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Double Hom-Associative Algebra and Double Hom–Lie Bialgebra

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Abstract. Recently, Hom-structures have been widely investigated in literature. In this paper, we introduce the conceptions of double Hom-associative algebras and double Hom–Lie bialgebras, and give a necessary and sufficient condition for double Hom-associative algebras to be Hom-associative algebras. Meanwhile, we characterize a classical Hom–Yang–Baxter equation in terms of both Hom–Lie algebra morphisms and Hom–Lie coalgebra morphisms. Last but not least, we introduce the notion of double Hom–Lie bialgebras, and prove that double Hom-associative algebras are indeed quasi-triangular Hom–Lie bialgebras.

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1. Introduction

As a generalization of Lie algebras, Hom–Lie algebras were introduced from the motivation for physics and deformations of Lie algebras, in particular Lie algebras of vector fields. The notion of Hom–Lie algebras was firstly introduced by Hartwig et al. [10] to describe the structure on certain q-deformations of the Witt and the Virasoro algebras. Indeed, Hom–Lie algebras are different from Lie algebras in the Jacobi identity, which is replaced by the twisted form by using an endomorphism. This twisted Jacobi identity is called Hom–Jacobi identity given by

$$[\alpha(x),[y,z]]+[\alpha(y),[z,x]]+[\alpha(z),[x,y]]=0.$$

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Recently, Hom–Lie structures have been studied extensively and further in a series of articles [1,2,4,11,12,17,23,25–27] by many scholars, such as Hom–Lie bialgebras, quasi-Hom–Lie algebras, Hom–Lie superalgebras, Hom–Lie color algebras, Hom–Lie admissible Hom-algebras, Hom–Nambu–Lie algebras and so on.

This twisting manner was applied in other algebra structures naturally. Then, many Hom-structures were introduced, such as Hom-associative algebras, Hom–Hopf algebras, Hom-alternative algebras, Hom–Jordan algebras, Hom–Poisson algebras, Hom–Leibniz algebras, infinitesimal Hom-bialgebras, Hom-power associative algebras, and quasi-triangular Hom-bialgebras [6,8, 9,14–16,24].

The Yang–Baxter equation (YBE) was twisted to be Hom-type called Hom–Yang–Baxter equation (HYBE) in [25]. The HYBE can be stated as

$$(\alpha \otimes B) \circ (B \otimes \alpha) \circ (\alpha \otimes B) = (B \otimes \alpha) \circ (\alpha \otimes B) \circ (B \otimes \alpha),$$

where α is an endomorphism of the vector space V, and $B : V^{\otimes 2} \to V^{\otimes 2}$ is a bilinear map that commutes with $\alpha^{\otimes 2}$. Meanwhile, Yau defined the CHYBE in the same manner and studied Hom–Lie bialgebras in [27].

In [28], associative D-bialgebras were studied, and a necessary and sufficient condition for an associative algebra A with comultiplication Δ into an associative D-bialgebra was given by Zhelyabin. In the same article, relations between some types of Jordan bialgebras and Lie bialgebras was investigated. In [5], Drinfel'd showed that a Lie algebra L with a comultiplication is a Lie bialgebra if and only if the double space $D(L) = L^* \oplus L$ is a Lie algebra. Majid introduced the classical double Lie bialgebra which was proved to be a quasi-triangular Lie bialgebra in [18].

Based on the above work and the close connection between Clifford algebra and Hopf algebras in [19,22], we want to investigate double Hom-associative algebras and double Hom–Lie bialgebras. This paper is organized as follows. In Sect. 2, we recall some basic definitions and make a summary of the fundamental properties concerning Hom-structures. In Sect. 3, we study the properties of Hom-associative algebras and introduce double Hom-associative algebras $D(A^*, A)$. In addition, we discover a necessary and sufficient condition for the double $D(A^*, A)$ to be a Hom-associative algebra. In Sect. 4, we recall some concepts and results about Hom–Lie bialgebras and show that Hom–Lie bialgebras are self-dual. Meanwhile, we characterize the CHYBE in terms of both Hom–Lie algebra morphisms and Hom–Lie coalgebra morphisms. That is under what condition a coboundary Hom–Lie bialgebra is quasi-triangular. In Sect. 5, we introduce the conception of double Hom–Lie bialgebras which generalizes double Lie bialgebras in [18], and prove that they are indeed quasi-triangular Hom–Lie bialgebras. As an immediate application, by example, we investigate the quasi-triangular Hom–Lie bialgebra structure on the Hom–Lie algebra $sl(2)_\alpha$. Last, we discuss the coquasi-triangular structure on the codouble Hom–Lie bialgebra $D(L)^*$.

Throughout the rest of this paper, let k be a field and char$(k) = 0$. Unless otherwise specified, vector spaces, algebras, linearity, modules and $\otimes$ are all meant over k. Sum symbols are always omitted by Sweedler's notation:

we write $\Delta(x) = x_1 \otimes x_2$ in which Δ is a comultiplication of the coalgebra C, for $x \in C$. Let ξ be the cyclic permutation (123), we denote the symbol $\circlearrowleft$ by the sum over id, ξ, ξ^2. Namely, we denote the Hom–Jacobi identity by $\circlearrowleft [\alpha(x), [y, z]] = 0$ in place of $[\alpha(x), [y, z]] + [\alpha(y), [z, x]] + [\alpha(z), [x, y]] = 0$. Any unexplained definitions and notation may be found in [20].

2. Preliminaries

In what follows, by [3], we recall some concepts and results used in this paper firstly.

Definition 2.1. A *Hom-associative algebra* is a triple (A, m, α) where A is a vector space,

$$m : A \otimes A \to A, \quad a \otimes b \mapsto ab,$$

and $\alpha : A \to A$ are k-linear maps satisfying

$$\alpha(a)(bc) = (ab)\alpha(c) \qquad \text{(Hom-associativity)}$$

for any $a, b, c \in A$.

The Hom-associative algebra (A, m, α) is called a *multiplicative Hom-associative algebra* if $\alpha(ab) = \alpha(a)\alpha(b)$, for any $a, b \in A$, and called an *involutive Hom-associative algebra* if $\alpha^2 = id$.

Let (A, α, m) and (A', α', m') be two Hom-associative algebras. A linear map $f : A \to A'$ is called a Hom-associative algebra morphism if

$$m' \circ (f \otimes f) = f \circ m, \quad f \circ \alpha = \alpha' \circ f.$$

It is obvious that the tensor product $(A \otimes A', \alpha \otimes \alpha', m \otimes m')$ of two Hom-associative algebras (A, α, m) and (A', α', m') is still a Hom-associative algebra.

Definition 2.2. A *Hom-coassociative coalgebra* is a triple (C, Δ, β) where C is a vector space,

$$\Delta : C \to C \otimes C, \quad c \mapsto c_1 \otimes c_2,$$

and $\beta : C \to C$ are linear maps satisfying

$$(\beta \otimes \Delta) \circ \Delta = (\Delta \otimes \beta) \circ \Delta, \qquad \text{(Hom-coassociativity)}$$

for any $c \in C$.

Using Sweedler's notation, the Hom-coassociativity can be restated as

$$\beta(c_1) \otimes (c_{21} \otimes c_{22}) = (c_{11} \otimes c_{12}) \otimes \beta(c_2).$$

The Hom-coassociative coalgebra (C, Δ, β) is called a *comultiplicative Hom-coassociative coalgebra* if $\Delta \circ \beta = (\beta \otimes \beta) \circ \Delta$, i.e., $\beta(c)_1 \otimes \beta(c)_2 = \beta(c_1) \otimes \beta(c_2)$, for all $c \in C$, and called an *involutive Hom-coassociative coalgebra* if $\beta^2 = id$.

A morphism f from a Hom-coassociative coalgebra (C, Δ, β) to another Hom-coassociative coalgebra (C', Δ', β') is a linear map satisfying

$$(f \otimes f) \circ \Delta = \Delta' \circ f, \quad f \circ \beta = \beta' \circ f.$$

In this paper, all considered Hom-associative algebras are without units and all Hom-coassociative coalgebras are without counits. In addition, all Hom-associative algebras considered are multiplicative Hom-associative algebras and all Hom-coassociative coalgebras are comultiplicative Hom-coassociative coalgebras.

Definition 2.3. Let (A, m, α) be a Hom-associative algebra. A *left A-Hom module* (M, μ, ψ) introduced in [19] is a vector space M together with linear maps $\mu : M \to M$ and $\psi : A \otimes M \to M$; $a \otimes m \mapsto a \cdot m$, satisfying the following conditions

$$\alpha(a) \cdot (b \cdot m) = (ab) \cdot \mu(m),$$
$$\mu(a \cdot m) = \alpha(a) \cdot \mu(m),$$

for any $a, b \in A$ and $m \in M$.

Let $(M, \mu), (N, \nu)$ be two left A-Hom modules. A morphism $f : M \to N$ is called left A-linear if

$$f(a \cdot m) = a \cdot f(m), \quad f \circ \mu = \nu \circ f \quad \text{for } a \in A,\, m \in M.$$

Similarly, we can define a *right A-Hom module* and a right A-module morphism.

Definition 2.4. Let (A, m, α) be a Hom-associative algebra, and (M, μ) both a left A-Hom module with the left action " $\rightharpoonup''$ and a right A-Hom module with the right action " $\leftharpoonup''$. We say the quadruple $(M, \mu, \rightharpoonup, \leftharpoonup)$ is an *A-Hom bimodule* if

$$(a \rightharpoonup m) \leftharpoonup \alpha(b) = \alpha(a) \rightharpoonup (m \leftharpoonup b),$$

for all $a, b \in A$, $m \in M$.

In fact, for any Hom-associative algebra (A, m, α), it is an A-Hom bimodule on itself through its multiplication. The Hom-associativity is just the compatibility condition of Hom-bimodule.

3. Double Hom-Associative Algebras

In this section, the Hom-associative algebras and Hom-coassociative coalgebras are all involutive. In the following, we introduce the definition of double Hom-associative algebras $D(A^*, A)$, and provide a necessary and sufficient condition for the double $D(A^*, A)$ to be a Hom-associative algebra.

Proposition 3.1. *Let (C, Δ, β) be a Hom-coassociative coalgebra. Consider the dual space (C^*, β^*) and define the multiplication on C^* by setting*

$$\langle fg, c\rangle = \langle f, c_1\rangle\langle g, c_2\rangle, \quad \beta^*(f) = f \circ \beta,$$

where $\Delta(c) = c_1 \otimes c_2$, for any $f, g \in C^*$ and $c \in C$. Then, the space (C^*, β^*) with the assigned multiplication is a Hom-associative algebra, which is called the dual of (C, Δ, β).

Proof. For any $f, g \in C^*$, $c \in C$,

$$\begin{aligned}\langle \beta^*(f)(gh), c\rangle &= \langle \beta^*(f), c_1\rangle\langle gh, c_2\rangle \\ &= \langle f, \beta(c_1)\rangle\langle g, c_{21}\rangle\langle h, c_{22}\rangle \\ &= \langle f, c_{11}\rangle\langle g, c_{12}\rangle\langle h, \beta(c_2)\rangle \\ &= \langle (fg)\beta^*(h), c\rangle.\end{aligned}$$

That is, $\beta^*(f)(gh) = (fg)\beta^*(h)$. Meanwhile,

$$\begin{aligned}\langle \beta^*(fg), c\rangle &= \langle fg, \beta(c)\rangle \\ &= \langle f, \beta(c_1)\rangle\langle g, \beta(c_2)\rangle \\ &= \langle \beta^*(f)\beta^*(g), c\rangle.\end{aligned}$$

So, (C^*, β^*) is a Hom-associative algebra. In addition, the involutivity of β^* is from the involutivity of β. □

Similarly, for any finite dimensional Hom-associative algebra, its dual is a Hom-coassociative coalgebra.

Proposition 3.2. *Let (C, Δ, β) be a Hom-coassociative coalgebra. The dual (C^*, β^*) of (C, Δ, β) determines a C^*-Hom bimodule on C, defined as follows*

$$f \rightharpoonup c = c_1\langle f, c_2\rangle, \quad c \leftharpoonup f = \langle f, c_1\rangle c_2,$$

for $f \in C^*, c \in C$. In addition,

$$\langle fg, c\rangle = \langle f, g \rightharpoonup c\rangle = \langle g, c \leftharpoonup f\rangle,$$

for any $f, g \in C^*$ and $c \in C$.

Proof. Firstly, the above actions "$\rightharpoonup$" and "$\leftharpoonup$" define a left and right C^*-Hom module on C.

In fact, by the Hom-coassociativity, involutivity and comultiplicativity, for any $f, g \in C^*$, $c \in C$,

$$\begin{aligned}(fg) \rightharpoonup \beta(c) &= \beta(c_1)\langle fg, \beta(c_2)\rangle \\ &= \beta(c_1)\langle f, \beta(c_{21})\rangle\langle g, \beta(c_{22})\rangle \\ &= c_{11}\langle f, \beta(c_{12})\rangle\langle g, c_2\rangle \\ &= \beta^*(f) \rightharpoonup (g \rightharpoonup c), \\ \beta(f \rightharpoonup c) &= \beta(c_1)\langle f, c_2\rangle \\ &= \beta(c_1)\langle \beta^*(f), \beta(c_2)\rangle \\ &= \beta^*(f) \rightharpoonup \beta(c).\end{aligned}$$

So, C is a left C^*-Hom module. Similarly, C is also a right C^*-Hom module.

Next, the compatibility condition of Hom-bimodule holds: for any $c \in C$, $f \in C^*$,

$$\begin{aligned}\beta^*(f) \rightharpoonup (c \leftharpoonup g) &= \langle g, c_1\rangle\beta^*(f) \rightharpoonup c_2 \\ &= \langle g, c_1\rangle c_{21}\langle \beta^*(f), c_{22}\rangle \\ &= \langle \beta^*(g), \beta(c_1)\rangle c_{21}\langle f, \beta(c_{22})\rangle\end{aligned}$$

$$= \langle \beta^*(g), c_{11}\rangle c_{12}\langle f, c_2\rangle$$
$$= (f \rightharpoonup c) \leftharpoonup \beta^*(g).$$

In addition, for any $f, g \in C^*$, $c \in C$,

$$\langle fg, c\rangle = \langle f, g \rightharpoonup c\rangle = \langle g, c \leftharpoonup f\rangle = \langle f, c_1\rangle\langle g, c_2\rangle.$$

□

Conversely, we can define the Hom-bimodule on the dual space A^* of some finite dimensional Hom-associative algebra (A, m, α). The following result and the proof are similar to the above proposition from the Hom-associativity, involutivity and the multiplicativity of Hom-associative algebras.

Proposition 3.3. *Let (A, m, α) be a finite dimensional Hom-associative algebra, and (A^*, α^*) be it's dual with the comultiplication $\Delta(f) = f_1 \otimes f_2$. The Hom-associative algebra A induces an A-Hom bimodule structure on A^*, defined by*

$$a \triangleright f = f_1\langle f_2, a\rangle, \quad f \triangleleft a = \langle f_1, a\rangle f_2,$$

i.e.,

$$\langle a \triangleright f, b\rangle = \langle f, ba\rangle, \quad \langle f \triangleleft a, b\rangle = \langle f, ab\rangle,$$

for any $a, b \in A$, $f \in A^*$. That is, $(A^*, \triangleright, \triangleleft, \alpha^*)$ is an A-Hom bimodule.

Assume that (A, m, α) is a finite dimensional Hom-associative algebra with a comultiplication Δ such that (A, Δ, α) is a Hom-coassociative coalgebra and (A^*, α^*) is it's dual. We consider the double space $D(A^*, A) = A^* \oplus A$. Define two linear maps the multiplication "$\star$" and the endomorphism α_D on $D(A^*, A)$ by

$$(f + a) \star (g + b) = (fg + f \triangleleft b + a \triangleright g) + (ab + f \rightharpoonup b + a \leftharpoonup g),$$
$$\alpha_D(f + a) = \alpha^*(f) + \alpha(a),$$

for all $f, g \in A^*, a, b \in A$, where the actions "$\leftharpoonup, \rightharpoonup, \triangleleft, \triangleright$" are defined as in Propositions 3.2 and 3.3.

In the following, we will provide a necessary and sufficient condition for the double $D(A^*, A)$ to be a Hom-associative algebra.

Proposition 3.4. *Under the assumption as in the above proposition,*

$$(D(A^*, A), \star, \alpha_D)$$

is a Hom-associative algebra if and only if the following equalities hold:

$$(cd)_1 \otimes \alpha_D((cd)_2) = c_1\alpha_D(d) \otimes c_2 + \alpha_D(d_1) \otimes c\alpha_D(d_2), \tag{3.1a}$$

i.e.,

$$\alpha_D((cd)_1) \otimes (cd)_2 = \alpha_D(c_1)d \otimes \alpha_D(c_2) + d_1 \otimes \alpha_D(c)d_2, \tag{3.1b}$$

$$\alpha_D(c)d_1 \otimes d_2 - \alpha_D(d_1) \otimes \alpha_D(d_2)c = c_2\alpha_D(d) \otimes c_1 - \alpha_D(c_2) \otimes d\alpha_D(c_1), \tag{3.2}$$

for any $c, d \in D(A^*, A)$.

Proof. Assume that the above equalities (3.1) and (3.2) hold. By the above propositions, for any $f, g, h \in A^*$, $a, b, c \in A$, we have

$$\begin{aligned}
&((f+a)(g+b))(\alpha^*(h)+\alpha(c)) - (\alpha^*(f)+\alpha(a))((g+b)(h+c))\\
&\quad= (fg)\alpha^*(h) + (f \triangleleft b)\alpha^*(h) + (a \triangleright g)\alpha^*(h) + (fg) \triangleleft \alpha(c)\\
&\qquad+ (f \triangleleft b) \triangleleft \alpha(c) + (a \triangleright g) \triangleleft \alpha(c) + (ab) \triangleright \alpha^*(h) + (f \rightharpoonup b) \triangleright \alpha^*(h)\\
&\qquad+ (a \leftharpoonup g) \triangleright \alpha^*(h) + (ab)\alpha(c) + (f \rightharpoonup b)\alpha(c) + (a \leftharpoonup g)\alpha(c)\\
&\qquad+ (fg) \rightharpoonup \alpha(c) + (f \triangleleft b) \rightharpoonup \alpha(c) + (a \triangleright g) \rightharpoonup \alpha(c) + (ab) \leftharpoonup \alpha^*(h)\\
&\qquad+ (f \rightharpoonup b) \leftharpoonup \alpha^*(h) + (a \leftharpoonup g) \leftharpoonup \alpha^*(h) - \alpha^*(f)(gh) - \alpha^*(f)(g \triangleleft c)\\
&\qquad- \alpha^*(f)(b \triangleright h) - \alpha^*(f) \triangleleft (bc) - \alpha^*(f) \triangleleft (g \rightharpoonup c) - \alpha^*(f) \triangleleft (b \leftharpoonup h)\\
&\qquad- \alpha(a) \triangleright (gh) - \alpha(a) \triangleright (g \triangleleft c) - \alpha(a) \triangleright (b \triangleright h) - \alpha(a)(bc)\\
&\qquad- \alpha(a)(g \rightharpoonup c) - \alpha(a)(b \leftharpoonup h) - \alpha^*(f) \rightharpoonup (bc) - \alpha^*(f) \rightharpoonup (g \rightharpoonup c)\\
&\qquad- \alpha^*(f) \rightharpoonup (b \leftharpoonup h) - \alpha(a) \leftharpoonup (gh) - \alpha(a) \leftharpoonup (g \triangleleft c) - \alpha(a) \leftharpoonup (b \triangleright h)\\
&\quad= (f \triangleleft b)\alpha^*(h) + (a \triangleright g)\alpha^*(h) + (fg) \triangleleft \alpha(c) + (f \rightharpoonup b) \triangleright \alpha^*(h)\\
&\qquad+ (a \leftharpoonup g) \triangleright \alpha^*(h) + (f \rightharpoonup b)\alpha(c) + (a \leftharpoonup g)\alpha(c) + (f \triangleleft b) \rightharpoonup \alpha(c)\\
&\qquad+ (a \triangleright g) \rightharpoonup \alpha(c) + (ab) \leftharpoonup \alpha^*(h) - \alpha^*(f)(g \triangleleft c) - \alpha^*(f)(b \triangleright h)\\
&\qquad- \alpha^*(f) \triangleleft (g \rightharpoonup c) - \alpha^*(f) \triangleleft (b \leftharpoonup h) - \alpha(a) \triangleright (gh) - \alpha(a)(g \rightharpoonup c)\\
&\qquad- \alpha(a)(b \leftharpoonup h) - \alpha^*(f) \rightharpoonup (bc) - \alpha(a) \leftharpoonup (g \triangleleft c) - \alpha(a) \leftharpoonup (b \triangleright h).
\end{aligned}$$

The equality (3.1a) implies that

$$\begin{aligned}
&(f \rightharpoonup b)\alpha(c) + (f \triangleleft b) \rightharpoonup \alpha(c) - \alpha^*(f) \rightharpoonup (bc) = 0,\\
&(a \triangleright g)\alpha^*(h) + (a \leftharpoonup g) \triangleright \alpha^*(h) - \alpha(a) \triangleright (gh) = 0,
\end{aligned}$$

and the equality (3.1b) implies that

$$\begin{aligned}
&(ab) \leftharpoonup \alpha^*(h) - \alpha(a)(b \leftharpoonup h) - \alpha(a) \leftharpoonup (b \triangleright h) = 0,\\
&(fg) \triangleleft \alpha(c) - \alpha^*(f)(g \triangleleft c) - \alpha^*(f) \triangleleft (g \rightharpoonup c) = 0.
\end{aligned}$$

Meanwhile, the equality (3.2) implies that

$$\begin{aligned}
&(f \triangleleft b)\alpha^*(h) + (f \rightharpoonup b) \triangleright \alpha^*(h) - \alpha^*(f)(b \triangleright h) - \alpha^*(f) \triangleleft (b \leftharpoonup h) = 0,\\
&(a \leftharpoonup g)\alpha(c) + (a \triangleright g) \rightharpoonup \alpha(c) - \alpha(a)(g \rightharpoonup c) - \alpha(a) \leftharpoonup (g \triangleleft c) = 0.
\end{aligned}$$

So, $((f+a)(g+b))(\alpha^*(h)+\alpha(c)) - (\alpha^*(f)+\alpha(a))((g+b)(h+c)) = 0$.

Conversely, since $(D(A^*, A)$ is a Hom-associative algebra, we have

$$(f \star a) \star \alpha(b) = \alpha^*(f) \star (a \star b),$$

for all $f \in A^*$, $a, b \in A$. This implies that

$$(f \triangleleft a) \rightharpoonup \alpha(b) + (f \rightharpoonup a)\alpha(b) = \alpha^*(f) \rightharpoonup (ab).$$

Let g, f be arbitrary elements in A^*. Then

$$\begin{aligned}
\langle \rho(g \otimes f), (ab)_1 \otimes \alpha_D((ab)_2)\rangle &= \langle g, \alpha^*(f) \rightharpoonup (ab)\rangle\\
&= \langle g, (f \triangleleft a) \rightharpoonup \alpha(b) + (f \rightharpoonup a)\alpha(b)\rangle\\
&= \langle g, \alpha(b_1)\langle f, a\alpha(b_2)\rangle + a_1\alpha(b)\langle f, a_2\rangle\rangle\\
&= \langle \rho(g \otimes f), \alpha(b_1) \otimes a\alpha(b_2) + a_1\alpha(b) \otimes a_2\rangle,
\end{aligned}$$

where $\rho : A^* \otimes A^* \to (A \otimes A)^*$ is dense. So,

$$(ab)_1 \otimes \alpha_D((ab)_2) = a_1\alpha(b) \otimes a_2 + \alpha(b_1) \otimes a\alpha(b_2),$$

which is the equality (3.1a) restricted on A. In the same way, we can get the equality (3.1a) restricted on A^* from $(a \star f) \star \alpha^*(g) = \alpha(a) \star (f \star g)$, for any $a \in A$, $f, g \in A^*$.

Similar to the equality (3.1a), (3.1b) holds because of $\alpha(a) \star (b \star f) = (a \star b) \star \alpha^*(f)$ and $\alpha^*(f) \star (g \star a) = (f \star g) \star \alpha(a)$. The third equality (3.2) is obtained by using the similar argument to $(a \star f) \star \alpha(b) = \alpha(a) \star (f \star b)$ and $(f \star a) \star \alpha^*(g) = \alpha^*(f) \star (a \star g)$, for any $a, b \in A$, $f, g \in A^*$. □

4. Hom–Lie Bialgebras

In this section, we mainly recall some concepts and results about Hom–Lie bialgebras and discuss the dual of Hom–Lie bialgebras. We characterize the CHYBE in terms of both Hom–Lie algebra morphisms and Hom–Lie coalgebra morphisms.

Definition 4.1. A Hom–Lie algebra in [10] is a triple $(L, [-,-], \alpha)$ consisting of vector space L, bilinear map $[-,-] : L^{\otimes 2} \to L$ and linear endomorphism $\alpha : L \to L$ satisfying

$$[x, y] + [y, x] = 0, \qquad \text{(anti-symmetry)}$$

$$\circlearrowleft [\alpha(x), [y, z]] = 0, \qquad \text{(Hom-Jacobi identity)}$$

for any $x, y, z \in L$.

These Hom–Lie algebras with additional property that α is a Lie-algebra homomorphism, i.e.,

$$\alpha([x, y]) = [\alpha(x), \alpha(y)],$$

are called multiplicative Hom–Lie algebras. In the rest of the paper, all the Hom–Lie algebras considered are multiplicative Hom–Lie algebras. Furthermore, if $\alpha^2 = id$, we call them the involutive Hom–Lie algebras.

A subspace M of L is a sub-Hom–Lie algebra of L if M is also a Hom–Lie algebra with the restricted maps

$$[-,-]|_M : M \otimes M \to M, \quad \alpha|_M : M \to M.$$

A morphism of Hom–Lie algebras

$$f : (L, [-,-], \alpha) \to (L', [-,-]', \alpha')$$

is a linear map such that $\alpha' \circ f = f \circ \alpha$ and $f([-,-]) = [-,-]' \circ f^{\otimes 2}$.

For any Lie algebra $(L, [-,-])$, we can construct a Hom–Lie algebra $L_\alpha = (L, [-,-]_\alpha = \alpha \circ [-,-], \alpha)$ by a Lie algebra endomorphism $\alpha : L \to L$. Then some classical examples of Hom–Lie algebras can be given in this way.

Example 4.2. Let W_1 be a one-side Witt algebra with basis $\{e_i\}_{i=-1}^{\infty}$, whose Lie bracket is defined by

$$[e_i, e_j] = (j - i)e_{i+j},$$

for all integers $i, j \geq -1$. W_1 may be identified with $\mathrm{Der}_k(k[x])$, the Lie algebra of k-derivations of the algebra $k[x]$ of polynomials $\sum_{i=0}^{N} k_i x^i$ in the indeterminate x with coefficients in k.

Suppose that there is a linear map $\alpha : \{e_i\} \to \{e_i\}$ such that it is a Lie algebra endomorphism given by

$$\alpha(e_i) = \tfrac{1}{2} e_{2i}.$$

Then we obtain a Hom–Lie algebra $(W_1, [-,-]_\alpha, \alpha)$ called one-side Hom-Witt algebra.

Definition 4.3. A Hom–Lie coalgebra in [16] is a triple (Γ, Δ, α) with a vector space Γ, a linear map $\Delta : \Gamma \to \Gamma^{\otimes 2}$ and a linear endomorphism $\alpha : \Gamma \to \Gamma$, such that

$$\Delta \circ \alpha = \alpha^{\otimes 2} \circ \Delta, \qquad \text{(comultiplicativity)}$$

$$\Delta + \tau \circ \Delta = 0, \qquad \text{(anti-symmetry)}$$

$$\circlearrowleft (\alpha \otimes \Delta) \circ \Delta = 0. \qquad \text{(Hom-coJacobi identity)}$$

The definition of sub-Hom–Lie coalgebra is analogous to sub-Hom–Lie algebra. A morphism of Hom–Lie coalgebra f from a Hom–Lie coalgebra (Γ, Δ, α) to a Hom–Lie coalgebra $(\Gamma', \Delta', \alpha')$ is a linear map such that

$$\alpha' \circ f = f \circ \alpha \quad \text{and} \quad \Delta' \circ f = f^{\otimes 2} \circ \Delta.$$

Let $(L, [-,-], \alpha)$ be a Hom–Lie algebra. For any $x \in L$ and integer number $n \geq 2$, we define the adjoint diagonal action $\mathrm{ad}_x : L^{\otimes n} \to L^{\otimes n}$ by

$$\mathrm{ad}_x(y_1 \otimes \cdots \otimes y_n) = \sum_{i=1}^{n} \alpha(y_1) \otimes \cdots \otimes \alpha(y_{i-1}) \otimes [x, y_i] \otimes \alpha(y_{i+1}) \cdots \otimes \alpha(y_n).$$

In particular, for $n = 2$, we have

$$\mathrm{ad}_x(y_1 \otimes y_2) = [x, y_1] \otimes \alpha(y_2) + \alpha(y_1) \otimes [x, y_2].$$

Definition 4.4. A Hom–Lie bialgebra introduced in [27], is a quadruple

$$(L, [-,-], \Delta, \alpha)$$

in which $(L, [-,-], \alpha)$ is a Hom–Lie algebra and (L, Δ, α) is a Hom–Lie coalgebra such that the following compatibility condition holds, for all $x, y \in L$,

$$\Delta([x, y]) = \mathrm{ad}_{\alpha(x)}(\Delta(y)) - \mathrm{ad}_{\alpha(y)}(\Delta(x)). \tag{4.1}$$

Explicitly, the compatibility condition can be restated as

$$\begin{aligned}\Delta([x, y]) = {} & [\alpha(x), y_1] \otimes \alpha(y_2) + \alpha(y_1) \otimes [\alpha(x), y_2] \\ & - [\alpha(y), x_1] \otimes \alpha(x_2) - \alpha(x_1) \otimes [\alpha(y), x_2].\end{aligned}$$

A Hom–Lie bialgebra is a Lie bialgebra with the trivial condition $\alpha = id$. Just similar to Lie bialgebras, the compatibility condition in Hom–Lie bialgebras says exactly that $\Delta \in C^1(L, L \otimes L)$ is a 1-cocycle in Hom–Lie algebra cohomology.

Let (Γ, Δ, α) be a Hom–Lie coalgebra, from direct checking, then the dual space $L^* = \mathrm{Hom}(L, k)$ is a Hom–Lie algebra under the following Lie bracket $[-,-]^\circ$ and linear endomorphism α^*:

$$\langle [\phi, \varphi]^\circ, x\rangle = \langle \phi \otimes \varphi, \Delta(x)\rangle, \quad \alpha^*(\phi) = \phi \circ \alpha,$$

for all $\phi, \varphi \in L^*$, $x \in L$. Conversely, we consider the dual of a Hom–Lie algebra in the following.

Let $(L, [-,-], \alpha)$ be a Hom–Lie algebra. Then,

$$[-,-]^* : L^* \to (L \otimes L)^* \quad \text{and} \quad \alpha^* : L^* \to L^*.$$

A subspace M of L^* is called good if $[-,-]^*(M) \subseteq M \otimes M$ and $\alpha^*(M) \subseteq M$, in which $M \otimes M \subseteq L^* \otimes L^* \subseteq (L \otimes L)^*$ and $\alpha^*(\phi) = \phi \circ \alpha$, for any $\phi \in L^*$. This means that there exist two linear maps

$$\Delta^\circ : M \to M \otimes M \quad \text{and} \quad \beta : M \to M$$

such that $\Delta^\circ(\phi) = \phi_1 \otimes \phi_2$ and $\beta(\phi) = \alpha^*(\phi)$ for $\phi, \phi_1, \phi_2 \in L^*$. Then,

$$\langle \Delta^\circ(\phi), x \otimes y\rangle = \langle \phi, [x, y]\rangle = \langle \phi_1, x\rangle\langle \phi_2, y\rangle,$$

and $\beta(\phi)(x) = \phi \circ \alpha(x)$ for $x, y \in L$.

Let $L^\circ = \sum M$ be the sum of all good subspaces of L^* and $\alpha^\circ \in \mathrm{End}(L^\circ)$ such that $\alpha^\circ(\phi) = \phi \circ \alpha$, for any $\phi \in L^\circ$. Then, L° is a sub-Hom–Lie coalgebra of L^* clearly. Furthermore, we obtain the following result.

Proposition 4.5. *Let $(L, [-,-], \Delta, \alpha)$ be a Hom–Lie bialgebra. Then,*

$$(L^\circ, [-,-]^\circ, \Delta^\circ, \alpha^\circ)$$

defined as above is also a Hom–Lie bialgebra.

Proof. Since L° is a good space of L^*, we know that L° is both a Hom–Lie algebra and a Hom–Lie coalgebra. So, we only need to check the compatibility condition (3.1) of L°.

As a matter of fact, for any $\phi, \varphi \in L^\circ$, $x, y \in L$,

$$\begin{aligned}
\langle \Delta^\circ[\phi, \varphi]^\circ, x \otimes y\rangle &= \langle \phi \otimes \varphi, \Delta[x, y]\rangle \\
&= \langle \phi \otimes \varphi, [\alpha(x), y_1] \otimes \alpha(y_2) + \alpha(y_1) \otimes [\alpha(x), y_2] - [\alpha(y), x_1] \otimes \alpha(x_2) \\
&\quad - \alpha(x_1) \otimes [\alpha(y), x_2]\rangle \\
&= \langle \phi_1 \otimes \phi_2 \otimes \varphi, \alpha(x) \otimes y_1 \otimes \alpha(y_2)\rangle + \langle \phi \otimes \varphi_1 \otimes \varphi_2, \alpha(y_1) \otimes \alpha(x) \otimes y_2\rangle \\
&\quad - \langle \phi_1 \otimes \phi_2 \otimes \varphi, \alpha(y) \otimes x_1 \otimes \alpha(x_2)\rangle - \langle \phi \otimes \varphi_1 \otimes \varphi_2, \alpha(x_1) \otimes \alpha(y) \otimes x_2\rangle \\
&= \langle \alpha^\circ(\phi_1) \otimes \phi_2 \otimes \alpha^\circ(\varphi), x \otimes y_1 \otimes y_2\rangle + \langle \alpha^\circ(\phi) \otimes \alpha^\circ(\varphi_1) \otimes \varphi_2, \\
&\quad y_1 \otimes x \otimes y_2\rangle - \langle \alpha^\circ(\phi_1) \otimes \phi_2 \otimes \alpha^\circ(\varphi), y \otimes x_1 \otimes x_2\rangle \\
&\quad - \langle \alpha^\circ(\phi) \otimes \alpha^\circ(\varphi_1) \otimes \varphi_2, x_1 \otimes y \otimes x_2\rangle \\
&= -\langle \alpha^\circ(\phi_1) \otimes [\alpha^\circ(\varphi), \phi_2]^\circ, x \otimes y\rangle + \langle \alpha^\circ(\varphi_1) \otimes [\alpha^\circ(\phi), \varphi_2]^\circ, x \otimes y\rangle \\
&\quad - \langle [\alpha^\circ(\varphi), \phi_1]^\circ \otimes \alpha^\circ(\phi_2), x \otimes y\rangle + \langle [\alpha^\circ(\phi), \varphi_1]^\circ \otimes \alpha^\circ(\varphi_2), x \otimes y\rangle \\
&= \langle \mathrm{ad}_{\alpha^\circ(\phi)}(\Delta^\circ(\varphi)) - \mathrm{ad}_{\alpha^\circ(\varphi)}(\Delta^\circ(\phi)), x \otimes y\rangle.
\end{aligned}$$

So,

$$\Delta^\circ[\phi,\varphi]^\circ = \mathrm{ad}_{\alpha^\circ(\phi)}(\Delta^\circ(\varphi)) - \mathrm{ad}_{\alpha^\circ(\varphi)}(\Delta^\circ(\phi)),$$

by using the compatibility condition of L and the anti-symmetry of L°. □

Note that Proposition 4.5 interprets the self-dual property of Hom–Lie bialgebras which is the generalization of the self-dual property of Lie bialgebras in [21].

Definition 4.6. A Hom–Lie bialgebra $(L,[-,-],\Delta,\alpha)$ is called a coboundary Hom–Lie bialgebra in [27] if there exists an element $r=\sum r_1\otimes r_2\in L\otimes L$ (the sum symbols always omitted), such that for any $x\in L$,

$$\alpha^{\otimes 2}(r)=r,\quad \Delta(x)=\mathrm{ad}_x(r).$$

Furthermore, if r satisfies the classical Hom–Yang–Baxter equation (abbreviated to CHYBE)

$$CH(r)\equiv[r^{12},r^{13}]+[r^{12},r^{23}]+[r^{13},r^{23}]=0,$$

then we call it a quasi-triangular Hom–Lie bialgebra. Here,

$$\begin{aligned}[r^{12},r^{13}]&=[r_1,r_1']\otimes\alpha(r_2)\otimes\alpha(r_2'),\\ [r^{12},r^{23}]&=\alpha(r_1)\otimes[r_2,r_1']\otimes\alpha(r_2'),\\ [r^{13},r^{23}]&=\alpha(r_1)\otimes\alpha(r_1')\otimes[r_2,r_2'],\end{aligned}$$

where $r^{12}=r\otimes 1=r_1\otimes r_2\otimes 1, r^{13}=(\tau\otimes id)(1\otimes r)=r_1\otimes 1\otimes r_2$, $r^{23}=1\otimes r=1\otimes r_1\otimes r_2$, and r' is another copy of r.

Note that for a coboundary Hom–Lie bialgebra $(L,[-,-],\Delta,\alpha,r)$, the symmetric part $r_+=r_1\otimes r_2+r_2\otimes r_1$ is ad invariant, i.e., $\mathrm{ad}_x(r_+)=0$, for all $x\in L$. This is equivalent to the fact that Δ is anti-symmetry.

Example 4.7. Let $sl(2)=\mathrm{span}\{H,X_\pm\}$ be the three-dimensional simple Lie algebra [27] with the bracket

$$[H,X_\pm]=\pm 2X_\pm,\quad [X_+,X_-]=H.$$

It becomes a Lie bialgebra when equipped with the cobracket $\Delta: sl(2)\to sl(2)\otimes sl(2)$ defined by

$$\Delta(H)=0,\quad \Delta(X_\pm)=\tfrac{1}{2}(X_\pm\otimes H-H\otimes X_\pm).$$

Setting

$$r=X_+\otimes X_-+\tfrac{1}{4}H\otimes H,$$

we obtain a quasi-triangular Lie bialgebra $(sl(2),[-,-],\Delta,r)$ in [13].

Consider a linear endomorphism of Lie bialgebra $\alpha: sl(2)\to sl(2)$ defined by

$$\alpha(H)=H,\quad \alpha(X_\pm)=c^\pm X_\pm,$$

for the two non-zero complex numbers $c^\pm\in\mathbf{C}$. Then, there is a Hom–Lie bialgebra

$$sl(2)_\alpha=(sl(2),\,[-,-]_\alpha=\alpha\circ[-,-],\,\Delta_\alpha=\Delta\circ\alpha,\alpha)$$

with the Lie bracket and the Lie cobracket given by

$$[H, X_\pm]_\alpha = \pm 2c^\pm X_\pm, \quad [X_+, X_-]_\alpha = H,$$
$$\Delta_\alpha(H) = 0, \quad \Delta_\alpha(X_\pm) = \tfrac{1}{2}c^\pm(X_\pm \otimes H - H \otimes X_\pm).$$

Furthermore, $sl(2)_\alpha$ is a quasi-triangular Hom–Lie bialgebra with the same r as in $sl(2)$ with $\alpha^{\otimes 2}(r) = r$ by direct computation.

In the following proposition, we characterize the CHYBE in terms of both Hom–Lie algebra morphism and Hom–Lie coalgebra morphisms, which tells us precisely when a coboundary Hom–Lie bialgebra is quasi-triangular.

Proposition 4.8. *Let $(L, [-,-], \Delta, \alpha, r)$ be an involutive coboundary Hom–Lie bialgebra with $r = r_1 \otimes r_2$. Then L is a quasi-triangular Hom–Lie bialgebra if and only if $s_1 : L^* \to L$ defined by $s_1(\phi) = \langle \phi, \alpha(r_1)\rangle r_2$ is a Hom–Lie algebra morphism. Likewise, if and only if $s_2 : L^* \to L$ defined by $s_2(\phi) = r_1\langle \phi, \alpha(r_2)\rangle$ is a Hom–Lie coalgebra morphism.*

Proof. From the involutivity and coboundary of Hom–Lie bialgebra, we have

$$\begin{aligned}\alpha \circ s_1(\phi) &= \langle \phi, \alpha(r_1)\rangle \alpha(r_2) = \langle \phi, r_1\rangle r_2\\ &= \langle \alpha^*(\phi), \alpha(r_1)\rangle r_2 = s_1 \circ \alpha^*(\phi),\end{aligned}$$

for all $\phi \in L^*$.

Then, to show that L is quasi-triangular if and only if s_1 is a Hom–Lie algebra morphism, is equivalent to show that $CH(r) = 0$ if and only if $s_1([\phi, \varphi]) = [s_1(\phi), s_1(\varphi)]$, for any $\phi, \varphi \in L^*$.

Indeed,

$$\begin{aligned}&s_1([\phi, \varphi]) - [s_1(\phi), s_1(\varphi)]\\ &\quad= \langle [\phi, \varphi], \alpha(r_1)\rangle r_2 - \langle \phi, \alpha(r_1)\rangle\langle \varphi, \alpha(r_1')\rangle [r_2, r_2']\\ &\quad= \langle \phi \otimes \varphi \otimes id, \Delta(\alpha(r_1)) \otimes r_2 - \alpha(r_1) \otimes \alpha(r_1') \otimes [r_2, r_2']\rangle\\ &\quad= \langle \phi \otimes \varphi \otimes id, [\alpha(r_1), r_1'] \otimes \alpha(r_2') \otimes r_2 + \alpha(r_1') \otimes [\alpha(r_1), r_2'] \otimes r_2\\ &\qquad- \alpha(r_1) \otimes \alpha(r_1') \otimes [r_2, r_2']\rangle\\ &\quad= \langle \phi \otimes \varphi \otimes id, [r_1, r_1'] \otimes \alpha(r_2') \otimes \alpha(r_2) + \alpha(r_1') \otimes [r_1, r_2'] \otimes \alpha(r_2)\\ &\qquad- \alpha(r_1) \otimes \alpha(r_1') \otimes [r_2, r_2']\rangle\\ &\quad= \langle \phi \otimes \varphi \otimes id, -CH(r)\rangle,\end{aligned}$$

where r' is another copy of r, so, L is a quasi-triangular Hom–Lie bialgebra if and only if s_1 is a Hom–Lie algebra morphism.

The proof for s_2 is strictly analogous. Similarly,

$$\alpha \circ s_2(\phi) = s_2 \circ \alpha^*(\phi) = r_1\langle s_2, r_2\rangle.$$

Meanwhile, $CH(r) = 0$ if and only if $\Delta \circ s_2(\phi) - (s_2 \otimes s_2) \circ \Delta(\phi) = 0$. In fact,

$$\begin{aligned}&\Delta \circ s_2(\phi) - (s_2 \otimes s_2) \circ \Delta(\phi)\\ &\quad= \Delta(r_1)\langle \phi, \alpha(r_2)\rangle - r_1 \otimes r_1'\langle \Delta(\phi), \alpha(r_2) \otimes \alpha(r_2')\rangle\\ &\quad= ([r_1, r_1'] \otimes \alpha(r_2') + \alpha(r_1') \otimes [r_1, r_2'])\langle \phi, \alpha(r_2)\rangle - r_1 \otimes r_1'\langle \phi, [\alpha(r_2), \alpha(r_2')]\rangle\\ &\quad= \langle id^{\otimes 2} \otimes \phi, [r_1, r_1'] \otimes \alpha(r_2') \otimes \alpha(r_2) + \alpha(r_1') \otimes [r_1, r_2'] \otimes \alpha(r_2)\end{aligned}$$

$$
\begin{aligned}
&- r_1 \otimes r_1' \otimes [\alpha(r_2), \alpha(r_2')]\rangle \\
&= \langle id^{\otimes 2} \otimes \phi, -[r_1, r_1'] \otimes \alpha(r_2) \otimes \alpha(r_2') - \alpha(r_1) \otimes [r_2, r_1'] \otimes \alpha(r_2') \\
&- \alpha(r_1) \otimes \alpha(r_1') \otimes [r_2, r_2']\rangle \\
&= \langle id^{\otimes 2} \otimes \phi, -CH(r)\rangle.
\end{aligned}
$$

So, L is a quasi-triangular Hom–Lie bialgebra if and only if s_2 is a Hom–Lie coalgebra morphism. □

Lemma 4.9. *Let $(L, [-,-], \alpha)$ be a Hom–Lie algebra and $r = r_1 \otimes r_2 \in L \otimes L$ such that $\alpha^{\otimes 2}(r) = r$. Set $\Delta = \mathrm{ad}(r) : L \to L \otimes L$. Then, it satisfies (3.1), i.e.,*

$$\Delta([x, y]) = \mathrm{ad}_{\alpha(x)}(\Delta(y)) - \mathrm{ad}_{\alpha(y)}(\Delta(x)),$$

which is the compatibility of Hom–Lie bialgebra.

Proof. By the Hom–Jacobi identity, anti-symmetry and $\alpha^{\otimes 2}(r) = r$, we have

$$
\begin{aligned}
&\mathrm{ad}_{\alpha(x)}(\Delta(y)) - \mathrm{ad}_{\alpha(y)}(\Delta(x)) \\
&\quad = \mathrm{ad}_{\alpha(x)}([y, r_1] \otimes \alpha(r_2) + \alpha(r_1) \otimes [y, r_2]) - (x \leftrightarrow y) \\
&\quad = [\alpha(x), [y, r_1]] \otimes \alpha^2(r_2) + \alpha[y, r_1] \otimes \alpha[x, r_2] + \alpha[x, r_1] \otimes \alpha[y, r_2] \\
&\qquad + \alpha^2(r_1) \otimes [\alpha(x), [y, r_2]] - (x \leftrightarrow y) \\
&\quad = [[x, y], \alpha(r_1)] \otimes \alpha^2(r_2) + \alpha^2(r_1) \otimes [[x, y], \alpha(r_2)] \\
&\quad = [[x, y], r_1] \otimes \alpha(r_2) + \alpha(r_1) \otimes [[x, y], r_2] \\
&\quad = \mathrm{ad}_{[x,y]}(r) = \Delta([x, y])
\end{aligned}
$$

for any $x, y \in L$, where $x \leftrightarrow y$ means swapping x for y in the forward expression. □

Proposition 4.10. *Let $(L, [-,-], \alpha)$ be an involutive Hom–Lie algebra and $r = r_1 \otimes r_2 \in L \otimes L$ such that $\alpha^{\otimes 2}(r) = r$, $r = -\tau(r)$. Set*

$$\Delta(x) = \mathrm{ad}_x(r) = [x, r_1] \otimes \alpha(r_2) + \alpha(r_1) \otimes [x, r_2].$$

Then, $\circlearrowleft (\alpha \otimes \Delta) \circ \Delta(x) = \mathrm{ad}_{\alpha(x)}(CH(r))$ for any $x \in L$.

Proof. From the fact $\alpha(r_1) \otimes \alpha(r_2) = r_1 \otimes r_2$ and L is involutive, we have

$$\alpha(r_1) \otimes r_2 = r_1 \otimes \alpha(r_2),$$

which is used in the following proof. By the definition of Δ and the properties of r, for any element $x \in L$,

$$
\begin{aligned}
&\mathrm{ad}_{\alpha(x)}(CH(r)) \\
&\quad = [\alpha(x), [r_1, r_1']] \otimes r_2 \otimes r_2' + \alpha([r_1, r_1']) \otimes [\alpha(x), \alpha(r_2)] \otimes r_2' \\
&\qquad + \alpha([r_1, r_1']) \otimes r_2 \otimes [\alpha(x), \alpha(r_2')] + [\alpha(x), \alpha(r_1)] \otimes \alpha([r_2, r_1']) \otimes r_2' \\
&\qquad + r_1 \otimes [\alpha(x), [r_2, r_1']] \otimes r_2' + r_1 \otimes \alpha([r_2, r_1']) \otimes [\alpha(x), \alpha(r_2')] \\
&\qquad + [\alpha(x), \alpha(r_1)] \otimes r_1' \otimes \alpha([r_2, r_2']) + r_1 \otimes [\alpha(x), \alpha(r_1') \otimes \alpha([r_2, r_2']) \\
&\qquad + r_1 \otimes r_1' \otimes [\alpha(x), [r_2, r_2']]
\end{aligned}
$$

$$
\begin{aligned}
&= \underbrace{[\alpha(x),[r_1,r_1']]\otimes r_2\otimes r_2'}_{(1)} + \underbrace{[r_1,\alpha(r_1')]\otimes[\alpha(x),r_2]\otimes r_2'}_{(2)}\\
&\quad + \underbrace{[\alpha(r_1),r_1']\otimes r_2\otimes[\alpha(x),r_2']}_{(3)} + \underbrace{[\alpha(x),r_1]\otimes[r_2,\alpha(r_1')]\otimes r_2'}_{(4)}\\
&\quad + \underbrace{r_1\otimes[\alpha(x),[r_2,r_1']]\otimes r_2'}_{(5)} + \underbrace{r_1\otimes[\alpha(r_2),r_1']\otimes[\alpha(x),r_2']}_{(6)}\\
&\quad + \underbrace{[\alpha(x),r_1]\otimes r_1'\otimes[r_2,\alpha(r_2')]}_{(7)} + \underbrace{r_1\otimes[\alpha(x),r_1']\otimes[\alpha(r_2),r_2']}_{(8)}\\
&\quad + \underbrace{r_1\otimes r_1'\otimes[\alpha(x),[r_2,r_2']]}_{(9)}.
\end{aligned}
$$

Meanwhile,

$$
\begin{aligned}
&\circlearrowleft (\alpha\otimes\Delta)\circ\Delta(x)\\
&=\circlearrowleft (\alpha\otimes\Delta)([x,r_1]\otimes\alpha(r_2)+\alpha(r_1)\otimes[x,r_2])\\
&=\circlearrowleft \alpha([x,r_1])\otimes[\alpha(r_2),r_1']\otimes\alpha(r_2')+\alpha([x,r_1])\otimes\alpha(r_1')\otimes[\alpha(r_2),r_2']\\
&\quad + r_1\otimes[[x,r_2],r_1']\otimes\alpha(r_2')+r_1\otimes\alpha(r_1')\otimes[[x,r_2],r_2']\\
&=\circlearrowleft [\alpha(x),r_1]\otimes[r_2,r_1']\otimes\alpha(r_2')+[\alpha(x),r_1]\otimes\alpha(r_1')\otimes[r_2,r_2']\\
&\quad + r_1\otimes[[x,r_2],r_1']\otimes\alpha(r_2')+r_1\otimes\alpha(r_1')\otimes[[x,r_2],r_2']\\
&= \underbrace{[\alpha(x),r_1]\otimes[r_2,r_1']\otimes\alpha(r_2')}_{(4)} + \underbrace{[\alpha(x),r_1]\otimes\alpha(r_1')\otimes[r_2,r_2']}_{(7)}\\
&\quad + \underbrace{r_1\otimes[[x,r_2],r_1']\otimes\alpha(r_2')}_{(5)} + \underbrace{r_1\otimes\alpha(r_1')\otimes[[x,r_2],r_2']}_{(9)}\\
&\quad + \underbrace{[r_2,r_1']\otimes\alpha(r_2')\otimes[\alpha(x),r_1]}_{(3)} + \underbrace{\alpha(r_1')\otimes[r_2,r_2']\otimes[\alpha(x),r_1]}_{(6)}\\
&\quad + \underbrace{[[x,r_2],r_1']\otimes\alpha(r_2')\otimes r_1}_{(1)} + \underbrace{\alpha(r_1')\otimes[[x,r_2],r_2']\otimes r_1}_{(5)}\\
&\quad + \underbrace{\alpha(r_2')\otimes[\alpha(x),r_1]\otimes[r_2,r_1']}_{(8)} + \underbrace{[r_2,r_2']\otimes[\alpha(x),r_1]\otimes\alpha(r_1')}_{(2)}\\
&\quad + \underbrace{\alpha(r_2')\otimes r_1\otimes[[x,r_2],r_1']}_{(9)} + \underbrace{[[x,r_2],r_2']\otimes r_1\otimes\alpha(r_1')}_{(1)}.
\end{aligned}
$$

We break these twelve terms into nine groups, which is equal to the nine terms of $\mathrm{ad}_{\alpha(x)}(CH(r))$ respectively. □

From Lemma 4.9 and Proposition 4.10, we have the main result of this section, which generalizes the result in [5].

Theorem 4.11. *Under the assumption of Proposition* 4.10, Δ *endows*

$$(L,[-,-],\Delta,\alpha)$$

with a coboundary Hom–Lie bialgebra structure if and only if $\mathrm{ad}_{\alpha(x)}(CH(r)) = 0$ *for all* $x \in L$.

Corollary 4.12. *Let* $(L, [-,-], \alpha)$ *be an involutive Hom–Lie algebra and* $r \in Im(id - \tau)$ *satisfy CHYBE. Set* $\Delta(x) = \mathrm{ad}_x(r)$*, for any* $x \in L$*. Then* $(L, [-,-], \Delta, \alpha)$ *is a coboundary Hom–Lie bialgebra.*

Proof. Since $r \in Im(id - \tau)$, we get the anti-symmetry of Δ easily. The compatibility of Hom–Lie bialgebra is from Lemma 4.9. In addition,

$$\circlearrowleft (\alpha \otimes \Delta) \circ \Delta(x) = \mathrm{ad}_{\alpha(x)}(CH(r)) = 0$$

from Proposition 4.10. So, by the above theorem, $(L, [-,-], \Delta, \alpha)$ is a coboundary Hom–Lie bialgebra. □

In Proposition 3.4, if (3.1) and (3.2) hold, then we would get a double Hom-associative algebra $(D(A^*, A), \star, \alpha_D)$ on the finite dimensional Hom-associative algebra (A, m, α). Define a Lie bracket on $(D(A^*, A)$ as follows

$$[f + a, g + b] = (f + a) \star (g + b) - (g + b) \star (f + a),$$

that is,

$$\begin{aligned}[f + a, g + b] &= (fg + f \triangleleft b + a \triangleright g) + (ab + f \rightharpoonup b + a \leftharpoondown g) \\ &\quad - (f \leftrightarrow g, a \leftrightarrow b) \\ &= fg - gf + f \triangleleft b - b \triangleright f + a \triangleright g - g \triangleleft a \\ &\quad + ab - ba + f \rightharpoonup b - b \leftharpoondown f + a \leftharpoondown g - g \rightharpoonup a,\end{aligned}$$

for any $f, g \in A^*, a, b \in A$, where $(f \leftrightarrow g, a \leftrightarrow b)$ means swapping f for g, and a for b in the formal expression.

Furthermore, if there exist two pairs of linearly independent elements f_0, g_0 in A^* and a_0, b_0 in A satisfying

$$\begin{aligned}&\alpha^{\otimes 2}((f_0 + a_0) \otimes (g_0 + b_0) - (g_0 + b_0) \otimes (f_0 + a_0)) \\ &\quad = (f_0 + a_0) \otimes (g_0 + b_0) - (g_0 + b_0) \otimes (f_0 + a_0),\end{aligned}$$

and,

$$\begin{aligned}&f_0 g_0 - g_0 f_0 + f_0 \triangleleft b_0 - b_0 \triangleright f_0 + a_0 \triangleright g_0 - g_0 \triangleleft a_0 = p\alpha^*(f_0), \\ &a_0 b_0 - b_0 a_0 + f_0 \rightharpoonup b_0 - b_0 \leftharpoondown f_0 + a_0 \leftharpoondown g_0 - g_0 \rightharpoonup a_0 = p\alpha(a_0),\end{aligned}$$

in which $0 \neq p \in k$.

Then, from Theorem 2.4 in [4], there is a quasi-triangular Hom–Lie bialgebra $(D(A^*, A), [-,-], \Delta = \mathrm{ad}(r), r, \alpha_D)$ by setting

$$r = (f_0 + a_0) \otimes (g_0 + b_0) - (g_0 + b_0) \otimes (f_0 + a_0).$$

In addition, if A is commutative, the Lie bracket becomes

$$[f + a, g + b] = fg - gf + f \rightharpoonup b - b \leftharpoondown f + a \leftharpoondown g - g \rightharpoonup a.$$

In this moment, we can obtain a Hom–Lie bialgebra under the following two conditions

$$\begin{aligned}&f_0 g_0 - g_0 f_0 = p\alpha^*(f_0), \\ &f_0 \rightharpoonup b_0 - b_0 \leftharpoondown f_0 + a_0 \leftharpoondown g_0 - g_0 \rightharpoonup a_0 = p\alpha(a_0).\end{aligned}$$

5. Double Hom–Lie Bialgebras

In this section, we introduce the notion of double Hom–Lie bialgebras which generalizes double Lie bialgebras in [18] and show that the double Hom–Lie bialgebras are indeed quasi-triangular Hom–Lie bialgebras. Meanwhile, we discuss the coquasi-triangular structure on the dual space-codouble Hom–Lie bialgebras.

We cite a useful lemma in [27] which plays an important role in Theorem 5.2.

Lemma 5.1. *Let $(L, [-,-], \alpha)$ be a Hom–Lie algebra and $r \in L^{\otimes 2}$ such that $\alpha^{\otimes 2}(r) = r$. Then $\Delta = ad(r) : L \to L^{\otimes 2}$ satisfies*

$$\mathrm{ad}_{[x,y]}(r) = \mathrm{ad}_{\alpha(x)}(\mathrm{ad}_y(r)) - \mathrm{ad}_{\alpha(y)}(\mathrm{ad}_x(r))$$

for any $x, y \in L$.

Theorem 5.2. *Let $(L, [-,-], \Delta, \alpha)$ be a finite-dimensional involutive Hom–Lie bialgebra with the dual space L^* given by the note after Proposition 4.5. Then, there is a quasi-triangular Hom–Lie bialgebra*

$$(D(L) = L^* \oplus L,\ [-,-]_D,\ \Delta_D,\ \alpha_D, r)$$

called double Hom–Lie bialgebra, built on $L^{*op} \oplus L$ as a vector space, with the following structures,

$$\begin{aligned}
&[\phi \oplus x, \varphi \oplus y]_D = [\varphi, \phi] \\
&\qquad + \varphi_1 \langle \varphi_2, x\rangle - \phi_1\langle \phi_2, y\rangle \oplus [x, y] + x_1\langle \varphi, x_2\rangle - y_1\langle \phi, y_2\rangle, \\
&\Delta_D(\phi + x) = \phi_1 \otimes \phi_2 + x_1 \otimes x_2, \\
&\alpha_D(\phi + x) = \alpha^*(\phi) + \alpha(x), \\
&r = \tfrac{1}{2} \sum_a (f^a \otimes \alpha(e_a) + \alpha^*(f^a) \otimes e_a),
\end{aligned}$$

for all $\phi, \varphi \in L^*$, $x, y \in L$. Here, L^{*op} and L are sub-Hom–Lie bialgebras, where $(-)^{op}$ denotes the opposite Lie bracket, the set $\{e_a\}$ is a basis of L and $\{f^a\}$ is its dual basis.

Proof. Noting that every element of direct sum has a unique decomposition into a vector in L^* and a vector in L, and from the definition of $D(L)$, we know that,

$$\begin{aligned}
&[\phi, \varphi]_D = -[\phi, \varphi], [x, y]_D = [x, y], \\
&[x, \phi]_D = \phi_1\langle \phi_2, x\rangle + x_1\langle \phi, x_2\rangle, \\
&\Delta_D(\phi) = \Delta(\phi), \Delta_D(x) = \Delta(x), \\
&\alpha_D(\phi) = \alpha^*(\phi), \alpha_D(x) = \alpha(x),
\end{aligned}$$

for all $\phi, \varphi \in L^*$, $x, y \in L$, where the right hand of the above equalities are in terms of the structures of L^* and L.

It is clear that $[-,-]_D$ is anti-symmetric and the Hom–Jacobi identity holds when we restrict all the elements on L^* or on L. So, we need to check the cross brackets.

In fact, for any $\phi, \varphi \in L^*, x \in L$,

$$\begin{aligned}
&\circlearrowleft [\alpha_D(x), [\phi,\varphi]_D]_D \\
&\quad = [\alpha(x), [\phi,\varphi]_D]_D + [\alpha^*(\phi), [\varphi, x]_D]_D + [\alpha^*(\varphi), [x,\phi]_D]_D \\
&\quad = [\alpha(x), [\phi,\varphi]_D]_D + [\alpha^*(\phi), [\varphi, x]_D]_D - [\alpha^*(\varphi), [\phi,x]_D]_D,
\end{aligned}$$

$$\begin{aligned}
&[\alpha(x), [\phi,\varphi]_D]_D \\
&\quad = -[\phi,\varphi]_1\langle[\phi,\varphi]_2, \alpha(x)\rangle - \alpha(x)_1\langle[\phi,\varphi], \alpha(x)_2\rangle \\
&\quad = -[\alpha^*(\phi), \varphi_1]\langle\alpha^*(\varphi_2), \alpha(x)\rangle - \alpha^*(\varphi)_1\langle[\alpha^*(\phi), \varphi_2], \alpha(x)\rangle \\
&\qquad - (\phi \leftrightarrow \varphi) - \langle id \otimes \alpha^*(\phi) \otimes \alpha^*(\varphi), (\alpha \otimes \Delta) \circ \Delta(x)\rangle \ (by \quad (4.1)) \\
&\quad = -[\alpha^*(\phi), \varphi_1]\langle\varphi_2, x\rangle - \alpha^*(\varphi_1)\langle[\phi, \alpha^*(\varphi_2)], x\rangle \\
&\qquad - (\phi \leftrightarrow \varphi) - \langle id \otimes \alpha^*(\phi) \otimes \alpha^*(\varphi), (\alpha \otimes \Delta) \circ \Delta(x)\rangle,
\end{aligned}$$

where $\phi \leftrightarrow \varphi$ means swapping ϕ for φ in the forward expression. On the other hand,

$$\begin{aligned}
&[\alpha^*(\phi), [\varphi, x]_D]_D - [\alpha^*(\varphi), [\phi, x]_D]_D \\
&\quad = [\alpha^*(\phi), [\varphi, x]_D]_D - (\phi \leftrightarrow \varphi) \\
&\quad = [\alpha^*(\phi), \varphi_1]\langle\varphi_2, x\rangle - [\alpha^*(\phi), x_1]_D\langle\varphi, x_2\rangle - (\phi \leftrightarrow \varphi) \\
&\quad = [\alpha^*(\phi), \varphi_1]\langle\varphi_2, x\rangle + \alpha^*(\phi_1)\langle\alpha^*(\phi_2), x_1\rangle\langle\varphi, x_2\rangle \\
&\qquad + \langle id \otimes \alpha^*(\phi) \otimes \varphi, (\Delta \otimes id) \circ \Delta(x)\rangle - (\phi \leftrightarrow \varphi).
\end{aligned}$$

Then, by merging the above two equalities, we have

$$\begin{aligned}
&\circlearrowleft [\alpha_D(x), [\phi,\varphi]_D]_D \\
&\quad = -\langle id \otimes \alpha^*(\phi) \otimes \alpha^*(\varphi), (\alpha \otimes \Delta) \circ \Delta(x)\rangle \\
&\qquad + \langle id \otimes \alpha^*(\phi) \otimes \varphi, (\Delta \otimes id) \circ \Delta(x)\rangle \\
&\qquad - \langle id \otimes \alpha^*(\varphi) \otimes \phi, (\Delta \otimes id) \circ \Delta(x)\rangle \\
&\quad = -\langle id \otimes \alpha^*(\phi) \otimes \alpha^*(\varphi), (\alpha \otimes \Delta) \circ \Delta(x)\rangle \\
&\qquad + \langle id \otimes \alpha^*(\phi) \otimes \alpha^*(\varphi), (\Delta \otimes \alpha) \circ \Delta(x)\rangle \\
&\qquad - \langle id \otimes \alpha^*(\varphi) \otimes \alpha^*(\phi), (\Delta \otimes \alpha) \circ \Delta(x)\rangle \\
&\quad = -\langle id \otimes \alpha^*(\phi) \otimes \alpha^*(\varphi), \circlearrowleft (\alpha \otimes \Delta) \circ \Delta(x)\rangle,
\end{aligned}$$

which is just the Hom–coJacobi identity for L. So, $\circlearrowleft [\alpha_D(x), [\phi,\varphi]_D]_D = 0$.

Similarly, from the Hom–coJacobi identity for L^*, we can prove that $\circlearrowleft [\alpha_D(x), [y,\phi]_D]_D = 0$, for any $x, y \in L, \phi \in L^*$.

Thus, $(D(L), [-,-]_D, \alpha_D)$ is a Hom–Lie algebra.

In addition, from the definition of Δ_D, we know that it satisfies the anti-symmetry and the Hom–coJacobi identity, so $(D(L), \Delta_D, \alpha_D)$ is a Hom–Lie coalgebra.

In the following proof we need two very useful identities:

$$\alpha^*(f^a) \otimes \alpha(e_{a_1})\langle\phi, \alpha(e_{a_2})\rangle = [f^a, \phi] \otimes e_a, \tag{5.1}$$

$$f^a_1\langle f^a_2, x\rangle \otimes e_a = \alpha^*(f^a) \otimes [\alpha(e_a), x], \tag{5.2}$$

for any $\phi \in L^*, x \in L$. These are true by using the fact of duality pairing $f^a\langle\phi, e_a\rangle = \phi$ and $\langle f^a, x\rangle e_a = x$, for all $\phi \in L^*$, $x \in L$. In fact, for any

$\phi, \varphi \in L^*$,

$$\begin{aligned}[f^a, \phi]\langle \varphi, e_a\rangle &= [\varphi, \phi] \\ &= \alpha^*(f^a)\langle \alpha^*[\varphi, \phi], e_a\rangle \\ &= \alpha^*(f^a)\langle \varphi, \alpha(e_{a_1})\rangle\langle \phi, \alpha(e_{a_2})\rangle,\end{aligned}$$

so, equation (5.1) holds.

In the same way, the second identity is satisfied too.

By the identity (5.1), for any $\phi \in L^*$, $x \in L$, we have,

$$\begin{aligned}\mathrm{ad}_\phi(r) &= \tfrac{1}{2}\mathrm{ad}_\phi(f^a \otimes \alpha(e_a) + \alpha^*(f^a) \otimes e_a) \\ &= \tfrac{1}{2}([\phi, f^a]_D \otimes e_a + \alpha^*(f^a) \otimes [\phi, \alpha(e_a)]_D + [\phi, \alpha^*(f^a)]_D \otimes \alpha(e_a) \\ &\quad + f^a \otimes [\phi, e_a]_D) \\ &= \tfrac{1}{2}([f^a, \phi] \otimes e_a - \alpha^*(f^a) \otimes \phi_1\langle \phi_2, \alpha(e_a)\rangle \\ &\quad - \alpha^*(f^a) \otimes \alpha(e_{a_1})\langle \phi, \alpha(e_{a_2})\rangle + [\alpha^*(f^a), \phi] \otimes \alpha(e_a) \\ &\quad - f^a \otimes \phi_1\langle \phi_2, e_a\rangle - f^a \otimes e_{a_1}\langle \phi, e_{a_2}\rangle) \\ &= \tfrac{1}{2}(-\phi_2 \otimes \phi_1 - \phi_2 \otimes \phi_1) \\ &= \Delta_D(\phi).\end{aligned}$$

Meanwhile, from the identity (5.2),

$$\begin{aligned}\mathrm{ad}_x(r) &= \tfrac{1}{2}\mathrm{ad}_x(f^a \otimes \alpha(e_a) + \alpha^*(f^a) \otimes e_a) \\ &= \tfrac{1}{2}([x, f^a]_D \otimes e_a + \alpha^*(f^a) \otimes [x, \alpha(e_a)]_D + [x, \alpha^*(f^a)]_D \otimes \alpha(e_a) \\ &\quad + f^a \otimes [x, e_a]_D) \\ &= \tfrac{1}{2}(f^a_1\langle f^a_2, x\rangle \otimes e_a + x_1\langle f^a, x_2\rangle \otimes e_a + \alpha^*(f^a) \otimes [x, \alpha(e_a)] \\ &\quad + \alpha^*(f^a_1)\langle \alpha^*(f^a_2), x\rangle \otimes \alpha(e_a) + x_1\langle \alpha^*(f^a), x_2\rangle \otimes \alpha(e_a) \\ &\quad + f^a \otimes [x, e_a]) \\ &= \tfrac{1}{2}(x_1 \otimes x_2 + x_1 \otimes x_2) \\ &= \Delta_D(x).\end{aligned}$$

So, $\Delta_D(d) = \mathrm{ad}_d(r)$, for any $d \in D(L)$.

In addition,

$$\alpha_D^{\otimes 2}(r) = \alpha_D^{\otimes 2}(f^a \otimes \alpha(e_a) + \alpha^*(f^a) \otimes e_a) = r,$$

from Lemma 5.1, the compatibility of Hom–Lie bialgebra

$$\Delta_D[c, d] = \mathrm{ad}_{\alpha(c)}(\Delta_D(d)) - \mathrm{ad}_{\alpha(d)}(\Delta_D(c))$$

holds, for any $c, d \in D(L)$. Thus, $(D(L), [-,-]_D, \Delta_D, \alpha_D, r)$ is a coboundary Hom–Lie bialgebra.

Last, r obeys the CHYBE. Since $r = \frac{1}{2}(f^a \otimes \alpha(e_a) + \alpha^*(f^a) \otimes e_a)$, by Definition 4.6,

$$\begin{aligned}r^{12} &= \tfrac{1}{2}(f^a \otimes \alpha(e_a) \otimes 1 + \alpha^*(f^a) \otimes e_a \otimes 1), \\ r^{13} &= \tfrac{1}{2}(f^a \otimes 1 \otimes \alpha(e_a) + \alpha^*(f^a) \otimes 1 \otimes e_a), \\ r^{23} &= \tfrac{1}{2}(1 \otimes f^a \otimes \alpha(e_a) + 1 \otimes \alpha^*(f^a) \otimes e_a),\end{aligned}$$

we have

$$\begin{aligned}
CH(r) = \tfrac{1}{4}(&\underbrace{[f^a, f^b]_D \otimes e_a \otimes e_b}_{(1)} + \underbrace{[f^a, \alpha^*(f^b)]_D \otimes e_a \otimes \alpha(e_b)}_{(2)} \\
&+ \underbrace{[\alpha^*(f^a), f^b]_D \otimes \alpha(e_a) \otimes e_b}_{(3)} + \underbrace{[\alpha^*(f^a), \alpha^*(f^b)]_D \otimes \alpha(e_a) \otimes \alpha(e_b)}_{(4)} \\
&+ \underbrace{\alpha^*(f^a) \otimes [\alpha(e_a), f^b]_D \otimes e_b}_{(1)} + \underbrace{\alpha^*(f^a) \otimes [\alpha(e_a), \alpha^*(f^b)]_D \otimes \alpha(e_b)}_{(2)} \\
&+ \underbrace{f^a \otimes [e_a, f^b]_D \otimes e_b}_{(3)} + \underbrace{f^a \otimes [e_a, \alpha^*(f^b)]_D \otimes \alpha(e_b)}_{(4)} \\
&+ \underbrace{\alpha^*(f^a) \otimes \alpha^*(f^b) \otimes [\alpha(e_a), \alpha(e_b)]_D}_{(1)} + \underbrace{\alpha^*(f^a) \otimes f^b \otimes [\alpha(e_a), e_b]_D}_{(2)} \\
&+ \underbrace{f^a \otimes \alpha^*(f^b) \otimes [e_a, \alpha(e_b)]_D}_{(3)} + \underbrace{f^a \otimes f^b \otimes [e_a, e_b]_D}_{(4)}),
\end{aligned}$$

which can be divided into four groups as above. One of the groups (1):

$$\begin{aligned}
&[f^a, f^b]_D \otimes e_a \otimes e_b + \alpha^*(f^a) \otimes [\alpha(e_a), f^b]_D \otimes e_b \\
&\quad + \alpha^*(f^a) \otimes \alpha^*(f^b) \otimes [\alpha(e_a), \alpha(e_b)]_D \\
&\quad = -[f^a, f^b] \otimes e_a \otimes e_b + \alpha^*(f^a) \otimes f^b_1 \langle f^b_2, \alpha(e_a)\rangle \otimes e_b \\
&\quad + \ \alpha^*(f^a) \otimes \alpha(e_{a_1})\langle f^b, \alpha(e_{a_2})\rangle \otimes e_b \\
&\quad + \alpha^*(f^a) \otimes \alpha^*(f^b) \otimes [\alpha(e_a), \alpha(e_b)] \\
&= 0,
\end{aligned}$$

by identities (5.1) and (5.2). In the same way, the other three groups are all zero too. So, $CH(r) = 0$.

Thus $D(L)$ is a quasi-triangular Hom–Lie bialgebra. □

Example 5.3. Let $sl(2)_\alpha$ be the Hom–Lie bialgebra and $sl(2)^*_\alpha$ be its dual Hom–Lie bialgebra given in Example 4.7. For the condition $\alpha^2 = id$ needed in Theorem 5.2, so, $c = \pm 1$ in Example 4.7. But if $c = 1$, then the Hom–Lie bialgebra $sl(2)_\alpha$ is just $sl(2)$. So, c can be only equal to -1. At this particular moment, the structure maps of $sl(2)_\alpha$ are given by

$$\begin{aligned}
&\alpha(H) = H, \quad \alpha(X_\pm) = -X_\pm, \\
&[H, X_\pm]_\alpha = \mp 2X_\pm, \quad [X_+, X_-]_\alpha = H, \\
&\Delta_\alpha(H) = 0, \quad \Delta_\alpha(X_\pm) = -\tfrac{1}{2}(X_\pm \otimes H - H \otimes X_\pm).
\end{aligned}$$

Respectively, the structures of $sl(2)^*_\alpha$ are as follows

$$\begin{aligned}
&\alpha^*(H^*) = H^*, \quad \alpha^*(X^*_\pm) = -X^*_\pm, \\
&[X^*_\pm, H^*]_\alpha = -\tfrac{1}{2}X^*_\pm, \quad [X^*_+, X^*_-]_\alpha = 0, \\
&\Delta_\alpha(X^*_\pm) = \mp 2(H^* \otimes X^*_\pm - X^*_\pm \otimes H^*), \quad \Delta_\alpha(H^*) = X^*_+ \otimes X^*_- - X^*_- \otimes X^*_+.
\end{aligned}$$

From direct computation, we obtain the double Hom–Lie bialgebra $D(sl(2)_\alpha)$ built on the vector space $sl(2)^*_\alpha \oplus sl(2)_\alpha$ with the following structures $[-,-]_D$, Δ_D, and α_D:

$$
\begin{aligned}
&[X^*_\pm, H^*]_D = \tfrac{1}{2}X^*_\pm, \quad [X^*_+, X^*_-]_D = 0, \quad [H, X_\pm]_D = \mp 2X_\pm,\\
&[X_+, X_-]_D = H, \quad [H, H^*]_D = 0, \quad [X_\pm, X^*_\pm]_D = \mp 2H^* + \tfrac{1}{2}H,\\
&[H, X^*_\pm]_D = \pm 2X^*_\pm, \quad [X_\pm, H^*]_D = -\tfrac{1}{2}X_\pm \mp X^*_\mp,\\
&\Delta_D(X^*_\pm) = \mp 2(H^* \otimes X^*_\pm - X^*_\pm \otimes H^*), \quad \Delta_D(H^*) = X^*_+ \otimes X^*_- - X^*_- \otimes X^*_+,\\
&\Delta_D(H) = 0, \quad \Delta_D(X_\pm) = -\tfrac{1}{2}(X_\pm \otimes H - H \otimes X_\pm),\\
&\alpha_D(H^*) = H^*, \quad \alpha_D(X^*_\pm) = -X^*_\pm, \quad \alpha_D(H) = H, \quad \alpha_D(X_\pm) = -X_\pm.
\end{aligned}
$$

In addition,

$$
\begin{aligned}
r &= \tfrac{1}{2}(H^* \otimes \alpha(H) + \alpha^*(H^*) \otimes H + X^*_+ \otimes \alpha(X_+)\\
&\quad + \alpha^*(X^*_+) \otimes X_+ + X^*_- \otimes \alpha(X_-) + \alpha^*(X^*_-) \otimes X_-)\\
&= H^* \otimes H - X^*_+ \otimes X_+ - X^*_- \otimes X_-.
\end{aligned}
$$

That is, the double $(D(sl(2)_\alpha), [-,-]_D, \Delta_D, \alpha_D, r)$ is a quasi-triangular Hom–Lie bialgebra.

Example 5.4. Working on the complex filed $\mathbb{C}$, we know that $sl(2)_\alpha$ and $sl(2)^*_\alpha$ defined as in Example 4.3 have another dual bases

$$
\begin{aligned}
&e_1 = -\tfrac{i}{2}(X_+ + X_-), \quad e_2 = -\tfrac{1}{2}(X_+ - X_-), \quad e_3 = -\tfrac{i}{2}H,\\
&f^1 = i(X^*_+ + X^*_-), \quad f^2 = -(X^*_+ - X^*_-), \quad f^3 = 2iH^*.
\end{aligned}
$$

We can easily check that $\langle f^a, e_b\rangle = \delta^a_b$ by the duality pairing relation. We can construct another quasi-triangular Hom–Lie bialgebra on $D(sl(2)_\alpha)$ with $[-,-]_D, \Delta_D, \alpha_D$ defined as above and r' given as follows:

$$
\begin{aligned}
r' &= \tfrac{1}{2}\sum_a (f^a \otimes \alpha(e_a) + \alpha^*(f^a) \otimes e_a)\\
&= \tfrac{1}{2}(f^1 \otimes \alpha(e_1) + \alpha^*(f^1) \otimes e_1 + f^2 \otimes \alpha(e_2) + \alpha^*(f^2) \otimes e_2\\
&\quad + f^3 \otimes \alpha(e_3) + \alpha^*(f^3) \otimes e_3)\\
&= -\tfrac{1}{2}((X^*_+ + X^*_-) \otimes (X_+ + X_-) + (X^*_+ - X^*_-) \otimes (X_+ - X_-)\\
&\quad - 2H^* \otimes H)\\
&= H^* \otimes H - X^*_+ \otimes X_+ - X^*_- \otimes X_-.
\end{aligned}
$$

We find that $r = r'$ clearly. That is, though $sl(2)_\alpha$ and $sl(2)^*_\alpha$ have different dual bases, $D(sl(2)_\alpha)$ has the same quasi-triangular structure.

We give a useful lemma in [27] which plays a key role in Proposition 5.6.

Lemma 5.5. *Let* $(L, [-,-], \Delta, \alpha)$ *be a Hom–Lie bialgebra and* $t \in L^{\otimes 2}$ *such that* $\alpha^{\otimes 2}(t) = t, t_{21} = -t$, *and*

$$\alpha^{\otimes 3}(\mathrm{ad}_x(CH(t) + \circlearrowleft (\alpha \otimes \Delta)(t))) = 0$$

for all $x \in L$. *Define the perturbed cobracket* $\Delta_t = \Delta + ad(t)$. *Then,* $L_t = (L, [-,-], \Delta_t, \alpha)$ *is a Hom–Lie bialgebra.*

Next, we consider the codouble Hom–Lie bialgebra $D(L)^*$ built on the vector space $L^{cop} \oplus L^*$ which is the dual to the double Hom–Lie bialgebra $D(L)$. From the note after Proposition 4.5, $D(L)^*$ has a Hom–Lie algebra structure and a complicated Lie cobracket, which is just analogous to the codouble Lie bialgebra in [18]. In addition, we know that the twisting map $\alpha_{D(L)^*} = \alpha + \alpha^*$. Then we have the following result.

Proposition 5.6. *Let $(L, [-,-], \Delta, \alpha)$ be a finite dimensional involutive Hom–Lie bialgebra. From the Lie cobracket of Hom–Lie bialgebra $L^{cop} \oplus L^*$, we define a perturbed Lie cobracket $\Delta_{L^{cop}\oplus L^*} + ad(t)$ which is exactly the Lie cobracket $\Delta_{D(L)^*}$ of codouble Hom–Lie bialgebra, where*

$$t = \tfrac{1}{2}\sum_a (\alpha^*(f^a) \otimes e_a - e_a \otimes \alpha^*(f^a) + f^a \otimes \alpha(e_a) - \alpha(e_a) \otimes f^a).$$

Here $\{e_a\}$ is a basis of L and $\{f^a\}$ is its dual basis, and the L^{cop} denotes the opposite cobracket.

Proof. Except for $\alpha_{L^{cop}\oplus L^*}(x \oplus \phi) = \alpha(x) \oplus \alpha^*(\phi)$, the Hom–Lie algebra structure on $L^{cop} \oplus L^*$ means that

$$[x \oplus \phi, y \oplus \varphi] = [x, y] \oplus [\phi, \varphi],$$

and the Hom–Lie coalgebra structure on $L^{cop} \oplus L^*$ means that

$$\Delta_{L^{cop}\oplus L^*}(x) = -\Delta(x); \quad \Delta_{L^{cop}\oplus L^*}(\phi) = \Delta(\phi),$$

for all $x, y \in L, \phi, \varphi \in L^*$, or equivalently, L, L^* are both sub-Hom–Lie algebras with $[x, \phi] = 0$ for the Lie bracket between them and sub-Hom–Lie coalgebras for the corresponding Lie cobrackets of them respectively.

The duality paring of the codouble is given by

$$\langle x \oplus \phi, \varphi \oplus y\rangle = \langle x, \varphi\rangle + \langle \phi, y\rangle.$$

Using this, we can obtain the Hom–Lie cobracket of codouble as follows

$$\begin{aligned}
&\langle \Delta_{D(L)^*}(x \oplus \phi), (\varphi \oplus y) \otimes (\psi \oplus z)\rangle \\
&\quad = \langle x \oplus \phi, [\varphi \oplus y, \psi \oplus z]_{D(L)}\rangle \\
&\quad = \langle x \oplus \phi, (-[\varphi, \psi] + \psi_1\langle \psi_2, y\rangle - \varphi_1\langle \varphi_2, z\rangle) \\
&\qquad \oplus\ ([y, z] + y_1\langle \psi, y_2\rangle - z_1\langle \varphi, z_2\rangle)\rangle \\
&\quad = \langle x, -[\varphi, \psi]\rangle + \langle x, \psi_1\langle \psi_2, y\rangle\rangle - \langle x, \varphi_1\langle \varphi_2, z\rangle\rangle \\
&\qquad +\ \langle \phi, [y, z]\rangle + \langle \phi, y_1\langle \psi, y_2\rangle\rangle - \langle \phi, z_1\langle \varphi, z_2\rangle\rangle \\
&\quad = \langle -\Delta(x), \varphi \otimes \psi\rangle + \langle [x, y], \psi\rangle - \langle [x, z], \varphi\rangle \\
&\qquad +\ \langle \Delta(\phi), y \otimes z\rangle + \langle [\phi, \psi], y\rangle - \langle [\phi, \varphi], z\rangle.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\langle \Delta_{L^{cop}\oplus L^*}(x \oplus \phi) + \mathrm{ad}_{x\oplus\phi}(t), (\varphi \oplus y) \otimes (\psi \oplus z)\rangle \\
&\quad = \langle \Delta_{L^{cop}\oplus L^*}(x \oplus \phi), (\varphi \oplus y) \otimes (\psi \oplus z)\rangle + \langle \mathrm{ad}_{x\oplus\phi}(t), (\varphi \oplus y) \otimes (\psi \oplus z)\rangle \\
&\quad = \langle -\Delta(x) + \Delta(\phi), (\varphi \oplus y) \otimes (\psi \oplus z)\rangle \\
&\qquad +\ \langle \tfrac{1}{2}([\phi, \alpha^*(f^a)] \otimes \alpha(e_a) + f^a \otimes [x, e_a] - [x, e_a] \otimes f^a
\end{aligned}$$

$$
\begin{aligned}
&- \alpha(e_a)\otimes[\phi,\alpha^*(f^a)] + [\phi,f^a]\otimes e_a + \alpha^*(f^a)\otimes[x,\alpha(e_a)]\\
&- [x,\alpha(e_a)]\otimes\alpha^*(f^a) - e_a\otimes[\phi,f^a]),(\varphi\oplus y)\otimes(\psi\oplus z)\rangle\\
&= \langle-\Delta(x),\varphi\otimes\psi\rangle + \langle\Delta(\phi),y\otimes z\rangle + \tfrac{1}{2}(\langle[\phi,\alpha^*(f^a)]\otimes\alpha(e_a)\\
&+ [\phi,f^a]\otimes e_a, y\otimes\psi\rangle + \langle f^a\otimes[x,e_a] + \alpha^*(f^a)\otimes[x,\alpha(e_a)], y\otimes\psi\rangle\\
&- \langle[x,e_a]\otimes f^a + [x,\alpha(e_a)]\otimes\alpha^*(f^a),\varphi\otimes z\rangle - \langle\alpha(e_a)\otimes[\phi,\alpha^*(f^a)]\\
&+ e_a\otimes[\phi,f^a],\varphi\otimes z\rangle)\\
&= \langle-\Delta(x),\varphi\otimes\psi\rangle + \langle\Delta(\phi),y\otimes z\rangle\\
&+ \langle[\phi,\psi],y\rangle + \langle[x,y],\psi\rangle - \langle[x,z],\varphi\rangle - \langle[\phi,\varphi],z\rangle,
\end{aligned}
$$

for any $x, y, z \in L$, $\phi, \varphi, \psi \in L^*$. So,

$$\Delta_{D(L)^*} = \Delta_{L^{cop}\oplus L^*} + \mathrm{ad}(t),$$

and the Lie cobracket of the codouble on the Lie subalgebras is

$$\Delta_{D(L)^*}(x) = -\Delta(x) + \tau\circ\mathrm{ad}_x - \mathrm{ad}_x,\quad \Delta_{D(L)^*}(\phi) = -\Delta(\phi) + \tau\circ\mathrm{ad}_\phi - \mathrm{ad}_\phi,$$

for all $x \in L, \phi \in L^*$, where we regard $\mathrm{ad}_x : L \to L$ and $\mathrm{ad}_\phi : L^* \to L^*$ as elements of $L \otimes L^*$ in the natural way.

Next we show that the stronger condition in Lemma 5.5 is satisfied. That is,

$$CH(t) + \circlearrowleft (\alpha_{L^{cop}\oplus L^*}\otimes\Delta_{L^{cop}\oplus L^*})(t) = 0.$$

In fact,

$$
\begin{aligned}
CH(t) = {}& \tfrac{1}{4}([\alpha^*(f^a),\alpha^*(f^b)]\otimes\alpha(e_a)\otimes\alpha(e_b) + [\alpha^*(f^a),f^b]\otimes\alpha(e_a)\otimes e_b\\
&+ [e_a,e_b]\otimes f^a\otimes f^b + [e_a,\alpha(e_b)]\otimes f^a\otimes\alpha^*(f^b)\\
&+ [f^a,\alpha^*(f^b)]\otimes e_a\otimes\alpha(e_b) + [f^a,f^b]\otimes e_a\otimes e_b\\
&+ [\alpha(e_a),e_b]\otimes\alpha^*(f^a)\otimes f^b + [\alpha(e_a),\alpha(e_b)]\otimes\alpha^*(f^a)\otimes\alpha^*(f^b)\\
&- f^a\otimes[e_a,e_b]\otimes f^b - f^a\otimes[e_a,\alpha(e_b)]\otimes\alpha^*(f^b)\\
&- \alpha(e_a)\otimes[\alpha^*(f^a),\alpha^*(f^b)]\otimes\alpha(e_b) - \alpha(e_a)\otimes[\alpha^*(f^a),f^b]\otimes e_b\\
&- \alpha^*(f^a)\otimes[\alpha(e_a),e_b]\otimes f^b - \alpha^*(f^a)\otimes[\alpha(e_a),\alpha(e_b)]\otimes\alpha^*(f^b)\\
&- e_a\otimes[f^a,\alpha^*(f^b)]\otimes\alpha(e_b) - e_a\otimes[f^a,f^b]\otimes e_b\\
&+ f^a\otimes f^b\otimes[e_a,e_b] + f^a\otimes\alpha^*(f^b)\otimes[e_a,\alpha(e_b)]\\
&+ \alpha(e_a)\otimes\alpha(e_b)\otimes[\alpha^*(f^a),\alpha^*(f^b)] + \alpha(e_a)\otimes e_b\otimes[\alpha^*(f^a),f^b]\\
&+ \alpha^*(f^a)\otimes f^b\otimes[\alpha(e_a),e_b] + \alpha^*(f^a)\otimes\alpha^*(f^b)\otimes[\alpha(e_a),\alpha(e_b)]\\
&+ e_a\otimes\alpha(e_b)\otimes[f^a,\alpha^*(f^b)] + e_a\otimes e_b\otimes[f^a,f^b])\\
={}& \circlearrowleft \tfrac{1}{4}([\alpha^*(f^a),\alpha^*(f^b)]\otimes\alpha(e_a)\otimes\alpha(e_b) + [\alpha^*(f^a),f^b]\otimes\alpha(e_a)\otimes e_b\\
&+ [e_a,e_b]\otimes f^a\otimes f^b + [e_a,\alpha(e_b)]\otimes f^a\otimes\alpha^*(f^b)\\
&+ [f^a,\alpha^*(f^b)]\otimes e_a\otimes\alpha(e_b) + [f^a,f^b]\otimes e_a\otimes e_b\\
&+ [\alpha(e_a),e_b]\otimes\alpha^*(f^a)\otimes f^b + [\alpha(e_a),\alpha(e_b)]\otimes\alpha^*(f^a)\otimes\alpha^*(f^b))\\
={}& \circlearrowleft \tfrac{1}{2}([\alpha^*(f^a),\alpha^*(f^b)]\otimes\alpha(e_a)\otimes\alpha(e_b) + [f^a,f^b]\otimes e_a\otimes e_b
\end{aligned}
$$

$$
\begin{aligned}
&+ [e_a, e_b] \otimes f^a \otimes f^b + [\alpha(e_a), \alpha(e_b)] \otimes \alpha^*(f^a) \otimes \alpha^*(f^b)) \\
= &\circlearrowleft \tfrac{1}{2}(\alpha^*(f^a) \otimes \Delta(\alpha(e_a)) + \alpha(e_a) \otimes \Delta(\alpha^*(f^a)) + f^a \otimes \Delta(e_a) \\
&+ e_a \otimes \Delta(f^a)) \\
= &- \circlearrowleft (\alpha_{L^{cop} \oplus L^*} \otimes \Delta_{L^{cop} \oplus L^*})(t),
\end{aligned}
$$

which is required. □

Furthermore, we consider the coquasi-triangular (dual quasi-triangular) structure on the codouble Hom–Lie bialgebra $D(L)^*$. In principle, all that we have to say in the following can be obtained by dualising along the lines in the usual way, by writing out the axioms of a quasi-triangular Hom–Lie bialgebra as diagrams and then turning all the arrows around.

Unlike the Hom–Lie bialgebra axioms themselves, the axioms of a quasi-triangular Hom–Lie bialgebra are clearly not self-dual. Since the quasi-triangular structure $r \in L \otimes L$ in quasi-triangular Hom–Lie bialgebra can be regarded as a map $k \to L \otimes L$, there is a map $r : G \otimes G \to k$ such that the Lie bracket has a special form

$$[x, y] = x_1 r(x_2 \otimes \alpha(y)) + y_1 r(\alpha(x) \otimes y_2)$$

and obeys the CHYBE in a dual form

$$
\begin{aligned}
&r(x_1 \otimes \alpha(y))r(x_2 \otimes \alpha(z)) + r(\alpha(x) \otimes y_1)r(y_2 \otimes \alpha(z)) \\
&\quad + r(\alpha(x) \otimes z_1)r(\alpha(y) \otimes z_2) = 0,
\end{aligned}
$$

where $x, y, z \in G$ and G is used to refer to the Hom–Lie bialgebra in the dual formation.

The symmetric part $2r_+$ is required to be invariant under the adjoint Lie coaction according to $r_+(\alpha(x) \otimes y_1)y_2 + r_+(x_1 \otimes \alpha(y))x_2 = 0$.

Here, the coquasi-triangular structure $r : D(L)^* \otimes D(L)^* \to k$ in the codouble Hom–Lie bialgebra $D(L)^*$ is

$$\tfrac{1}{2} \sum_a (r(f^a \otimes \alpha(e_a)) + r(\alpha^*(f^a) \otimes e_a)).$$

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