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Born–Infeld Gravity from the MacDowell–Mansouri Action and Its Associated β-Term

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Dedicated to the memory of Professor Waldyr A. Rodrigues Jr..

Abstract. In this work a generalization of a Born–Infeld theory of gravity with a topological β-term is proposed. These type of Born–Infeld actions were found from the theory introduced by MacDowell and Mansouri. This theory known as MacDowell–Mansouri (MM) gravity was one of the first attempts to construct a gauge theory of gravitation, and within this framework it was introduced in the action a topological β-term relevant for quantization purposes in an analogous way as in Yang–Mills theory. By the use of the self-dual and antiself-dual actions of MM gravity, we further define a Born–Infeld gravity generalization corresponding to MM gravity with the β-term.
Keywords. Born–Infeld gravity, MacDowell–Mansouri gravity, Self-dual gravity.

1. Introduction

Several modifications to the undoubtedly successful classical theory of gravitation, namely General Relativity (GR), have been considered since GR fails to describe field situations where quantum effects are supposed to be important. The classical formulation as GR as such does not allow to be consistently quantized by any of the well known methods of quantization. We can find in the literature several types of modifications to GR that have been proposed in order to get a possibly consistent quantum version of gravity. Even in the classical formulation, GR have always had the problem of leading to singularities. We expect that a consistent quantum theory of gravitation should avoid such singularities. The classical reformulation of electromagnetism of Born–Infeld (BI) [6], which was very successful in regularizing the singularity

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related to the self energy due to the electric field of the point particle, has inspired similar reformulations of GR [20] to avoid singularities in the classical field theory and it is also remarkable that the BI action appears in the realm of string theories [32]. The first work in that direction was done by Deser–Gibbons (DG) [10] where they considered the basic determinantal form of the electrodynamic field action of BI and constructed the natural generalization of the corresponding covariant gravitational field action. We can find other type of actions constructed by the same principle of avoiding singularities but in the metric affine approach considered by Eddington in [11], these kind of modifications to the original work of BI have the name Eddington-Inspired Born–Infeld gravities [4,38]. We also find natural non-Abelian generalizations to the original Abelian BI theory [37] but no gravitational generalizations in the spirit of DG has been made. Although all the previously mentioned Born–Infeld type theories share the same square root structure of its Lagrangian, remarkable enough is the work developed in [36] where an equivalent theory to Born's original proposal was presented and without the square root. This work inspired a corresponding non-Abelian generalization in [29]. There are in general several BI theories of gravity that can be subdivided basically in theories in the metric formalism [9,10,12,17,41] and those in the affine-metric formalism [4,11,38] but other classification was stated in [20] where some criteria were applied to put all those theories in just five classes taking into account mainly theories with ghost instabilities and other ones in which no additional degrees of freedom are present or considering the inclusion of extra degrees of freedom.

Another interesting purely geometrical classical theory of gravitation is the one of MacDowell–Mansouri [21,22]. This theory is one of the closest approaches of a gauge theory of gravity (see also [5] and references therein). Formulated in terms of the $SO(4)$ gauge group valued connection for gravity, this formalisms allows to describe supergravity in the same fashion and for the $SO(4,1)$ group. This formulation also poses the question whether it can be quantized in the same way as pure Yang–Mills (YM) theories for Lie-group valued fields. In regards of quantization of a gauge theory of gravity, it has been put in the discussion the relevance of the topological term, also known as β-term, that can be added to the usual Yang–Mills action which is in principle harmless to the classical dynamics but could be relevant to take it into account in the quantization process. In the expansion of the MM action with the additional β-term we are considering, among other topological terms, it also arises the term corresponding to the so called Immirzi parameter [18,24] which is necessary in the canonical formulation of quantum gravity [1–3,23,33,34]. The one parameter ambiguity related to this term in canonical quantum gravity and ambiguities in Yang–Mills theories are in some sense similar [13]. Therefore from the point of view of the MM proposal with the additional β-term, the Immirzi parameter naturally comes into play [24]. In order to make a connection with requirements in canonical quantum gravity it is consistent and consequently necessary to include the β-term as it already takes into account the Immirzi parameter. The relation between MM gravity and the LQG formulation is given through the self-dual formulation of MM gravity [19,27,35]. In this work we will later consider a non-trivial connection between BI gravity and MM gravity. Following the procedure proposed in [25], it can be shown that particular DG actions can be deduced from the determinantal structure of the MM action. Based on the kind of actions proposed in [25] we will construct the corresponding BI gravity related to the MM action with a β-term, to achieve this goal we will make use of the self-dual antiself-dual formulation of MM gravity [27].

The structure of the paper is as follows, first in Sect. 2, we briefly introduce the general features of MM gravity and explain the basic objects appearing in the MM action. In Sect. 3 we describe the structure of the gravitational Born–Infeld generalization of Deser–Gibbons, afterwards in Sect. 4 we discuss in detail the relation between a MM type action and a particular DG action, formalism which was developed in [25]. We will then introduce the self-dual and anti-selfdual MM actions to generate a more general MM action containing a β-term and subsequently we construct the generalization of the DG action corresponding to the MM action with the β-term included. Finally in Sect. 5 we make some concluding remarks and draw attention to questions related to future work.

2. MacDowell–Mansouri Gravity

When considering modifications to general relativity with the purpose of making a unified description of fundamental forces, it is natural to think of interactions described by a fundamental connection associated with an internal symmetry group. Among a few attempts of describing gravitation as a gauge theory [8,16,39], we will focus on the elegant formalism developed by MacDowell and Mansouri [21]. They introduced a field theory for which the action depends on a Lie group valued connection. The theory is constructed by considering the gauge potential $\omega_\mu^{AB}(x)$ where the indices $\mu = 0,1,2,3$ correspond to space-time, with x^α local coordinates and the indices $A, B = 0,1,2,3,4$ are associated to the fiber bundle De Sitter group $SO(4,1)$ later broken into a $SO(3,1)$, in this way the group valued one form connection is $\omega_\mu^{ab}(x)$ and the tetrad field is identified with $\omega_\mu^{4a}(x) = e^a{}_\mu$. Explicitly, we have that the gauge fields associated to the generators of the Poincaré group are the spin connection for the Lorentz group and the tetrads for the translations. The way in which these fields transform, using the corresponding covariant derivative, requires the vanishing of the torsion if we want reparametrization invariance [31]. The gauge invariant objects are then, the ten gauge fields and the corresponding field strength. The field strength associated to the gauge potential ω_μ^{AB} is given by

$$\mathcal{R}_{\mu\nu}^{AB} = \partial_\mu \omega_\nu^{AB} - \partial_\nu \omega_\mu^{AB} + \frac{1}{2} f_{[[CD][EF]]}^{[AB]} \omega_\mu^{CD} \omega_\nu^{EF}, \tag{2.1}$$

where $f_{[[CD][EF]]}^{[AB]}$ are the structure constants and the group generators satisfy $S_{AB} = -S_{BA}$ and

$$[S_{AB}, S_{CD}] = f_{[[AB][CD]]}^{[EF]} S_{EF}.$$

The structure constants are given by

$$f^{[EF]}_{[[AB][CD]]} = \frac{1}{2}\left[\eta_{AC}\delta^E_B\delta^F_D - \eta_{AD}\delta^E_B\delta^F_C + \eta_{BD}\delta^E_A\delta^F_C - \eta_{BC}\delta^E_A\delta^F_D\right] - (E \leftrightarrow F). \tag{2.2}$$

Once the original symmetry is broken, the curvature breaks up into the Lorentz and translational parts, the last related to the torsion that we impose to vanish [40]. The final action proposed in [21] is given by

$$S_{_{MM}} = \int d^4x \epsilon^{\mu\nu\alpha\beta}\epsilon_{abcd}\mathcal{R}^{ab}_{\mu\nu}\mathcal{R}^{cd}_{\alpha\beta}\ , \tag{2.3}$$

where the curvature is

$$\begin{aligned} \mathcal{R}_{\mu\nu}^{\ ab} &= R_{\mu\nu}^{\ ab} + \Sigma_{\mu\nu}^{\ ab}, \\ \Sigma_{\mu\nu}^{\ ab} &= e^a_\nu e^b_\mu - e^a_\mu e^b_\nu. \end{aligned} \tag{2.4}$$

In the action (2.3) $\epsilon^{\mu\nu\rho\sigma}$ is the completely antisymmetric tensor associated to space-time, with $\epsilon^{0123} = 1$, while ϵ_{abcd} is the corresponding antisymmetric tensor associated to the internal group $SO(3,1)$, with $\epsilon_{0123} = -1$. We assume that the internal metric is given by $\eta_{ab} = (-1,1,1,1)$. With the group structure defined in this way, the action (2.3) is equivalent to a gravitational action given by three contributions [26]; the Einstein–Hilbert action, the cosmological constant term and the Euler topological invariant.

We have already discussed the relevance of the addition of a θ-term to the MM action in analogy with Yang–Mills theories. In the MM formalism such term has been usually called β-term [30] and the action takes the following form

$$S_{_{MM\beta}} = \int d^4x \epsilon^{\mu\nu\alpha\beta}\epsilon_{abcd}\mathcal{R}^{ab}_{\mu\nu}\mathcal{R}^{cd}_{\alpha\beta} + \beta\int d^4x \epsilon^{\mu\nu\alpha\beta}\mathcal{R}^{ab}_{\mu\nu}\mathcal{R}_{\alpha\beta ab}\ . \tag{2.5}$$

The mathematical structure of MM can also be generalized to a Yang–Mills type of supergravity [26] also taking into account the corresponding β-term and its corresponding supersymmetric term [30]. In this formulation, the corresponding supersymmetric Immirzi parameter, among other couplings, arises naturally by the symmetry breaking mechanism [24,30].

3. Born–Infeld Gravity

Born–Infeld gravity was inspired by the non-linear version of electrodynamics proposed in [6], which action takes the following form

$$S_{_{BI}} = -b^2\left[\int d^4x\sqrt{-\det\left(\eta_{\mu\nu} + \frac{1}{b}F_{\mu\nu}\right)} - 1\right], \tag{3.1}$$

where $\eta_{\mu\nu}$ is the Minkowski metric and $F_{\mu\nu}$ is the usual electromagnetic strength tensor. The expansion of the determinant allows to write this action in the more appealing form

$$S_{_{BI}} = -b^2\left[\int d^4x\sqrt{1 + \frac{1}{b^2}F_{\mu\nu}F^{\mu\nu} - \frac{1}{16b^4}\left(F_{\mu\nu}{}^*F^{\mu\nu}\right)^2} - 1\right]. \tag{3.2}$$

We have introduced the dual tensor ${}^*F_{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}$, and in that way the action is written in terms of invariants formed with the strength tensor and its corresponding electromagnetic dual. It was Eddington who considered first the kind of action (3.1) as it transforms consistently under the diffeomorphism group. The gauge invariance of this action is evident as the functions under the square root are indeed gauge invariant scalars. Inspired by this structure, Deser and Gibbons proposed a similar action of the type

$$S_{DG} = \int d^4x\sqrt{-\det\left(ag_{\mu\nu} + bR_{\mu\nu} + cX_{\mu\nu}\right)}\ . \tag{3.3}$$

Originally this action analogous to (3.1) considered the constant $c = 0$ but ghost instabilities lead to consider the introduction of $X_{\mu\nu}$ to determine *a posteriori* the analytic form of such fudge tensor in order to avoid instabilities [10, 20]. These kind of DG actions can be deduced from the mathematical structure of MM gravity, as we will see in the following section.

4. BI Gravity from a MacDowell–Mansouri Type Action

A Deser–Gibbons type action was proposed in [25] and is given by

$$S_{DG} = -\frac{\lambda}{4!}\int d^4x\sqrt{-\det\left(g_{\mu\nu} + \Lambda R_{\mu\nu} + \frac{\Lambda^2}{4}R_{\mu\alpha}R_{\nu}{}^{\alpha}\right)}\ , \tag{4.1}$$

where $\Lambda = \frac{\lambda}{2}$. The starting point in [25] is the proposition of the following d-dimensional modified MM type action

$$S_{mMM} = -\frac{1}{d!}\int d^dx\epsilon^{\mu_1\mu_2\ldots\mu_d}\epsilon_{a_1a_2\ldots a_d}\mathcal{R}_{\mu_1}{}^{a_1}\mathcal{R}_{\mu_2}{}^{a_2}\ldots\mathcal{R}_{\mu_d}{}^{a_d}\ , \tag{4.2}$$

which in $d = 4$ takes the explicit form

$$S_{mMM} = -\frac{1}{4!}\int d^4x\epsilon^{\mu\nu\alpha\beta}\epsilon_{abcd}\mathcal{R}_{\mu}{}^{a}\mathcal{R}_{\nu}{}^{b}\mathcal{R}_{\alpha}{}^{c}\mathcal{R}_{\beta}{}^{d}\ . \tag{4.3}$$

In this expression $\mathcal{R}_{\mu}{}^{a}$ is given as the contraction $\mathcal{R}_{\mu}{}^{a} = e_b{}^{\nu}\mathcal{R}_{\mu\nu}^{ab}$ which can be written with the help of (2.4) as

$$\mathcal{R}_{\mu}{}^{a} = e_b^{\nu}\left(R_{\mu\nu}^{ab} + \Sigma_{\mu\nu}^{ab}\right) = R_{\mu}{}^{a} + \lambda e_{\mu}^{a}\ , \tag{4.4}$$

with $\lambda = (1-d)$. With the use of $\mathcal{R}_{\mu}{}^{a}$, the action (4.3) can be written as the determinant of the contracted curvature

$$S_{mMM} = -\frac{1}{4!}\int d^4x\det\left(\mathcal{R}_{\mu}{}^{a}\right). \tag{4.5}$$

It can be seen that from the transformations of the gauge fields, namely the spin connection and the tetrads, the contracted $\mathcal{R}_{\mu}{}^{a}$ and consequently the integrand in (4.5) is gauge invariant and it will also be the case for the self(anti-self) dual actions defined below. The link between Eqs. (4.3) and (4.1) is given by the following definition

$$G_{\mu\nu} = \frac{1}{\lambda^2}\mathcal{R}_{\mu}{}^{a}\mathcal{R}_{\nu}{}^{b}\eta_{ab}\ , \tag{4.6}$$

from which it follows that

$$\det\left(\mathcal{R}_{\mu}{}^{a}\right) = \lambda\sqrt{-\det\left(G_{\mu\nu}\right)}\ , \tag{4.7}$$

and using expansion of $\det(G_{\mu\nu})$ with the help of (4.4) the action (4.1) follows directly. Regarding the gauge invariance of (4.7), notice that since both factors of the contraction (4.4) are gauge invariant, $\mathcal{R}_{\mu}{}^{a}$ is itself a gauge invariant quantity and in the quadratic definition (4.6) two gauge indices have been consistently contracted, or traced, which leaves a purely spacetime but gauge invariant tensor. Inserting the last equation into (4.5) leaves us with a pure DG action which is also gauge invariant in the same way as the MacDowell–Mansouri action.

Now, we can ask if this kind of connection between MM gravity and BI gravity can be achieved by adding to the MM action the contribution of the topological term in Eq. (2.5). Before we proceed in that direction, we remember that the action (2.5) can be written with the help of the self-dual ${}^{+}\mathcal{R}_{\mu\nu}^{\ ab}$ and antiself-dual curvature ${}^{-}\mathcal{R}_{\mu\nu}^{\ ab}$ which are defined as follows

$${}^{\pm}\mathcal{R}_{\mu\nu}^{\ ab} = \frac{1}{2}\mathcal{R}_{\mu\nu}^{\ ab} \mp \frac{i}{4}\epsilon^{ab}_{\ \ cd}\mathcal{R}_{\mu\nu}^{\ cd}\ . \tag{4.8}$$

Let us then consider the following dual action

$$S_{_{DMM}} = \int d^4x\epsilon^{\mu\nu\alpha\beta}\epsilon_{abcd}\left[{}^{+}\tau\,{}^{+}\mathcal{R}_{\mu\nu}^{\ ab}\,{}^{+}\mathcal{R}_{\alpha\beta}^{\ cd} - {}^{-}\tau\,{}^{-}\mathcal{R}_{\mu\nu}^{\ ab}\,{}^{-}\mathcal{R}_{\alpha\beta}^{\ cd}\right], \tag{4.9}$$

where ${}^{+}\tau$ and ${}^{-}\tau$ are both constants. This linear combination of the self-dual and antiself-dual actions were considered in [14,15,28] in the search of a gravitational duality and to construct a supergravity gauge theory in [26]. We will take the advantage of action (4.9) as it is equivalent to the original MM action plus a β-term. In particular, it was shown in [26] that the self-dual action can be decomposed in four terms; the Ashtekar action [19,35], the cosmological constant contribution and the Euler and Pontrjagin topological invariants. The action (4.9) can be written as

$$S_{_{DMM}} = \int d^4x\epsilon^{\mu\nu\alpha\beta}\epsilon_{abcd}\left[({}^{-}\tau - {}^{+}\tau)\mathcal{R}_{\mu\nu}^{\ ab}\mathcal{R}_{\alpha\beta}^{\ cd} + ({}^{+}\tau + {}^{-}\tau)\mathcal{R}_{\mu\nu}^{\ ab}\ {}^{*}\mathcal{R}_{\alpha\beta}^{\ cd}\right], \tag{4.10}$$

where we have introduced the dual (star) curvature defined

$${}^{*}\mathcal{R}_{\alpha\beta}^{\ cd} = \frac{i}{2}\epsilon^{ab}_{\ \ cd}\mathcal{R}_{\alpha\beta}^{\ cd}\ . \tag{4.11}$$

By introducing the last definition in (4.10) and using the proportionally between the product of antisymmetric tensors and Kronecker deltas $\epsilon_{abcd}\epsilon^{abef} \propto (\delta^e_c\delta^f_d - \delta^e_d\delta^f_c)$ it is found to be

$$\begin{aligned} S_{_{DMM}} = ({}^{-}\tau - {}^{+}\tau)\int d^4x\epsilon^{\mu\nu\alpha\beta}\epsilon_{abcd}\mathcal{R}_{\mu\nu}^{\ ab}\mathcal{R}_{\alpha\beta}^{\ cd} \\ +i({}^{-}\tau + {}^{+}\tau)\int d^4x\epsilon^{\mu\nu\alpha\beta}\mathcal{R}_{\mu\nu}^{\ ab}\mathcal{R}_{\alpha\beta ab}\ , \end{aligned} \tag{4.12}$$

action which is equivalent to (2.5) and the second term is identified with the β-term [15,28]. This combination of the self-dual and anti-selfdual actions was used to derive a non-trivial relation between the Immirzi parameter and its counterpart in the gauge theory of (2+1) gravity [7]. It is now possible to apply the procedure followed in [25] to find the generalization of the Deser–Gibbons action related to MM gravity with the β-term. For that purpose let us propose the following action which hereafter we call DGβ action

$$S_{DG\beta} = {}^{+}\tau \int d^4x \epsilon^{\mu\nu\alpha\beta} \epsilon_{abcd} {}^{+}\mathcal{R}_{\mu}{}^{a} {}^{+}\mathcal{R}_{\nu}{}^{b} {}^{+}\mathcal{R}_{\alpha}{}^{c} {}^{+}\mathcal{R}_{\beta}{}^{d} - {}^{-}\tau \int d^4x \epsilon^{\mu\nu\alpha\beta} \epsilon_{abcd} {}^{-}\mathcal{R}_{\mu}{}^{a} {}^{-}\mathcal{R}_{\nu}{}^{b} {}^{-}\mathcal{R}_{\alpha}{}^{c} {}^{-}\mathcal{R}_{\beta}{}^{d}, \tag{4.13}$$

such action can be written as

$$S_{DG\beta} = {}^{+}\tau \int d^4x \det({}^{+}\mathcal{R}_{\mu}{}^{a}) - {}^{-}\tau \int d^4x \det({}^{-}\mathcal{R}_{\mu}{}^{a}), \tag{4.14}$$

and now we can define in an analogous way as in Eq. (4.6) the following self-dual and antiself-dual quantities

$$\begin{aligned} {}^{+}G_{\mu\nu} &= \frac{4}{\lambda^2} {}^{+}\mathcal{R}_{\mu}{}^{a} {}^{+}\mathcal{R}_{\nu}{}^{b} \eta_{ab} , \\ {}^{-}G_{\mu\nu} &= \frac{4}{\lambda^2} {}^{-}\mathcal{R}_{\mu}{}^{a} {}^{-}\mathcal{R}_{\nu}{}^{b} \eta_{ab} . \end{aligned} \tag{4.15}$$

These definitions allow to find

$$\begin{aligned} \det({}^{+}\mathcal{R}_{\mu}{}^{a}) &= \frac{\lambda}{2} \sqrt{-\det({}^{+}G_{\mu\nu})} , \\ \det({}^{-}\mathcal{R}_{\mu}{}^{a}) &= \frac{\lambda}{2} \sqrt{-\det({}^{-}G_{\mu\nu})} , \end{aligned} \tag{4.16}$$

which upon substitution in (4.17) leads to

$$S_{DG\beta} = \frac{\lambda}{2} {}^{+}\tau \int d^4x \sqrt{-\det({}^{+}G_{\mu\nu})} - \frac{\lambda}{2} {}^{-}\tau \int d^4x \sqrt{-\det({}^{-}G_{\mu\nu})} . \tag{4.17}$$

We will write this action in terms of the curvatures starting with

$${}^{\pm}\mathcal{R}_{\mu\nu}^{ab} = \frac{1}{2} \left[\mathcal{R}_{\mu\nu}^{ab} \mp {}^{*}\mathcal{R}_{\mu\nu}^{ab} \right] , \tag{4.18}$$

where we have used (4.8) and the definition of the dual (star) curvature. We can define in an analogous way the contracted gauge invariant self-dual and antiself-dual curvatures

$$\begin{aligned} {}^{+}\mathcal{R}_{\mu}{}^{a} &= e_{b}{}^{\nu} {}^{+}\mathcal{R}_{\mu\nu}^{ab} , \\ {}^{-}\mathcal{R}_{\mu}{}^{a} &= e_{b}{}^{\nu} {}^{-}\mathcal{R}_{\mu\nu}^{ab} , \end{aligned} \tag{4.19}$$

and it can be easily seen that these can be written as

$$\begin{aligned} {}^{+}\mathcal{R}_{\mu}{}^{a} &= R_{\mu}{}^{a} + \lambda e_{\mu}^{a} - {}^{*}\mathcal{R}_{\mu}{}^{a} , \\ {}^{-}\mathcal{R}_{\mu}{}^{a} &= R_{\mu}{}^{a} + \lambda e_{\mu}^{a} + {}^{*}\mathcal{R}_{\mu}{}^{a} , \end{aligned} \tag{4.20}$$

where we have defined

$${}^{*}\mathcal{R}_{\mu}{}^{a} = e_{b}{}^{\nu} {}^{*}\mathcal{R}_{\mu\nu}^{ab} . \tag{4.21}$$

In this way we have the following

$$^{\pm}G_{\mu\nu} = \frac{1}{\lambda^2}\left[R_{\mu}{}^{a}R_{\nu a} + 2\lambda R_{\mu\nu} + \lambda^2 g_{\mu\nu} + {}^{*}R_{[\mu}{}^{a*}R_{\nu]a} \mp 2\lambda^{*}R_{\mu\nu} \mp 2R_{[\mu}{}^{a*}R_{\nu]a}\right] \quad (4.22)$$

and finally we write

$$\begin{aligned} S_{DG\beta} = \frac{\lambda}{2}{}^{+}\tau \int d^4x \sqrt{-\det\left(g_{\mu\nu} + \Lambda R_{\mu\nu} + \frac{\Lambda^2}{4}R_{\mu\alpha}R_{\nu}{}^{\alpha} + {}^{*}R_{\mu}{}^{a*}R_{\nu a} + \frac{{}^{+}A_{\mu\nu}}{\lambda^2}\right)} \\ -\frac{\lambda}{2}{}^{-}\tau \int d^4x \sqrt{-\det\left(g_{\mu\nu} + \Lambda R_{\mu\nu} + \frac{\Lambda^2}{4}R_{\mu\alpha}R_{\nu}{}^{\alpha} + {}^{*}R_{\mu}{}^{a*}R_{\nu a} + \frac{{}^{-}A_{\mu\nu}}{\lambda^2}\right)}\,, \end{aligned} \quad (4.23)$$

where in order to make the expressions shorter we have defined $^{\pm}A_{\mu\nu}$ as follows

$$^{\pm}A_{\mu\nu} = \mp 2\lambda^{*}R_{\mu\nu} \mp 2R_{[\mu}{}^{a*}R_{\nu]a}\;, \quad (4.24)$$

where the pair of indices between square brackets means symmetrization. The action (4.23) corresponds to the Deser–Gibbons generalization of the MM action with a β-term (2.5) and it is the closest in structure analogue to the determinantal form of the action (3.1). The determinant of this action can be expanded to get a form equivalent to Eq. (3.2) where the dual curvature will also appear under the argument of the square root analogous to the pure BI theory.

In the action proposed in [10], an ad-hoc fudge tensor $X_{\mu\nu}$ was proposed which form would prevent the action from having instabilities. In a similar manner, one would need to add a fudge tensor to the action (4.23). This fudge tensor could be adjusted to avoid the corresponding instabilities [10]. We can then redefine the action (4.23) in the following manner

$$S_{DG\beta} = \frac{\lambda}{2}{}^{+}\tau \int d^4x \sqrt{-\det\left({}^{+}G_{\mu\nu} + X_{\mu\nu}\right)} - \frac{\lambda}{2}{}^{-}\tau \int d^4x \sqrt{-\det\left({}^{-}G_{\mu\nu} + X_{\mu\nu}\right)}\,. \quad (4.25)$$

The inclusion of this additional tensor does not affect the diffeomorphism invariance, however the form of this term will be strongly constrained if we want to maintain the gauge invariance and the dual structure of the action.

5. Final Remarks

Non-Abelian generalizations of the BI theory have been considered [29,37] but it is not clear if these approaches allow for gravitational generalizations of the DG type. MM gravity represents the closest theoretic approach to a Yang–Mills type gravitational theory and it already can be related to DG gravities by a similar procedure as that followed in [25]. In this work we have defined the DG gravity related to the modified version of MM gravity that contains the topological β-term (2.5). As it was stated before, the MM gravity action with the additional β-term takes into account the term corresponding the so called Immirzi parameter relevant in the canonical formulation of quantum gravity [18,24]. If we want to make contact with this quantum gravity formulation, it is consistent and necessary to consider the complete Yang–Mills action, namely the MM action with the additional

β-term. It is also remarkable that the addition of the β-term in the generalization of gauge supergravity, the corresponding supersymmetric Immirzi parameter arises naturally [30]. Therefore, we considered the complete MM action (2.5) and as we have seen, the procedure followed in [25] can be applied by the use of the combination of the self-dual and anti-selfdual MM actions. It is possible to construct independently the DG actions of the selfdual (anti-selfdual) actions respectively [25] but the combination of both is necessary in order to have a equivalent action to (2.5). The approach followed by us is then supported by the equivalence between the actions in Eqs. (2.5) and (4.9). This was not considered in [25].

Guided by the same motivation that inspired the construction of DG gravity it is interesting to search if the DG gravity represented by action (4.23) could avoid singularities classically or a possible quantum version of it could do. Using the supersymmetric extensions of MM gravity [26,30], a supersymmetric generalization of the procedure followed here could be possible. It will also be relevant to make the analysis of mode propagation in order to determine if our action could be ghost free. We notice that even when some DG gravities in the metric formalisms have ghost instabilities, some ghost free DG gravity theories have been constructed [17]; These, and other interesting topics will be presented in a future work.

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