1,\, 1)/Sp^*(1) \times Sp^*(1)$ can be realized as the quaternion upper half plane:

$$\mathbb{H}_+ = \{p \in \mathbb{H} :\ \mathrm{Re}(p) > 0\}, \quad \text{where} \quad \mathrm{Re}(p) = \frac{p + p^\star}{2}.$$

The group $Sp(1,\, 1)$ acts transitively on $\mathbb{H}_+$ by $g \cdot p = \frac{a_1p+b_1}{a_2p+b_2}$, where $g = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \in Sp(1,\, 1)$; see, for example, [12,13]. In order to describe the Iwasawa decomposition of $\mathfrak{g}$, we need the restricted root of $\mathfrak{g}$, which is of C_1-type. Let $H := \mathrm{diag}(1,\, -1)$. Then $\mathfrak{a} := \mathbb{R}H$ forms a maximal abelian subalgebra of $\mathfrak{p}$. Let α be the element in $\mathfrak{a}^*$ such that $\alpha(H) = 1$. The root system $\Delta(\mathfrak{a},\, \mathfrak{g})$ of $(\mathfrak{g},\, \mathfrak{a})$ is then given as $\{\pm 2\alpha\}$ and the positive root spaces are

$$\mathfrak{g}_{2\alpha} = \mathbb{R}\begin{pmatrix} 0 & e_1 \\ 0 & 0 \end{pmatrix} \oplus \mathbb{R}\begin{pmatrix} 0 & e_2 \\ 0 & 0 \end{pmatrix} \oplus \mathbb{R}\begin{pmatrix} 0 & e_3 \\ 0 & 0 \end{pmatrix}.$$

Write $\mathfrak{n} = \mathfrak{g}_{2\alpha}$. We see at once that

$$\mathfrak{n} = \left\{ \begin{pmatrix} 0 & p \\ 0 & 0 \end{pmatrix} :\ p = x_1e_1 + x_2e_2 + x_3e_3,\, x_1,\, x_2,\, x_3 \in \mathbb{R} \right\}. \tag{2.1}$$

By this, we obtain the Iwasawa decomposition $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{k}$. Let

$$N = \exp(\mathfrak{n}). \tag{2.2}$$

The subgroup N can be realized as:

$$\mathcal{H} := \{p \in \mathbb{H} :\ \mathrm{Re}(p) = 0\} = \{p \in \mathbb{H} :\ p^\star = -p\}.$$

Suppose $\{e_1, e_2, e_3\}$ now is an orientation of $\mathcal{H}$, that is, $\mathcal{H}{:=}(\mathcal{H}, \{e_1, e_2, e_3\})$ is an oriented vector space. Motivated by [11, p. 9], a normal oriented map ν from $\mathbb{R}^3$ to $\mathcal{H}$ can be defined as

$$\nu :\ x \mapsto \nu(x). \tag{2.3}$$

The following definition implies that the between vector spaces $\mathcal{H}$ and $\mathbb{R}^3$ are isomorphism (see [14] for more details).

Definition 2.1. The pair $(\mathcal{H}, |\nu(\cdot)|)$ is said to be an oriented 3-dimensional manifold for $(\mathbb{R}^3, |\cdot|)$, where $|\cdot|$ denotes the usual Euclidean norm in $\mathbb{R}^3$, when

(i) $\mathcal{H}$ is generated as a vector space by $\{\nu(x) :\ x \in \mathbb{R}^3\}$;
(ii) $|\nu(x)|^2 = |x|^2$ for any $x \in \mathbb{R}^3$.

Definition 2.2. Let $(\mathcal{H}, |\nu(\cdot)|)$ be as in Definition 2.1. Then, for any $\nu(x) \in \mathcal{H}$, the volume element on $\mathcal{H}$ (the invariant measure of N) is defined by $d\nu(x) = dx_1 \wedge dx_2 \wedge dx_3$.

Remark 2.3. For the general case, we define the oriented map ν from $\mathbb{R}^3$ to $\mathcal{H}$ by

$$\nu :\ (x_1, x_2, x_3) \mapsto \sum_{j=1}^{3} x^j(x_1, x_2, x_3)e_j.$$

Define a positive chart of $(\mathcal{H}, |\nu(\cdot)|)$ by (ϕ, U), where U is an open subset of $\mathcal{H}$ and ϕ is C^∞ diffeomorphism maps from U to $\phi(U) \subset \mathbb{R}^3$. Let (ϕ, U) be a positive chart with $\phi = (x^1, x^2, x^3)$. Then, for any $p = \sum_{j=1}^{3} x^j e_j \in \mathcal{H}$, the volume element on $\mathcal{H}$ is defined by $dp\big|_U = \sqrt{M}\, dx^1 \wedge dx^2 \wedge dx^3$, where $M := \det[m_{st}]_{1\le s,\, t\le 3} \in C^\infty(U)$ denotes the Gram determinant and m_{st} is an Euclidean inner product given by $m_{st}(p) := \mathrm{tr}(\partial_s\big|_p, \partial_t\big|_p)$, for $1 \le s, t \le 3$ and $p \in U$. We only restrict our the discussion

$$m_{st}(p) := \mathrm{tr}\left(\partial_s\big|_p, \partial_t\big|_p\right) = \begin{cases} 1, & \text{if } s = t; \\ 0, & \text{otherwise.} \end{cases}$$

In this case, we have $M = 1$ and dp is as in Definition 2.2.

Definition 2.4. Mapping $f :\ \mathcal{H} \to \mathbb{H}$ is a quaternion-valued function of the form, for any $x \in \mathbb{R}^3$,

$$f(\nu(x)) = f_0(x) + \sum_{j=1}^{3} f_j(x)e_j, \tag{2.4}$$

where $\nu(x) \in \mathcal{H}$ is as in (2.3), and the component functions f_0, f_j are real-valued functions.

Together with Definition 2.2 and (2.4), we introduce the Lebesgue spaces $L^2(\mathcal{H}, \mathbb{H})$. For a similar definition we refer the reader to [22, p. 174]. A quaternionic valued function f belongs to $L^2(\mathcal{H}, \mathbb{H})$ if

$$\|f\|^2_{L^2(\mathcal{H}, \mathbb{H})} = \int_{\mathcal{H}} |f(\nu(x))|^2\, d\nu(x) = \sum_{j=0}^{3} \int_{\mathbb{R}^3} |f_j(x)|^2\, dx < \infty,$$

where $f_j \in L^3(\mathbb{R}^3)$ for each $j \in \{0, 1, 2, 3\}$. The space $L^2(\mathcal{H}, \mathbb{H})$ is a Hilbert space with inner product

$$\langle f, g\rangle = \int_{\mathcal{H}} f(\nu(x))g(\nu(x))^\star \, d\nu(x) = \sum_{j=0}^{3} \int_{\mathbb{R}^3} f_j(x) g_j(x)\, dx.$$

Moveover, for any $\nu(\xi) \in \mathcal{H}$ and any f in $L^2(\mathcal{H}, \mathbb{H})$, we define the left and right Fourier transforms by

$$\mathscr{F}_L(f)(\nu(\xi)) := \frac{1}{(2\pi)^{3/2}} \int_{\mathcal{H}} e^{-i\nu(\xi)\nu(x)^\star} f(\nu(x))\, d\nu(x) \tag{2.5}$$

and

$$\mathscr{F}_R(f)(\nu(\xi)) := \frac{1}{(2\pi)^{3/2}} \int_{\mathcal{H}} f(\nu(x)) e^{-i\nu(\xi)\nu(x)^\star}\, d\nu(x). \tag{2.6}$$

For that same definitions will be applied by the left and right inverse Fourier transforms, the details being omitted.

Remark 2.5. (i) Because of the non-commutativity of quaternions, the left Fourier transform is unequal to right Fourier transform as in (2.5) and (2.6), namely,

$$e^{-i\nu(\xi)\nu(x)^\star} f(\nu(x)) \neq f(\nu(x)) e^{-i\nu(\xi)\nu(x)^\star}.$$

(ii) Moveover, these Fourier transforms (2.5) and (2.6) are described by Kirillov's orbit method (see [7,16] for more details). For the sake of argument, that it is necessary to have the details. Let $\mathfrak{n}$ be as in (2.1). We note that $[X, X] = 0$ for any $X \in \mathfrak{n}$, then $\mathfrak{n}$ is abelian. Denote the dual of $\mathfrak{n}$ by $\mathfrak{n}^*$. It is easy to see that $\mathfrak{n}^* = \mathfrak{n}$. Let N be as in (2.2). The product of elements of N are given by $n(x)n(y) = n(x+y)$, where $n(x), n(y) \in N$. Then N also is abelian. N acts on $\mathfrak{n}^*$ the coadjoint map Ad^*, namely, for any $X \in \mathfrak{n}$, $\ell \in \mathfrak{n}^*$ and $n(x) \in N$, $((\mathrm{Ad}^* n(x))\ell)(X) = \ell((\mathrm{Ad} n(x)^{-1})X)$. Similar to the construction of [28], we also have the Vergne polarizing subalgebra by $\mathfrak{n}$. In fact, for any $n(x) \in N$ and $X \in \mathfrak{n}$, we obtain

$$\left(\mathrm{Ad}(n(x))^{-1}\right) X = \begin{pmatrix} 1 & -n(x) \\ 0 & 1 \end{pmatrix} X \begin{pmatrix} 1 & n(x) \\ 0 & 1 \end{pmatrix} = X.$$

One see that $R_\ell = \{n(x) \in N : \ (\mathrm{Ad}^*(n(x)))\ell = \ell\}$ is a stabilizer of N associated with $\ell \in \mathfrak{n}^*$. For any $\ell \in \mathfrak{n}^*$, by using [7, Lemma 1.3.1], we have $N = R_\ell$, which implies that the Vergne polarizing subalgebra must be $\mathfrak{n}$. For some $\xi \in \mathfrak{n}^*$ and any $X \in \mathfrak{n}$, we let $\ell_\xi \in \mathfrak{n}^*$ satisfy $\ell_\xi(X) = \xi X^\star$. Then we have a one-dimensional representation $N \to \mathbb{S}^1$ (since $\ell[\mathfrak{n}, \mathfrak{n}] = 0$) defined as $e^{i\ell_\xi(x)} \in \{z \in \mathbb{C} : z \cdot \bar{z} = 1\}$. Notice that Ad^* is the identity map for all $n(x) \in N$, $\mathfrak{n}^*/\mathrm{Ad}^*(N) = \mathcal{H}$. Furthermore, let $\widehat{N}$ be the dual of N. From [16, Theorems 5.1 and 5.2], it follows that there is a bijection such that $\widehat{N} \simeq \{\ell_\xi : \ \xi \in \mathcal{H}\}$. Hence, for any $\eta_\xi \in \widehat{N}$, we get the characters $e^{i\ell_\xi(\cdot)}$. By using Remark 2.5(i), we obtain Fourier transforms of the classical function f as in (2.5) and (2.6), respectively.

The following Plancherel theorem is just [25], which is a slight modification.

Theorem 2.6. *For any* $f \in L^2(\mathcal{H}, \mathbb{H})$*, we have*

$$\|\mathscr{F}_L(f)\|_{L^2(\mathcal{H},\mathbb{H})} = \frac{1}{(2\pi)^{3/2}}\|f\|_{L^2(\mathcal{H},\mathbb{H})}.$$

The same is true of right Fourier transform.

3. Main Results

First, we recall the concepts of $\mathbb{R}$-linear, $\mathbb{R}$-differentiable and Fueter regular function.

Definition 3.1. A function $\ell : \mathbb{H} \to \mathbb{H}$ is said to be $\mathbb{R}$-linear if:

(i) $\ell(q+p) = \ell(q) + \ell(p)$ for all $q, p \in \mathbb{H}$;
(ii) $\ell(\lambda q) = \lambda \ell(q)$ for all $q \in \mathbb{H}$ and all $\lambda \in \mathbb{R}$.

Any $\mathbb{R}$-linear function in $\mathbb{H}$ has the form

$$\ell(q) = \alpha q + \beta q^\star, \text{ where } \alpha, \beta \in \mathbb{H}.$$

Definition 3.2. A function $F : U \to \mathbb{H}$, where U is a neighborhood of $q \in \mathbb{H}$, is said to be $\mathbb{R}$-differentiable at q if

$$F(q+h) = F(q) + \ell(h) + o(h),$$

where ℓ is as in Definition 3.1 and $o(h)/|h| \to 0$ as $h \to 0$.

The following Cauchy–Riemann–Fueter condition is just [27, Proposition 3].

Definition 3.3. Let $q = x_0 + x_1e_1 + x_2e_2 + x_3e_3 \in \mathbb{H}$, where $x_0, x_1, x_2, x_3 \in \mathbb{R}$. A $\mathbb{R}$-differentiable function $F = F_0 + e_1F_1 + e_2F_2 + e_3F_3$ is Fueter left or right regular at the neighborhood U of q if and only if F_j for $j = 0, 1, 2, 3$ are differentiable functions and

$$\frac{\partial F}{\partial x_0} + \sum_{j=1}^{3} e_j \frac{\partial F}{\partial x_j} = 0 \quad \text{or} \quad \frac{\partial F}{\partial x_0} + \sum_{j=1}^{3} \frac{\partial F}{\partial x_j} e_j = 0.$$

Definition 3.4. A Hilbert space $H^2(\mathbb{H}_+, \mathbb{H})$ is said to be Hardy space if it is the set of all Fueter left (or right) regular functions F satisfying

$$\|F\|^2_{H^2(\mathbb{H}_+,\mathbb{H})} = \sup_{x_0>0} \int_{\mathcal{H}} |F(x_0+\nu(x))|^2 \, d\nu(x) < \infty.$$

Another approach aries from the observation that a function, for $x \in \mathbb{R}^3$ and $\nu(x) \in \mathcal{H}$,

$$F(x_0+\nu(x)) = F_0(x_0,x) + \sum_{j=1}^{3} e_j F_j(x_0,x)$$

and

$$F(x_0+\nu(x)) = F_0(x_0,x) + \sum_{j=1}^{3} F_j(x_0,x) e_j$$

are Fueter type left and right conjugate harmonic functions, respectively, if F satisfies left and right Cilmmino systems (see [3] for more details). The left and right Cilmmino systems, respectively, are given by

$$\begin{cases} \frac{\partial F_0}{\partial x_0} - \frac{\partial F_1}{\partial x_1} - \frac{\partial F_2}{\partial x_2} - \frac{\partial F_3}{\partial x_3} = 0; \\ \frac{\partial F_0}{\partial x_1} + \frac{\partial F_1}{\partial x_0} - \frac{\partial F_2}{\partial x_3} + \frac{\partial F_3}{\partial x_2} = 0; \\ \frac{\partial F_0}{\partial x_2} + \frac{\partial F_1}{\partial x_3} + \frac{\partial F_2}{\partial x_0} - \frac{\partial F_3}{\partial x_1} = 0; \\ \frac{\partial F_0}{\partial x_3} - \frac{\partial F_1}{\partial x_2} + \frac{\partial F_2}{\partial x_1} + \frac{\partial F_3}{\partial x_0} = 0, \end{cases} \quad \text{and} \quad \begin{cases} \frac{\partial F_0}{\partial x_0} - \frac{\partial F_1}{\partial x_1} - \frac{\partial F_2}{\partial x_2} - \frac{\partial F_3}{\partial x_3} = 0; \\ \frac{\partial F_0}{\partial x_1} + \frac{\partial F_1}{\partial x_0} + \frac{\partial F_2}{\partial x_3} - \frac{\partial F_3}{\partial x_2} = 0; \\ \frac{\partial F_0}{\partial x_2} - \frac{\partial F_1}{\partial x_3} + \frac{\partial F_2}{\partial x_0} + \frac{\partial F_3}{\partial x_1} = 0; \\ \frac{\partial F_0}{\partial x_3} + \frac{\partial F_1}{\partial x_2} - \frac{\partial F_2}{\partial x_1} + \frac{\partial F_3}{\partial x_0} = 0. \end{cases} \tag{3.1}$$

Moveover, we also assume that there exists a positive constant C such that

$$\|F\|^2_{H^2(\mathbb{H}_+,\, \mathbb{H})} = \sum_{j=0}^{3} \int_{\mathbb{R}^3} |F_j(x_0,\, x)|^2 \, dx \le C < \infty.$$

Thus, an equivalent formulation of the spaces $H^2(\mathbb{H}_+,\, \mathbb{H})$ as in Definition 3.4 is obtained.

Definition 3.5. Let $f \in L^2(\mathcal{H},\, \mathbb{H})$. For any $x_0 > 0$, $x \in \mathbb{R}^3$ and $\nu(x) \in \mathcal{H}$, the Poisson integral of f is defined by,

$$F(x_0 + \nu(x)) = P_{x_0} * f\,(\nu(x)) = \frac{2^{3/2}}{\sqrt{\pi}} \int_{\mathcal{H}} \frac{x_0}{(x_0^2 + |\nu(x) - p|^2)^2} f(p)dp. \tag{3.2}$$

Remark 3.6. (i) By Definition 2.1(ii), for any $x_0 > 0$, $x \in \mathbb{R}^3$ and $\nu(x) \in \mathcal{H}$, we have

$$P_{x_0}\,(\nu(x)) = \frac{2^{3/2}}{\sqrt{\pi}} \frac{x_0}{(x_0^2 + |\nu(x)|^2)^2} = \frac{2^{3/2}}{\sqrt{\pi}} \frac{x_0}{(x_0^2 + |x|^2)^2} = P_{x_0}\,(x)\,.$$

Moveover, it is easy to check that

$$\widehat{e^{-x_0|\cdot|}}(x) = \frac{2^{3/2}}{\sqrt{\pi}} \frac{x_0}{(x_0^2 + |x|^2)^2} = P_{x_0}(x),$$

where $\widehat{\cdot}$ is the classical Fourier transform. Then

$$\widehat{e^{-x_0|\cdot|}}(x) = \mathscr{F}_L(e^{-x_0|\nu(\cdot)|})(\nu(x)) = \mathscr{F}_R(e^{-x_0|\nu(\cdot)|})(\nu(x)),$$

where $\mathscr{F}_L$ and $\mathscr{F}_R$ denote as in (2.5) and (2.6), respectively.

(ii) The study of Cauchy–Fueter formulae was initiated by the celebrated paper of Fueter in 1934 (see [9] or [27] for the definition of Cauchy–Fueter formulae), which is a generalization of the classical Cauchy formula for holomorphic functions. Because of the non-commutativity of quaternions, the Cauchy–Fueter formulae comes in two versions, one for each analogue of the complex holomorphic functions. For any $q \in \mathbb{H}$, the Cauchy–Fueter kernel is given by

$$K(q) = \frac{1}{2\pi^2} \frac{q^\star}{|q|^4}. \tag{3.3}$$

The Cauchy–Fueter integral on $\mathbb{H}_+$ is defined by, for any $q \in \mathbb{H}_+$, F is Fueter left or right regular function on $\mathbb{H}_+$ and f is a quaternion-valued function square integral function on $\mathcal{H}$,

$$F_L(q) = \int_{\mathcal{H}} K(q-p)f(p)dp \quad \text{and} \quad F_R(q) = \int_{\mathcal{H}} f(p)K(q-p)dp, \tag{3.4}$$

where $K(\cdot)$ is as in (3.3). Here we remark that it may be found other weaker boundary condition than $L^2(\mathcal{H}, \mathbb{H})$. In this paper, we only consider the boundary condition as in (3.4) for function $f \in L^2(\mathcal{H}, \mathbb{H})$. Let $x_0 > 0$, $\nu(x) \in \mathcal{H}$ and $q = x_0 + \nu(x) \in \mathbb{H}_+$. Then, by (3.3), we obtain

$$K(q) = \frac{1}{2\pi^2}\frac{x_0}{|q|^4} - \frac{1}{2\pi^2}\frac{\nu(x)}{|q|^4},$$

which, together with (3.4) and the left convolution

$$f *_L g(p) = \int_{\mathcal{H}} f(p-p')g(p')\,dp' \quad \text{for all } p \in \mathcal{H},$$

implies that

$$F_L(q) = \frac{1}{2\pi^2}\int_{\mathcal{H}} \frac{q^\star - p^\star}{|q-p|^4} f(p)dp = cP_{x_0} *_L f\left(\nu(x)\right) + cQ_{x_0} *_L f\left(\nu(x)\right),$$

where $c \equiv \frac{1}{2(2\pi)^{3/2}}$,

$$P_{x_0}\left(\nu(x)\right) = \frac{2^{3/2}}{\sqrt{\pi}}\frac{x_0}{(x_0^2 + |\nu(x)|^2)^2} \quad \text{and} \quad Q_{x_0}\left(\nu(x)\right) = \frac{2^{3/2}}{\sqrt{\pi}}\frac{\nu(x)}{(x_0^2 + |\nu(x)|^2)^2}.$$

Similarly, we also consider the right situation as in (3.4), the details being omitted.

Proposition 3.7. *Let F be as in (3.2). Then, for any $f \in L^2(\mathcal{H}, \mathbb{H})$,*

$$\lim_{x_0 \to 0_+} \|F(x_0 + \cdot) - f(\cdot)\|_{L^2(\mathcal{H}, \mathbb{H})} = 0.$$

Proof. We only need to show that

$$\lim_{x_0 \to 0_+} F(x_0 + \nu(x)) = f\left(\nu(x)\right) \quad \text{a.e. } \nu(x) \in \mathcal{H}. \tag{3.5}$$

By using (2.4), we have $f(\nu(x)) = f_0(x) + \sum_{j=1}^3 f_j(x)e_j$ and

$$P_{x_0} *_L f(\nu(x)) = P_{x_0} * f_0(x) + \sum_{j=1}^{3} P_{x_0} * f_j(x)e_j,$$

which, together with $f_j \in L^2(\mathbb{R}^3)$ and

$$\lim_{x_0 \to 0_+} P_{x_0} * f_j(x) = f_j(x) \quad \text{a.e. } x \in \mathbb{R}^3, \tag{3.6}$$

implies that (3.5) holds. Moveover, we note that (3.6) holds in the sense of L^2-Lebesgue points, and hence (3.5) also holds almost everywhere on $\mathcal{H}$. □

Let $Q_{x_0}(\nu(x))$ be as in Remark 3.6(ii). Then, as $x_0 \to 0_+$, we have the following definition:

Definition 3.8. For any $x = (x_1,\, x_2,\, x_3) \in \mathbb{R}^3$, let

$$K(x) := \sum_{j=1}^{3} K_j(x) e_j, \tag{3.7}$$

where, for $j \in \{1,\, 2,\, 3\}$,

$$K_j(x) = \frac{(2\pi)^{3/2}}{\pi^2} \frac{x_j}{|x|^4}. \tag{3.8}$$

Then the Riesz transform $\mathscr{R}(g)$ is defined by, for the real-valued function $g \in L^2(\mathbb{R}^3)$,

$$\mathscr{R}(g)(x) := K *_L g(x) = \text{p.v.} \int_{\mathbb{R}^3} K(x-y) g(y)\, dy. \tag{3.9}$$

This integral exists in the sense of Cauchy principal value.

Lemma 3.9. *Let K be as in (3.7). Then, for $\nu(\xi) \in \mathcal{H}$,*

$$\mathscr{F}_L\left(K\right)\left(\nu(\xi)\right) = -\frac{\nu(\xi)}{|\nu(\xi)|} i \tag{3.10}$$

and

$$\mathscr{F}_R\left(K\right)\left(\nu(\xi)\right) = -i \frac{\nu(\xi)}{|\nu(\xi)|} \tag{3.11}$$

hold in the sense of the distribution of principal value.

Proof. For $j \in \{1,\, 2,\, 3\}$, we let K_j be as in (3.8). Similar to the proof of [26, Theorem 2.6, Chapter VI], we have

$$e_j \widehat{K}_j(\xi) = -e_j \frac{\xi_j}{|\xi|} i \quad \text{and} \quad \widehat{K}_j(\xi) e_j = -i \frac{\xi_j}{|\xi|} e_j,$$

where $\widehat{\cdot}$ denotes the classical Fourier transform in the sense of the distribution of principal value. From this and (3.7), it follows that

$$\mathscr{F}_L(K)(\nu(\xi)) = \sum_{j=1}^{3} e_j \widehat{K}_j(\xi) = -\sum_{j=1}^{3} e_j \frac{\xi_j}{|\xi|} i = -\frac{\nu(\xi)}{|\nu(\xi)|} i,$$

where $\mathscr{F}_L$ is as in (2.5). And hence (3.10) holds. Similarly, we see that (3.11) holds. □

Proposition 3.10. *Let $\mathscr{R}$ be as in (3.9). Then, for any $g \in L^2(\mathbb{R}^3)$,*

$$\|\mathscr{R}(g)\|_{L^2(\mathbb{R}^3)} = \|g\|_{L^2(\mathbb{R}^3)}.$$

Proof. For any $g \in L^2(\mathbb{R}^3)$, by using (3.9), (3.10) and (3.11), we obtain

$$\mathscr{F}_L(\mathscr{R}(g))(\nu(\xi)) = -\frac{\nu(\xi)}{|\nu(\xi)|} i \widehat{g}(\xi) \quad \text{and} \quad \mathscr{F}_R(\mathscr{R}(g))(\nu(\xi)) = -i \widehat{g}(\xi) \frac{\nu(\xi)}{|\nu(\xi)|}.$$

From this and Theorem 2.6, it follows that $\|\mathscr{R}(g)\|_{L^2(\mathbb{R}^3)} = \|g\|_{L^2(\mathbb{R}^3)}$. □

Theorem 3.11. *Let $f = f_0 + \sum_{j=1}^{3} f_j e_j$, where f_0, f_1, f_2 and $f_3 \in L^2(\mathbb{R}^3)$. Suppose F is the Poisson integral of f as in (3.2). Then F is a Fueter conjugate harmonic function in $\mathbb{H}_+$ if and only if, for any $x \in \mathbb{R}^3$,*

$$\sum_{j=1}^{3} f_j(x)e_j = -\mathscr{R}(f_0)(x), \tag{3.12}$$

where $\mathscr{R}$ is as in (3.9).

Proof. Let $f_0 \in L^2(\mathbb{R}^3)$. Then, by using Remark 3.6(i), for any $x_0 > 0$, we have

$$\frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{-x_0|\xi|} \widehat{f_0}(\xi) e^{ix\cdot\xi}\, d\xi = \int_{\mathbb{R}^3} P_{x_0}(x-\xi) f_0(\xi)\, d\xi := F_0(x_0,\, x). \tag{3.13}$$

For $j \in \{1,\, 2,\, 3\}$, by (3.7), (3.8), (3.9) and (3.12), we see that

$$f_j(x)e_j = -c \int_{\mathbb{R}^3} \frac{x_j - y_j}{|x-y|^4} e_j f_0(y) dy,$$

where $c = \frac{(2\pi)^{3/2}}{\pi^2}$. From this, (3.13) and $e_j^2 = -1$, it follows that,

$$\begin{aligned} F_j(x_0,\, x) \ &:= \tfrac{1}{(2\pi)^{3/2}} \textstyle\int_{\mathbb{R}^3} e^{-x_0|\xi|} \widehat{f_j}(\xi) e^{ix\cdot\xi}\, d\xi \\ &= \tfrac{i}{(2\pi)^{3/2}} \textstyle\int_{\mathbb{R}^3} \frac{\xi_j}{|\xi|} e^{-x_0|\xi|} \widehat{f_0}(\xi) e^{ix\cdot\xi}\, d\xi, \end{aligned} \tag{3.14}$$

which, together with (3.1), implies that $(F_0,\, F_1,\, F_2,\, F_3)$ satisfies the first equation of Cilmmino systems as in (3.1). We also discuss similarly to the second equation of Climmino systems as in (3.1), the details being ommited. Thus $F = F_0 + \sum_{j=1}^{3} F_j$ is a Fueter conjugate harmonic function.

Conversely, we let $(F_0,\, F_1,\, F_2,\, F_3)$ satisfy the first equation of Cilmmino systems (3.1) and (3.14). By this, (3.1) and the uniqueness of the classical Fourier transform, we obtain (3.12). This finishes the proof of Theorem 3.11. □

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