



Gastineau-Hills' Quasi-Clifford Algebras and Plug-In Constructions for Hadamard Matrices

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Abstract. The quasi-Clifford algebras as described by Gastineau-Hills in 1980 and 1982, should be better known, and have only recently been rediscovered. These algebras and their representation theory provide effective tools to address the following problem arising from a plug-in construction for Hadamard matrices: Given λ , a pattern of amicability/anti-amicability, with $\lambda_{j,k} = \lambda_{k,j} = \pm 1$, find a set of n monomial $\{-1, 0, 1\}$ matrices D of minimal order such that

$$D_j D_k^T - \lambda_{j,k} D_k D_j^T = 0 \quad (j \neq k).$$

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1. Introduction

The work of Gastineau-Hills on quasi-Clifford algebras [5, 6] should be better known. In particular, as at April 2018, his paper on the subject [6] had only four citations other than self-citations, according to Google, [3, 12, 16, 22], and only one of these [16] discusses quasi-Clifford algebras and their representation theory to any depth.

Since the quasi-Clifford algebras are a fundamental and natural generalization of the Clifford algebras, these algebras, or some subset of them, as well as their representation theory, have been rediscovered or partially rediscovered a number of times. The rediscoveries include

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- da Rocha and Vaz' extended Clifford algebras [2] which are doubled real Clifford algebras, including the simplest case of a quasi-Clifford algebra that is not itself a Clifford algebra;
- Rajan and Rajan's extended Clifford algebras [20, 21], which are a subset of the special quasi-Clifford algebras; and
- Marchuk's extended Clifford algebras [16], which correspond to the special quasi-Clifford algebras over the field of real numbers.

The partial rediscoveries include near misses, such as the rediscoveries of the finite groups generated by the basis elements of the quasi-Clifford algebras, and their representations:

- The finite groups generated by the basis elements of the quasi-Clifford algebras are signed groups and have real monomial representations of the order of a power of two [1].
- The non-abelian extensions of C_2 by C_2^k as classified by de Launey and Flannery's Theorem 21.2.3 [3, Section 21.2] and the finite groups generated by the basis elements of the quasi-Clifford algebras are connected by their relationship to the finite groups generated by the basis elements of Clifford algebras. The correspondence between the two deserves further investigation. See in particular, Lam and Smith's classification of the finite groups generated by the basis elements of Clifford algebras [11]. See also the classifications given by de Launey and Smith [4].

The original application of quasi-Clifford algebras and their representation theory was to systems of orthogonal designs [5, 6, 22]. The current paper applies quasi-Clifford algebras and their representation theory to the study of some plug-in constructions for Hadamard matrices described by the author in 2014 [12]. The key question addressed is: Given λ , a pattern of amicability/anti-amicability, with $\lambda_{j,k} = \lambda_{k,j} = \pm 1$, find a set of n monomial $\{-1, 0, 1\}$ matrices D of minimal order such that

$$D_j D_k^T - \lambda_{j,k} D_k D_j^T = 0 \quad (j \neq k).$$

Specifically, this paper contains a new proof of Theorem 5 of [12] that answers Question 1 of that paper.

The remainder of the paper is organized as follows. Section 2 outlines Gastineau-Hills' theory of quasi-Clifford algebras. Section 3 revises the plug-in constructions for Hadamard matrices. Section 4 uses the theory of real Special quasi-Clifford algebras to address questions related to those constructions.

2. Quasi-Clifford Algebras

Humphrey Gastineau-Hills fully developed the theory of quasi-Clifford algebras in his thesis of 1980 [5] and published the key results in a subsequent paper [6]. The paper describes the theory of quasi-Clifford algebras in full generality for fields of characteristic other than 2. This paper uses only the properties of quasi-Clifford algebras over the real field, and after giving the

general definition, this section presents a summary of Gastineau-Hills' constructions and results in this case.

Definition 1. [6, (2.1)] Let F be a commutative field of characteristic not 2, m a positive integer, (κ_i) , $1 \leq i \leq m$ a family of non-zero elements of F , and $(\delta_{i,j})$, $1 \leq i \leq j \leq m$ a family of elements from $\{0, 1\}$. The quasi-Clifford, or QC, algebra $\mathcal{C} = \mathcal{C}_F[m, (\kappa_i), (\delta_{i,j})]$ is the algebra (associative, with a 1) over F on m generators $\alpha_1, \dots, \alpha_m$, with defining relations

$$\alpha_i^2 = \kappa_i, \quad \alpha_j \alpha_i = (-1)^{\delta_{i,j}} \alpha_i \alpha_j \quad (i < j) \tag{1}$$

(where κ_i of F is identified with κ_i times the 1 of \mathcal{C}).

If all $\delta_{i,j} = 1$ we have a Clifford algebra corresponding to some non-singular quadratic form on F^m [10]. If in addition each $\kappa_i = \pm 1$ we have the Special Clifford algebras studied by Kawada and Imahori [9], Porteous [17, 18] and Lounesto [15] amongst others.

Theorem 1. [6, (2.3)] *The QC algebra \mathcal{C} of Definition 1 has dimension 2^m as a vector space over F , and a basis is $\{\alpha_1^{\epsilon_1} \dots \alpha_m^{\epsilon_m}, \epsilon_i = 0 \text{ or } 1\}$.*

This paper concentrates on the QC algebras for which each $\kappa_i = \pm 1$. Gastineau-Hills call such algebras Special quasi-Clifford, or SQC, algebras.

Also, from this point onwards, the field F is the real field \mathbb{R} , and Gastineau-Hills' key theorems and constructions are summarised for this case. Additionally, the real Special Clifford algebras are referred to simply as Clifford algebras, and the notation of Porteous [17, 18] and Lounesto [15] is used for these algebras and their representations.

Gastineau-Hills [6] uses the notation $[\alpha_1, \dots, \alpha_m]$ for the QC algebra generated by $\alpha_1, \dots, \alpha_m$, and the following notation for two special cases. In the case of a single generator, $\mathbb{C}_b := [\beta]$ where $\beta^2 = b$. For a pair of anti-commuting generators, $\mathbb{Q}_{c,d} := [\gamma, \delta]$ where $\gamma^2 = g$, $\delta^2 = d$. This notation yields the following isomorphisms between these low dimensional real SQC algebras and their corresponding Clifford algebras [6, (2.2)].

$$\begin{aligned} \mathbb{C}_{-1} &\simeq \mathbb{R}_{0,1} \simeq \mathbb{C}, \\ \mathbb{C}_1 &\simeq \mathbb{R}_{1,0} \simeq {}^2\mathbb{R}, \\ \mathbb{Q}_{-1,-1} &\simeq \mathbb{R}_{0,2} \simeq \mathbb{H}, \\ \mathbb{Q}_{-1,1} &\simeq \mathbb{R}_{1,1} \simeq \mathbb{R}(2), \\ \mathbb{Q}_{1,-1} &\simeq \mathbb{R}_{1,1} \simeq \mathbb{R}(2), \\ \mathbb{Q}_{1,1} &\simeq \mathbb{R}_{2,0} \simeq \mathbb{R}(2). \end{aligned} \tag{2}$$

Gastineau-Hills first decomposition theorem in the special case of real SQC algebras is as follows.

Theorem 2. [6, (2.7)] *Any real SQC algebra $\mathcal{C}[m, (\kappa_i), (\delta_{i,j})] = [\alpha_1, \dots, \alpha_m]$ is expressible as a tensor product over \mathbb{R} :*

$$\begin{aligned} \mathcal{C} &\simeq \mathbb{C}_{b_1} \otimes \dots \otimes \mathbb{C}_{b_r} \otimes \mathbb{Q}_{c_1,d_1} \otimes \dots \otimes \mathbb{Q}_{c_s,d_s} \\ &= [\beta_1] \otimes \dots \otimes [\beta_r] \otimes [\gamma_1, \delta_1] \otimes \dots \otimes [\gamma_s, \delta_s] \end{aligned} \tag{3}$$

where $r, s \geq 0$, $r + 2s = m$, and each b_i, c_j, d_k is ± 1 .

Each $\beta_i, \gamma_j, \delta_k$ (where $\beta_i^2 = b_i, \gamma_j^2 = c_j, \delta_k^2 = d_k$ and all pairs commute except $\delta_i \gamma_i = -\gamma_i \delta_i, 1 \leq i \leq s$) is, to within multiplication by ± 1 , one of the basis elements $\alpha_1^{\epsilon_1} \dots \alpha_m^{\epsilon_m}$ of \mathcal{C} . Conversely each $\alpha_1^{\epsilon_1} \dots \alpha_m^{\epsilon_m}$ is, to within division by ± 1 , one of

$$\beta_1^{\theta_1} \dots \beta_r^{\theta_r} \gamma_1^{\phi_1} \delta_1^{\psi_1} \dots \gamma_s^{\phi_s} \delta_s^{\psi_s}$$

(each $\theta_i, \phi_j, \psi_k = 0$ or 1). Thus the latter $2^{r+2s} = 2^m$ elements form a new basis of \mathcal{C} , and $\{\beta_i, \gamma_j, \delta_k\}$ is a new set of generators.

Here the tensor product \otimes is the real tensor product of real algebras, as per Porteous [17].

Gastineau-Hills [6] goes on to investigate the Wedderburn structure of the real SQC algebras by first determining the centre of each algebra, and then determining the irreducible representations.

Lemma 1. [6, (2.8)] *The centre of $\mathcal{C} = [\beta_1] \otimes \dots \otimes [\beta_r] \otimes [\gamma_1, \delta_1] \otimes \dots \otimes [\gamma_s, \delta_s]$ ($\beta_i, \gamma_j, \delta_k$ as in Theorem 2) is the 2^r -dimensional subalgebra $[\beta_1] \otimes \dots \otimes [\beta_r]$.*

Remark 1. [6, (2.9)] The converse of Theorem 2 is obviously also true: any algebra of the form (3) is a QC algebra. Indeed, regarded as an algebra on the generators $\{\beta_i, \gamma_j, \delta_k\}$, \mathcal{C} of the form (3) is the QC algebra $\mathcal{C}[r+2s, (\kappa_i), (\delta_{i,j})]$ where $\kappa_1, \dots, \kappa_{r+2s} = b_1, \dots, b_r, c_1, d_1, \dots, c_s, d_s$, respectively, and all $\delta_{i,j} = 0$ except $\delta_{r+2i-1, r+2i} = 1$ for $1 \leq i \leq s$.

Theorem 3. [6, (2.10)] *The class of SQC algebras over \mathbb{R} is the smallest class which is closed under tensor products over \mathbb{R} and which contains the Clifford algebras. It is the smallest class which is closed under tensor products over \mathbb{R} and contains the algebras $\mathbb{C}_b, \mathbb{Q}_{c,d}$ ($b, c, d = \pm 1$). The Clifford algebras are the QC algebras with 1- or 2-dimensional centres (general QC algebras can have 2^r -dimensional centres, r any non-negative integer).*

Theorem 4. [6, (2.11)] *Every real SQC algebra $\mathcal{C}[m, (\kappa_i), (\delta_{i,j})]$ is semi-simple.*

Remark 2. [6, (3.2)] There are irreducible representations of $\mathbb{C}_b, \mathbb{Q}_{c,d}$ ($b, c, d = \pm 1$) in which β, γ, δ are each represented by monomial $\{-1, 0, 1\}$ matrices.

Remark 3. [6, (3.3)] Following from (2) the decomposition of a real SQC algebra takes (possibly after reordering the factors) the form:

$$\begin{aligned} \mathcal{C} &= [\alpha_1, \dots, \alpha_m] \\ &\simeq {}^2\mathbb{R} \otimes \dots \otimes {}^2\mathbb{R} \otimes \mathbb{C} \otimes \dots \otimes \mathbb{C} \otimes \mathbb{H} \otimes \dots \otimes \mathbb{H} \otimes \mathbb{R}(2) \otimes \dots \otimes \mathbb{R}(2) \\ &\simeq [\beta_1] \otimes \dots \otimes [\beta_r] \otimes [\gamma_1, \delta_1] \otimes \dots \otimes [\gamma_s, \delta_s] \end{aligned} \tag{4}$$

where each $\beta_i, \gamma_j, \delta_k$ is plus or minus a product of the α_i , and conversely each α_i is plus or minus a product of the $\beta_i, \gamma_j, \delta_k$. In general, each of ${}^2\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{R}(2)$ may appear zero or more times in the tensor product (4).

We now come to a well known lemma used in the representation theory of real and complex Clifford algebras.

Lemma 2. [6, (3.4)] [17, Prop. 10.44] [18, Prop. 11.9]

(i) $\mathbb{C} \otimes \mathbb{C} \simeq {}^2\mathbb{R} \otimes \mathbb{C} \simeq {}^2\mathbb{C}$.

- (ii) $\mathbb{C} \otimes \mathbb{H} \simeq \mathbb{C} \otimes \mathbb{R}(2) \simeq \mathbb{C}(2)$.
- (iii) $\mathbb{H} \otimes \mathbb{H} \simeq \mathbb{R}(2) \otimes \mathbb{R}(2) \simeq \mathbb{R}(4)$.

Remark 3 and the repeated application of Lemma 2 lead to the following result.

Theorem 5. [6, (3.7)]

The Wedderburn structure of a real SQC algebra $\mathcal{C}[m, (\kappa_i), (\delta_{i,j})]$ as a direct sum of full matrix algebras over division algebras is (depending on $m, (\kappa_i), (\delta_{i,j})$) one of

- (i) $2^r \mathbb{R}(2^s)$,
- (ii) $2^{r-1} \mathbb{C} \otimes \mathbb{R}(2^s)$, or
- (iii) $2^r \mathbb{H} \otimes \mathbb{R}(2^{s-1})$,

where in each case $r + 2s = m$, and 2^r is the dimension of the centre. Conversely (as in Remark 1) any such algebra (i), (ii) or (iii) is an SQC algebra $\mathcal{C}[r + 2s, (\kappa_i), (\delta_{i,j})]$ with respect to certain generators. Also (as in Theorem 3) the subclass of algebras with structures (i), (ii) or (iii) for which $r \leq 1$ is precisely the class of algebras isomorphic to Clifford algebras on $r + 2s$ generators.

Corollary 3. [6, (3.8)] In case (i) of Theorem 5 there are 2^r inequivalent irreducible representations, of order 2^s ; in case (ii) 2^{r-1} of order 2^{s+1} , and in case (iii) 2^r of order 2^{s+1} . Any representation must be of order a multiple of (i) 2^s , (ii) 2^{s+1} , (iii) 2^{s+1} respectively.

As a result of the well-known constructions that lead to Remark 2, Gastineau-Hills establishes the following result.

Theorem 6. [6, (3.10)] Each representation of a real SQC algebra $\mathcal{C}[m, (\kappa_i), (\delta_{i,j})]$ on generators (α_i) is equivalent to a matrix representation in which each α_i corresponds to a monomial $\{-1, 0, 1\}$ matrix, which is therefore orthogonal.

3. Plug-In Constructions for Hadamard Matrices

A recent paper of the author [12] describes a generalization of Williamson's construction for Hadamard matrices [23] using the real monomial representation of the basis elements of the Clifford algebra $\mathbb{R}_{m,m}$. (Recall that $\mathbb{R}_{p,q}$ is the real universal Clifford algebra of the 2^{p+q} dimensional real quadratic space $\mathbb{R}^{p,q}$, with $p+q$ anticommuting generators, $e_{\{-q\}}, \dots, e_{\{-1\}}, e_{\{1\}}, \dots, e_{\{p\}}$ with $e_{\{k\}}^2 = -1$ if $k < 0$, $e_{\{k\}}^2 = 1$ if $k > 0$, and that $\mathbb{R}_{m,m} \simeq \mathbb{R}(2^m)$, the algebra of real matrices with 2^m rows and 2^m columns [12, 15, 17].)

Briefly, the general construction uses some

$$A_k \in \{-1, 0, 1\}^{n \times n}, \quad B_k \in \{-1, 1\}^{b \times b}, \quad k \in \{1, \dots, n\},$$

where the A_k are monomial matrices, and constructs

$$H := \sum_{k=1}^n A_k \otimes B_k, \tag{H0}$$

such that

$$H \in \{-1, 1\}^{nb \times nb} \quad \text{and} \quad HH^T = nbI_{(nb)}, \tag{H1}$$

i.e. H is a Hadamard matrix of order nb . The paper [12] focuses on a special case of the construction, satisfying the conditions

$$\begin{aligned} A_j * A_k &= 0 \quad (j \neq k), \quad \sum_{k=1}^n A_k \in \{-1, 1\}^{n \times n}, \\ A_k A_k^T &= I_{(n)}, \\ A_j A_k^T + \lambda_{j,k} A_k A_j^T &= 0 \quad (j \neq k), \\ B_j B_k^T - \lambda_{j,k} B_k B_j^T &= 0 \quad (j \neq k), \\ \lambda_{j,k} &\in \{-1, 1\}, \\ \sum_{k=1}^n B_k B_k^T &= nbI_{(b)}, \end{aligned} \tag{5}$$

where $*$ is the Hadamard (element-by-element) matrix product. (That is, $(M * N)_{i,j} := M_{i,j} N_{i,j}$ for all pair of matrices M, N of the same shape.)

If, in addition, we stipulate that $A_j^2 = \kappa_j = \pm 1$ for j from 1 to n , we can now recognize that the n matrices A_1 to A_n are also the images, under a real representation of order n , of the generators of a real special quasi-Clifford algebra, with $\lambda_{j,k} = \kappa_j \kappa_k (-1)^{1+\delta_{j,k}}$. Thus n must be a power of 2 large enough for this representation to exist, or a multiple of such a power.

In Sect. 3 of the paper [12], it is noted that the Clifford algebra $\mathbb{R}(2^m) \simeq \mathbb{R}_{m,m}$ has a canonical basis consisting of 4^m real monomial matrices with the following properties:

Pairs of basis matrices either commute or anticommute. Basis matrices are either symmetric or skew, and so the basis matrices A_j, A_k satisfy

$$A_k A_k^T = I_{(2^m)}, \quad A_j A_k^T + \lambda_{j,k} A_k A_j^T = 0 \quad (j \neq k), \quad \lambda_{j,k} \in \{-1, 1\}. \tag{6}$$

Additionally, for $n = 2^m$, we can choose a transversal of n canonical basis matrices that satisfies conditions (5) on the A matrices,

$$A_j * A_k = 0 \quad (j \neq k), \quad \sum_{k=1}^n A_k \in \{-1, 1\}^{n \times n}. \tag{7}$$

4. Special Quasi-Clifford Algebras Applied to the Plug-In Constructions

The properties of the real SQC algebras yield an alternate proof of Theorem 5 of [12], and provide an answer to the question of whether the order of the B matrices used in that proof can be improved [12, Question 1]. That theorem is restated here as a proposition.

Proposition 1. [12, Theorem 5] *If n is a power of 2, the construction (H0) with conditions (5) can always be completed, in the following sense. If an n -tuple of A matrices which produce a particular λ is obtained by taking a*

transversal of canonical basis matrices of the Clifford algebra $\mathbb{R}_{m,m}$, an n -tuple of B matrices with a matching λ can always be found.

Proof. 1. For some sufficiently large order b , form an n -tuple (D_1, \dots, D_n) of $\{-1, 0, 1\}$ monomial matrices whose amicability/anti-amicability graph is the edge-colour complement of that of (A_1, \dots, A_n) . To be precise,

$$D_j D_k^T - \lambda_{j,k} D_k D_j^T = 0 \quad (j \neq k),$$

where λ is given by (6). This can be done because D_1, \dots, D_n are the images of generators of some real SQC algebra \mathcal{C} , and therefore b can be taken to be the order of an irreducible real representation of \mathcal{C} , which, by Corollary 3 is a power of 2.

2. Since b is a power of 2, we can find a Hadamard matrix S of order b . The Sylvester Hadamard matrix of order b will do. The n -tuple $(D_1 S, \dots, D_n S)$ of $\{-1, 1\}$ matrices of order b has the same amicability/anti-amicability graph as that of (D_1, \dots, D_n) .
3. The n -tuple of Hadamard matrices $(B_1, \dots, B_n) = (D_1 S, \dots, D_n S)$ of order b satisfies conditions (5) on the B matrices, and completes the construction (H0). □

The theory of SQC algebras is described by Gastineau-Hills [5,6] with enough detail to enable a concrete construction of the type given in the proof of Proposition 1 to be carried out for any given pattern of amicability/anti-amicability λ , and any arbitrary assignment κ of squares of generators.

For example, consider the cases where all of the A matrices are pairwise amicable, that is $\lambda_{j,k} = -1$ for $j \neq k$. We thus require an n -tuple of mutually anti-amicable $\{-1, 0, 1\}$ matrices (D_1, \dots, D_n) .

Consider the generators $\beta_{-q}, \dots, \beta_{-1}, \beta_1, \dots, \beta_p$ where $p + q = n$, $\beta_j^2 = \kappa_j$, with $\kappa_j = -1$ if $j < 0$, $\kappa_j = 1$ if $j > 0$, and

$$\beta_j \beta_k + \kappa_j \kappa_k \beta_k \beta_j = 0.$$

Thus generators whose squares have the same sign anticommute, and generators whose squares have opposite signs commute. For any real monomial representation ρ , we have

$$\rho(\beta_j)^T = \kappa_j \rho(\beta_j),$$

so that

$$\rho(\beta_j) \rho(\beta_k)^T = \kappa_k \rho(\beta_j) \rho(\beta_k) = -\kappa_j \rho(\beta_k) \rho(\beta_j) = -\rho(\beta_k) \rho(\beta_j)^T.$$

Thus any representation gives a set of mutually anti-amicable matrices.

We have split the set of n generators into disjoint subsets of size p and q , where the generators within each subset pairwise anti-commute, and each pair of generators, where one is taken from each subset, commute. The whole set of generators thus generates the algebra $\mathbb{R}_{p,0} \otimes \mathbb{R}_{0,q}$, whose faithful representations are given by Table 1.

The relevant representations and the dimensions of the corresponding irreducible real monomial representations for $p + q = 2, 4$ and 8 are given

TABLE 1. Tensor products of real Clifford algebras $\mathbb{R}_{p,0} \otimes \mathbb{R}_{0,q}$

p	$q \rightarrow$								
	0	1	2	3	4	5	6	7	8
0	\mathbb{R}	\mathbb{C}	\mathbb{H}	${}^2\mathbb{H}$	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	${}^2\mathbb{R}(8)$	$\mathbb{R}(16)$
1	${}^2\mathbb{R}$	${}^2\mathbb{C}$	${}^2\mathbb{H}$	${}^4\mathbb{H}$	${}^2\mathbb{H}(2)$	${}^2\mathbb{C}(4)$	${}^2\mathbb{R}(8)$	${}^4\mathbb{R}(8)$	${}^2\mathbb{R}(16)$
2	$\mathbb{R}(2)$	$\mathbb{C}(2)$	$\mathbb{H}(2)$	${}^2\mathbb{H}(2)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$	$\mathbb{R}(16)$	${}^2\mathbb{R}(16)$	$\mathbb{R}(32)$
3	$\mathbb{C}(2)$	${}^2\mathbb{C}(2)$	$\mathbb{C}(4)$	${}^2\mathbb{C}(4)$	$\mathbb{C}(8)$	${}^2\mathbb{C}(8)$	$\mathbb{C}(16)$	${}^2\mathbb{C}(16)$	$\mathbb{C}(32)$
4	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	${}^2\mathbb{R}(8)$	$\mathbb{R}(16)$	$\mathbb{C}(16)$	$\mathbb{H}(16)$	${}^2\mathbb{H}(16)$	$\mathbb{H}(32)$
5	${}^2\mathbb{H}(2)$	${}^2\mathbb{C}(4)$	${}^2\mathbb{R}(8)$	${}^4\mathbb{R}(8)$	${}^2\mathbb{R}(16)$	${}^2\mathbb{C}(16)$	${}^2\mathbb{H}(16)$	${}^4\mathbb{H}(16)$	${}^2\mathbb{H}(32)$
6	$\mathbb{H}(4)$	$\mathbb{C}(8)$	$\mathbb{R}(16)$	${}^2\mathbb{R}(16)$	$\mathbb{R}(32)$	$\mathbb{C}(32)$	$\mathbb{H}(32)$	${}^2\mathbb{H}(32)$	$\mathbb{H}(64)$
7	$\mathbb{C}(8)$	${}^2\mathbb{C}(8)$	$\mathbb{C}(16)$	${}^2\mathbb{C}(16)$	$\mathbb{C}(32)$	${}^2\mathbb{C}(32)$	$\mathbb{C}(64)$	${}^2\mathbb{C}(64)$	$\mathbb{C}(128)$
8	$\mathbb{R}(16)$	$\mathbb{C}(16)$	$\mathbb{H}(16)$	${}^2\mathbb{H}(16)$	$\mathbb{H}(32)$	$\mathbb{C}(64)$	$\mathbb{R}(64)$	${}^2\mathbb{R}(128)$	$\mathbb{R}(256)$

TABLE 2. Tensor products of real Clifford algebras with $p + q = 2$

Algebra	Faithful representation	Irreducible dimension
$\mathbb{R}_{2,0} \otimes \mathbb{R}_{0,0}$	$\mathbb{R}(2)$	2
$\mathbb{R}_{1,0} \otimes \mathbb{R}_{0,1}$	${}^2\mathbb{C}$	2
$\mathbb{R}_{0,0} \otimes \mathbb{R}_{0,2}$	\mathbb{H}	4

TABLE 3. Tensor products of real Clifford algebras with $p + q = 4$

Algebra	Faithful representation	Irreducible dimension
$\mathbb{R}_{4,0} \otimes \mathbb{R}_{0,0}$	$\mathbb{H}(2)$	8
$\mathbb{R}_{3,0} \otimes \mathbb{R}_{0,1}$	${}^2\mathbb{C}(2)$	4
$\mathbb{R}_{2,0} \otimes \mathbb{R}_{0,2}$	$\mathbb{H}(2)$	8
$\mathbb{R}_{1,0} \otimes \mathbb{R}_{0,3}$	${}^4\mathbb{H}$	4
$\mathbb{R}_{0,0} \otimes \mathbb{R}_{0,4}$	$\mathbb{H}(2)$	8

by Tables 2, 3 and 4 respectively. Due to the periodicity of 8 of real representations of real Clifford algebras, in general, for $n = 2^m$, for $m > 2$, there exists a real special quasi-Clifford algebra with an irreducible real monomial representation of order $2^{n/2-1}$ containing n pairwise anti-amicable $\{-1, 0, 1\}$ matrices.

Hurwitz–Radon theory. The following definition and proposition are taken from the author’s recent paper on twin bent functions and Hurwitz–Radon theory [14].

A set of real orthogonal matrices $\{A_1, A_2, \dots, A_s\}$ is called a *Hurwitz–Radon family* [7, 8, 19] if

1. $A_j^T = -A_j$ for all $j = 1, \dots, s$, and

2. $A_j A_k = -A_k A_j$ for all $j \neq k$.

The Hurwitz-Radon function ρ is defined by

$$\rho(2^{4d+c}) := 2^c + 8d, \quad \text{where } 0 \leq c < 4.$$

As stated by Geramita and Pullman [7, Theorem A], Radon proved the following result [19].

Proposition 2. *Any Hurwitz–Radon family of order N has at most $\rho(N) - 1$ members.*

As an immediate consequence of this proposition, at most $\rho(N)$ monomial $\{-1, 0, 1\}$ matrices can be mutually anti-amicable. The construction above in the case where all of the A matrices are pairwise amicable, $n = 2^m$ and $m > 2$ corresponds to the case $c = 3, d = 2^{m-3} - 1$, since

$$\begin{aligned} 4d + c &= 2^{m-1} + 4 - 1 = n/2 - 1, \\ 2^c + 8d &= 8 + 2^m - 8 = n. \end{aligned}$$

5. Discussion

The construction used in the proof of Proposition 1 is a special case of the construction (H0) with conditions (5). All of the low order cases investigated so far have been of this form. This prompts two questions:

1. Are all instances of construction (H0) with conditions (5) given by the special construction used in the proof of Proposition 1?
2. Must the order of the B matrices used in construction (H0) with conditions (5) always be a power of 2?

Perhaps a deeper study of the representation theory of Gastineau-Hills quasi-Clifford algebras could be used to address these questions.

TABLE 4. Tensor products of real Clifford algebras with $p + q = 8$

Algebra	Faithful representation	Irreducible dimension
$\mathbb{R}_{8,0} \otimes \mathbb{R}_{0,0}$	$\mathbb{R}(16)$	16
$\mathbb{R}_{7,0} \otimes \mathbb{R}_{0,1}$	${}^2\mathbb{C}(8)$	16
$\mathbb{R}_{6,0} \otimes \mathbb{R}_{0,2}$	$\mathbb{R}(16)$	16
$\mathbb{R}_{5,0} \otimes \mathbb{R}_{0,3}$	${}^4\mathbb{R}(8)$	8
$\mathbb{R}_{4,0} \otimes \mathbb{R}_{0,4}$	$\mathbb{R}(16)$	16
$\mathbb{R}_{3,0} \otimes \mathbb{R}_{0,5}$	${}^2\mathbb{C}(8)$	16
$\mathbb{R}_{2,0} \otimes \mathbb{R}_{0,6}$	$\mathbb{R}(16)$	16
$\mathbb{R}_{1,0} \otimes \mathbb{R}_{0,7}$	${}^4\mathbb{R}(8)$	8
$\mathbb{R}_{0,0} \otimes \mathbb{R}_{0,8}$	$\mathbb{R}(16)$	16

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