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Factorization and Generalized Roots of Dual Complex Matrices with Rodrigues' Formula

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Abstract. The paper provides an efficient method for obtaining powers and roots of dual complex 2×2 matrices based on a far reaching generalization of De Moivre's formula. We also resolve the case of normal 3×3 and 4×4 matrices using polar decomposition and the direct sum structure of $\mathfrak{so}_4$. The compact explicit expressions derived for rational powers formally extend (with loss of periodicity) to real, complex or even dual ones, which allows for defining some classes of transcendent functions of matrices in those cases without referring to infinite series or alternatively, obtain the sum of those series (explicit examples may be found in the text). Moreover, we suggest a factorization procedure for $\mathrm{M}(n, \mathbb{C}[\varepsilon])$, $n \leq 4$ based on polar decomposition and generalized Euler type procedures recently proposed by the author in the real case. Our approach uses dual biquaternions and their projective version referred to in the Euclidean setting as Rodrigues' vectors. Restrictions to certain subalgebras yield interesting applications in various fields, such as screw geometry extensively used in classical mechanics and robotics, complex representations of the Lorentz group in relativity and electrodynamics, conformal mappings in computer vision, the physics of scattering processes and probably many others. Here we only provide brief comments on these subjects with several explicit examples to illustrate the method.

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Introduction

Generalized Euler and Wigner decompositions began to emerge properly carried out in scientific literature surprisingly late (see for example [3,5,7,14]). Higher-dimensional extensions inevitably face the obstacle that invariant axes are not available in the generic case (see [2] for a detailed study). However, the complex vector-parameter construction introduced by Fedorov [10] as a generalization of a much earlier idea due to Rodrigues, allows for a treatment of the proper Lorentz group very similarly to the group of rotations in $\mathbb{R}^3$. We may extend this construction even further (see for example [1]) and take advantage of the convenience it provides in both theoretical and practical terms. In particular, here it is used as a base for generalizing the classical Rodrigues' rotation formula to the dual complex setting. The latter is then used for the derivation of factorization results analogous to those provided in [5,7]; dual axes and angles are interpreted geometrically in the spirit of Plücker coordinates, ruled surfaces and abelian subgroups. Particular real forms are discussed in the context of screw kinematics, quantum-mechanical scattering and special relativity. We also use the famous Lie algebra isomorphism $\mathfrak{so}_4 \cong \mathfrak{so}_3 \oplus \mathfrak{so}_3$ in order to extend our results to the group $\mathrm{O}(4, \mathbb{C}[\varepsilon])$.

As far as rational powers are concerned this paper is inspired by a recent study due to Özdemir [12] focused on the use of De Moivre's formula in the real setting. Here we point out that De Moivre's formula is a specific case of Rodrigues' rotation formula, which is more convenient to use in this context as it allows uniform treatment and natural transition from $\mathbb{R}$ to $\mathbb{C}$ and $\mathbb{C}[\varepsilon]$. For the latter we resort once more on the dual extension of the complex vector-parameter construction. The connection to the generalized Rodrigues' rotation formula is given by Cayley's transform and Euler's trigonometric substitution, both extended to dual complex axis-angle parameters. The formulas in all those cases, including the isotropic one which is considered separately, are far more general and at the same time simpler compared to previous studies confined only to the real setting. Concrete numerical examples and a brief discussion on certain practical applications are provided as well.

Preliminaries

In this short preliminary section we recall some basic facts about vectorial parametrization of $\mathrm{SO}(3)$, $\mathrm{ISO}(3)$ and $\mathrm{SO}^+(3,1)$ transformations—the curious reader may find more details in [1,10]. It is natural to begin our construction with an invertible quaternion $\zeta = (\zeta_\circ, \boldsymbol{\zeta}) \in \mathbb{H}^\times$, where the scalar and bivector parts are separated for convenience, after which perform the usual central projection to obtain the Rodrigues' parametrization[1] for $\mathrm{SO}(3)$

$$\zeta = (\zeta_\circ, \boldsymbol{\zeta}) \in \mathbb{H}^\times \cong \mathbb{R}^4 \ \xrightarrow{\pi}\ \mathbf{c} = \frac{\boldsymbol{\zeta}}{\zeta_\circ} \in \mathrm{SO}(3) \cong \mathbb{RP}^3. \tag{0.1}$$

[1]Here we mean only topological isomorphisms.

Quaternion multiplication then projects nicely on SO(3) and we have the composition law of vector-parameters expressed as

$$\langle\, \mathbf{c}_2, \mathbf{c}_1 \rangle = \frac{\mathbf{c}_2 + \mathbf{c}_1 + \mathbf{c}_2 \times \mathbf{c}_1}{1 - \mathbf{c}_2 \cdot \mathbf{c}_1} \tag{0.2}$$

where $\cdot$ and $\times$ denote respectively the dot and cross products in $\mathbb{R}^3$. More generally, for an arbitrary number of invertible quaternions $\zeta_i \in \mathbb{H}^\times$ we have

$$\langle \mathbf{c}_n, \mathbf{c}_{n-1}, \ldots, \mathbf{c}_1 \rangle = \frac{\langle \zeta_n \zeta_{n-1} \cdots \zeta_1 \rangle_2}{\langle \zeta_n \zeta_{n-1} \cdots \zeta_1 \rangle_0} \tag{0.3}$$

where $\langle \cdot \rangle_k$ denotes grade projection in the Clifford algebra. The corresponding matrix representation in $\mathbb{R}^3$ is given by means of the Cayley transform as[2]

$$\mathcal{R}(\mathbf{c}) = \frac{\mathcal{I} + \mathbf{c}^\times}{\mathcal{I} - \mathbf{c}^\times} = \frac{(1 - \mathbf{c}^2)\,\mathcal{I} + 2\,\mathbf{c}\mathbf{c}^t + 2\,\mathbf{c}^\times}{1 + \mathbf{c}^2}\cdot \tag{0.4}$$

The above expressions remain valid for complex vector-parameters $\mathbf{c} \in \mathbb{CP}^3$, which is linked with a similar construction to the even Liefschitz subgroup $\mathrm{GL}(2, \mathbb{C})$ of the space-time algebra $\mathrm{Cliff}_{3,1}(\mathbb{R}) \cong \mathrm{Cliff}_3(\mathbb{C})$, which yields the complex representation of the proper Lorentz group in special relativity

$$\mathrm{SO}^+(3, 1) \cong \mathrm{SO}_3(\mathbb{C}) \cong \mathrm{PSL}_2(\mathbb{C}). \tag{0.5}$$

The extension to the dual case is then performed using the transfer principle.

1. Powers and Roots of Complex Matrices

We begin by pointing out than the general linear group in $\mathbb{C}^2$ decomposes as $\mathbb{C}^\times \otimes \mathrm{SL}(2, \mathbb{C})$, so the proper Lorentz group is essential in factorization problems for GL_2 elements. In particular, the n-th root construction can be applied separately for the numerical pre-factor using De Moivre's formula and the Lorentzian component by means of a generalized Rodrigues' construction.

To proceed with the latter we recall that in the classical (real) case the link is given by the famous Euler's trigonometric substitution, namely substituting

$$\mathbf{c} = \tau \mathbf{n}, \qquad \tau = \tan \frac{\varphi}{2}, \qquad \mathbf{n} \in \mathbb{S}^2$$

into the Cayley transform (0.4), one easily gets the familiar

$$\mathcal{R}(\mathbf{n}, \varphi) = \cos \varphi\, \mathcal{I} + (1 - \cos \varphi)\, \mathbf{n}\mathbf{n}^t + \sin \varphi\, \mathbf{n}^\times \tag{1.1}$$

which may be written also as

$$\mathcal{R}(\mathbf{n}, \varphi) = \mathcal{P}_n^{\|} + \left(\cos \varphi + \sin \varphi\, \mathbf{n}^\times\right) \mathcal{P}_n^{\perp}$$

where $\mathcal{P}_n^{\|} = \mathbf{n}\mathbf{n}^t$ and $\mathcal{P}_n^{\perp} = \mathcal{I} - \mathbf{n}\mathbf{n}^t$ denote respectively the parallel and orthogonal projector in the direction determined by the unit vector $\mathbf{n}$. Note that since $\mathbf{n}^\times$ introduces a complex structure in the plane orthogonal to $\mathbf{n}$ (where it squares to $-\mathcal{I}$), fixing the invariant axis in (1.1) one obtains an action of $\mathbb{S}^1$ in $\mathbb{C}$ with the related De Moivre's formula. Analogous construction may be carried out in the complex setting as long as $\mathbf{c}^2 \neq 0$. Generally speaking, in

[2] Here $\mathcal{I}$ denotes the identity in $\mathbb{R}^3$ and $\mathbf{c}^\times$ the Hodge dual to $\mathbf{c}$, i.e., $\mathbf{c}^\times \mathbf{a} = \mathbf{c} \times \mathbf{a}\ \forall \mathbf{a} \in \mathbb{R}^3$.

this case $\mathcal{P}_n^{\|}$ and $\mathcal{P}_n^{\perp}$ lose their neat geometric interpretation while preserving their algebraic definition. Similarly, $\mathbf{n}$ is no longer associated with a unit sphere, but we still have $\mathbf{n}^2 = 1$ and finally, the "rotation angle" φ becomes complex as well. In certain cases it may be real or purely imaginary, which corresponds to the Wigner little groups associated with time-like and space-like momenta, respectively, while the isotropic case $\mathbf{c}^2 = 0$ is linked to the so-called "front form little group" E(2). A generic element of $\mathrm{SO}^+(3,1)$, however, would not be associated with an invariant plane and for its description we may use an additional phase, say $\arg\tau$, which describes that transition.

Coordinate Representation

Using the standard 2×2 matrix basis for Hamilton's quaternion units

$$\mathbf{i} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad \mathbf{j} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \mathbf{k} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \tag{1.2}$$

and the entries $\{\alpha_{ij}\}$ in the $\mathrm{SL}(2,\mathbb{C})$ representation

$$\zeta \quad \to \quad \mathcal{N} = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} = \begin{pmatrix} \zeta_\circ + i\zeta_1 & \zeta_2 + i\zeta_3 \\ i\zeta_3 - \zeta_2 & \zeta_\circ - i\zeta_1 \end{pmatrix}$$

it is straightforward to determine the complex vector-parameter as (see [4])

$$\mathbf{c} = \frac{-i}{\alpha_{11} + \alpha_{22}} \begin{pmatrix} \alpha_{11} - \alpha_{22} \\ i\alpha_{12} - i\alpha_{21} \\ \alpha_{12} + \alpha_{21} \end{pmatrix} \cdot \tag{1.3}$$

Certainly, (1.2) is equivalent to the basis of Pauli matrices over $\mathbb{C}$, but we choose to work with the former as it allows for borrowing the dot and cross product from the real setting so that formulas (0.2) and (0.4) apply directly. Next, we need to determine the complex axis and angle in the representation (1.1) corresponding to the orthogonal (with respect to the Killing form) transformation $X \to \zeta X \bar{\zeta}$ in $\mathrm{Mat}(2,\mathbb{C})$. It is clear from (0.1) and (1.3) that for a unimodular bi-quaternion $\zeta \in \mathrm{SL}(2,\mathbb{C})$, i.e., $\zeta_\circ^2 + \boldsymbol{\zeta}^2 = 1$, we have

$$1 + \mathbf{c}^2 = \zeta_\circ^{-2} = \frac{4}{\mathrm{tr}^2\zeta} \quad \Longrightarrow \quad \tau = \left(\zeta_\circ^{-2} - 1\right)^{\frac{1}{2}} = \left(4\,\mathrm{tr}^{-2}\zeta - 1\right)^{\frac{1}{2}}$$

which yields for the generalized complex angle

$$\varphi = 2\arctan\tau = -i\ln\frac{1 + i\tau}{1 - i\tau} \cdot \tag{1.4}$$

Note that the multi-valuedness of the complex logarithm may be ignored as it does not lead to new solutions. However the double-valuedness of the square root needs to be taken into consideration when determining the complex vector $\mathbf{n}$ in analogy with the agreement of orientation in the real case. Furthermore, we have a few very specific cases like the isotropic one $\tau = 0$ (for $\mathbf{c} \neq 0$) which is left for a separate treatment, or the infinite one $\tau = \infty$ which in the real setting corresponds to a half-turn, while here it may also be assigned to a tachyon (time-reversing) transforation in the analytic continuation of (0.1) to the entire Lorentz group. There are also forbidden values such as $\tau = \pm i$ (or in the isotropic case $\tau = \infty$) which has to do with the well-posedness of (0.4), hence the causal structure of Minkowski space-time.

Now, assuming $\tau \neq 0$ it is straightforward to determine the complex axis[3]

$$\mathbf{n} = \tau^{-1}\mathbf{c} = \left(1-\zeta_\circ^2\right)^{-\frac{1}{2}}\boldsymbol{\zeta} = \frac{-i\sigma}{\left(4-\mathrm{tr}^2\zeta\right)^{\frac{1}{2}}}\begin{pmatrix}\alpha_{11}-\alpha_{22}\\ i\alpha_{12}-i\alpha_{21}\\ \alpha_{12}+\alpha_{21}\end{pmatrix}, \qquad \sigma = \pm 1. \tag{1.5}$$

Note that since $\mathbf{n}^2 = 1$, the real and imaginary part of the above vector are normal to each other and this property is preserved in $\mathbf{c}$ as long as the factor τ is real or purely imaginary, but then $\zeta_\circ$ must be either real or purely imaginary too, or in other words, the central element does not mix scalar and pseudo-scalar components. These are the cases, in which one ends up with a plane transformation in the induced $\mathbb{R}^{3,1}$ representation of the proper Lorentz group, i.e., elements of a Wigner "little group", for which the corresponding generator lies within the image of a Plücker embedding (see [10] or [2] for details). In all other cases there are no proper invariant subspaces. These details play an important role in the generalized Euler and Wigner decomposition problems, but we shall first deal with the simpler task of applying the Rodrigues' formula for determining the fractional powers of an arbitrary complex 2×2 matrix $\mathcal{M}$, after which, obtain the dual extension of our results.

The Algorithm

The first step would be to write $\mathcal{M} = \lambda\mathcal{Q}$ with $\lambda \in \mathbb{C}^\times$ and $\mathcal{Q} \in \mathrm{SL}(2,\mathbb{C})$, which allows us to apply the classical De Moivre's formula

$$\lambda^\gamma_{(k)} = |\lambda^\gamma|\mathrm{e}^{i\gamma\psi}\varepsilon^k_\gamma, \qquad \psi = \arg\lambda$$

for the scalar factor, where the ε^k_γ's stand for the roots of unity

$$\varepsilon^k_\gamma = \mathrm{e}^{2ik\pi\gamma}, \qquad k = 0, 1\ldots\Delta\gamma - 1$$

and $\Delta\gamma$ denotes the denominator of γ. Then, we use the construction described above in order to express the Rodriges' transformation (1.1) corresponding to the unit biquaternion ζ. Certainly, it would be necessary to first check whether the so-obtained vector-parameter in formula (1.3) has a vanishing square. If not, one may proceed with the observation that (1.1) acts within the range of $\mathcal{P}_n^\perp$, where $\mathbf{n}^\times$ plays the role of a complex structure, as mentioned before, and we may apply the De Moivre's formula directly to determine all solutions (as usual, k varies from 0 to $\Delta\gamma - 1$) in the form

$$\mathcal{R}_{(k)} = \mathcal{R}(\mathbf{n}, \varphi_k), \qquad \varphi_k = \gamma(\varphi + 2k\pi)$$

with the obvious property $\mathcal{R}^{\Delta\gamma}_{(k)} = \mathcal{R}^{\gamma\Delta\gamma}(\mathbf{n}, \varphi)$. Now, we only need to "translate" the $\mathcal{R}_{(k)}$'s into $\mathrm{SL}(2,\mathbb{C})$ elements, say $\zeta^{(k)}$, using the obvious relation

$$\zeta_{\circ\pm}^{(k)} = \pm(1+\mathbf{c}^2_{(k)})^{-\frac{1}{2}}, \qquad \boldsymbol{\zeta}^{(k)}_\pm = \zeta_{\circ\pm}^{(k)}\mathbf{c}_{(k)}, \qquad \mathbf{c}_{(k)} = \tan\frac{\varphi_k}{2}\mathbf{n}$$

[3]The choice of complex root τ is linked to orientation via σ that is $\mathrm{sgn}\,\mathrm{tr}\zeta$ in the real case.

where the sign of the square root is chosen to agree with $\operatorname{tr}\zeta$. The explicit quaternion basis (1.2) allows for writing the above result in components as

$$
\begin{aligned}
&\zeta_{\pm}^{(k)} = \pm\left(\cos\frac{\varphi_k}{2},\ \sin\frac{\varphi_k}{2}\mathbf{n}\right) \ \to \\
&\mathcal{Q}_{(k)} = \csc\frac{\varphi}{2}\begin{bmatrix} \sin\dfrac{\varphi-\varphi_k}{2} + \alpha_{11}\sin\dfrac{\varphi_k}{2} & \alpha_{12}\sin\dfrac{\varphi_k}{2} \\ \alpha_{21}\sin\dfrac{\varphi_k}{2} & \sin\dfrac{\varphi-\varphi_k}{2} + \alpha_{22}\sin\dfrac{\varphi_k}{2} \end{bmatrix}
\end{aligned} \tag{1.6}
$$

where the parameter φ is given by (1.4) and one needs to be careful with signs in order to remain in the correct orbit, finally obtaining the solutions

$$
\mathcal{M}^{\gamma}_{(j,k)} = \lambda^{\gamma}_{(j)}\mathcal{Q}_{(k)}, \qquad j,k = 0,1\ldots\Delta\gamma - 1. \tag{1.7}
$$

In the isotropic setting $\mathbf{c}^2 = 0$ where the construction of $\mathbf{n}$ is pointless due to scale invariance, instead we work directly with formula (0.4) that reduces to

$$
\mathcal{R}(\mathbf{c})_{\mathbf{c}^2=0} = \exp(2\,\mathbf{c}^{\times}) = \mathcal{I} + 2\,\mathbf{c}^{\times} + 2(\mathbf{c}^{\times})^2 \tag{1.8}
$$

emphasising on the fact that $\mathbf{c}^{\times}$ is a nilpotent element of order three, which squares to $\mathbf{c}\mathbf{c}^t$. But then, all analytic functions of $\mathbf{c}^{\times}$ in the algebra generated by it have their Taylor series expansions reduced to quadratic polynomials

$$
f(\mathbf{c}^{\times}) = f(0)\,\mathcal{I} + f'(0)\,\mathbf{c}^{\times} + \frac{1}{2}f''(0)\,\mathbf{c}\mathbf{c}^t
$$

and in particular, for power functions $f = \mathcal{R}^{\gamma}(\mathbf{c})$ the chain rule yields

$$
\mathcal{R}^{\gamma}_{\mathbf{c}^2=0} = \exp(2\gamma\,\mathbf{c}^{\times}) = \mathcal{I} + 2\gamma\,\mathbf{c}^{\times} + 2\gamma^2\mathbf{c}\mathbf{c}^t, \qquad \gamma\in\mathbb{Q}. \tag{1.9}
$$

Note that $\exp(a)\exp(b) = \exp(a+b)$ holds in the nilpotent case too, but the exponential mapping loses its periodicity, so multiple roots in this setting come only from the scalar factor λ and the general solution has the form

$$
\mathcal{M}^{\gamma}_{(k)} = \lambda^{\gamma}_{(k)}\begin{bmatrix} \gamma\alpha_{11} - \gamma + 1 & \gamma\alpha_{12} \\ \gamma\alpha_{21} & \gamma\alpha_{22} - \gamma + 1 \end{bmatrix} \tag{1.10}
$$

where we take into account that in the isotropic case $|\operatorname{tr}\mathcal{M}| = 2$ and the vector-parameter of $\mathcal{R}^{\gamma}$ is simply $\gamma\mathbf{c}$. This completes the solution in the regular case. For singular 2×2 complex matrices $\mathcal{M}\in\operatorname{Mat}(2,\mathbb{C})$, on the other hand, is rather straightforward to see, either by Hamilton–Cayley theorem or using the proportionality of rows in a direct computation, that one has

$$
\mathcal{M}^2 = (\operatorname{tr}\mathcal{M})\,\mathcal{M} \quad \Rightarrow \quad \mathcal{M}^{\gamma} = (\operatorname{tr}\mathcal{M})^{\gamma-1}\,\mathcal{M} \tag{1.11}
$$

at least for γ positive integer. To prove that the above holds also for positive rational powers it is sufficient to substitute $\gamma = \dfrac{p}{q}$ in (1.11) and then

$$
\mathcal{M}^{\frac{p}{q}} = (\operatorname{tr}\mathcal{M})^{\frac{p-q}{q}}\,\mathcal{M} \quad \Rightarrow \quad \mathcal{M}^{p} = (\operatorname{tr}\mathcal{M})^{p-q}\,\mathcal{M}^{q} = (\operatorname{tr}\mathcal{M})^{p-1}\,\mathcal{M}.
$$

Since the rational powers of the complex number $\operatorname{tr}\mathcal{M}$ are multi-valued by De Moivre's formula, one generally has $q = \Delta\gamma$ solutions as long as $\operatorname{tr}\mathcal{M}\neq 0$. If the trace vanishes, $\mathcal{M}$ becomes nilpotent and (1.11) is no longer relevant.

The Dual Extension

Sometimes, e.g. in the screw formulation of kinematics, it is useful to add another nilpotent element ε (the so-called dual unit such that $\varepsilon^2 = 0$) as an extension to the ring of scalars. Thus we have dual numbers $\mathbb{C}[\varepsilon]$ defined as

$$\underline{w} = u + \varepsilon v, \qquad u, v \in \mathbb{C}$$

which immediately yields

$$\underline{w}^\gamma = u^\gamma + \varepsilon\gamma u^{\gamma-1} v \tag{1.12}$$

at least for positive integer values of γ and can be extended to arbitrary powers only for algebraically invertible elements, i.e., as long as $u \neq 0$. Furthermore, whenever the righthand side is well-defined we can generalize the above formula to Taylor series of analytic functions

$$f\left(\underline{w}\right) = f\left(u\right) + \varepsilon f'\left(u\right) v. \tag{1.13}$$

Similarly, one may define dual quaternions $\mathbb{H}[\varepsilon]$ or dual matrices in general

$$\underline{\mathcal{M}} = \mathcal{M} + \varepsilon\mathcal{N}, \qquad \mathcal{M}, \mathcal{N} \in \mathrm{Mat(n,\mathbb{C})}$$

but the relation (1.12) only holds in the commutative case $[\mathcal{M}, \mathcal{N}] = 0$. Moreover, whether it works also for negative and rational values depends on $\mathcal{M}$, namely if $\det\mathcal{M} = 0$, then $\gamma > 0$ and if also $\operatorname{tr}\mathcal{M} = 0$, the restriction is $\gamma \in \mathbb{N}$. This needs to be taken into account for the validity of (1.13) as well.

Next, we consider rational powers of $\underline{\mathcal{M}} = \mathcal{M} + \varepsilon\mathcal{N}$ with $[\mathcal{M}, \mathcal{N}] \neq 0$, first pointing out that in this more general setting instead of (1.12) one has

$$\underline{\mathcal{M}}^n = \mathcal{M}^n + \varepsilon\mathcal{M}^{n-1} \circ \mathcal{N}, \qquad n \in \mathbb{N} \tag{1.14}$$

where the operation $\circ$ is given by the anti-commutator $A \circ B = \{A, B\}$ as

$$\mathcal{M}^n \circ \mathcal{N} = \underbrace{\{\mathcal{M}, \{\mathcal{M}, \{\ldots\{\mathcal{M}}_{n}, \mathcal{N}\}\}\ldots\}$$

which may also be defined recursively (assuming as usual that $\mathcal{M}^0\circ = \mathrm{id}$)

$$\mathcal{M}^n \circ \mathcal{N} = \mathcal{M} \circ \left(\mathcal{M}^{n-1} \circ \mathcal{N}\right), \qquad \mathcal{M}^1 \circ \mathcal{N} = \mathcal{M}\mathcal{N} + \mathcal{N}\mathcal{M}$$

As long as $\mathcal{M}$ is invertible, the above can be expressed via the adjoint action

$$\mathrm{Ad}_{\mathcal{M}} \colon \mathcal{N} \to \mathcal{M}\mathcal{N}\mathcal{M}^{-1}$$

namely as

$$\mathcal{M}^n \circ \mathcal{N} = \mathcal{M}^n \left(\mathcal{I} + \mathrm{Ad}_{\mathcal{M}}^{-1}\right)^n \mathcal{N} = \left(\mathcal{I} + \mathrm{Ad}_{\mathcal{M}}\right)^n \mathcal{N}\mathcal{M}^n$$

where we make use the symmetry in Newton's binomial formula

$$\mathcal{M}^n \circ \mathcal{N} = \sum_{k=0}^{n} \binom{n}{k} \mathcal{M}^k \mathcal{N} \mathcal{M}^{n-k} = \sum_{k=0}^{n} \binom{n}{k} \mathcal{M}^{n-k} \mathcal{N} \mathcal{M}^{k}.$$

Extending this even only to negative powers is problematic, e.g. if -1 is in the spectrum of $\mathrm{Ad}_{\mathcal{M}}$, and it yields the wrong result, e.g. for $n = -1$ one has

$$\underline{\mathcal{M}}^{-1} = \mathcal{M}^{-1} - \varepsilon\mathcal{M}^{-1}\mathcal{N}\mathcal{M}^{-1}$$

which may be used to obtain

$$\underline{\mathcal{M}}^{-n} = \left(\underline{\mathcal{M}}^{-1}\right)^n = \mathcal{M}^{-n} - \varepsilon\mathcal{M}^{-1}\left(\mathcal{M}^{-1}\right)^{n-1} \circ \mathcal{N}\,\mathcal{M}^{-1} = \left(\underline{\mathcal{M}}^n\right)^{-1}.$$

As for fractional powers, let us first begin with the n-th root

$$\mathcal{P}+\varepsilon\mathcal{Q}=(\mathcal{M}+\varepsilon\mathcal{N})^{\frac{1}{n}} \;\Leftrightarrow\; (\mathcal{P}+\varepsilon\mathcal{Q})^n=\mathcal{M}+\varepsilon\mathcal{N}$$

that yields $\mathcal{P}=\mathcal{M}^{\frac{1}{n}}$ and

$$\mathcal{P}^{n-1}\circ\mathcal{Q}=\left(\mathcal{I}+\mathrm{Ad}^{\frac{1}{n}}_{\mathcal{M}}\right)^{n-1}\mathcal{M}^{1-\frac{1}{n}}\mathcal{Q}=\mathcal{N}$$

$$\Rightarrow\quad \mathcal{Q}=\mathcal{M}^{\frac{1}{n}-1}\left(\mathcal{I}+\mathrm{Ad}^{\frac{1}{n}}_{\mathcal{M}}\right)^{1-n}\mathcal{N}$$

so we may write the result for $\gamma=n^{-1}$ formally as

$$(\mathcal{M}+\varepsilon\mathcal{N})^{\gamma}=\mathcal{M}^{\gamma}+\varepsilon\mathcal{M}^{\gamma-1}\left(\mathcal{I}+\mathrm{Ad}^{\gamma}_{\mathcal{M}}\right)^{1-\frac{1}{\gamma}}\mathcal{N}. \tag{1.15}$$

Since $\mathrm{Ad}^{\gamma}_{\mathcal{M}}=\mathrm{Ad}_{\mathcal{M}^{\gamma}}$ whenever $\mathcal{M}^{\gamma}$ exists and the adjoint action is norm-preserving, we may use the binomial series expansion

$$\left(\mathcal{I}+\mathrm{Ad}^{\gamma}_{\mathcal{M}}\right)^{\alpha}=\sum_{k=0}^{\infty}\binom{\alpha}{k}\mathrm{Ad}^{k}_{\mathcal{M}^{\gamma}} \tag{1.16}$$

converging whenever $\Re(\alpha)>0$, or even for $\Re(\alpha)>-1$ as long as $-1\notin\mathrm{Ad}^{\gamma}_{\mathcal{M}}$.

The Unit Dual Sphere and Screw Motion

Although we have formally resolved the problem of rational powers for dual matrices (provided the solution for their real part is known), working with infinite operator series is not always the most practical way to do things. One particulary efficient alternative in the two-dimensional setting is to first factorize the initial matrix into a scalar component and a unit dual quaternion, in analogy with the real case

$$\mathrm{GL}(2,\mathbb{C}[\varepsilon])\;\ni\;\underline{\mathcal{M}}=\underline{\lambda}\underline{\mathcal{Q}},\qquad \underline{\lambda}\in\mathbb{C}[\varepsilon]/\{0\},\qquad \underline{\mathcal{Q}}\in\mathbb{S}^3[\varepsilon]$$

where the unit dual sphere is defined as the set

$$\mathbb{S}^3[\varepsilon]=\{\mathcal{Q}\in\mathbb{H}[\varepsilon],\;|\underline{\mathcal{Q}}|^2=\underline{\mathcal{Q}\bar{\mathcal{Q}}}=1\}$$

and one may parameterize it with pairs of unit and pure quaternions as

$$\underline{\mathcal{Q}}=\left(1+\frac{1}{2}\varepsilon t\right)r,\qquad |r|^2=1,\quad \bar{t}=-t$$

which guarantees both the proper normalization of $\underline{\mathcal{Q}\bar{\mathcal{Q}}}$ and the vanishing of the dual part. In mechanical context, r yields the rotational component of a rigid displacement[4], while t is related to the translation vector expressed in the quaternion basis (1.2). Since the rational powers of $\underline{\lambda}$ are easily dealt with using (1.12), we shall concentrate solely on the dual sphere resorting on a generalized De Moivre's formula (see [8] for detail), but for that we need to first introduce the dual angle $\underline{\varphi}=\varphi+\varepsilon d$ and axis vector $\underline{\mathbf{n}}=\mathbf{n}+\varepsilon\mathbf{m}\in\mathbb{S}^2[\varepsilon]$, i.e., $\mathbf{n}^2=1$ and $\mathbf{m}\perp\mathbf{n}$, associated with $\underline{\mathcal{Q}}$ and consider grade projections

$$\langle\underline{\mathcal{Q}}\rangle_0\;\Leftrightarrow\; r_0-\frac{1}{2}\varepsilon\,\mathbf{r}\cdot\mathbf{t}=\cos\frac{\underline{\varphi}}{2},\qquad \langle\underline{\mathcal{Q}}\rangle_2\;\Leftrightarrow\;\mathbf{r}+\frac{1}{2}\varepsilon\left(r_0\mathbf{t}+\mathbf{t}\times\mathbf{r}\right)=\underline{\mathbf{n}}\sin\frac{\underline{\varphi}}{2}$$

[4] Assuming rotation is applied first, otherwise the two factors need to switch places.

using the correspondence between vectors and pure quaternions $\langle r\rangle_2 \Leftrightarrow \mathbf{r}$. The polar representation $r_0 = \cos\frac{\varphi}{2}$, $\mathbf{r} = \sin\frac{\varphi}{2}\,\mathbf{n}$ and trigonometric identities

$$\sin(\varphi + \varepsilon d) = \sin\varphi + \varepsilon d\cos\varphi, \qquad \cos(\varphi + \varepsilon d) = \cos\varphi - \varepsilon d\sin\varphi \tag{1.17}$$

provide the screw displacement d and moment $\mathbf{m}$ in terms of $\mathbf{n}$, φ and $\mathbf{t}$

$$d = \mathbf{t}\cdot\mathbf{n}, \qquad \mathbf{m} = \frac{1}{2}\left(\cot\frac{\varphi}{2}\,\mathcal{I} - \mathbf{n}^{\times}\right)\mathcal{P}^{\perp}_{\mathbf{n}}\mathbf{t} \tag{1.18}$$

and thus, Plücker coordinates of the screw axis $\underline{\mathbf{n}}$ given by Mozzi-Chasles theorem (see [11]) and the dual angle that takes into account the $\mathbf{n}$-projection of the translation vector $\mathbf{t}$. Now, the generalized Euler's formula asserts that

$$\left(\cos\frac{\underline{\varphi}}{2},\ \underline{\mathbf{n}}\sin\frac{\underline{\varphi}}{2}\right)^k \Leftrightarrow \left(\cos\frac{k\underline{\varphi}}{2},\ \underline{\mathbf{n}}\sin\frac{k\underline{\varphi}}{2}\right), \qquad k\in\mathbb{Z} \tag{1.19}$$

and as in the classical case, its extension to rational powers $\gamma\in\mathbb{Q}$ is obtained by simply substituting $k\underline{\varphi}$ above with $\underline{\varphi_k} = \gamma(\underline{\varphi}+4k\pi)$ for $k = 0, 1, \ldots \Delta\gamma - 1$.

Similarly, in the 3×3 representation we work with the dual extensions of (0.4) and (1.1) that are rather straightforward if we use the screw axis-angle notation (see [9,13] for details), e.g. for $\varphi\neq 0$ one has

$$\underline{\mathbf{c}} = \tan\frac{\varphi+\varepsilon d}{2}\,(\mathbf{n}+\varepsilon\mathbf{m}) = \tan\frac{\varphi}{2}\left(1 + \frac{\varepsilon\, d}{\sin\varphi}\right)\underline{\mathbf{n}} \tag{1.20}$$

with displacement d and moment $\mathbf{m}$ provided by (1.18). While the geometric meaning of the former is pretty clear, for the latter we have a mechanical analogy encoded in the identity $\mathbf{m} = \mathbf{r}\times\mathbf{n}$, where $\mathbf{r}$ denotes the radius-vector of the screw line. Thus, it is easy to express $\mathbf{r} = \mathbf{n}\times\mathbf{m}$. Note, however, that both the screw pitch $p = d\csc\varphi$ in (1.20) and the moment $\mathbf{m}$ in (1.18) are ill-defined in the case of trivial rotation $\varphi = 0$. A pure translational screw acts on lines in $\mathbb{E}^3$ expressed with their Plücker coordinates as $\ell = \boldsymbol{\nu} + \varepsilon\boldsymbol{\mu}$ as $\mathcal{I} + \varepsilon\mathbf{t}^{\times}$, thus preserving the orientation $\boldsymbol{\nu}$ and adds $\mathbf{t}\times\boldsymbol{\nu}$ to the moment $\boldsymbol{\mu}$, i.e., the radius-vector is altered with $\mathcal{P}^{\perp}_{\boldsymbol{\nu}}\mathbf{t}$ as it should be. So, one has

$$\varphi = 0 \quad \Leftrightarrow \quad \underline{\mathbf{c}} = \frac{\varepsilon}{2}\,\mathbf{t}, \qquad \mathbf{n}\times\mathbf{t} = \mathbf{m} = 0 \tag{1.21}$$

and with that clarification both the composition law (0.2) and Rodrigues' rotational formula (1.1) apply for the dual vector-parameter (1.20) and respectively, the dual axis-angle coordinates. So does the above described approach would work also in this case, keeping in mind the singularity (1.21). Naturally, our construction works also over $\mathbb{C}[\varepsilon]$ although the kinematical interpretation is not that straightforward: dual bi-quaternions do not parameterize the Poincaré group in the way Euclidean motions are represented in the real setting. Note also that there are two nilpotent elements in the complex isotropic setting: one is ε itself and the other is $\underline{\mathbf{c}}^{\times}$ whenever $\underline{\mathbf{c}}^2 = 0$. Some of these cases are illustrated with examples at the end of this section.

A Hierarchy of Trigonometric Identities

The solutions (1.6) in the regular non-isotropic setting can be written also in the form (for convenience of notation we have doubled all the angles)

$$\mathcal{Q}^{\gamma} = \sin^{-1}\varphi\,\left(\sin\left(\varphi - \varphi_l\right)\mathcal{I} + \sin\varphi_l\,\mathcal{Q}\right), \qquad \varphi_l = \gamma(\varphi + 2l\pi).$$

Denotind $\gamma = \frac{m}{n} \in \mathbb{Q}$ and applying Newton's binomial formula to the equality $(\mathcal{Q}^\gamma)^n = \mathcal{Q}^m$ yields for $l = 0, 1, \dots n-1$ a series of $2n$ trigonometric identities in the form (here we assume $m \neq 0$ and $n \geq 2$)

$$\sum_{k=0}^{n} \binom{n}{k} \sin^k\varphi_l \, \sin^{n-k}(\varphi - \varphi_l) \cos k\varphi = \sin^n\varphi \, \cos m\varphi$$
$$\sum_{k=0}^{n} \binom{n}{k} \sin^k\varphi_l \, \sin^{n-k}(\varphi - \varphi_l) \sin k\varphi = \sin^n\varphi \, \sin m\varphi \qquad (1.22)$$

obtained via separation of the real and imaginary parts and De Moivre's formula for the matrix $\mathcal{Q} \in \mathrm{SU}(2)$. Note that some roots φ_l lead to identical formulas, e.g. in the case $n = 2$ one has the same result for $l = 0$ and $l = 1$. Finally, there is no problem in principle to apply the solutions (1.6) in the case of arbitrary (real or complex) parameter $\gamma \neq 0, 1$. Thus we lose the periodicity and are left with only $\varphi_0 = \gamma\varphi$, leads to identities involving infinite binomial series:

$$\sum_{k=0}^{\infty} \binom{\alpha}{k} \sin^k\varphi_0 \, \sin^{\alpha-k}(\varphi - \varphi_0) \cos k\varphi = \sin^\alpha\varphi \, \cos\varphi$$
$$\sum_{k=0}^{\infty} \binom{\alpha}{k} \sin^k\varphi_0 \, \sin^{\alpha-k}(\varphi - \varphi_0) \sin k\varphi = \sin^\alpha\varphi \, \sin\varphi \qquad (1.23)$$

with $\varphi_0 = \alpha^{-1}\varphi \neq \varphi$, e.g. for $\alpha = -1$ and $\varphi \in \mathbb{R}$ one has

$$\sum_{k=1}^{\infty} \frac{1}{2^k} \frac{\cos k\varphi}{\cos^k\varphi} = \cos 2\varphi, \qquad \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{\sin k\varphi}{\cos^k\varphi} = \sin 2\varphi, \qquad \varphi \in \Big(-\frac{\pi}{3}, \frac{\pi}{3}\Big).$$

Let us also point out that the above trigonometric identities may be extended to the dual complex setting via the transfer principle: (1.22) and (1.23) apply in the case $\varphi, \gamma \in \mathbb{C}[\varepsilon]$ and $\varphi_0 = \gamma\varphi$, with the aid of (1.12) and (1.17), e.g. one has $x^{\varepsilon d} = 1 + \varepsilon d x^{-1}$ for $x \neq 0$ etc., although 2π-periodicity is possible only if $\gamma \in \mathbb{Q}[\varepsilon]$ or $\gamma \in i\mathbb{Q}[\varepsilon]$.

There are subtle questions around series convergence, analytic continuation and multi-valuedness related to the general identities (1.23). Their detailed treatment, however, although quite interesting, goes beyond the scope of the present paper is therefore left to the reader's curiosity.

Extension to 3×3 and 4×4 Normal Matrices

Next, we consider the 3D and 4D complex settings using (left) polar decomposition that allows us to represent an arbitrary $\mathcal{M} \in \mathrm{Mat}(n, \mathbb{C})$ as

$$\mathcal{M} = \mathcal{S}\mathcal{R}, \qquad \mathcal{R} \in \mathrm{SO}_n(\mathbb{C})$$

with $\mathcal{R}$ orthogonal and $\mathcal{S} = \sqrt{\mathcal{M}\mathcal{M}^t}$-symmetric. Alternatively, we may use the right decomposition $\mathcal{M} = \mathcal{R}\mathcal{S}'$ where $\mathcal{S}' = \sqrt{\mathcal{M}^t\mathcal{M}}$. Although the very definition of $\mathcal{S}$ and $\mathcal{S}'$ already involves fractional powers, one can easily obtain it in their canonical bases. Moreover, if $\mathcal{M}$ is normal, i.e., $[\mathcal{M}, \mathcal{M}^t] = 0$, then obviously $\mathcal{S} = \mathcal{S}'$ and it is straightforward to proceed as

$$[\mathcal{M}, \mathcal{M}^t] = 0 \quad \Rightarrow \quad (\mathcal{S}\mathcal{R})^\gamma = \mathcal{S}^\gamma\mathcal{R}^\gamma.$$

The left factor is dealt with using eigenvalues and eigenvectors, while for the right one we extract the angle φ and use (1.1) with $\varphi_k = \gamma(\varphi + 2k\pi)$, $k = 0, 1, \dots \Delta\gamma - 1$, starting with (0.4), which in the case $\mathcal{R}^t \neq \mathcal{R}$ yields

$$\mathbf{c}^\times = \frac{\mathcal{I} - \mathcal{R}}{\mathcal{I} + \mathcal{R}} = \frac{\mathcal{R} - \mathcal{R}^t}{1 + \operatorname{tr}\mathcal{R}}\cdot \tag{1.24}$$

Thus, one may derive $\mathbf{c}$ via Hodge duality, while the scalar parameter τ is more directly given by the identity $1 + \mathbf{c}^2 = \det(\mathcal{I} - \mathbf{c}^\times)$ and formula (1.4). In the symmetric setting we clearly have an involution since $\mathcal{R}^t = \mathcal{R}^{-1}$, so the only possible eigenvalues are ± 1 and the invariant axis is derived as an eigenvector of $\mathcal{R} = 2\mathbf{n}\mathbf{n}^t - \mathcal{I}$, while in the isotropic case $\mathbf{c}^2 = 0$ we simply multiply $\mathbf{c}$ in (1.8) by γ, rather than dealing with the generalized angle φ.

The 4×4 setting still allows for using a vector-parameters technique due to the direct sum structure of the corresponding Lie algebra $\mathfrak{so}_4 \cong \mathfrak{so}_3 \oplus \mathfrak{so}_3$, which yields for the orthogonal group, at least locally $\mathbf{c} \otimes \tilde{\mathbf{c}} \in \mathbb{CP}^3 \otimes \mathbb{CP}^3$. The composition law (0.2) and Rodrigues' formula (1.1) apply to each copy in the tensor product and one can easily relate them to the 4×4 block-matrix

$$\mathcal{R}(\mathbf{c} \otimes \tilde{\mathbf{c}}) = \chi^{-1} \begin{pmatrix} 1 - \mathbf{c}\cdot\tilde{\mathbf{c}} + \mathbf{c}\,\tilde{\mathbf{c}}^t + \tilde{\mathbf{c}}\,\mathbf{c}^t + (\mathbf{c} + \tilde{\mathbf{c}})^\times & \mathbf{c} - \tilde{\mathbf{c}} + \tilde{\mathbf{c}}\times\mathbf{c} \\ (\tilde{\mathbf{c}} - \mathbf{c} + \tilde{\mathbf{c}}\times\mathbf{c})^t & 1 + \mathbf{c}\cdot\tilde{\mathbf{c}} \end{pmatrix} \tag{1.25}$$

with $\chi = \sqrt{(1 + \mathbf{c}^2)(1 + \tilde{\mathbf{c}}^2)}$ that may be obtained from the entries of the skew-symmetric matrix $\tilde{\mathcal{R}} = \mathcal{R} - \mathcal{R}^t$ in the form (see [4] for details)

$$\mathbf{c} = \frac{1}{\operatorname{tr}\mathcal{R}} \begin{pmatrix} \tilde{\mathcal{R}}_{32} + \tilde{\mathcal{R}}_{14} \\ \tilde{\mathcal{R}}_{13} + \tilde{\mathcal{R}}_{24} \\ \tilde{\mathcal{R}}_{21} + \tilde{\mathcal{R}}_{34} \end{pmatrix}, \qquad \tilde{\mathbf{c}} = \frac{1}{\operatorname{tr}\mathcal{R}} \begin{pmatrix} \tilde{\mathcal{R}}_{32} - \tilde{\mathcal{R}}_{14} \\ \tilde{\mathcal{R}}_{13} - \tilde{\mathcal{R}}_{24} \\ \tilde{\mathcal{R}}_{21} - \tilde{\mathcal{R}}_{34} \end{pmatrix}\cdot \tag{1.26}$$

Finally, with the dual extension $\mathbb{C} \to \mathbb{C}[\varepsilon]$ one needs to first perform polar decomposition in $\operatorname{Mat}(n, \mathbb{C}[\varepsilon])$ factorizing $\underline{\mathcal{M}} = \underline{\mathcal{S}\mathcal{R}}$ into a dual symmetric and dual orthogonal factor, respectively. For the former we have from (1.14)

$$\underline{\mathcal{S}} = (\mathcal{M}\mathcal{M}^t)^{\frac{1}{2}} + \frac{1}{2}(\mathcal{M}\mathcal{M}^t)^{-\frac{1}{2}}\varepsilon(\mathcal{M}\mathcal{N}^t + \mathcal{N}\mathcal{M}^t).$$

while $\underline{\mathcal{R}}$ is constructed using the eigenvector basis of the symmetric operator $\underline{\mathcal{M}\mathcal{M}^t}$ in the usual way. For computational purposes, however, it is convenient to use Newton's iterative method that yields the successive approximations

$$\underline{\mathcal{R}}_{k+1} = \frac{1}{2}\left(\underline{\mathcal{R}}_k + \underline{\mathcal{R}}_k^{-t}\right), \qquad \underline{\mathcal{R}}_0 = \underline{\mathcal{M}}$$

which afterwards gives also the symmetric part simply $\mathcal{S} = \mathcal{M}\mathcal{R}^t$.

Applying the Transfer Principle

The transfer principle provides an efficient means of extending results to a broader context. Initially formulated for the purposes of non-standard analysis, it was meant to justify well-known theorems of real calculus to the hyperreal extension of $\mathbb{R}$, where new (infinitely small and large) elements are introduced. Although the case of dual numbers—$\mathbb{R}[\varepsilon]$ or $\mathbb{C}[\varepsilon]$, is slightly different, there are some similarities as well, e.g. the nilpotent element ε may be used in the definition and computation of the first derivative[5] via formula (1.13). Here we are going to use the transfer principle in a slightly different manner: considering all results obtained so far as the zeroth-order term of a Taylor series in a dual-valued functional equality. More precisely, the dual extensions of the matrices, quaternions, generalized rotational axes and angles considered above need to satisfy the same relations as the classical ones, e.g. (1.6), (1.10) and (1.11) extend nicely to the dual case $\underline{\mathcal{M}} \in \mathrm{Mat}(2, \mathbb{C}[\varepsilon])$ with the geometric interpretation of the screw axis $\underline{\mathbf{n}}$ and angle $\underline{\varphi}$ given above. A similar approach proves fruitful also in the case of 3×3 and 4×4 normal dual matrices if we combine it with polar decomposition as already explained. Higher-dimensional generalizations of our technique, however, are not always possible even in the normal setting and the reason is that orthogonal transformations are usually not decomposable, i.e., there are non-trivial Plücker relations to be satisfied. As discussed in [2], the vectorial parametrization allows for expressing them as $\Im\, \mathbf{c}^2 = 0$ and standard projective-geometric incidence relations allow for explicitly obtaining the embedding of the SL_2 group representation into higher-dimensional space. Yet, the construction applies both to the real and complex settings, together with their dual extensions. Consider for instance one such extension of formula (1.6) in the particular case of a square root, in which it reduces to

$$\sqrt{\underline{\mathcal{M}}} = \pm\frac{1}{2}\sec\frac{\underline{\varphi}}{4}\,(\mathcal{I} + \underline{\mathcal{M}}) = \pm\frac{1}{2\cos\frac{\varphi}{4}}\left[\left(1 + \frac{\varepsilon d}{4}\tan\frac{\varphi}{4}\right)(\mathcal{I} + \mathcal{M}) + \varepsilon\mathcal{N}\right]$$

where we have in quaternion terms

$$\mathcal{M} \sim \left(\cos\frac{\varphi}{2},\, \sin\frac{\varphi}{2}\mathbf{n}\right), \qquad \mathcal{N} \sim \frac{d}{2}\left(-\sin\frac{\varphi}{2},\, \cos\frac{\varphi}{2}\mathbf{n}\right) + \left(0,\, \sin\frac{\varphi}{2}\mathbf{m}\right)\cdot$$

Similarly, in the nilpotent case $\underline{\mathbf{c}}^2 = 0$, (1.10) yields for the square root

$$\sqrt{\underline{\mathcal{M}}} = \pm\frac{\sqrt{\lambda}}{2}\,(\mathcal{I} + \underline{\mathcal{M}})$$

while in the singular setting $\det \underline{\mathcal{M}} = 0$ from (1.11) one has

$$\sqrt{\underline{\mathcal{M}}} = \pm\,(\mathrm{tr}\,\underline{\mathcal{M}})^{-\frac{1}{2}}\,\underline{\mathcal{M}}.$$

Note that the pre-factor used for normalization is generally in $\mathbb{C}[\varepsilon]$, which is not a division ring, so we face certain difficulties in cases such as $\det \underline{\mathcal{M}} = 0$ with $\mathrm{tr}\,\underline{\mathcal{M}} \in \mathbb{C}[\varepsilon]/\mathbb{C}$ or $\det \underline{\mathcal{M}} \in \mathbb{C}[\varepsilon]/\mathbb{C}$: the former is obviously nilpotent, so only positive integer powers exist, while for the latter both the dual quaternion construction and the more general solution given by formula (1.15) fail.

Numerical Examples

We begin with the relatively simple square root: consider the matrix

$$\underline{\mathcal{M}} = \begin{pmatrix} \sqrt{3} + (3i-4)\varepsilon & 3i \\ 3i & \sqrt{3} - (3i+4)\varepsilon \end{pmatrix}, \qquad \lambda^2 = \det \underline{\mathcal{M}} = 12 - 8\sqrt{3}\varepsilon$$

[5] This approach is sometimes referred to in literature as "automatic differentiation".

which, upon division with $\lambda = 2(\sqrt{3} - \varepsilon)$, yields the unit dual biquaternion

$$\underline{\mathcal{Q}} = \frac{1}{2}\begin{pmatrix} 1 + \varepsilon\sqrt{3}(i-1) & i(\sqrt{3}+\varepsilon) \\ i(\sqrt{3}+\varepsilon) & 1 - \varepsilon\sqrt{3}(i+1) \end{pmatrix}$$

and we can use (1.3) to determine the Rodrigues' vector in the basis (1.2)

$$\underline{\mathbf{c}} = \frac{1}{1-\varepsilon\sqrt{3}}\left(\varepsilon\sqrt{3},\, 0,\, \varepsilon + \sqrt{3}\right)^t, \qquad \tau = \sqrt{\underline{\mathbf{c}}^2} = \sqrt{3} + 4\varepsilon.$$

Hence, from (1.4) and (1.20) we easily retrieve the screw axis and angle

$$\underline{\mathbf{n}} = (\varepsilon,\, 0,\, 1)^t, \qquad \underline{\varphi} = \frac{2\pi}{3} + 2\varepsilon$$

which yields the solutions based on formula (1.6) explained in previous section

$$\sqrt{\underline{\mathcal{M}}} = \pm\frac{\sqrt{\lambda}}{2}\left[\begin{pmatrix} \sqrt{3} & i \\ i & \sqrt{3} \end{pmatrix} + \frac{\varepsilon}{2}\begin{pmatrix} 2i-1 & i\sqrt{3} \\ i\sqrt{3} & -2i-1 \end{pmatrix}\right].$$

Next, we illustrate the isotropic setting with $\underline{\mathbf{c}} = (-i,\, \varepsilon,\, 1)^t$ and using (1.10) obtain (note that the sine and cosine of the matrix logarithm appear below)

$$\begin{pmatrix} 2 & i+\varepsilon \\ i-\varepsilon & 0 \end{pmatrix}^{\frac{7}{8}} = \frac{\sqrt[8]{1}}{8}\begin{pmatrix} 15 & 7i+7\varepsilon \\ 7i-7\varepsilon & 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & i+\varepsilon \\ i-\varepsilon & 0 \end{pmatrix}^{-i} = \begin{pmatrix} 1-i & 1-i\varepsilon \\ 1+i\varepsilon & 1+i \end{pmatrix}$$

where $\sqrt[k]{1}$ are the complex roots given by the classical De Moivre's formula.

Finally, let us consider a 4×4 dual complex orthogonal matrix in the form[6]

$$\underline{\mathcal{R}} = \begin{pmatrix} -1 & -1 & -i & \varepsilon \\ 1 & -1 & & i \\ -i & \varepsilon & 1 & 1 \\ \varepsilon & i & -1 & 1 \end{pmatrix} \quad \Leftrightarrow \quad \underline{\mathbf{c}} \otimes \underline{\tilde{\mathbf{c}}} = \begin{pmatrix} 0 \\ 0 \\ \infty \end{pmatrix} \otimes \begin{pmatrix} -i \\ \varepsilon \\ 1 \end{pmatrix}$$

so the parametrization of $\underline{\mathcal{R}}^{\frac{1}{3}}$ is given by $\frac{1}{\sqrt{3}}(0,\, 0,\, 1)^t \otimes \frac{1}{3}(-i,\, \varepsilon,\, 1)^t$ since $\underline{\mathbf{c}}$ stands for a half-turn ($\varphi = \pi$) while $\underline{\tilde{\mathbf{c}}}^2 = 0$. Hence, applying once more formula (1.25) we easily obtain

$$\sqrt[3]{\underline{\mathcal{R}}} = \frac{\sqrt[3]{1}}{6}\begin{pmatrix} 3\sqrt{3}-1 & -\sqrt{3}-3 & \varepsilon\sqrt{3}-i & i\sqrt{3}+\varepsilon \\ \sqrt{3}+3 & 3\sqrt{3}-1 & i\sqrt{3}+\varepsilon & i-\varepsilon\sqrt{3} \\ -i-\varepsilon\sqrt{3} & \varepsilon-i\sqrt{3} & 3\sqrt{3}+1 & 3-\sqrt{3} \\ \varepsilon-i\sqrt{3} & i+\varepsilon\sqrt{3} & \sqrt{3}-3 & 3\sqrt{3}+1 \end{pmatrix}.$$

[6] We use l'Hôpital's rule since $\operatorname{tr}\underline{\mathcal{R}} = 0$ and thus (1.26) cannot be applied directly here.

2. Generalized Euler Decompositions

Although the generalized Rodrigues' formula considered above is just one of the many ways to approach the problem of rational powers, it really shines beyond comparison when it comes to Euler-type decompositions considered below in this section. We begin by briefly revising and extending the technique used in [5,7] to the complex and dual settings, pointing out that apart from the obvious similarities to the real case, there are significant differences as well. For instance, in dimensions higher than three we talk about rotation planes rather than rotation axes and their existence is ensured only on a zero measure set given by the famous Plücker embedding (see [2]). As for the complex case, where the geometric intuition may often be misleading, we consider the linear span of a three-vector, which may be thought of as a point in $\mathbb{CP}^2$, and associate it with a commutative subgroup of $\mathrm{SO}(3,\mathbb{C})$ given by (0.4) and (1.1). We shall assume for now that these axis vectors $\mathbf{a}_i$ are non-isotropic so one can normalize $\mathbf{a}_i \to \hat{\mathbf{a}}_i$ with $\hat{\mathbf{a}}_i^2 = 1$ and of course, it makes sense to demand also that $\mathbf{a}_2$ is not aligned with $\mathbf{a}_1$ or $\mathbf{a}_3$, although for the latter two no such restriction holds. Moreover, we assign to the vectors $\hat{\mathbf{a}}_i$ the Gram matrix G with entries $g_{ij} = \hat{\mathbf{a}}_i \cdot \hat{\mathbf{a}}_j$ and its adjoint $G^{\#}$ with entries g^{ij} equal to the co-factors of g_{ij}. We also define the adjoint system of vectors

$$\hat{\mathbf{a}}^i = -\frac{1}{2}\varepsilon^{ijk}\hat{\mathbf{a}}_j \times \hat{\mathbf{a}}_k$$

where ε^{ijk} are the components of the skew-symmetric Levi–Civita symbol and Einstein summation is assumed. Note that the $\hat{\mathbf{a}}^i$'s defined in this manner are not unit in general and one has $g^{ij} = \hat{\mathbf{a}}^i \cdot \hat{\mathbf{a}}^j$. This construction resembles the one used for conjugate bases but $\{\hat{\mathbf{a}}_i\}$ and $\{\hat{\mathbf{a}}^i\}$ may not be bases as the volume forms $\omega = (\hat{\mathbf{a}}_1, \hat{\mathbf{a}}_2, \hat{\mathbf{a}}_3)$ and $\omega^{\#} = (\hat{\mathbf{a}}^1, \hat{\mathbf{a}}^2, \hat{\mathbf{a}}^3)$ are allowed to vanish.

Next, we denote $r_{ij} = \hat{\mathbf{a}}_i \cdot \mathcal{R}\hat{\mathbf{a}}_j$ and similarly $r^{ij} = \hat{\mathbf{a}}^i \cdot \mathcal{R}\hat{\mathbf{a}}^j$. For the mixed entries, however, one needs to be careful in the way indices are ordered since $\mathcal{R}$ is neither symmetric, nor skew-symmetric, so to avoid confusion, we shall use proper spacing, e.g. $r^i_{\ j} = \hat{\mathbf{a}}^i \cdot \mathcal{R}\hat{\mathbf{a}}_j$ and $r_i^{\ j} = \hat{\mathbf{a}}_i \cdot \mathcal{R}\hat{\mathbf{a}}^j$. We also consider the system of invariant vectors $\{\hat{\mathbf{a}}'_i\}$ for a given decomposition in the body rotating frame. The two are linked by a curious relation (see [7] for details)

$$\mathcal{R} = \mathcal{R}_3\mathcal{R}_2\mathcal{R}_1 = \mathcal{R}'_1\mathcal{R}'_2\mathcal{R}'_3$$

where $\mathcal{R}_i$ and $\mathcal{R}'_i$ leave invariant $\hat{\mathbf{a}}_i$ and $\hat{\mathbf{a}}'_i$, respectively. Applying the transformations consecutively we obtain for the vectors $\hat{\mathbf{a}}'_i$ defining the frame in motion (note that, as shown in [7], we have for the scalar parameters $\tau'_i = \tau_i$)

$$\hat{\mathbf{a}}'_3 = \hat{\mathbf{a}}_3, \qquad \hat{\mathbf{a}}'_2 = \mathcal{R}_1\,\hat{\mathbf{a}}_2, \qquad \hat{\mathbf{a}}'_1 = \mathcal{R}\,\hat{\mathbf{a}}_1$$

so the associated Gram matrix $\tilde{G}$ has the same entries as G except for

$$\tilde{g}_{13} = \tilde{g}_{31} = r_{31}.$$

We also denote $g = \det G$ with the obvious relation $\omega = \sqrt{g}$, which holds also for $\tilde{g}$ and $\tilde{\omega}$ as well. Next, following [5,7], one may express different matrix entries of $\mathcal{R}$ in the $\{\hat{\mathbf{a}}_i\}$ frame using formula (0.4), thus obtaining a system of quadratic equations for the scalar parameters τ_i of the $\mathcal{R}_i$'s, starting with

$$(g^{31}+\tilde{g}^{31})\tau_2^2 - 2\omega\tau_2 + g_{31} - r_{31} = 0$$

where the discriminant is $4\tilde{\omega}^2$ and we apply the identities

$$\tilde{g} - g = (\tilde{\omega}-\omega)(\tilde{\omega}+\omega) = (r_{31}-g_{31})(\tilde{g}^{31}+g^{31})$$

to obtain the two solutions (assuming in the second equality $r_{31} \neq g_{31}$)

$$\tau_2^{\pm} = \frac{\omega \pm \tilde{\omega}}{g^{31}+\tilde{g}^{31}} = \frac{g_{31}-r_{31}}{\omega \mp \tilde{\omega}}$$

at least for the regular case, in which both the numerator and the denominator above are non-vanishing. For the other two parameters it is convenient to use respectively r_{32} and r_{21} noting that the Gramm matrix $\tilde{G}$ can actually be obtained equivalently if in the system of vectors $\{\hat{\mathbf{a}}_i\}$ we only substitute $\hat{\mathbf{a}}_1 \to \mathcal{R}_2\hat{\mathbf{a}}_1$. Similarly, changing instead $\hat{\mathbf{a}}_3 \to \mathcal{R}^t\hat{\mathbf{a}}_3$ or $\hat{\mathbf{a}}_1 \to \mathcal{R}\hat{\mathbf{a}}_1$ one obtains respectively $G_{(1)}$ and $G_{(3)}$ used below

$$(g^{23}+g^{23}_{(1)})\tau_1^2 + 2\tilde{\omega}\tau_1 + g_{32} - r_{32} = 0$$
$$(g^{12}+g^{12}_{(3)})\tau_3^2 + 2\tilde{\omega}\tau_3 + g_{21} - r_{21} = 0$$

and with the notation $\omega_1 = r_3^3$ and $\omega_3 = r_1^1$ (note that $\omega_i^2 = g_{(i)}$) we have

$$g_{(1)} - \tilde{g} = (\omega_1-\tilde{\omega})(\omega_1+\tilde{\omega}) = (r_{32}-g_{32})(g^{32}_{(1)}+\tilde{g}^{32})$$
$$g_{(3)} - \tilde{g} = (\omega_3-\tilde{\omega})(\omega_3+\tilde{\omega}) = (r_{21}-g_{21})(g^{21}_{(3)}+\tilde{g}^{21}).$$

Next, we denote for convenience $g^{ij}_{(2)} = g^{ij}$ (as well as $\omega_2 = \omega$) and using a simple continuity argument (see [5] for details) obtain the solutions[7]

$$\tau_i^{\pm} = \frac{\omega_i \mp \tilde{\omega}}{g^{jk}_{(i)}+\tilde{g}^{jk}} = \varepsilon^{ijk}\,\frac{g_{jk}-r_{jk}}{\omega_i \pm \tilde{\omega}}, \qquad j>k. \tag{2.1}$$

Note that the necessary and sufficient condition from the real case $\tilde{g} \geq 0$ can be dropped here, while the calculations remain identical. There are some specific cases, for which the proper solution needs to be discussed separately, e.g. assuming $\mathcal{R}=\mathcal{I}$ from the first expression in formula (2.1) we easily obtain

$$\tau_i^- = \frac{\omega}{g^{jk}}, \qquad \tau_i^+ = 0, \qquad j>k$$

that yields only the trivial solution in the case $\omega = 0$ unless $\hat{\mathbf{a}}_1$ and $\hat{\mathbf{a}}_3$ are proportional, which might be important if one wishes to consider infinitesimal transformations as it has been done in [6]. Similarly, whenever

$$r_{jk} = g_{jk}, \qquad j>k$$

one may decompose as $\mathcal{R} = \mathcal{R}_j\mathcal{R}_k$ with scalar parameters

$$\tau_j = \varepsilon^{ijk}\frac{\varsigma^i}{g_{j[j}\varsigma_{k]}}, \qquad \tau_k = \varepsilon^{ijk}\frac{\varsigma^i}{g_{k[k}\varsigma_{j]}} \tag{2.2}$$

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[7] The indices i, j, k are assumed different and summation is not implied.

where $\varsigma_i = \mathbf{c}\cdot\hat{\mathbf{a}}_i$ and $\varsigma^j = \mathbf{c}\cdot\hat{\mathbf{a}}^j$, while $a_{[i}b_{j]} = a_i b_j - a_j b_i$ denotes the usual alternator of indices. In the gimbal lock setting, on the other hand

$$\hat{\mathbf{a}}_3 = \pm\mathcal{R}\hat{\mathbf{a}}_1$$

formula (2.1) yields indefinite expressions for τ_1 and τ_3 but their sum or difference is determined, namely

$$\tau_2 = \frac{\varsigma^3}{g_{2[1}\varsigma_{2]}}, \qquad \frac{\tau_1 \pm \tau_3}{1 \mp \tau_1\tau_3} = \frac{\varsigma^3}{g_{1[2}\varsigma_{1]}} \tag{2.3}$$

or in other words, for the generalized angles one has

$$\varphi_2 = 2\arctan\frac{\varsigma^3}{g_{2[1}\varsigma_{2]}}, \qquad \varphi_1 \pm \varphi_3 = 2\arctan\frac{\varsigma^3}{g_{1[2}\varsigma_{1]}}\cdot$$

Note that since we work in a projective space it possible to hit the plane at infinity every now and then. In these cases one may use l'Hôpital's rule to determine the corresponding complex direction, e.g. whenever $\omega_k = \pm\tilde{\omega}$, one of the solutions lives on the complex plane at infinity. The corresponding quaternion is given as $(\zeta_\circ, \boldsymbol{\zeta}) = (0, \mathbf{n})$ and as mentioned earlier, this is an analytic continuation of a real-valued half-turn expressed as $\mathcal{R}(\infty\,\mathbf{n}) = 2\mathbf{n}\mathbf{n}^t - \mathcal{I}$. We shall denote such transformations associated with infinite elements with $\mathcal{O}(\mathbf{n})$ since they resemble reflections, e.g. one has the involution property $\mathcal{O}^{-1} = \mathcal{O}$ and like in the real case, generate the whole group. Since they are purely symmetric, however, the vector $\mathbf{n}$ can no longer be derived from the matrix coefficients in the usual way (1.24), but should be obtained as the eigenvector, corresponding to the only unit eigenvalue, i.e., $\mathbf{n} \in \ker(\mathcal{O} - \mathcal{I})$.

The Isotropic Singularity

One major distinction from the real setting is related to the presence of isotropic directions, for which the rules seem to be somewhat different: on the one hand, it is not possible to normalize and on the other, a proper invariant subspace appears—an effect that may be referred to as isotropic singularity (see Lemma 1 in [5]), namely

Lemma 2.1. *For each null vector* $\mathbf{c}_\circ$ $(\mathbf{c}_\circ^2 = 0)$ *the normal complement*

$$\mathbf{c}_\circ^\perp = \{\mathbf{c} \,\in\, \mathbb{CP}^3 :\, \mathbf{c}\cdot\mathbf{c}_\circ = 0\}$$

is an invariant subspace for all orthogonal transformations (0.4) *with* $\mathbf{c} \in \mathbf{c}_\circ^\perp$. *Moreover, the linear span* $\{\mathbf{c}_\circ\}$ *is closed under* (0.2) *and in particular*

$$\langle \mathbf{c}_\circ, \cdot\rangle : \quad \tilde{\mathbf{c}} \;\to\; \tilde{\mathbf{c}} + \lambda\mathbf{c}_\circ, \qquad \forall\, \tilde{\mathbf{c}} \in \mathbf{c}_\circ^\perp.$$

Proof. To see this, one only needs to show first that $\mathbf{c} \in \mathbf{c}_\circ^\perp \;\Rightarrow\; \mathbf{c}\times\mathbf{c}_\circ \sim \mathbf{c}_\circ$, which is quite obvious since on the one hand the cross product lies in $\mathbf{c}_\circ^\perp$ and on the other, it is isotropic itself, but $\{\mathbf{c}_\circ\}$ is the only isotropic direction in $\mathbf{c}_\circ^\perp$ as it is of complex dimension two. But then, since $\mathbf{c} \in \mathbf{c}_\circ^\perp \;\Rightarrow\; \mathbf{c}\mathbf{c}^t :\, \mathbf{c}_\circ \to 0$, we have that $\mathcal{R}(\mathbf{c})$ indeed preserves $\{\mathbf{c}_\circ\}$. Now, since for an arbitrary $\tilde{\mathbf{c}} \in \mathbf{c}_\circ^\perp$

$$\tilde{\mathbf{c}} = \lambda\mathbf{c}_\circ + \mu\mathbf{c} \quad\Rightarrow\quad \mathcal{R}(\mathbf{c}) :\, \tilde{\mathbf{c}} \to \lambda'\mathbf{c}_\circ + \mu\mathbf{c} \;\in\; \mathbf{c}_\circ^\perp$$

so $\mathbf{c}_\circ^\perp$ is an invariant subspace. Finally, suppose that $\mathbf{c}_{1,2} \in \mathbf{c}_\circ^\perp$, then we can express $\mathbf{c}_2 = \lambda \mathbf{c}_\circ + \mu \mathbf{c}_1$ and thus

$$\langle \mathbf{c}_2, \mathbf{c}_1 \rangle = \frac{\lambda(\mathbf{c}_\circ + \mathbf{c}_\circ \times \mathbf{c}_1) + (1+\mu)\mathbf{c}_1}{1 - \mu \mathbf{c}_1^2} = \lambda' \mathbf{c}_\circ + \mu' \mathbf{c}_1 \ \in \ \mathbf{c}_\circ^\perp$$

and in particular, for $\mu = 0$ one has $\mu' = 1$, which completes the proof. □

With this in mind we are now ready to approach the decomposition problem:

Proposition 2.2. *Let $\hat{\mathbf{a}}_j \in \mathbf{c}_\circ^\perp$ for some isotropic $\mathbf{c}_\circ \in \mathbb{C}^3$. The decomposition*

$$\mathcal{R}(\mathbf{c}) = \mathcal{R}_k \cdots \mathcal{R}_1, \qquad k \geq 2$$

exists if and only if $\mathbf{c} \in \mathbf{c}_\circ^\perp$, while the solutions depend on $k-2$ parameters.

Proof. The necessity follows from the fact that $\mathbf{c}_\circ^\perp$ is closed under group composition. To prove the sufficiency we note that unless $\mathbf{c} \sim \mathbf{c}_\circ$ one has

$$\hat{\mathbf{a}}_k = \lambda_k \mathbf{c}_\circ + \mu_k \mathbf{c} \quad \Rightarrow \quad \mathcal{R}(\mathbf{c})\, \hat{\mathbf{a}}_k = \lambda_k' \mathbf{c}_\circ + \mu_k \mathbf{c}$$

but as $\mathbf{c}_\circ$ gives zero contribution to the scalar product in $\mathbf{c}_\circ^\perp$, the condition $r_{jk} = g_{jk}$ holds for any pair of different $\hat{\mathbf{a}}_i$'s. For $\mathcal{R}(\mathbf{c}_\circ) = \mathcal{I} + 2\,\mathbf{c}_\circ^\times + 2\,\mathbf{c}_\circ \mathbf{c}_\circ^t$ we use a different, non-isotropic $\mathbf{c}$ above and the action on the $\mathbf{c}_k$'s remains the same. Since we can always decompose in two factors, $\mathbf{c} = \langle \mathbf{c}_3, \mathbf{c}_2, \mathbf{c}_1 \rangle = \langle \tilde{\mathbf{c}}, \mathbf{c}_1 \rangle$ with $\tilde{\mathbf{c}} = \langle \mathbf{c}_3, \mathbf{c}_2 \rangle$ holds for a one-parameter subgroup determining the arbitrary in $\mathbf{c}_\circ^\perp$ direction of $\tilde{\mathbf{c}}$. For $k > 3$ the result follows easily by induction using the same technique. □

Note that the above lemma and proposition follow closely a result obtained in [5] for the real hyperbolic case and so does the following construction. Although the existence of a given decomposition in the isotropic setting is quite easy to prove, obtaining the particular solutions for the scalar parameters if not trivial. The problem is that if we apply the methods used so far, ultimately based on carefully chosen projections, we always end up with undetermined expressions of the type $\frac{0}{0}\cdot$ To avoid this inconvenience in the two-axes setting, we express the decomposition using (0.2) in the two forms

$$\mathbf{c}_1 = \langle -\mathbf{c}_2, \mathbf{c} \rangle, \qquad \mathbf{c}_2 = \langle \mathbf{c}, -\mathbf{c}_1 \rangle$$

after which, act on the first equality on the left with $\hat{\mathbf{a}}_1^\times$ and on the second one—with $\hat{\mathbf{a}}_2^\times$. Finally, we take the hermitian scalar product of the result with the isotropic vector in this subspace $\mathbf{c}_\circ$ (as the usual one vanishes). Finally, denoting $\mathbf{x}^\circ = \mathbf{x} \cdot \bar{\mathbf{c}}_\circ$ for each vector $\mathbf{x} \in \mathbf{c}_\circ^\perp$ we can express the solutions as

$$\tau_1 = \frac{(\mathbf{c} \times \hat{\mathbf{a}}_2)^\circ}{(\hat{\mathbf{a}}_1 \times \hat{\mathbf{a}}_2)^\circ + \varsigma_2 \hat{\mathbf{a}}_1^\circ - g_{12}\mathbf{c}^\circ}, \qquad \tau_2 = \frac{(\hat{\mathbf{a}}_1 \times \mathbf{c})^\circ}{(\hat{\mathbf{a}}_1 \times \hat{\mathbf{a}}_2)^\circ + \varsigma_1 \hat{\mathbf{a}}_2^\circ - g_{12}\mathbf{c}^\circ} \tag{2.4}$$

where the numerators are non-vanishing unless one of the $\hat{\mathbf{a}}_i$'s is proportional to $\mathbf{c}$, in which case the decomposition is trivial. As for the case of three factors, one may proceed in a similar manner this time with

$$\mathbf{c}_1 = \langle -\mathbf{c}_2, -\mathbf{c}_3, \mathbf{c} \rangle, \qquad \mathbf{c}_3 = \langle \mathbf{c}, -\mathbf{c}_1, -\mathbf{c}_2 \rangle$$

left-multiplying respectively with $\hat{\mathbf{a}}_1$ and $\hat{\mathbf{a}}_3$, then taking the hermitian product with $\mathbf{c}_\circ$ to obtain the scalar expressions for $\tau_{1,3}$ as fractional–linear functions of the free parameter τ_2 (as a generalization of formula (61) in [5])

$$\tau_1 = \frac{(\sigma_{23} - g_{23}\mathbf{c}^\circ + \varsigma_3\hat{\mathbf{a}}_2^\circ)\tau_2 + \rho_3}{(g_{3[1}\hat{\mathbf{a}}_{2]}^\circ + \sigma_{3[1}\varsigma_{2]} - g_{12}\rho_3)\tau_2 - \varsigma_3\hat{\mathbf{a}}_1^\circ + g_{31}\mathbf{c}^\circ + \sigma_{31}}$$

$$\tau_3 = \frac{(\sigma_{21} + g_{21}\mathbf{c}^\circ - \varsigma_1\hat{\mathbf{a}}_2^\circ)\tau_2 + \rho_1}{(g_{1[2}\hat{\mathbf{a}}_{3]}^\circ + \sigma_{1[3}\varsigma_{2]} - g_{23}\rho_1)\tau_2 + \varsigma_1\hat{\mathbf{a}}_3^\circ - g_{13}\mathbf{c}^\circ + \sigma_{13}}$$

where we denote $\sigma_{ij} = (\hat{\mathbf{a}}_i \times \hat{\mathbf{a}}_j)^\circ$ and $\rho_k = (\hat{\mathbf{a}}_k \times \mathbf{c})^\circ$. Naturally, each pair of parameters τ_i, τ_j can be expressed in a similar way from the third one and when it is set to zero, we obtain the usual two-factor decomposition, as long as $\hat{\mathbf{a}}_i \times \hat{\mathbf{a}}_j \neq 0$. Such fractional–linear relations hold in the general case as well, but there are no free parameters there except in the singular gimbal lock setting (see [5,7] for more details).

Note also that we might have some isotropic vectors in the general setting as well, e.g. $\mathbf{c} \sim \mathbf{c}_\circ$ or $\hat{\mathbf{a}}_k \sim \mathbf{c}_\circ$, one can still use the standard construction in these cases and in particular, formula (2.1). However, the normalization along null direction is arbitrary, e.g. we could use the hermitian one to set $\mathbf{n}^\circ = 1$ or $\hat{\mathbf{a}}_k^\circ = 1$ respectively. Another major difference is the trigonometric interpretation of the scalar parameter τ as in this case Euler's substitution yields $\varphi = 2\tau$ rather than $\varphi = 2\arctan\tau$.

Other Decomposition Schemes

Other factorizations, such as the ones due to Iwasawa, Wigner and Bargman, are broadly used as a means of group parametrization in geometry and physics, similarly to Euler angles. We begin with a straightforward complex generalization of a decomposition into a pair of transformations $\mathcal{R} = \mathcal{R}_2\mathcal{R}_1$ proposed in [3], pointing out that the necessary and sufficient condition for it $r_{21} = g_{21}$ is always satisfied if we chose $\hat{\mathbf{a}}_j$ arbitrary and (as long as $r_{11} \neq 0$)

$$\hat{\mathbf{a}}_2 = \lambda\, \hat{\mathbf{a}}_1 \times \mathcal{R}(\mathbf{c})\, \hat{\mathbf{a}}_1, \qquad \lambda = (1 - r_{11}^2)^{-1/2} \tag{2.5}$$

which together with formula (2.2) yields the solutions

$$\tau_1 = \varsigma_1, \qquad \tau_2 = \varsigma_2 = \sqrt{\frac{1 - r_{11}}{1 + r_{11}}} \tag{2.6}$$

that may easily be expressed in terms of the corresponding generalized angles

$$\phi_1 = 2\arctan\varsigma_1, \qquad \phi_2 = \arccos r_{11}.$$

Note that similar solutions may be constructed with an arbitrary g_{12} as

$$\phi_1 = 2\arctan\left(\frac{\varsigma_1 - g_{12}\tau_2}{1 + g_{12}\varsigma_1\tau_2}\right), \qquad \phi_2 = \arccos\left(\frac{r_{11} - g_{12}^2}{1 - g_{12}^2}\right) \tag{2.7}$$

where the second axis is determined from $\hat{\mathbf{a}}_2 = \tau_2^{-1}\langle \mathbf{c}, -\tau_1\hat{\mathbf{a}}_1\rangle$ provided that

$$|\arccos r_{11}| \leq 2|\arccos g_{12}|$$

is satisfied, which in the orthogonal case is by default. Finally, $r_{11}^2 = 1$ means that $\mathcal{R}\hat{\mathbf{a}}_1 = \pm\hat{\mathbf{a}}_1$, which for the positive sign yields a trivial decomposition and for the negative one—a symmetric form of $\mathcal{R}$ with $\mathbf{n} \perp \hat{\mathbf{a}}_1$ that allows for choosing $\hat{\mathbf{a}}_2 \in \hat{\mathbf{a}}_1^{\perp}$ arbitrary so that $\mathcal{R}_1$ is expressed as a product of two symmetric transformations (half-turns in the real case), namely $\mathcal{R}_1 = \mathcal{O}(\hat{\mathbf{a}}_2)\mathcal{O}(\mathbf{n})$. As pointed out in [3], this might be viewed as an analogue of the well-known gimbal lock singularity in the Euler-type decompositions. As for the isotropic singularity, it is clear that as long as both $\hat{\mathbf{a}}_1$ and $\mathbf{n}$ belong to $\mathbf{c}_\circ^{\perp}$ for some null vector $\mathbf{c}_\circ \in \mathbb{C}^3$, the construction yields that $\hat{\mathbf{a}}_2 \in \mathbf{c}_\circ^{\perp}$ unless $\hat{\mathbf{a}}_1$ is aligned with the isotropic direction, in which case $\hat{\mathbf{a}}_2$ cannot be determined from (2.5), so we may choose $\hat{\mathbf{a}}_2 \in \mathbf{c}_\circ^{\perp}$ arbitrarily and obtain the solutions via formula (2.4).

Note that except for the isotropic singularity, the solutions in the complex case are derived in the same way as the real ones. For the applications it is often preferable to choose the first transformation to be decomposable, i.e., representing an element of the corresponding Wigner little group that preserves an elementary particle's relativistic momentum. To do so, however, we need to make sure that $\varsigma_1^2 \in \mathbb{R}$, which gives us three separate cases: $\mathcal{R}_1$ is a rotation for $\varsigma_1^2 > 0$, a hyperbolic Lorentz boost ($\varsigma_1^2 < 0$) and an isotropic transformation, i.e., an element of the light-cone preserving (so-called "front form") little group, that is known to be isomorphic to the group of Euclidean motions in the plane, for $\varsigma_1^2 = 0$. This, however, does not guarantee that the second transformation in (2.6) will be decomposable as well, but if we use the more general construction (2.7) instead, it is possible to choose the parameter g_{12} in such a way, that we end up with both τ_1^2 and τ_2^2 real. A similar choice is provided in the singular isotropic setting as well. This is straightforward algorithm to present an $\mathrm{SO}^+(3,1)$ pseudo-rotation as a composition of two decomposable transformations, which in theory is known to be always possible. There are variations to the factorization problem proposed above, demanding a given fixed value for ϕ_2 without imposing any restrictions on $\hat{\mathbf{a}}_2$.

Similarly, in the 4×4 matrix case one uses formula (1.26) to determine the two separate vector-parameters. This time the half-turn setting is recognized by the infinite value of the corresponding scalar parameter—τ or $\tilde{\tau}$. Then, we use formula (1.4) for the angles φ and $\tilde{\varphi}$, which we later multiply with γ in order to obtain the scalar (respectively vector-) parameters that provide us with the solution using the corresponding matrix representation above. Next, fixing the orbits in both copies of SO_3, we may consider the decomposition

$$\mathbf{c} \otimes \tilde{\mathbf{c}} = \langle \mathbf{c}_3 \otimes \tilde{\mathbf{c}}_3, \mathbf{c}_2 \otimes \tilde{\mathbf{c}}_2, \mathbf{c}_1 \otimes \tilde{\mathbf{c}}_1 \rangle = \langle \mathbf{c}_3, \mathbf{c}_2, \mathbf{c}_1 \rangle \otimes \langle \tilde{\mathbf{c}}_3, \tilde{\mathbf{c}}_2, \tilde{\mathbf{c}}_1 \rangle$$

using the technique described in the previous section and then translate back the results into matrix form with the above formula. The results obtained in [4] for the real forms $\mathrm{SO}(4)$, $\mathrm{SO}(3,1)$, $\mathrm{SO}(2,2)$ and $\mathrm{SO}^*(4)$ can be derived from the complex case using different involutions. For instance, the first two, which are most common, correspond to narrowing the field of scalars to $\mathbb{R}$ and fixing $\tilde{\mathbf{c}} = \bar{\mathbf{c}}$, respectively. Note, however, that even in the real case one cannot associate invariant planes in $\mathbb{R}^4$ or $\mathbb{R}^{3,1}$ to the commutative subgroups determined by the generalized axes $\{\mathbf{c}_k\} \otimes \{\tilde{\mathbf{c}}_k\}$ unless the Plücker embedding relation, namely $\mathbf{c}_k \perp \tilde{\mathbf{c}}_k$, is satisfied (see [2] for details). This remark refers also to representations of SL_2 in higher-dimensional spaces studied in [2], where the embedding is realized via plane transformations satisfying the Plücker conditions mentioned above and a geometric construction determining the invariant subspaces is available both in the real and complex settings. Moreover, it is well-known that an arbitrary special orthogonal transformation in $\mathbb{R}^n$ or $\mathbb{R}^{p,q}$ with $p+q=n$ may be decomposed into $\left[\frac{n}{2}\right]$ plane rotations or pseudo-rotations. The invariant subspaces may be derived from the Jordan decomposition of the compound transformation and one can perform the procedure described above in each of them separately. Needless to say, this holds also in the complex and dual settings with only minor changes.

Back to the transfer principle: As it was pointed out in [15], the transfer principle applies also in the decomposition setting. However, like in the case of generalized powers, singular configurations, such as gimbal lock or isotropic singularity cannot be resolved directly by this method if they have a non-vanishing dual part, e.g. $\underline{\mathbf{c}}^2=\varepsilon$. Here one needs to be careful also with conditions for parallel and orthogonal vectors which hold only modulo ε, but the kinematical interpretation, for which we refer to [9,13], is quite helpful.

More Examples

Let us begin this time with a regular 3×3 dual complex matrix, to which we apply polar decomposition into a symmetric and orthogonal component

$$\begin{pmatrix} -3\varepsilon & i(2+\varepsilon) & 1+3\varepsilon \\ i\varepsilon & 1-3\varepsilon & i(2-\varepsilon) \\ -1 & i\varepsilon & \varepsilon \end{pmatrix} = \begin{pmatrix} 1+\varepsilon & 2i & \varepsilon \\ 2i & 1-\varepsilon & 0 \\ \varepsilon & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & i\varepsilon & 1 \\ i\varepsilon & 1 & -i\varepsilon \\ -1 & i\varepsilon & 0 \end{pmatrix}$$

and then use formula (1.24) to determine the Rodrigues' parametrization for the latter as $\underline{\mathbf{c}} = (i\varepsilon,\,1,\,0)^t$. Suppose we want to decompose the orthogonal factor into three consecutive transformations $\underline{\mathcal{Q}} = \underline{\mathcal{Q}}_3\underline{\mathcal{Q}}_2\underline{\mathcal{Q}}_1$ with generalized screw axes given by the unit dual complex vectors

$$\hat{\underline{\mathbf{a}}}_1 = (\varepsilon,\,0,\,1)^t, \qquad \hat{\underline{\mathbf{a}}}_2 = (1,\,0,\,i\varepsilon)^t, \qquad \hat{\underline{\mathbf{a}}}_3 = (0,\,-i\varepsilon,\,1)^t.$$

A straightforward application of formula (2.1) and substitution in (0.4) yields

$$\underline{\mathcal{Q}} = \begin{pmatrix} -i\varepsilon & -1 & \varepsilon \\ 1 & -i\varepsilon & -\varepsilon \\ \varepsilon & \varepsilon & 1 \end{pmatrix}\begin{pmatrix} 1 & i\varepsilon & i\varepsilon \\ -i\varepsilon & (i-1)\varepsilon & 1 \\ i\varepsilon & -1 & (i-1)\varepsilon \end{pmatrix}\begin{pmatrix} \varepsilon & 1 & i\varepsilon \\ -1 & \varepsilon & -i\varepsilon \\ -i\varepsilon & -i\varepsilon & 1 \end{pmatrix}$$

and the other solution takes the form $\underline{\mathcal{Q}} =$

$$\begin{pmatrix} -i\varepsilon & 1 & \varepsilon \\ -1 & -i\varepsilon & \varepsilon \\ \varepsilon & -\varepsilon & 1 \end{pmatrix}\begin{pmatrix} 1 & -i\varepsilon & i\varepsilon \\ i\varepsilon & -(1+i)\varepsilon & -1 \\ i\varepsilon & 1 & -(1+i)\varepsilon \end{pmatrix}\begin{pmatrix} (1+2i)\varepsilon & -1 & -i\varepsilon \\ 1 & (1+2i)\varepsilon & -i\varepsilon \\ i\varepsilon & -i\varepsilon & 1 \end{pmatrix}.$$

Our next example illustrates the decomposition (2.5) in a 2×2 setting, namely

$$\underline{\mathcal{M}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1+i\nu\varepsilon & i-\nu\varepsilon \\ i+\nu\varepsilon & 1-i\nu\varepsilon \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C}[\varepsilon]), \qquad \nu = \frac{1+i}{2}$$

from which we easily derive $\underline{\mathbf{c}} = (\nu\varepsilon,\,-\nu\varepsilon,\,1)^t$ and choosing the direction vector $\hat{\underline{\mathbf{a}}}_1 = (1,\,0,\,i\varepsilon)^t$, work as in the 3×3 representation (0.4) ending up with $r_{11} = 0$, so from (2.5) and (2.6) it is straightforward to see that

$\tau_1 = (\nu + i)\varepsilon$ and $\tau_2 = 1$ with $\underline{\hat{\mathbf{a}}}_2 = (-i\varepsilon,\, -(2\nu+i)\varepsilon,\, 1)^t$, which allows for decomposing into a pair of dual complex matrices in the following way

$$\underline{\mathcal{M}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1+\varepsilon & -(2\nu+i)\varepsilon \\ (2\nu+i)\varepsilon & 1-\varepsilon \end{pmatrix} \begin{pmatrix} 1+(i\nu-1)\varepsilon & 0 \\ 0 & 1+(1-i\nu)\varepsilon \end{pmatrix}.$$

We encourage the reader to try other examples while being on the alert for pseudo-singularities (modulo ε), in which the complex and dual parts of the decomposition disagree, so the above construction cannot be applied directly.

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References

[1] Brezov, D.: Projective bivector parametrization of isometries in low dimensions. Geom. Integr. Quant. **20**, 91–104 (2018)

[2] Brezov, D.: Higher-dimensional representations of SL_2 and its real forms via Plücker embedding. Adv. Appl. Clifford Algebras **27**, 2375–2392 (2017)

[3] Brezov, D., Mladenova, C., Mladenov, I.: Factorizations in special relativity and quantum scattering on the Line II, AIP Conference Proceedings, vol. 1789, pp. 020009-1–020009 -10 (2016)

[4] Brezov, D., Mladenova, C., Mladenov, I.: Generalized Euler decompositions of some six-dimensional lie groups. AIP Conf. Proc. **1631**, 282–291 (2014)

[5] Brezov, D., Mladenova, C., Mladenov, I.: A decoupled solution to the generalized Euler decomposition problem in $\mathbb{R}^3$ and $\mathbb{R}^{2,1}$. J. Geom. Symmetry Phys. **33**, 47–78 (2014)

[6] Brezov, D., Mladenova, C., Mladenov, I.: On the Decomposition of the Infinitesimal (Pseudo-)Rotations. CR Acad. Bulg. Sci. **67**, 1337–1344 (2014)

[7] Brezov, D., Mladenova, C., Mladenov, I.: Some new results on three-dimensional rotations and pseudo-rotations. AIP Conf. Proc. **1561**, 275–288 (2013)

[8] Cho, E.: De Moivre's formula for quaternions. Appl. Math. Lett. **11**, 33–35 (1998)

[9] Condurache, D., Burlacu, A.: Dual tensors based solutions for rigid body motion parameterization. Mech. Mach. Theory **74**, 390–412 (2014)

[10] Fedorov, F.: The Lorentz Group. Science, Moscow (1979) (in Russian)

[11] Kotelnikov A.: Screw calculus and some applications to geometry and mechanics (in Russian). Annuals of the Imperial University of Kazan (1895)

[12] Özdemir, M.: Finding n-th roots of a 2 × 2 matrix using De Moivre's formula. Adv. Appl. Clifford Algebras **29**, 625–638 (2019)

[13] Parkin, I.: Alternative forms for displacement screws and their pitches. In: Lenarčič, J., Wenger, P. (eds.) Advances in Robot Kinematics: Analysis and Design, pp. 193–202. Springer, Dordrecht (2008)

[14] Piovan, G., Bullo, F.: On coordinate-free rotation decomposition euler angles about arbitrary axes. IEEE Trans. Robot. **28**, 728–733 (2012)

[15] Wittenburg, J., Lilov, L.: Decomposition of a finite rotation into three rotations about given axes. Multibody Syst. Dyn. **9**, 353–375 (2003)

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