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Subalgebras of the Split Octonions

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Abstract. We classify the subalgebras of the split octonions, paying particular attention to the null subalgebras and their extensions.
Keywords. Octonions, Division algebras, Null subalgebras.

1. Introduction

The octonions appear to play an important role in the modern description of fundamental particles, although the full scope of this role is not yet known. There are clear indications that the octonions are related to supersymmetry [10,11,20,21] and Grand Unified Theories [8,14–16,18,22]. Furthermore, the octonions are at the heart of any description of the exceptional Lie groups [1,4,6,9,12,19,24]. In all of these applications, the octonions generalize the notion of a Clifford algebra, as typified by the underlying quaternionic nature of the Clifford algebra in two Euclidean dimensions. The structure of the octonions is therefore of considerable interest to both mathematicians and physicists working on these and related topics.

The subalgebras of the octonions $\mathbb{O}$ are well-known: According to the Hurwitz theorem [17], the only proper subalgebras are the reals $\mathbb{R}$, the complex numbers $\mathbb{C}$, and the quaternions $\mathbb{H}$. In its usual formulation, the Hurwitz theorem asserts that these are the only *Euclidean* composition algebras (over $\mathbb{R}$). However, there is also a non-Euclidean cousin of the octonions, namely the split octonions $\mathbb{O}'$. Both the Hurwitz theorem and the Cayley–Dickson process [5] lead to the construction of split complex numbers $\mathbb{C}'$ and split quaternions $\mathbb{H}'$, but these turn out not to be the only proper subalgebras of $\mathbb{O}'$.

Maximal subalgebras of Cayley algebras have been classified over arbitrary fields of characteristic other than two [23]. In the split case, the maximal subalgebras are either four- or six-dimensional; the quaternions, or the sextonions, both discussed below. Implicit in this work is a classification of all subalgebras, but so far as we are aware no such list has appeared in print. Over finite fields, subalgebras of split Cayley algebras have indeed been fully

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classified [13], but again, these results do not appear to have been explicitly applied to the real case, which is of greatest interest in physical applications.

We present here a complete classification of the subalgebras of the split octonions, that is, the split Cayley algebra over the reals. We work throughout in an explicit basis, therefore also providing an explicit example of each type of subalgebra. It is straightforward to show that our classification is in fact complete, but we omit a formal proof, as we deem the explicit list likely to be more useful in applications.

2. The Split Octonions

We follow Dray and Manogue [7] in writing the basis of the quaternions as $\{1, I, J, K\}$, so that

$$\mathbb{H} = \langle 1, I, J, K\rangle \tag{1}$$

where of course

$$I^2 = J^2 = K^2 = -1, \tag{2}$$

$$IJ = K = -JI. \tag{3}$$

The split octonions can then be written as

$$\mathbb{O}' = \mathbb{H} \oplus \mathbb{H}L \tag{4}$$

where direct sums always refer to the underlying vector space structure, and where

$$L^2 = +1. \tag{5}$$

Like the octonions, the split octonions are not associative, but they are alternative; the full multiplication table is shown in Table 1. Octonionic conjugation is defined as usual via

$$\overline{x} = 2\,\mathrm{Re}(x) - x \tag{6}$$

and there is a norm, given by

$$|x|^2 = x\overline{x}, \tag{7}$$

which can be polarized to yield a non-degenerate inner product

$$x \cdot y = \frac{1}{2}(x\overline{y} + y\overline{x}) = \frac{1}{2}(\overline{x}y + \overline{y}x) \tag{8}$$

with signature $(4, 4)$, since $|x|^2 > 0$ for $x \in \mathbb{H}$, but $|x|^2 < 0$ for $x \in \mathbb{H}L$. The split octonions are indeed a composition algebra, that is, we have

$$|xy|^2 = |x|^2|y|^2 \tag{9}$$

for any $x, y \in \mathbb{O}'$.

It is useful to regard $\mathbb{O}'$ as the sum of four Lorentzian subspaces labeled by the basis elements of $\mathbb{H}$. The inner product has signature $(1, 1)$ on each such subspace, but each such subspace also contains elements of norm 0, and in fact admits a basis of such elements. The null elements so obtained fall into two categories: there are (multiples of) *idempotents*, such as $\frac{1}{2}(1 \pm L)$, which square to themselves, and *nilpotents*, such as $\frac{1}{2}(I \pm IL)$, which square to 0.

TABLE 1. The split octonionic multiplication table

	$\mathbf{1}$	$\boldsymbol{I}$	$\boldsymbol{J}$	$\boldsymbol{K}$	$\boldsymbol{KL}$	$\boldsymbol{JL}$	$\boldsymbol{IL}$	$\boldsymbol{L}$
$\mathbf{1}$	1	I	J	K	KL	JL	IL	L
$\boldsymbol{I}$	I	-1	K	$-J$	JL	$-KL$	$-L$	IL
$\boldsymbol{J}$	J	$-K$	-1	I	$-IL$	$-L$	KL	JL
$\boldsymbol{K}$	K	J	$-I$	-1	$-L$	IL	$-JL$	KL
$\boldsymbol{KL}$	KL	$-JL$	IL	L	1	$-I$	J	K
$\boldsymbol{JL}$	JL	KL	L	$-IL$	I	1	$-K$	J
$\boldsymbol{IL}$	IL	L	$-KL$	JL	$-J$	K	1	I
$\boldsymbol{L}$	L	$-IL$	$-JL$	$-KL$	$-K$	$-J$	$-I$	1

In the presence of null elements, we must extend the usual definition of signature. A vector subspace of $\mathbb{O}'$, henceforth simply called a *subspace*, has signature (p, m, n) if it contains an n-dimensional subspace on which the inner product is identically zero, and a $(p+m)$-dimensional subspace on which the inner product is nondegenerate with signature (p, m).

The Hurwitz theorem is sufficient to classify the *nondegenerate* (proper) subalgebras of $\mathbb{O}'$, yielding as expected $\mathbb{R}$, $\mathbb{C}$, $\mathbb{C}'$, $\mathbb{H}$, and $\mathbb{H}'$. But what about *degenerate* subalgebras?

Furthermore, the nondegenerate subalgebras of $\mathbb{O}'$ contain no proper, nondegenerate ideals. But the composition property means that degenerate null subalgebras can be ideals, as we now show.

Lemma 1. *The product of a null element of $\mathbb{O}'$ with any other element of $\mathbb{O}'$ is null.*

Proof. If $q \in \mathbb{O}'$ is null, then

$$|q|^2 = 0 \tag{10}$$

by definition, and

$$|qx|^2 = |q|^2|x|^2 = 0 \tag{11}$$

for any $x \in \mathbb{O}'$, so qx is also null. □

Lemma 2. *The null elements of any subalgebra of $\mathbb{O}'$ close under multiplication.*

Proof. Let $\mathbb{K} \subset \mathbb{O}'$ be a subalgebra of $\mathbb{O}'$, and let $\mathbb{N}$ be the set of all null elements of $\mathbb{K}$. Since $\mathbb{K}$ is a subalgebra, it closes under multiplication, so $qx \in \mathbb{K}$ for any $q \in \mathbb{N}$ and $x \in \mathbb{K}$. But qx is null by Lemma 1, completing the proof. □

It is tempting to conclude that the null elements of a subalgebra $\mathbb{K} \subset \mathbb{O}'$ form an ideal of $\mathbb{K}$, but this statement is false; an algebra ideal must also be closed under addition. For example, the null elements of a Lorentzian vector space span the entire space, not merely the null subset. However, this example correctly suggests that *degenerate* subalgebras contain null ideals.

We first need the following lemma, which we state without proof.

Lemma 3. *The inner product* (8) *on* $\mathbb{O}'$ *satisfies the associator identity*

$$xy \cdot z - x \cdot yz = 2\,\mathrm{Re}(x)\, y \cdot z - 2\,\mathrm{Re}(z)\, x \cdot y. \tag{12}$$

We now define the *degenerate subspace*, also denoted $\mathbb{N}$, of a subalgebra $\mathbb{K} \subset \mathbb{O}'$ to be the set of elements orthogonal to *all* elements of $\mathbb{K}$. Thus, not only is the inner product identically zero on $\mathbb{N}$ (so that its elements are null), but elements of $\mathbb{N}$ are also orthogonal to any non-degenerate elements of $\mathbb{K}$.

Lemma 4. *Let* $\mathbb{N}$ *be the degenerate subspace of the subalgebra* $\mathbb{K} \subset \mathbb{O}'$*. If* $\mathbb{N}$ *closes under multiplication, then* $\mathbb{N}$ *is an ideal of* $\mathbb{K}$*.*

Proof. If $\mathbb{N} = \mathbb{K}$, we're done, so assume $\mathbb{N} \neq \mathbb{K}$. Then $\mathbb{K}$ contains at least one nondegenerate element x, satisfying $0 \neq x\overline{x} = x(2\,\mathrm{Re}(x) - x)$. Since $\mathbb{K}$ is an algebra, the right-hand side of this expression is in $\mathbb{K}$, and hence $\mathbb{R} \subset \mathbb{K}$. Thus, $q \in \mathbb{N}$ forces $q \perp \mathbb{R}$, so that $\mathrm{Re}(q) = 0$.

For any $p \in \mathbb{K}$, $pq = \alpha + \beta$, with $\alpha \in \mathbb{K}\backslash\mathbb{N}$, $\beta \in \mathbb{N}$. If $0 \neq \alpha \notin \mathbb{N}$, then there exists $s \in \mathbb{K}$ such that $s \cdot \alpha \neq 0$, so that $s \cdot (pq) = s \cdot (\alpha + \beta) = s \cdot \alpha + 0 \neq 0$. Now use (12) with $x = s$, $y = p$, $z = q$, noting that $y \cdot z = 0 = \mathrm{Re}(z)$. Therefore, $(sp) \cdot q \neq 0$, which contradicts the assumed degeneracy of q, which in turn forces $\alpha = 0$. Thus, $pq \in \mathbb{N}$, as desired. □

Finally, we can characterize the nondegenerate part of any proper subalgebra $\mathbb{K} \subset \mathbb{O}'$. Using the Cayley–Dickson process, we can construct a maximal nondegenerate subalgebra $\mathbb{X} \subset \mathbb{K}$. Thus, we can assume without loss of generality that (the basis is chosen such that) the nondegenerate subspace $\mathbb{X} \subset \mathbb{K}$ is in fact a subalgebra of $\mathbb{K}$; the Cayley–Dickson process then assures us that $\mathbb{X}$ is one of $\mathbb{R}$, $\mathbb{C}$, $\mathbb{C}'$, $\mathbb{H}$, or $\mathbb{H}'$

With these tools, we are ready to classify the subalgebras of $\mathbb{O}'$.

3. One-Dimensional Subalgebras

One-dimensional subalgebras of $\mathbb{O}'$ could have signature $(1, 0, 0)$, $(0, 1, 0)$, or $(0, 0, 1)$. However, the nondegenerate part of the subalgebra must itself be a subalgebra, which rules out signature $(0, 1, 0)$. Signature $(1, 0, 0)$ corresponds to $\mathbb{R} = \langle 1 \rangle$, but what about signature $(0, 0, 1)$?

If $x\overline{x} = 0$, then $x^2 = 2x\mathrm{Re}(x)$ from (6). There are two cases: $\mathrm{Re}(x) = 0$, in which case $x^2 = 0$ and x is *nilpotent*, and $\mathrm{Re}(x) \neq 0$, in which case we can rescale x so that $\mathrm{Re}(x) = \frac{1}{2}$ and $x^2 = x$, and x is *idempotent*. In either case, x generates a 1-dimensional null subalgebra.

An example of an idempotent null subalgebras is

$$\mathbb{I}^{\pm} = \left\langle \frac{1}{2}(1 \pm L) \right\rangle = \langle 1 \pm L \rangle, \tag{13}$$

and an example of a nilpotent null subalgebra is

$$\mathbb{N}^{\pm}_{K} = \langle K \pm KL \rangle. \tag{14}$$

As with the other examples discussed below, these examples are generic; our use of an explicit basis notwithstanding, it is straightforward to show that

TABLE 2. The split octonionic multiplication table in a null basis, using the abbreviations $1_\pm = \frac{1}{2}(1 \pm L)$, $I_\pm = \frac{1}{2}(I \pm IL)$, $J_\pm = \frac{1}{2}(J \pm JL)$, $K_\pm = \frac{1}{2}(K \pm KL)$

	$\mathbf{1}_+$	$\boldsymbol{I}_+$	$\boldsymbol{J}_+$	$\boldsymbol{K}_-$	$\boldsymbol{K}_+$	$\boldsymbol{J}_-$	$\boldsymbol{I}_-$	$\mathbf{1}_-$
$\mathbf{1}_+$	1_+	0	0	K_-	0	J_-	I_-	0
$\boldsymbol{I}_+$	I_+	0	K_-	0	$-J_-$	0	-1_-	0
$\boldsymbol{J}_+$	J_+	$-K_-$	0	0	I_-	-1_-	0	0
$\boldsymbol{K}_-$	0	0	0	0	-1_+	$-I_+$	J_+	K_-
$\boldsymbol{K}_+$	K_+	J_-	$-I_-$	-1_-	0	0	0	0
$\boldsymbol{J}_-$	0	0	-1_+	I_+	0	0	$-K_+$	J_-
$\boldsymbol{I}_-$	0	-1_+	0	$-J_+$	0	K_+	0	I_-
$\mathbf{1}_-$	0	I_+	J_+	0	K_+	0	0	1_-

all null 1-dimensional subalgebras of $\mathbb{O}'$ are isomorphic to one of these two cases.

4. Two-Dimensional Subalgebras

The only nondegenerate two-dimensional subalgebras of $\mathbb{O}'$ are $\mathbb{C}$ and $\mathbb{C}'$; we take the particular examples $\mathbb{C} = \langle 1, I\rangle$ and $\mathbb{C}' = \langle 1, L\rangle$. Examples of degenerate two-dimensional subalgebras are

$$\mathbb{I}_K^\pm = \mathbb{I}^\pm \oplus \mathbb{N}_K^- = \langle 1 \pm L, K - KL\rangle, \tag{15}$$

$$\mathbb{N}_{JK} = \mathbb{N}_J^+ \oplus \mathbb{N}_K^- = \langle J + JL, K - KL\rangle. \tag{16}$$

Although both of these subalgebras have completely degenerate inner products, they are not isomorphic, as only $\mathbb{N}_{JK}$ also has a completely degenerate multiplication table. (The multiplication table for $\mathbb{O}'$ in a null basis is shown in Table 2.) Notice that $\mathbb{N}_J^+$ and $\mathbb{N}_K^-$ are both ideals of $\mathbb{N}_{JK}$, but only $\mathbb{N}_K^-$ is an ideal of $\mathbb{I}_K^\pm$.

There are also mixed two-dimensional subalgebras of $\mathbb{O}'$, which must take the form $\mathbb{R} + \mathbb{N}$, with $\mathbb{N}$ degenerate. An example of such a subalgebra is

$$\mathbb{R} \oplus \mathbb{N}_K^\pm = \langle 1, K \pm KL\rangle, \tag{17}$$

of which $\mathbb{N}_K^\pm$ must be an ideal by Lemma 4.

5. Three-Dimensional Subalgebras

There are no nondegenerate three-dimensional composition algebras over $\mathbb{R}$, so any three-dimensional subalgebra of $\mathbb{O}'$ must contain a degenerate piece. Beginning with the completely degenerate case, we have the remarkable subalgebra

$$\mathbb{N}_{IJK} = \mathbb{N}_I^+ \oplus \mathbb{N}_J^+ \oplus \mathbb{N}_K^- = \langle I + IL, J + JL, K - KL\rangle, \tag{18}$$

which closes since the product of the first two generators is (a multiple of) the third. We can also combine the null two-dimensional subalgebras in the previous section, obtaining

$$\mathbb{I}_\pm + \mathbb{N}_{JK} = \langle 1 \pm L, J + JL, K - KL\rangle, \tag{19}$$

with a quite different multiplication table. Turning to mixed algebras, the only possibilities have the form $\mathbb{R} \oplus \mathbb{N}$, $\mathbb{C}' \oplus \mathbb{N}$, or $\mathbb{C} \oplus \mathbb{N}$, with $\mathbb{N}$ degenerate. Examples of the first two types are

$$\mathbb{R} \oplus \mathbb{N}_{JK} = \langle 1, J + JL, K - KL\rangle, \tag{20}$$

$$\mathbb{C}' \oplus \mathbb{N}_K^\pm = \langle 1, L, K \pm KL\rangle, \tag{21}$$

but it is not possible to extend $\mathbb{C}$ to a three-dimensional subalgebra, as the subspace will not close. Subalgebras of $\mathbb{O}'$ with signature $(1, 1, 1)$ such as $\mathbb{C}' \oplus \mathbb{N}_K^\pm$ are called *ternions* ($\mathbb{T}$).

It is easy to see that $\mathbb{N}_J^+$, $\mathbb{N}_K^-$, and $\mathbb{N}_{JK}$ are ideals of $\mathbb{N}_{IJK}$, but the only ideal of $\mathbb{I}_\pm \oplus \mathbb{N}_{JK}$ is $\mathbb{N}_{JK}$. As before, $\mathbb{N}_{JK}$ must be an ideal of $\mathbb{R} \oplus \mathbb{N}_{JK}$, and $\mathbb{N}_K^\pm$ must be an ideal of $\mathbb{C}' \oplus \mathbb{N}_K^\pm$, by Lemma 4.

6. Four-Dimensional Subalgebras

The only nondegenerate four-dimensional subalgebras of $\mathbb{O}'$ are $\mathbb{H}$ and $\mathbb{H}'$; we take the particular examples $\mathbb{H} = \langle 1, I, J, K\rangle$ and $\mathbb{C}' = \langle 1, I, IL, L\rangle$. The only degenerate four-dimensional subalgebras of $\mathbb{O}'$ have the form

$$\mathbb{I}^\pm \oplus \mathbb{N}_{IJK} = \langle 1 \pm L, I + IL, J + JL, K - KL\rangle, \tag{22}$$

with ideals $\mathbb{N}_K^-$, $\mathbb{N}_{JK}$, and $\mathbb{N}_{IJK}$. A mixed subalgebra must have the same general form as given above in the three-dimensional case; examples of each case are:

$$\mathbb{R} \oplus \mathbb{N}_{IJK} = \langle 1, I + IL, J + JL, K - KL\rangle, \tag{23}$$

$$\mathbb{C} \oplus \mathbb{N}_{JK} = \langle 1, I, J + JL, K - KL\rangle, \tag{24}$$

$$\mathbb{C}' \oplus \mathbb{N}_{JK} = \langle 1, L, J + JL, K - KL\rangle, \tag{25}$$

where in each case the degenerate subspace is in fact a subalgebra of $\mathbb{O}'$ and hence an ideal by Lemma 4.

7. Higher-Dimensional Subalgebras

We refer to the remaining cases, with dimension greater than four, as *higher-dimensional subalgebras* of $\mathbb{O}'$. There are no proper, nondegenerate, higher-dimensional subalgebras of $\mathbb{O}'$, nor can there be more than four independent, orthogonal null directions. Therefore, any higher-dimensional subalgebra of $\mathbb{O}'$ must be mixed, although now we must include $\mathbb{H}$ and $\mathbb{H}'$ as possibilities for the nondegenerate part of the subalgebra.

There is no higher-dimensional subalgebra of $\mathbb{O}'$ of the form $\mathbb{R} \oplus \mathbb{N}$, with $\mathbb{N}$ degenerate, since $\mathbb{N}$ would have to be four-dimensional, leaving no room for any further elements. Put differently, adding any additional element to the null, four-dimensional subalgebra $\mathbb{I}^{\pm} \oplus \mathbb{N}_{IJK}$ would change the signature to $(1,1,3)$. But this signature is possible, an example being

$$\mathbb{C}' \oplus \mathbb{N}_{IJK} = \langle 1, L, I + IL, J + JL, K - KL \rangle \tag{26}$$

with $\mathbb{N}_{IJK}$ as an ideal by Lemma 4. But the same argument shows that there can be no larger subalgebras containing $\mathbb{C}'$.

Higher-dimensional subalgebras containing $\mathbb{C}$ or $\mathbb{H}$ are not possible for similar reasons; there are not enough Lorentzian planes left over to find a sufficient number of orthogonal null elements.

That leaves the possibility of a subalgebra of $\mathbb{O}'$ of the form $\mathbb{H}' \oplus \mathbb{N}$, with $\mathbb{N}$ degenerate. This case is possible, an example being

$$\mathbb{H}' \oplus \mathbb{N}_{JK} = \langle 1, I, IL, L, J + JL, K - KL \rangle \tag{27}$$

with, yet again, $\mathbb{N}_{JK}$ as an ideal by Lemma 4. Subalgebras of $\mathbb{O}'$ with signature $(2,2,2)$ such as $\mathbb{H}' \oplus \mathbb{N}_{JK}$ are called *sextonions* ($\mathbb{S}$). Both the ternions and sextonions have recently been used to generalize the Freudenthal–Tits magic square [3].

8. Summary

By examining each case, we have in fact shown that any subalgebra of $\mathbb{O}'$ can be decomposed into orthogonal subalgebras, one degenerate and the other nondegenerate.

Theorem 1. *Any subalgebra* $\mathbb{K} \subset \mathbb{O}'$ *can be decomposed as* $\mathbb{K} = \mathbb{X} \oplus \mathbb{N}$*, where* $\mathbb{X}$, $\mathbb{N}$ *are orthogonal subalgebras of* $\mathbb{K}$*, with* $\mathbb{N}$ *degenerate and* $\mathbb{X}$ *nondegenerate. Furthermore,* $\mathbb{N}$ *is an ideal of* $\mathbb{K}$*.*

An exhaustive list of the possible signatures, together with examples of each case, is given in Table 3. As enumerated above, each null subalgebra of dimension greater than one contains one or more null subalgebras that are ideals, and Lemma 4 guarantees that each subalgebra with mixed signature contains an ideal, namely its degenerate subalgebra. Are these the only subalgebras with ideals, that is, are the remaining algebras simple?

The answer is, not quite. The Euclidean subalgebras $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$ are of course simple, as are the one-dimensional null subalgebras, but what about the split subalgebras? It is easy to verify that $\mathbb{H}'$ and $\mathbb{O}'$ itself are indeed simple, but $\mathbb{C}'$ is not, as it can be written as the sum of two idempodent null subalgebras, namely

$$\mathbb{C}' = \mathbb{I}^{+} \oplus \mathbb{I}^{-}, \tag{28}$$

and each of these subalgebras is an ideal. The simple subalgebras of $\mathbb{O}'$ are therefore $\mathbb{I}^{\pm}$, $\mathbb{N}_K^{\pm}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{H}'$, and of course $\mathbb{O}'$ itself.

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Table 3. The possible signatures for proper subalgebras of $\mathbb{O}'$, with examples for each case

Signature	**Example(s)**
(1,0,0)	$\mathbb{R} = \langle 1 \rangle$
(0,0,1)	$\mathbb{I}^{\pm} = \langle 1 \pm L \rangle; \quad \mathbb{N}_K^{\pm} = \langle K \pm KL \rangle$
(2,0,0)	$\mathbb{C} = \langle 1, I \rangle$
(1,1,0)	$\mathbb{C}' = \langle 1, L \rangle$
(1,0,1)	$\mathbb{R} \oplus \mathbb{N}_K^{\pm} = \langle 1, K \pm KL \rangle$
(0,0,2)	$\mathbb{I}_K^{\pm} = \langle 1 \pm L, K - KL \rangle;$ $\mathbb{N}_{JK} = \langle J + JL, K - KL \rangle$
(0,0,3)	$\mathbb{I}^{\pm} \oplus \mathbb{N}_{JK} = \langle 1 \pm L, J + JL, K - KL \rangle;$ $\mathbb{N}_{IJK} = \langle I + IL, J + JL, K - KL \rangle$
(1,0,2)	$\mathbb{R} \oplus \mathbb{N}_{JK} = \langle 1, J + JL, K - KL \rangle$
(1,1,1)	$\mathbb{C}' \oplus \mathbb{N}_K^{\pm} = \langle 1, L, K \pm KL \rangle$
(4,0,0)	$\mathbb{H} = \langle 1, I, J, K \rangle$
(2,2,0)	$\mathbb{H}' = \langle 1, I, IL, L \rangle$
(0,0,4)	$\mathbb{I}^{\pm} \oplus \mathbb{N}_{IJK} = \langle 1 \pm L, I + IL, J + JL, K - KL \rangle$
(1,0,3)	$\mathbb{R} \oplus \mathbb{N}_{IJK} = \langle 1, I + IL, J + JL, K - KL \rangle$
(2,0,2)	$\mathbb{C} \oplus \mathbb{N}_{JK} = \langle 1, I, J + JL, K - KL \rangle$
(1,1,2)	$\mathbb{C}' \oplus \mathbb{N}_{JK} = \langle 1, L, J + JL, K - KL \rangle$
(1,1,3)	$\mathbb{C}' \oplus \mathbb{N}_{IJK} = \langle 1, L, I + IL, J + JL, K - KL \rangle$
(2,2,2)	$\mathbb{H}' \oplus \mathbb{N}_{JK} = \langle 1, I, IL, L, J + JL, K - KL \rangle$

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