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Modeling 3D Geometry in the Clifford Algebra $R(4,4)$

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Abstract. We flesh out the affine geometry of $\mathbb{R}^3$ represented inside the Clifford algebra $\mathbb{R}(4,4)$. We show how lines and planes as well as conic sections and quadric surfaces are represented in this model. We also investigate duality between different representations of points, lines, and planes, and we show how to represent intersections between these geometric elements. Formulas for lengths, areas, and volumes are also provided.

Keywords. Mother algebra, Affine geometry, Computer graphics.

1. Introduction

Clifford algebra is a powerful paradigm for investigating geometry. The separation of operators and operands, the availability of versors and rotors for computing transformations, the use of meets and joins for representing intersections and unions, the large array of additional computational tools such as the Clifford product, the inner product, and the outer product, the built-in orientations, dimensions, and duality, and the geometric insights gained from this powerful algebra often lead to cleaner, leaner, more robust algorithms, programs, and code as well as the ability to solve challenging technical problems not always possible using classical matrix methods [9,19,22].

The algebra we use depends on the geometry we wish to investigate. Let $\mathbb{R}(n,m)$ denote the algebra with n basis vectors that square to $+1$ and m basis vectors that square to -1. The Clifford algebra $\mathbb{R}(3,0)$ contains vectors representing oriented line segments, bivectors representing oriented plane sectors, and a subalgebra of quaternions representing rotations in 3-dimensions. Thus $\mathbb{R}(3,0)$ is the model of choice for investigating rotations of

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lines and planes through the origin as well as tangent vectors and normal vectors in 3-dimensions [9]. Similarly, the conformal model $\mathbb{R}(4,1)$ contains flats-oriented lines and planes—and rounds-oriented circles and spheres—as well as all the 3-dimensional conformal transformations—translation, rotation, reflection, uniform scaling, and spherical inversion—as versors and rotors. Thus $\mathbb{R}(4,1)$ is the model of choice for investigating conformal geometry and has been successfully applied to solve challenging problems in kinematics and robotics [1,19].

The success of these two Clifford algebras has motivated a search for a Clifford algebra appropriate for the study of affine and projective geometry in 3-dimensions, a Clifford algebra that could serve as an algebraic foundation for computer graphics, computer vision, and geometric modeling. Such an algebra must incorporate all the 3-dimensional affine and projective transformations—translation, rotation, reflection, uniform and non-uniform scaling, classical and scissor shear, orthogonal and perspective projection—as versors and rotors. To compete with matrix algebra, this Clifford algebra must also be able to model lines and planes as well as conic sections and quadric surfaces.

Two potential candidates are currently under investigation as the foundation for affine and projective geometry in 3-dimensions: $\mathbb{R}(3,3)$ [5,8,16,17,20] and $\mathbb{R}(4,4)$ [4,13]. So far only the affine and projective transformations in each of these algebras have been thoroughly investigated; Dorst [8] gives an extended comparison between the transformations available in $\mathbb{R}(3,3)$ and $\mathbb{R}(4,4)$. The purpose of this paper is to flesh out the geometry of one of these models, what Doran et al. call the *mother algebra* $\mathbb{R}(4,4)$ [4].

This paper makes the following contributions. In an earlier paper [13], we show how to use versors and rotors in $\mathbb{R}(4,4)$ to perform all the affine and projective transformations needed in computer graphics. To flesh out the geometry of $\mathbb{R}(4,4)$ into a more complete model for computer graphics, here we show how

1. points, vectors, lines and planes are represented in $\mathbb{R}(4,4)$;
2. conic sections and quadric surfaces are represented in $\mathbb{R}(4,4)$;
3. duality works in $\mathbb{R}(4,4)$;
4. intersections are computed in $\mathbb{R}(4,4)$;
5. lengths, areas, and volumes are computed in $\mathbb{R}(4,4)$.

To verify our results, we implemented and tested the formulas in this paper using Gaigen [12] to generate a C++ library for $\mathbb{R}(4,4)$.

Two other models share some of the features of our model. Gunn develops the Clifford algebra model $P(R^*(3,0,1))$ for 3-dimensional euclidean space and using alternative derivations establishes similar formulas to measure distances and angles between geometric elements [6,7]. The even subalgebra of $P(R^*(3,0,1))$ is isomorphic to the dual quaternions [6,7], but $P(R^*(3,0,1))$ lacks representations for shears and non-uniform scaling, as well as quadric surfaces and conic sections, all of which are features in our $\mathbb{R}(4,4)$ model and all of which are needed for computer graphics.

Easter and Hitzer [10,11] use the double conformal model to represent quadric surfaces and conic sections in addition to points and planes; further, they can represent some quartic surfaces, including tori. Their represention of such surfaces has similarities to our representation of quadric surfaces. However, this model is limited to conformal transformations and therefore also lacks representations for shears and non-uniform scaling.

Thus the primary advantage of the Clifford algebra $\mathbb{R}(4,4)$ with the geometric interpretation presented here and in [13] is that so far $\mathbb{R}(4,4)$ is the only model of Clifford algebra supporting the full range of geometry needed for computer graphics. Additionally, in this geometric interpretation of $\mathbb{R}(4,4)$, objects and their duals live in different subspaces. Thus, generally one can tell if an object is meant to be a primal object or its dual, which is an advantage in software implementations.

We organize our results in the following fashion. In Sect. 2.1 we introduce the Witt basis, the basis that is used to represent geometry in $\mathbb{R}(4,4)$. Thus we construct $\mathbb{R}(4,4)$ by starting with $R^8 = W \oplus W^*$, where W,W^* are two dual 4-dimensional subspaces of null vectors and then taking the Clifford algebra of this 8-dimensional vector space. Standard representations for geometric elements in $\mathbb{R}^3$, including not only points, lines, and planes, but also quadric surfaces and conic curves are presented in Sect. 2[1]. Intersection formulas are presented in Sect. 3 and formulas for barycentric coordinates are provided in Sect. 4. Since the treatment of flats in this paper is similar to the treatment of flats in other papers on Clifford algebra [9], we exclude metric formulas for flats from the main body of this paper, and instead place formulas for length, area and volume in Tables 3, 4, 5. We also include two appendices. In Appendix A we verify that an object and its dual have the same metric properties; in Appendix B we present derivations of the formulas for distances and angles between a pair of lines or a pair of planes, since these derivations are somewhat more complicated in $\mathbb{R}(4,4)$ than in other Clifford algebras.

2. Geometry in $\mathbb{R}(4,4)$

In this section, we show how to use the mother algebra $\mathbb{R}(4,4)$ to represent geometric elements in R^3, including not only points, lines, and planes, but also quadric surfaces and conic curves.

In the subsequent discussion, we will invoke the following well known formulas [14] for the interaction between the inner product and the outer product:

$$(a_1 \wedge \ldots \wedge a_r) \cdot (b_1 \wedge \ldots \wedge b_s) = \begin{cases} ((a_1 \wedge \ldots \wedge a_r) \cdot b_1) \cdot (b_2 \wedge \ldots \wedge b_s) & r \geq s \\ (a_1 \wedge \ldots \wedge a_{r-1}) \cdot (a_r \cdot (b_1 \wedge \ldots \wedge b_s)) & r < s \end{cases} \tag{1}$$

[1] Some people use E^3 instead of $\mathbb{R}^3$ to denote affine 3-space, reserving $\mathbb{R}^3$ for the vector space of 3-dimensions. We use $\mathbb{R}^3$ instead of E^3, since $\mathbb{R}^3$ is still the standard notation for affine 3-space in the field of computer graphics.

and

$$(a_1 \wedge ... \wedge a_r) \cdot b = \sum_{i=1}^{r} (-1)^{r-i} a_1 \wedge ... \wedge a_{i-1} \wedge (a_i \cdot b) \wedge a_{i+1} \wedge ... \wedge a_r$$
$$a \cdot (b_1 \wedge ... \wedge b_s) = \sum_{i=1}^{s} (-1)^{i-1} b_1 \wedge ... \wedge b_{i-1} \wedge (a \cdot b_i) \wedge b_{i+1} \wedge ... \wedge b_s.$$

In particular,

$$a \cdot (b \wedge c) = (a \cdot b)c - (a \cdot c)b \tag{2}$$
$$(b \wedge c) \cdot a = (a \cdot c)b - (a \cdot b)c \tag{3}$$
$$(a \wedge b) \cdot (c \wedge d) = (b \cdot c)(a \cdot d) - (b \cdot d)(a \cdot c). \tag{4}$$

2.1. The Witt Basis

The mother algebra $\mathbb{R}(4,4)$ is a geometric algebra for the 8-dimensional vector space generated by four basis vectors e_0, e_1, e_2, e_3 that square to $+1$, and four additional basis vectors $\bar{e}_0$, $\bar{e}_1$, $\bar{e}_2$, $\bar{e}_3$ that square to -1 [4]. Thus

$$e_i^2 = 1, \quad \bar{e}_i^2 = -1, \quad (i = 0, 1, 2, 3).$$

Moreover, $\{e_i, \bar{e}_i\}$ is an orthonormal basis, so

$$e_i \cdot e_j = 0, \ \bar{e}_i \cdot \bar{e}_j = 0, \quad i \neq j$$
$$e_i \cdot \bar{e}_j = 0, \quad \text{for all} \quad i, j.$$

Therefore

$$e_i e_j = -e_j e_i, \quad \bar{e}_i \bar{e}_j = -\bar{e}_j \bar{e}_i, \quad i \neq j.$$
$$e_i \bar{e}_j = -\bar{e}_j e_i, \quad \text{for all } i, j$$

However, this basis is used mainly for computing, but not for representing geometry [13]. To represent points and vectors in 3-dimensions as well as affine and projective transformations on $\mathbb{R}^3$, we use the Witt basis [3]

$$w_i = \frac{e_i + \bar{e}_i}{2}, \quad w_i^* = \frac{e_i - \bar{e}_i}{2}, \quad (i = 0, 1, 2, 3).$$

From these definitions, it follows easily that

$$w_i \cdot w_j = 0, \quad w_i^* \cdot w_j^* = 0, \quad w_i^* \cdot w_j = \frac{1}{2}\delta_{i,j}. \tag{5}$$

In particular, all the Witt basis vectors in $\mathbb{R}(4,4)$ are null vectors. Therefore

$$w_i \wedge w_j = w_i w_j = -w_j w_i, \quad w_i^* \wedge w_j^* = w_i^* w_j^* = -w_j^* w_i^*,$$

$$w_i^2 = w_i \wedge w_i = 0, \quad (w_i^*)^2 = w_i^* \wedge w_i^* = 0.$$

In this model of $\mathbb{R}(4,4)$, $W = span\{w_0, w_1, w_2, w_3\}$ are vectors, and $W^* = span\{w_0^*, w_1^*, w_2^*, w_3^*\}$ are dual functionals (1-forms). That is, $W \cong R^4$ is a vector space, and $W^* \cong (R^4)^*$ is the space of dual functionals (1-forms). So in our algebra we use one copy of $\mathbb{R}^4$ and one copy of $(R^4)^*$ and we denote this algebra by $W \oplus W^*$. Further note that the subalgebra generated by the vector space W has signature $(0,0,0,0)$; thus W is completely degenerate and equivalent to the 4D Grassmann algebra; this is also the case for the subalgebra generated by W^*. The importance of Eq. 5 now becomes clear: $w_i \cdot w_i^* \neq 0$ is necessary to introduce metric properties.

In Eq. 5, the factor of $\frac{1}{2}$ in $w_i^* \cdot w_j = \frac{1}{2}\delta_{i,j}$ leads, unfortunately, to powers of $\frac{1}{2}$ appearing in many of the formulas in this paper. A variation of this definition of the inner product would remove this factor of $\frac{1}{2}$ (and thus the powers of $\frac{1}{2}$ in the other formulas in this paper). However, changing the definition of the inner product in this paper would require rederiving the formulas in the earlier paper on transformations using versors and rotors in $\mathbb{R}(4,4)$ [13], and these rederived formulas would have powers of 2 where no such powers of 2 currently appear. In fact, the factor of $\frac{1}{2}$ in Eq. 5 was chosen in order to remove these powers of 2 from the transformation formulas. In any event, since the two papers must be used together to form a computational model for computer graphics, we have kept the factor of $\frac{1}{2}$ in the inner product.

2.2. Points and Planes

Before launching into a general treatment of the geometric algebra $\mathbb{R}(4,4)$, we consider first how this homogeneous model characterizes the simplest objects and relationships in Euclidean geometry.

We let $w_1, w_2, w_3 \in W$ represent basis vectors along the coordinate axes of $\mathbb{R}^3$. A point p in $\mathbb{R}^3$ is represented homogeneously in the usual manner by using w_0 to carry the homogeneous coordinate so that

$$p = w_0 + p_1 w_1 + p_2 w_2 + p_3 w_3.$$

In homogeneous coordinates, a weighted point or a mass-point has the form

$$p = p_0 w_0 + p_1 w_1 + p_2 w_2 + p_3 w_3,$$

where the weight $p_0 \neq 0$. In this case p represents the point located at $p_0 + \frac{p_1}{p_0}p_1 + \frac{p_2}{p_0}p_2 + \frac{p_3}{p_0}p_3$. We write $p \equiv q$ if $p = kq$ for some non-zero scalar k because p and q are located at the same position; they differ only in their masses. In particular, if $p_0 \neq 0$

$$p_0 w_0 + p_1 w_1 + p_2 w_2 + p_3 w_3 \equiv w_0 + \frac{p_1}{p_0} w_1 + \frac{p_2}{p_0} w_2 + \frac{p_3}{p_0} w_3.$$

All points in this paper are unweighted points unless noted otherwise.

An arbitrary free vector v in $\mathbb{R}^3$ is written as

$$v = v_1 w_1 + v_2 w_2 + v_3 w_3.$$

Remark 1. In the limit, as the weight goes to zero, we get a point at infinity. Thus $v \in span\{w_1, w_2, w_3\}$ can represent either a free vector or a point at infinity, depending on the context. Similarly the outer product of two vectors $u \wedge v$ can represent either a free parallelogram[2] or a line at infinity, depending on the context.

[2] *Free* means the object can move freely in 3-dimensions parallel to itself.

Next, consider a plane in 3-dimensions with the homogeneous implicit equation

$$S(x_0, x_1, x_2, x_3) \equiv s_0x_0 + s_1x_1 + s_2x_2 + s_3x_3 = 0,$$

where s_1, s_2, s_3, s_0 are constants. Let

$$\pi_s^* = s_0w_0^* + s_1w_1^* + s_2w_2^* + s_3w_3^*.$$

Then for any homogeneous point $p = x_0w_0 + x_1w_1 + x_2w_2 + x_3w_3$

$$p \cdot \pi_s^* = \frac{1}{2}S(x_0, x_1, x_2, x_3).$$

Thus $p \cdot \pi_s^* = 0 \Leftrightarrow S(x_0, x_1, x_2, x_3) = 0$. Therefore we use the vector $\pi_s^* \in W^*$ to represent the plane $S(x_0, x_1, x_2, x_3)$.

Proposition 2. *For any plane* $\pi_s^* = s_0w_0^* + s_1w_1^* + s_2w_2^* + s_3w_3^* \in W^*$, *the normal to the plane is* $n_s^* = s_1w_1^* + s_2w_2^* + s_3w_3^*$.

Proof. For any two points $p = w_0 + p_1w_1 + p_2w_2 + p_3w_3$ and $q = w_0 + q_1w_1 + q_2w_2 + q_3w_3$ on the plane $\pi_s^* = s_0w_0^* + s_1w_1^* + s_2w_2^* + s_3w_3^*$,

$$n_s^* \cdot (q - p) = \frac{1}{2}(S(1, q_1, q_2, q_3) - S(1, p_1, p_2, p_3)) = 0.$$

Conversely a vector v not in the plane π_s^* can be represented as $q - p$ where $p = w_0+p_1w_1+p_2w_2+p_3w_3$ is in the plane π_s^* and $q = w_0+q_1w_1+q_2w_2+q_3w_3$ is not in the plane π_s^*. Then

$$n_s^* \cdot (q - p) = \frac{1}{2}(S(1, q_1, q_2, q_3) - S(1, p_1, p_2, p_3)) = \frac{1}{2}S(1, q_1, q_2, q_3) \neq 0.$$

□

Notice that $n_s^* = s_1w_1^* + s_2w_2^* + s_3w_3^*$ has two meanings in W^* : n_s^* represents a plane through the origin as well as a normal vector to any plane π_s^* in W^* of the form $\pi_s^* = s_0w_0^* + s_1w_1^* + s_2w_2^* + s_3w_3^*$. In our subsequent discussions, the notation n_s^* is used exclusively to represent the normal vector to a plane π_s^*. Similarly, the notation $n_1^* \wedge n_2^*$ represents a plane sector, epitomized by the parallelogram determined by the normals to two planes.

In addition, throughout this paper, lower case letter without stars typically represent points or vectors in W and lower case letters with stars typically represent normal vectors or planes in W^*. We shall also adopt the following notation: For any free vector $v = v_1w_1 + v_2w_2 + v_3w_3$ in $\mathbb{R}^3$, we define

$$v^* = v_1w_1^* + v_2w_2^* + v_3w_3^*$$

and $(v^*)^* = v$. This definition of * is a minor abuse of notation, since we have already used π^* to represent planes and now we are using v^* as an operator. However, both uses of star result in planes in W^*; furthermore, this notation is unambiguous since, by definition, $(w_j)^* = w_j^*$, $j = 1, 2, 3$. We extend this star operator to all of $\wedge W$ by setting

$$(a_1 \wedge a_2 \cdots \wedge a_k)^* = (a_k^* \wedge \cdots \wedge a_2^* \wedge a_1^*) = (-1)^{k(k-1)/2}\, a_1^* \wedge a_2^* \cdots \wedge a_k^*$$

and extending by linearity. Note that our *-operator differs from standard use in that it maps between W and W^*. In Sect. 2.7, we shall define a Hodge dual using a pre-* operator.

2.3. Geometry in $\wedge W$: Lines and Planes

Here we shall use outer products of homogeneous points in W to represent lines and planes in $\mathbb{R}^3$. We call the set of all points P such that $P \wedge E = 0$ the *outer product null space* (OPNS) of E. If $P \wedge E = 0$ if and only if $P \in x$ then we say that x is represented by E.

Two points determine a line. Thus a line l in $\mathbb{R}^3$ can be represented in $\mathbb{R}(4, 4)$ by the outer product (the join) of two distinct points in W. Moreover, the Plücker version of a line in projective space can be represented in $\mathbb{R}(4, 4)$ by the outer product (the join) of a homogeneous point and a free vector in W. Similarly, a plane π in $\mathbb{R}^3$ can be represented in $\mathbb{R}(4, 4)$ by the outer product (the join) of three non-collinear points in W. Formally we have the following results.

Theorem 3. *For any two distinct points p_1 and p_2 in W, their outer product $l = p_1 \wedge p_2$ represents the line determined by p_1 and p_2.*

Proof. Any point $q \in W$ on the line determined by the points p_1 and p_2 can be written as

$$q = (1 - t)p_1 + tp_2.$$

Therefore from the definition of OPNS, l represents the line determined by p_1 and p_2 since

$$\begin{aligned} q \wedge l &= ((1 - t)p_1 + tp_2) \wedge (p_1 \wedge p_2) \\ &= (1 - t)p_1 \wedge (p_1 \wedge p_2) + tp_2 \wedge (p_1 \wedge p_2) = 0. \end{aligned}$$

Moreover, if the point q is not on the line determined by the points p_1, p_2, then the points q, p_1 and p_2 are affinely independent. Therefore

$$q \wedge p_1 \wedge p_2 \neq 0.$$

♢

Note that the points p_1, p_2 in Theorem 3 can be weighted points, one (but not both) of the points can be a point at infinity, giving, for example, Plücker coordinates, and two points at infinity generate a line at infinity.

Corollary 4. *Given a vector v and a point p in W, the expression $l = p \wedge v$ is the Plücker representation of the line determined by p and v.*

Proof. Since $l = p \wedge v = p \wedge (p + v)$, this result follows immediately from Theorem 3. ♢

Theorem 5. *For any three non-collinear points p_1, p_2, p_3 in W, their outer product $\pi = p_1 \wedge p_2 \wedge p_3$ represents the plane determined by the points p_1, p_2, p_3.*

Theorem 6. *For any four non-collinear points p_1, p_2, p_3, p_4 in W, their outer product $\pi = p_1 \wedge p_2 \wedge p_3 \wedge p_4$ represents the solid sector determined by the points p_1, p_2, p_3, p_4.*

The proofs of Theorem 5 and Theorem 6 are similar to the proof of Theorem 3.

Notice that $p_1 \wedge p_2$ is oriented, so in fact $p_1 \wedge p_2$ represents an oriented line. Moreover, $p_1 \wedge p_2$ also has an associated length, the distance from p_1 to p_2 (see Table 3 in Appendix A). Since $p_1 \wedge p_2$ is free to slide along the associated line

$$p_1 \wedge p_2 = (p_1 + c(p_2 - p_1)) \wedge (p_2 + c(p_2 - p_1))$$

$p_1 \wedge p_2$ is more like a vector than a line. In fact, $p_1 \wedge p_2$ is really a vector attached to the line determined by the points p_1 and p_2. Thus we call $p_1 \wedge p_2$ a *line vector* in contrast to the *free vectors* $v = v_1w_1 + v_2w_2 + v_3w_3$ in W. For example, both w_1 and $w_0 \wedge (w_0 + w_1) = w_0 \wedge w_1$ are unit vectors parallel to the x-axis. But the free vector w_1 is free to roam anywhere in 3-space as long as it remains parallel to the x-axis, whereas the line vector $w_0 \wedge w_1$ is constrained to roam only along the x-axis.

Similarly, $p_1 \wedge p_2 \wedge p_3$ is oriented, so $p_1 \wedge p_2 \wedge p_3$ represents an oriented plane. Moreover, $p_1 \wedge p_2 \wedge p_3$ has an associated area, the area of $\Delta p_1p_2p_3$ (see Table 3). Again $p_1 \wedge p_2 \wedge p_3$ is free to slide along the plane determined by the points p_1, p_2, p_3; thus $p_1 \wedge p_2 \wedge p_3$ is an oriented plane sector, similar to a bivector in the Clifford algebra $\mathbb{R}(3, 0)$.

Finally $p_1 \wedge p_2 \wedge p_3 \wedge p_4$ has an associated orientation and volume, the orientation and the volume of the tetrahedron determined by the four ordered points p_1, p_2, p_3, p_4 (see again Table 3). Thus $p_1 \wedge p_2 \wedge p_3 \wedge p_4$ represents an oriented volume in 3-dimensions.

2.4. Geometry in $\wedge W^*$: Points and Lines

Dually, besides their representation in W as the join of two points, lines can also be represented in W^* as the intersection of two planes. Similarly, points are represented in W^* as the intersection of three planes.

Here then we shall use outer products of planes in W^* to represent lines and points as intersections of planes in $\mathbb{R}^3$. We call the set of all points P such that $P \cdot E = 0$ the *inner product null space* (IPNS) of E. If $P \cdot E = 0$ if and only if $P \in x$ then we say that x is represented by E.

Theorem 7. *For any two distinct planes π_1^* and π_2^* in W^*, their outer product $l^* = \pi_1^* \wedge \pi_2^*$ represents the line determined by the intersection of π_1^* and π_2^*.*

Proof. For any point $q \in W$

$$q \cdot l^* = q \cdot (\pi_1^* \wedge \pi_2^*) = (q \cdot \pi_1^*)\pi_2^* - (q \cdot \pi_2^*)\pi_1^*.$$

Therefore, since π_1^* and π_2^* are linearly independent, it follows that

$$q \cdot l^* = q \cdot (\pi_1^* \wedge \pi_2^*) = 0 \Leftrightarrow q \cdot \pi_1^* = 0, q \cdot \pi_2^* = 0.$$

◊

Theorem 8. *For any three linearly independent planes π_1^*, π_2^*, π_3^* in W^*, their outer product $p^* = \pi_1^* \wedge \pi_2^* \wedge \pi_3^*$ represents the point determined by the intersection of the three planes π_1^*, π_2^*, π_3^*. Note that p^* may be a homogeneous point.*

Proof. The proof is similar to the proof of Theorem 7. ◊

2.5. Quadrics and Dual Quadrics

Using the method of Parkin [18], Goldman and Mann [13] show that the quadric surface represented by the polynomial $F(x_0, x_1, x_2, x_3) = \sum_{i,j=0}^{3} \lambda_{ij} x_i x_j$, where $\lambda_{ij} = \lambda_{ji}$, corresponds to the bivector $b_F = \sum_{i,j=0}^{3} \lambda_{ij} w_i^* \wedge w_j$ in the vector space $W^* \wedge W$.

Indeed for any point p

$$p(p_0, p_1, p_2, p_3) \cdot b_F \cdot p^*(p_0, p_1, p_2, p_3) = \frac{1}{4} F(p_0, p_1, p_2, p_3), \tag{6}$$

where

$$p(p_0, p_1, p_2, p_3) = p_0 w_0 + p_1 w_1 + p_2 w_2 + p_3 w_3$$
$$p^*(p_0, p_1, p_2, p_3) = p_0 w_0^* + p_1 w_1^* + p_2 w_2^* + p_3 w_3^*.$$

Notice that Eq. 6 does not represent an IPNS nor does it represent an OPNS; instead Eq. 6 represents a *double inner product null space* (DIPNS).

Goldman and Mann also observe that the tangent plane to the quadric surface b_F at the point $p(p_0, p_1, p_2, p_3)$ is given by

$$\pi_s^* = b_F \cdot p^*(p_0, p_1, p_2, p_3).$$

Thus the dual quadric [21] is represented by the bivector

$$b_F^* = \sum_{i,j=0}^{3} \lambda_{ij}^* w_i^* \wedge w_j$$

where λ_{ij}^* is the adjoint of λ_{ij}. A plane π^* lies on the dual quadric if and only if

$$\pi(s_0, s_1, s_2, s_3) \cdot b_F^* \cdot \pi^*(s_0, s_1, s_2, s_3) = 0$$

where

$$\pi^*(s_0, s_1, s_2, s_3) = s_0 w_0^* + s_1 w_1^* + s_2 w_2^* + s_3 w_3^*$$

$$\pi(s_0, s_1, s_2, s_3) = s_0 w_0 + s_1 w_1 + s_2 w_2 + s_3 w_3$$

For more details and proofs concerning quadric surfaces in the Clifford algebra $\mathbb{R}(4, 4)$ see [13].

2.6. Conic Sections: The Meet of a Quadric and a Plane

Not only quadric surfaces, but also conic curves play an important role in applications. Next we show how to represent conic curves in $\mathbb{R}(4, 4)$ as the intersection of a quadric surface with a plane—that is, the meet of a quadric and a plane. This intersection is an implicit way to represent conic curves in 3-dimensions.

Theorem 9. *For any quadric surface b_F in $W^* \wedge W$ and any plane π_s^* in W^*, the outer product $C = \pi_s \wedge b_F \wedge \pi_s^*$ represents the conic curve given by the intersection of the quadric surface and the plane. A point p in W lies on the conic curve C if and only if $p \cdot C \cdot p^* = p \cdot (\pi_s \wedge b_F \wedge \pi_s^*) \cdot p^* = 0$.*

Proof. Since $p \cdot \pi_s = p^* \cdot \pi_s^* = 0$, it follows from Eq. 1 that the test equation $p \cdot C \cdot p^* = p \cdot (\pi_s \wedge b_F \wedge \pi_s^*) \cdot p^*$ can be written as

$$\begin{aligned} & p \cdot (\pi_s \wedge b_F \wedge \pi_s^*) \cdot p^* \\ &= [(p \cdot \pi_s) \wedge b_F \wedge \pi_s^* - \pi_s \wedge (p \cdot b_F) \wedge \pi_s^* + \pi_s \wedge b_F \wedge (p \cdot \pi_s^*)] \cdot p^* \\ &= (p \cdot b_F \cdot p^*)(\pi_s \wedge \pi_s^*) - (p \cdot b_F) \wedge (\pi_s \cdot p^*) \wedge \pi_s^* + (p \cdot b_F) \wedge \pi_s \wedge (\pi_s^* \cdot p^*) \\ &\quad -(p \cdot \pi_s^*) \wedge (\pi_s \cdot p^*) \wedge b_F + (p \cdot \pi_s^*) \wedge \pi_s \wedge (b_F \cdot p^*) \\ &= -(\pi_s \cdot p^*) \wedge (p \cdot b_F) \wedge \pi_s^* + \pi_s \wedge (p \cdot b_F \cdot p^*) \wedge \pi_s^* \\ &\quad -(\pi_s \cdot p^*) \wedge b_F \wedge (p \cdot \pi_s^*) + \pi_s \wedge (b_F \cdot p^*) \wedge (p \cdot \pi_s^*) \\ &= (p \cdot b_F \cdot p^*)(\pi_s \wedge \pi_s^*) \\ &\quad -(\pi_s \cdot p^*)((p \cdot b_F) \wedge \pi_s^* - \pi_s \wedge (b_F \cdot p^*) + b_F \wedge (p \cdot \pi_s^*)). \end{aligned} \tag{7}$$

Since $\pi_s \cdot p^* = \pi_s^* \cdot p$, it follows that

$$p \cdot \pi_s^* = 0,\ p \cdot b_F \cdot p^* = 0 \ \Rightarrow\ p \cdot (\pi_s \wedge b_F \wedge \pi_s^*) \cdot p^* = 0.$$

To prove the converse, wedge both sides of Eq. 7 with π_s on the left and by π_s^* on the right. Then we find that

$$\begin{aligned} & \pi_s \wedge [p \cdot (\pi_s \wedge b_F \wedge \pi_s^*) \cdot p^*] \wedge \pi_s^* \\ &= \pi_s \wedge [(p \cdot b_F \cdot p^*)(\pi_s \wedge \pi_s^*) - (\pi_s \cdot p^*)((p \cdot b_F) \wedge \pi_s^* \\ &\quad -\pi_s \wedge (b_F \cdot p^*) + b_F \wedge (p \cdot \pi_s^*))] \wedge \pi_s^* \\ &= -(\pi_s \cdot p^*)(p \cdot \pi_s^*)(\pi_s \wedge b_F \wedge \pi_s^*). \end{aligned}$$

Hence

$$\begin{aligned} p \cdot (\pi_s \wedge b_F \wedge \pi_s^*) \cdot p^* = 0 &\Rightarrow \pi_s \wedge [p \cdot (\pi_s \wedge b_F \wedge \pi_s^*) \cdot p^*] \wedge \pi_s^* = 0 \\ &\Rightarrow (\pi_s \cdot p^*)(p \cdot \pi_s^*)(\pi_s \wedge b_F \wedge \pi_s^*) = 0 \\ &\Rightarrow p \cdot \pi_s^* = 0. \end{aligned}$$

When $p \cdot \pi_s^* = 0$ Eq. 7 reduces to

$$p \cdot (\pi_s \wedge b_F \wedge \pi_s^*) \cdot p^* = (p \cdot b_F \cdot p^*)(\pi_s \wedge \pi_s^*).$$

Therefore

$$p \cdot (\pi_s \wedge b_F \wedge \pi_s^*) \cdot p^* = 0 \Rightarrow p \cdot b_F \cdot p^* = 0.$$

Thus, $p \cdot (\pi_s \wedge b_F \wedge \pi_s^*) \cdot p^* = 0 \Leftrightarrow p \cdot \pi_s^* = 0, p \cdot b_F \cdot p^* = 0$, so any conic curve can be represented in $W \wedge W^* \wedge W \wedge W^*$ by the intersection of a quadric surface with a plane. ♢

2.7. Duality

As we have seen in Sects. 2.3 and 2.4, there are two ways to represent points, lines, and planes: either as joins of points in $\wedge W$ or as intersections of planes in $\wedge W^*$. In this section we investigate duality between the geometry in the spaces $\wedge W$ and $\wedge W^*$.

This duality is mediated by the pseudo-scalars I in $\wedge W$ and I^* in $\wedge W^*$:

$$\begin{aligned} I &= w_0 \wedge w_1 \wedge w_2 \wedge w_3 \\ I^* &= w_0^* \wedge w_1^* \wedge w_2^* \wedge w_3^*. \end{aligned}$$

The inner product of a point or a line or a plane in W with I^* introduces a factor of $(1/2)^{d+1}$, where d is the dimension of the object in W (e.g., $d = 0$ for points, $d = 1$ for lines, and $d = 2$ for planes). Similarly, taking the inner product of an object in W^* with I introduces a factor of $(1/2)^{3-d}$. Since we would like $f = dual(dual(f))$, we define the dual as

$$dual(f) = 2^{dim(f)+1} f \cdot I^* \quad \text{for} \quad f \in \wedge W$$
$$dual(f^*) = 2^{3-dim(f^*)} f^* \cdot I \quad \text{for} \quad f^* \in \wedge W^*.$$

To shorten and simplify our notation, we shall also write $^*x$ to denote the dual of x.

The following three lemmas give some insight into the algebraic nature of duality in terms of the dot product. We shall use these lemmas to prove that an object and its dual represent the same geometry. We give the proof of Lemma 11; the proofs of Lemmas 10 and 12 are similar to the proof of Lemma 11.

Lemma 10. *Let* $^*p = 2(p \cdot I^*)$*, where* p *is a point in* W*. Then* $p \cdot {}^*p = 0$.

Lemma 11. *Let* $^*l = 4(l \cdot I^*)$*, where* l *is a line in* $\wedge W$*. Then* $l \cdot {}^*l = 0$.

Lemma 12. *Let* $^*\pi = 8(\pi \cdot I^*)$*, where* π *is a plane in* $\wedge W$*. Then* $\pi \cdot {}^*\pi = 0$.

Proof of Lemma 11. Suppose that l is the line segment along the x-axis from w_0 to $w_0 + w_1$. Then

$$l = w_0 \wedge (w_0 + w_1) = w_0 \wedge w_1.$$

In this case

$$^*l = 4(l \cdot I^*) = (w_2^* \wedge w_3^*).$$

Therefore by Eq. 4, $l \cdot {}^*l = 0$.

Now for any arbitrary line segment l', we can always apply an affine transformation A consisting of translation, rotation, and scaling to map the line segment l along the x-axis to the arbitrary line segment l'. Therefore the general result follows, since A is an inner automorphism [13]. ♢

Theorem 13. *The line* l *in* $\wedge W$ *and the dual line* $^*l = 4(l \cdot I^*)$ *in* $\wedge W^*$ *represent the same line.*

Proof. From Lemma 11, $l \cdot {}^*l = 0$. Hence for any point p

$$^*l \cdot (p \wedge l) = ({}^*l \cdot p) \wedge l - ({}^*l \cdot l) \wedge p = ({}^*l \cdot p) \wedge l.$$

Therefore

$$p \wedge l = 0 \Leftrightarrow p \cdot {}^*l = 0.$$

♢

Example 14. Consider the line $l = w_0 \wedge w_2$ in $\wedge W$, which represents the y-axis. The expression $^*l = w_1^* \wedge w_3^*$ in $\wedge W^*$ also represents the y-axis. The

Table 1. The geometric entities in $\mathbb{R}(4,4)$

Entity	$\wedge W$ space	$\wedge W^*$ space
Point	$p = p_0 w_0 + p_1 w_1 + p_2 w_2 + p_3 w_3$	${}^*p = \pi_1^* \wedge \pi_2^* \wedge \pi_3^*$
Line	$l = p_1 \wedge p_2$	${}^*l = \pi_1^* \wedge \pi_2^*$
Plane	$\pi = p_1 \wedge p_2 \wedge p_3$	${}^*\pi = s_0 w_0^* + s_1 w_1^* + s_2 w_2^* + s_3 w_3^*$
Quadric surface	$b_F = \sum_{i,j=0}^{3} \lambda_{ij} w_i^* \wedge w_j$	
Conic curve in plane π	$C = \pi \wedge b_F \wedge \pi^*$	

lines l and ${}^*l$ are dual; they represent same line in two different ways and the inner product with the pseudo-scalars I and I^* mediates this duality:

$$\begin{aligned} 4(w_0 \wedge w_2) \cdot I^* &= 4(w_0 \wedge w_2) \cdot (w_0^* \wedge w_1^* \wedge w_2^* \wedge w_3^*) \\ &= 4w_0 \cdot (w_2 \cdot (w_0^* \wedge w_1^* \wedge w_2^* \wedge w_3^*)) = 2w_0 \cdot (w_0^* \wedge w_1^* \wedge w_3^*) \\ &= w_1^* \wedge w_3^*. \end{aligned}$$

Similarly

$$\begin{aligned} 4(w_1^* \wedge w_3^*) \cdot I &= 4(w_1^* \wedge w_3^*) \cdot (w_0 \wedge w_1 \wedge w_2 \wedge w_3) \\ &= 4w_1^* \cdot (w_3^* \cdot (w_0 \wedge w_1 \wedge w_2 \wedge w_3)) = -2w_1^* \cdot (w_0 \wedge w_1 \wedge w_2) \\ &= w_0 \wedge w_2. \end{aligned}$$

The fact that there are two different sets of line coordinates, one in terms of points and another in terms planes was known already to 19th century mathematicians; see [15, pp. 82, 83].

The proofs of the following duality results for points and planes are similar to the proof of duality for lines in Theorem 13.

Theorem 15. *The point p in $\wedge W$ and the dual point ${}^*p = 2(p \cdot I^*)$ in $\wedge W^*$ represent the same point.*

Theorem 16. *The plane π in $\wedge W$ and the dual plane ${}^*\pi = 8(\pi \cdot I^*)$ in $\wedge W^*$ represent the same plane.*

We list the geometric entities of $\mathbb{R}(4,4)$ including points, lines, planes, quadric surfaces, and conic sections in Table 1. Several of these elements—points, lines and planes—have two different algebraic representations: the representation in $\wedge W$ and the representation in $\wedge W^*$.

3. Intersections in $\mathbb{R}(4,4)$

Intersections between geometric elements are widely used in computer graphics and geometric modeling. For example, ray tracing a surface requires computing the intersection points of an arbitrary line with the surface as well as calculating the corresponding parameter values along the line of these intersection points. In this section, we show how to compute intersections of lines with planes, planes with planes, and lines with quadric surfaces.

3.1. Line-Plane and Plane-Plane Intersections

We start by intersecting lines with planes and planes with planes. There are two different ways to represent lines: one as the join of two points, the other as the intersection of two planes [15]. By representing one object in primal form and the other in dual form, we can compute their intersection using the inner product.

Theorem 17. *Consider a line l and a plane π^*. Then $l \cdot \pi^* = 0$ if and only if l lies in π^*. Otherwise $l \cdot \pi^*$ is the (possibly weighted) point of intersection of the line l and the plane π^*. Note that if l is parallel to π^* then $l \cdot \pi^*$ is the direction of l, i.e., a point at infinity.*

Proof. Let p_1 and p_2 be two distinct points and let $l = p_1 \wedge p_2$.

- If $l \cdot \pi^* = 0$, then

$$0 = l \cdot \pi^* = (p_1 \wedge p_2) \cdot \pi^* = (\pi^* \cdot p_2)p_1 - (\pi^* \cdot p_1)p_2.$$

  Thus, $l \cdot \pi^*$ is a (possibly weighted) point, and $l \cdot \pi^* = 0 \Leftrightarrow p_1 \cdot \pi^* = 0$, $p_2 \cdot \pi^* = 0$, which means that the points p_1 and p_2 lie on the plane π^*. Therefore the line l is also on the plane π^*.
- If $l \cdot \pi^* \neq 0$, then

$$\begin{aligned}(l \cdot \pi^*) \wedge l &= \big((p_1 \wedge p_2) \cdot \pi^*\big) \wedge (p_1 \wedge p_2)\\ &= ((p_1 \cdot \pi^*)p_2) \wedge (p_1 \wedge p_2) - ((p_2 \cdot \pi^*)p_1) \wedge (p_1 \wedge p_2)\\ &= 0.\end{aligned}$$

  Hence by the definition of OPNS, the point $l \cdot \pi^*$ lies on the line l. Moreover

$$\begin{aligned}(l \cdot \pi^*) \cdot \pi^* &= \big((p_1 \wedge p_2) \cdot \pi^*\big) \cdot \pi^*\\ &= (p_1 \cdot \pi^*)(p_2 \cdot \pi^*) - (p_2 \cdot \pi^*)(p_1 \cdot \pi^*) = 0.\end{aligned}$$

  Hence from the definition of IPNS, the point $l \cdot \pi^*$ is on the plane π^*. Therefore, the point $l \cdot \pi^*$ is the intersection of the line l and the plane π^*.
- If the line l is parallel to the plane π^*, then $\pi^* \cdot (p_2 - p_1) = 0$. Hence

$$l \cdot \pi^* = (p_1 \wedge p_2) \cdot \pi^* = (\pi^* \cdot p_2)p_1 - (\pi^* \cdot p_1)p_2 = (\pi^* \cdot p_1)(p_1 - p_2)$$

  Therefore, in this case, $l \cdot \pi^*$ is the point at infinity in the direction parallel to l.

$\diamondsuit$

Theorem 18. *Consider a plane π in $\wedge W$ and a plane π^* in $\wedge W^*$. Then $\pi \cdot \pi^* = 0$ if and only if the planes are the same plane. Otherwise $\pi \cdot \pi^*$ is the line of intersection of π and π^*.*

Proof. The proof is similar to the proof of Theorem 17.

3.2. Line-Quadric Intersections

Our goal is to find the intersection of any line with an arbitrary quadric surface. We begin with the special case where the line is the z-axis. Later we shall show how to reduce all other cases to this special case. In this section, we will use $\Gamma = w_0 \wedge w_1 \wedge w_2 \wedge w_3 \wedge w_0^* \wedge w_1^* \wedge w_2^* \wedge w_3^*$ to represent the pseudo-scalar of $\mathbb{R}(4,4)$.

Lemma 19. *Let b_F be a bivector representing a quadric surface and let $l^* = w_2^* \wedge w_1^*$ represent the z-axis with $l = (l^*)^* = w_1 \wedge w_2$. Then*

$$P_p \equiv (l^* \wedge b_F \wedge l) \cdot \Gamma = c(p_1 \wedge p_2^* + p_2 \wedge p_1^*),$$

where c is a scalar and p_1, p_2 are the intersection points of the quadric b_F and the line l^* in complex projective space.

Proof. For any quadric surface $b_F = \sum_{i,i=0}^{3} \lambda_{ij} w_i^* \wedge w_j$,

$$\begin{aligned} P_p &= (l^* \wedge b_F \wedge l) \cdot \Gamma = (w_2^* \wedge w_1^* \wedge b_F \wedge w_1 \wedge w_2) \cdot \Gamma \\ &= (\lambda_{00} w_0 \wedge w_1 \wedge w_2 \wedge w_0^* \wedge w_1^* \wedge w_2^* + \lambda_{03} w_1 \wedge w_2 \wedge w_3 \wedge w_0^* \wedge w_1^* \wedge w_2^* \\ &\quad + \lambda_{30} w_0 \wedge w_1 \wedge w_2 \wedge w_1^* \wedge w_2^* \wedge w_3^* + \lambda_{33} w_1 \wedge w_2 \wedge w_3 \wedge w_1^* \wedge w_2^* \wedge w_3^*) \cdot \Gamma \\ &= -\frac{1}{64}(\lambda_{00} w_3 \wedge {w_3}^* - \lambda_{30} w_0 \wedge {w_3}^* - \lambda_{03} w_3 \wedge {w_0}^* + \lambda_{33} w_0 \wedge {w_0}^*). \end{aligned} \tag{8}$$

Now consider the polynomial

$$F(z) = \lambda_{33} z^2 + \lambda_{30} z + \lambda_{03} z + \lambda_{00}.$$

Let a, b be the (possibly complex) roots of $F(z) = 0$ and set $p_1 = w_0 + a w_3$ and $p_2 = w_0 + b w_3$. Expanding $p_1 \wedge p_2^* + p_2 \wedge p_1^*$ we find that

$$p_1 \wedge p_2^* + p_2 \wedge p_1^* = 2ab\, w_3 \wedge w_3^* + (a+b) w_0 \wedge w_3^* + (a+b) w_3 \wedge w_0^* + 2 w_0 \wedge w_0^*.$$

Now compare this result to Eq. 8 using the formulas for the sum and product of the roots of $F(z)$ ($a + b = -\frac{\lambda_{30}+\lambda_{03}}{\lambda_{33}}$ and $ab = \frac{\lambda_{00}}{\lambda_{33}}$). Recalling that $\lambda_{30} = \lambda_{03}$, we observe that $P_p = -\frac{\lambda_{33}}{128}(p_1 \wedge p_2^* + p_2 \wedge p_1^*)$. From the definition of p_1 and p_2, it follows easily that $p_1 \cdot w_1^* = p_1 \cdot w_2^* = p_2 \cdot w_1^* = p_2 \cdot w_2^* = 0$. Moreover, since $F(a) = F(b) = 0$,

$$\begin{aligned} p_1 \cdot b_F \cdot p_1^* &= \lambda_{00} - a\lambda_{03} - a\lambda_{30} + a^2 \lambda_{33} = 0 \\ p_2 \cdot b_F \cdot p_2^* &= \lambda_{00} - b\lambda_{03} - b\lambda_{30} + b^2 \lambda_{33} = 0 \end{aligned}$$

Thus, p_1 and p_2 are two points lying on the quadric surface b_F and the line $l^* = w_2^* \wedge w_1^*$ in complex projective space. ♢

Remark 20. In real affine space, which is the primary concern of computer graphics, Lemma 19 can be expanded into four cases:

1. $\lambda_{33} \neq 0$ and $\lambda_{30}^2 - \lambda_{00}\lambda_{33} > 0$. In this case, the roots a, b of $F(z)$ are real numbers and the line and the quadric intersect in two points.
2. $\lambda_{33} \neq 0$ and $\lambda_{30}^2 - \lambda_{00}\lambda_{33} < 0$. In this case, the roots a, b of $F(z)$ are complex numbers and the line and the quadric do not intersect.
3. $\lambda_{33} \neq 0$ and $\lambda_{30}^2 - \lambda_{00}\lambda_{33} = 0$. In this case, the roots a, b of $F(z)$ are equal and the line and the quadric are tangent.

4. $\lambda_{33} = 0$. In this case, $F(z)$ is linear and one of the two intersection points of the line and the quadric lies at infinity.

From Lemma 19, we can extract the intersection points between a line and a quadric in the following fashion.

Theorem 21. *Let* $P_p = c(p_1 \wedge p_2^* + p_2 \wedge p_1^*)$.

- *If* $(P_p \cdot w_0) \cdot (P_p \cdot w_0^*) = 0$, *then* p_1 *and* p_2 *are two points at infinity.*
- *If* $(P_p \cdot w_0) \cdot (P_p \cdot w_0^*) \neq 0$ *but* $P_p \cdot (w_0 \wedge w_0^*) = 0$, *then* p_1 *is a point at infinity, with*

$$p_1 = 2P_p \cdot w_0 = u (\text{for } u \in W)$$

and

$$p_2 = \frac{p_1 \cdot P_p - \frac{p_1 \cdot P_p \cdot p_1^*}{2 p_1 \cdot p_1^*} p_1}{p_1 \cdot p_1^*}.$$

- *If* $(P_p \cdot w_0) \cdot (P_p \cdot w_0^*) \neq 0$ *and* $P_p \cdot (w_0 \wedge w_0^*) \neq 0$, *then*

$$\begin{aligned} p_{1,2} = P_p \cdot w_0 \pm \Big(& \sqrt{(w_1^* \wedge w_1) \cdot A}\, w_1 + \\ & (-1)^{r_1} \sqrt{(w_2^* \wedge w_2) \cdot A}\, w_2 + (-1)^{r_2} \sqrt{(w_3^* \wedge w_3) \cdot A}\, w_3 \Big) \end{aligned} \tag{9}$$

where $A = 4w_0 \cdot (P_p \wedge (w_0^* \cdot P_p))$, *and*

$$r_1 = \begin{cases} 0 & \text{if } (w_1 \wedge w_2^*) \cdot A \geq 0 \\ 1 & \text{if } (w_1 \wedge w_2^*) \cdot A < 0 \end{cases} \quad r_2 = \begin{cases} 0 & \text{if } (w_1 \wedge w_3^*) \cdot A \geq 0 \\ 1 & \text{if } (w_1 \wedge w_3^*) \cdot A < 0 \end{cases}$$

- *If* $(w_1^* \wedge w_1) \cdot A < 0$ *or* $(w_2^* \wedge w_2) \cdot A < 0$ *or* $(w_3^* \wedge w_3) \cdot A < 0$, *then* p_1, p_2 *are not real points.*

Proof. We know $p_1 = w_0 + u$ or $p_1 = u$, and $p_2 = w_0 + v$ or $p_2 = v$, for $u, v \in W$. So

$$\begin{aligned} P_p &= (w_0 + u) \wedge (w_0^* + v^*) + (w_0 + v) \wedge (w_0^* + u^*) \\ &= 2w_0 \wedge w_0^* + w_0 \wedge (u^* + v^*) + (u + v) \wedge w_0^* + u \wedge v^* + v \wedge u^* \end{aligned}$$

with some terms missing from this expression if $p_1 = u$ and/or $p_2 = v$. Expanding the expressions in the conditions in each of the three bullets of this theorem shows that

- $(P_p \cdot w_0) \cdot (P_p \cdot w_0^*) = 0$ if and only if P_p has no $w_0 \wedge (u^* + v^*)$ term (meaning that both p_1 and p_2 are vectors rather than finite points);
- $(P_p \cdot w_0) \cdot (P_p \cdot w_0^*) \neq 0$ and $P_p \cdot (w_0 \wedge w_0^*) = 0$ if and only if P_p has a non-zero $w_0 \wedge (u^* + v^*)$ term but does not have a $w_0 \wedge w_0^*$ term, meaning that either p_1 or p_2 is a vector (i.e., a point at infinity) and the other is a finite point; and
- $(P_p \cdot w_0) \cdot (P_p \cdot w_0^*) \neq 0$ and $P_p \cdot (w_0 \wedge w_0^*) \neq 0$ if and only if P_p has both a $w_0 \wedge w_0^*$ term and a $w_0 \wedge (u^* + v^*)$ term (meaning that both p_1 and p_2 are finite points).

Expanding $P_p = c(p_1 \wedge p_2^* + p_2 \wedge p_1^*)$ in the equations for p_1, p_2 in the statement of this theorem completes the proof. ♢

Remark 22. In Eq. 9, $P_p \cdot w_0 = (p_1 + p_2)/2$, while the remaining portion of Eq. 9 is a vector offset from this midpoint of $\overline{p_1 p_2}$ to the points p_1 and p_2.

To intersect an arbitrary line l with a quadric surface b_F, we have two choices. First, we could proceed in the following fashion:

1. Find the rigid motion $\mathbb{R}$ that maps the line l to the z-axis.
2. Apply the transformation $\mathbb{R}$ to the quadric surface b_F using the techniques in [13].
3. Intersect this transformed quadric surface with the z-axis, using Lemma 19 and Theorem 21.
4. Transform the points found in step 3 by the inverse of the transformation $\mathbb{R}$ found in step 1.

This approach—transforming the problem to a canonical location where the problem is easy to solve—is a standard technique in computer graphics. Alternatively, we can find these intersection points without performing any transformations using the following theorem.

Theorem 23. *Let b_F be a bivector representing a quadric surface and let $l^* = \pi_2^* \wedge \pi_1^*$ represent a line with $l = (l^*)^* = \pi_1 \wedge \pi_2$. Then $(l^* \wedge b_F \wedge l) \cdot \Gamma = c(p_1 \wedge p_2^* + p_2 \wedge p_1^*)$, where c is a scalar and p_1, p_2 are the intersection points of the quadric b_F and the line l^*.*

Proof. By an affine transformation A, any arbitrary planes π_1^* and π_2^* can be mapped to the planes w_1^* and w_2^* . In this way, the general case of line-quadric intersection can be reduced to the special case in Lemma 19 where the result is valid. Now this general result follows by applying the affine transformation A^{-1} to the result in Lemma 19 and recalling that both A and A^{-1} are inner and outer automorphisms [13]. $\diamondsuit$

Using Theorem 23 together with Theorem 21 we can find the intersection points of an arbitrary line and a quadric surface without performing any transformations.

4. Barycentric Coordinates

Barycentric coordinates are used extensively in computer graphics. For example, barycentric coordinates for triangles appear in both ray tracing and scan line algorithms such as Gouraud and Phong shading [2], to compute colors or normals inside a triangle by interpolating colors or normals given at the triangle vertices. Here we show how to apply formulas from Sect. 2.3 to compute barycentric coordinates in $\mathbb{R}(4, 4)$.

Barycentric coordinates represent a given point as an affine combination of related points. A point p on a line can be represented as an affine combination of the end points of a segment $\overline{p_1 p_2}$ along the line:

$$p = b_1\, p_1 + b_2\, p_2, \quad b_1 + b_2 = 1.$$

Similarly, a point p in a plane can be represented as an affine combination of the vertices of a triangle $\triangle p_1 p_2 p_3$ in the same plane:

$$p = b_1\, p_1 + b_2\, p_2 + b_3\, p_3, \quad b_1 + b_2 + b_3 = 1.$$

Finally, a point p in 3-dimensions can be represented as an affine combination of the vertices of a tetrahedron $\triangle p_1 p_2 p_3 p_4$:

$$p = b_1\, p_1 + b_2\, p_2 + b_3\, p_3 + b_4\, p_4, \quad b_1 + b_2 + b_3 + b_4 = 1.$$

Coefficients in these affine combinations are called *barycentric coordinates* and are the ratios of lengths, areas, and volumes in 1, 2, and 3-dimensions. In Sect. 2.3, we showed that in $\mathbb{R}(4,4)$ line segments can be represented by the outer product of two distinct points; triangles by the outer product of three non-collinear points; and tetrahedra by the outer product of four non-coplanar points. Here we shall show that the ratios of such outer products can be used to represent barycentric coordinates.

Theorem 24. *Consider a point p that lies along the same line as two distinct points p_1, p_2. Then for $ij \in \{12, 21\}$ the ith barycentric coordinate b_i of p relative to the points p_1, p_2 is given by*

$$b_i = \frac{(-1)^{i+1}(p \wedge p_j) \cdot (p_1 \wedge p_2)^*}{(p_1 \wedge p_2) \cdot (p_1 \wedge p_2)^*}.$$

Proof. Since p lies in the line determined by the points p_1, p_2, we have $p = b_1\, p_1 + b_2\, p_2$. Therefore for $j \neq i$

$$\frac{(-1)^{i+1}(p \wedge p_j) \cdot (p_1 \wedge p_2)^*}{(p_1 \wedge p_2) \cdot (p_1 \wedge p_2)^*} = \frac{(-1)^{i+1}(b_1\, p_1 + b_2\, p_2) \wedge p_j) \cdot (p_1 \wedge p_2)^*}{(p_1 \wedge p_2) \cdot (p_1 \wedge p_2)^*} = b_i.$$

◊

The proofs of the following theorems for barycentric coordinates in planes and in 3-dimensional space are similar to the proof of Theorem 24.

Theorem 25. *Consider a point p that lies in the same plane as three non-collinear points p_1, p_2, p_3. Then the ith barycentric coordinate b_i of p relative to $\triangle p_1 p_2 p_3$ is given by*

$$b_i = \frac{(p \wedge p_j \wedge p_k) \cdot (p_1 \wedge p_2 \wedge p_3)^*}{(p_1 \wedge p_2 \wedge p_3) \cdot (p_1 \wedge p_2 \wedge p_3)^*},$$

for $ijk \in \{123, 231, 312\}$.

Theorem 26. *Consider a point p that lies in 3-dimensional space and let p_1, p_2, p_3, p_4 be four non-coplanar points. Then the ith barycentric coordinate b_i of p relative to the tetrahedron $\triangle p_1 p_2 p_3 p_4$ is given by*

$$b_i = \frac{(p \wedge p_j \wedge p_k \wedge p_\ell) \cdot (p_1 \wedge p_2 \wedge p_3 \wedge p_4)^*}{(p_1 \wedge p_2 \wedge p_3 \wedge p_4) \cdot (p_1 \wedge p_2 \wedge p_3 \wedge p_4)^*},$$

for $ijk\ell \in \{1234, 2314, 3412, 4132\}$.

5. Conclusions

In this paper, we have shown how to represent lines, planes, quadric surfaces, and conic sections in $\mathbb{R}(4,4)$. Together with our earlier work on affine and projective transformations [13], we have shown that $\mathbb{R}(4,4)$ can model all the standard geometric entities and projective transformations typically represented in computer graphics by matrix methods. As derivations of metric formulas for flats in $\mathbb{R}(4,4)$ are essentially the same as the derivations of metric formulas for flats in other Clifford algebras, we have omitted these derivations from the main body of this paper; however, since the metric formulas in $\mathbb{R}(4,4)$ differ somewhat from those metric formulas in other Clifford algebras, we have collected these metric formulas for flats (including formulas for lengths, areas, volumes, distances, and angles) in Tables 3, 4, and 5. Table 2 contains some useful identities, which can be used to derive the formulas in Tables 3, 4, and 5. We have also included two appendices: Appendix A verifies that an object and its dual have the same metric properties; Appendix B focuses on deriving distances and angles between a pair of lines or a pair of planes, which are a bit more complicated in $\mathbb{R}(4,4)$ than in other Clifford algebras.

The Clifford algebra $\mathbb{R}(4,4)$ has several advantages over other potential Clifford algebras for computer graphics. The versors and rotors in $\mathbb{R}(4,4)$ can model all the affine and projective transformations needed for computer graphics, including shears and non-uniform scaling, which are not readily available in other models of Clifford algebra. Moreover, as we have shown, conic sections and quadric surfaces are also readily modeled in $\mathbb{R}(4,4)$, but are not so readily available in other Clifford algebras. Additionally, in $\mathbb{R}(4,4)$ primal objects and their duals live in a different spaces, which leads to greater clarity and is often an advantage in software implementations.

The main disadvantage of $\mathbb{R}(4,4)$ is its enormous size: $2^8 = 256$ dimensions seems way out of proportion to what is needed for 3-dimensional computer graphics, and indeed many of the multi-vectors of $\mathbb{R}(4,4)$ appear to have no clear geometric interpretation. Whether these additional multi-vectors have useful geometric meanings or whether a smaller Clifford algebra will ultimately prevail remain open questions for future research.

Appendix A: Equivalent Metric Properties of Objects and their Duals

Here we verify that a line and its dual have the same length and that a plane and its dual have the same area.

Theorem 27. *Let l represent a line in $\wedge W$ with dual representation $^*l = 4(l \cdot I^*)$ in $\wedge W^*$. Then the lengths*

$$\|l\| = \|^*l\| .$$

Table 2. Standard algebraic identities in Clifford algebra

$\|u_1 \wedge ... \wedge u_k\|^2 = 2^k (u_1 \wedge ... \wedge u_k) \cdot (u_k^* \wedge ... \wedge u_1^*)$
$\|u_1^* \wedge ... \wedge u_k^*\|^2 = 2^k (u_1^* \wedge ... \wedge u_k^*) \cdot (u_k \wedge ... \wedge u_1)$
$\|v\|^2 = 2(v \cdot v^*)$
$2(v \cdot u^*) = \|v\| \, \|u\| \cos(\theta)$
$\|n_s^*\|^2 = 2(n_s \cdot n_s^*)$
$2 \, n_1 \cdot n_2^* = \|n_1^*\| \|n_2^*\| \cos(\theta)$
$u \times v = -4(u^* \wedge v^*) \cdot (w_1 \wedge w_2 \wedge w_3)$
$\|u \wedge v\|^2 = -4(u \wedge v) \cdot (v \wedge u)^* = 4(u \wedge v) \cdot (v^* \wedge u^*) = \|u \times v\|^2$
$\|{n_1}^* \wedge {n_2}^*\|^2 = 4(n_1 \wedge n_2) \cdot (n_2^* \wedge n_1^*) = \|n_1^* \times n_2^*\|^2$
$\|w_1 \wedge w_2 \wedge w_3\| = 1$
$\|u \wedge v \wedge w\|^2 = det(u, v, w)^2$

Proof. Let $l = aw_0 \wedge w_1 + bw_0 \wedge w_2 + cw_0 \wedge w_3 + dw_1 \wedge w_2 + ew_1 \wedge w_3 + fw_2 \wedge w_3$. From Table 3, $\|l\|^2 \equiv 8(w_0^* \cdot l) \cdot (l^* \cdot w_0)$; therefore

$$\begin{aligned}\|l\|^2 &\equiv 8(w_0^* \cdot l) \cdot (l^* \cdot w_0) \\ &= 2(aw_1 + bw_2 + cw_3) \cdot (aw_1^* + bw_2^* + cw_3^*) \\ &= a^2 + b^2 + c^2.\end{aligned}$$

Moreover, using Eq. 1, the dual representation of l is

$$\begin{aligned}{}^*l &= 4l \cdot I^* \\ &= -aw_2^* \wedge w_3^* + bw_1^* \wedge w_3^* - cw_1^* \wedge w_2^* - dw_0^* \wedge w_3^* + ew_0^* \wedge w_2^* - fw_0^* \wedge w_1^*.\end{aligned}$$

Since ${}^*l$ is a line, we can find two planes π_1^* and π_2^* in W^* such that their intersection is equal to the dual line—that is, ${}^*l = \pi_1^* \wedge \pi_2^*$. Let n_1^* and n_2^* be the normal vectors to the planes π_1^* and π_2^*. Then

$$\begin{aligned}n_1^* \wedge n_2^* &= 2w_0 \cdot (w_0^* \wedge (d_1 w_0^* + n_1^*) \wedge (d_2 w_0^* + n_2^*)) \\ &= 2w_0 \cdot (w_0^* \wedge \pi_1^* \wedge \pi_2^*).\end{aligned}$$

Therefore by Table 2

$$\begin{aligned}\|n_1^* \wedge n_2^*\|^2 &= 4(n_1 \wedge n_2) \cdot (n_2^* \wedge n_1^*) \\ &= 8(w_0^* \cdot (w_0 \wedge \pi_1 \wedge \pi_2)) \cdot (w_0 \cdot (w_0^* \wedge \pi_2^* \wedge \pi_1^*)) \\ &= a^2 + b^2 + c^2 = \|l\|^2.\end{aligned}$$

From Table 3 it follows that

$$\|{}^*l\| = \|n_1^* \wedge n_2^*\| = \|l\|.$$

◇

Theorem 28. *Let π represent a plane in $\wedge W$ with dual representation ${}^*\pi = 8\pi \cdot I^*$ in $\wedge W^*$. Then the areas*

$$\|\pi\| = \|{}^*\pi\| .$$

Table 3. Formulas for length, area, and volume of the line segment, triangle, and tetrahedron associated to the corresponding blades

Object	Squared length, area, volume
Line segment $l = p_1 \wedge p_2$	$\|\|l\|\|^2 = 8(w_0^* \cdot l) \cdot (l^* \cdot w_0)$
Triangle $\pi = p_1 \wedge p_2 \wedge p_3$	$\|\|\pi\|\|^2 = -4(w_0^* \cdot \pi) \cdot (\pi^* \cdot w_0)$
Tetrahedron $\Delta = p_1 \wedge p_2 \wedge p_3 \wedge p_4$	$\|\|\Delta\|\|^2 = \frac{16}{3}(w_0^* \cdot \Delta) \cdot (\Delta^* \cdot w_0)$
Dual plane	$\\|\pi^*\\| = \frac{1}{2}\\|n^*\\|$
Dual tetrahedron	$\\|l^*\\| = \\|n_1^* \wedge n_2^*\\|$

Proof. Let $\pi = aw_0 \wedge w_1 \wedge w_2 + bw_0 \wedge w_1 \wedge w_3 + cw_0 \wedge w_2 \wedge w_3 + dw_1 \wedge w_2 \wedge w_3$. From Table 3, $\|\pi\|^2 \equiv 4(w_0^* \cdot \pi) \cdot (\pi^* \cdot w_0)$; therefore

$$\|\pi\|^2 = \frac{1}{4}(a^2 + b^2 + c^2).$$

Moreover, using Eq. 1, the dual representation of π is

$${}^*\pi = 8\pi \cdot I^* = dw_0^* - cw_1^* + bw_2^* - aw_3^*.$$

Let n^* be the normal vector to the dual plane ${}^*\pi$. Then $n^* = -cw_1^* + bw_2^* - aw_3^*$. Therefore by Table 2

$$\|n^*\|^2 = a^2 + b^2 + c^2 = 4\|\pi\|^2.$$

From Table 3 it follows that

$$\|{}^*\pi\| = \frac{1}{2}\|n^*\| = \|\pi\|.$$

♢

Appendix B: Measurement in $\mathbb{R}(4,4)$: Distances and Angles

Besides intersections between geometric elements, measurements such as distances and angles between geometric elements are also widely used in many areas of computer graphics, geometric modeling and robotics. For example, to solve the inverse kinematics problem for a simple robot, we need to compute not only the distance between two points or a point and a plane for path planning, but also the angle between two lines or even between two planes to calculate rotation angles.

In $\mathbb{R}(4,4)$, the inner product can be used to measure distances and angles. Table 4 shows the inner product can be used for tasks such as finding the distance between two points, or a point and a line, or a point and a plane. These results are easy to derive. The formula for the distance between two points p_1 and p_2 is similar to the formula for the length of a line segment: both are equal to the norm of the vector from p_1 to p_2. The formula for the distance between a point and a line simply divides the area of the triangle generated by the point and a line segment by the length of the base of the triangle which is half the height of the triangle or equivalently exactly half

TABLE 4. Distance between two points, or a point and a line, or a point and a plane in $\mathbb{R}(4,4)$

Distance between points p_1 and p_2	$dist^2(p_1,p_2) = 2(p_2 - p_1)\cdot(p_2^* - p_1^*)$
Distance between point p and line ℓ	$dist(p,\ell) = \frac{2\|p\wedge\ell\|}{\|\ell\|}$
Signed distance between point p and plane π^*	$sdist(p,\pi^*) = \frac{p\cdot\pi^*}{\|\pi^*\|}$

the distance from the point to the line. Finally the formula for the distance between a point and a plane is simply the projection onto the normal to the plane of the vector from the given point to a point on the plane.

Remark 29. In the formula for the distance between a point p and a plane π^*, the sign of the distance appears because p may be above or below the plane so the direction of the normal vector n^* may point towards or away from the point p.

Two lines in 3-dimensions are either skew, parallel, or intersect. Similarly, two planes in 3-dimensions are either parallel or intersect. Angles between two lines or two planes can be computed using the inner product of the normalized OPNS representation of the lines or planes. To proceed, however, we need to extract some basic geometry from a line l in W.

We shall adopt the following notation: For any line l, we define

$$v(l) = w_0^* \cdot l, \quad p(l) = {}^*({}^*l \wedge (v(l))^*)$$

Proposition 30. *Let l be a line in $\wedge W$. Then $v(l) = w_0^* \cdot l$ is a vector parallel to l.*

Proof. Let $l = q \wedge r$. Then $w_0^* \cdot l = (r-q)/2$. ♢

Proposition 31. *Let l be a line in $\wedge W$. Then $p(l) = {}^*({}^*l \wedge (v(l))^*)$ is the point on the line l closest to w_0.*

Proof. By Proposition 30, the vector $v(l)$ is parallel to the line l. Therefore the plane $(v(l))^*$ is perpendicular the line l and hence to the dual line ${}^*l$, since the normal vector to the plane $(v(l))^*$ is $(v(l))^*$. Thus the dual point ${}^*l \wedge (v(l))^*$ lies on the intersection of the line ${}^*l$ and the plane $(v(l))^*$. Hence the point $p(l) = {}^*({}^*l \wedge (v(l))^*)$ lies on the original line l, since the point $p(l)$ is at the same location as the dual point ${}^*l \wedge (v(l))^*$ and the line l is the same line as the dual line ${}^*l$. ♢

Remark 32. The idea behind Proposition 31 is that the outer product of 3 planes generates a point, so to get a point we need to compute an outer product in $\wedge W^*$ and dualize to get a point in $\wedge W$.

Theorem 33. *Classification of lines. Consider two lines l_1, l_2 in $\wedge W$.*

1. *l_1, l_2 skew $\Leftrightarrow l_1 \wedge l_2 \neq 0$.*
2. *l_1, l_2 parallel $\Leftrightarrow v(l_1) \wedge v(l_2) = 0$.*

Table 5. Distances and angles between two lines and two planes in $\mathbb{R}(4,4)$

Distance between two skew lines l_1, l_2	$dist(l_1, l_2) = \frac{\lvert p(l_2) \cdot {}^*\pi \rvert}{\lVert {}^*\pi \rVert}$ where $\pi = p(l_1) \wedge v(l_1) \wedge v(l_2)$
Distance between two parallel lines l_1, l_2	$dist(l_1, l_2) = dist(p(l_2), l_1)$
Angle θ between oriented lines l_1 and l_2	$\cos(\theta) = \frac{2(w_0^* \cdot l_1)\cdot(w_0 \cdot l_2^*)}{\lVert w_0^* \cdot l_1 \rVert \lVert w_0 \cdot l_2^* \rVert}$
Signed distance between parallel planes π_1, π_2	$sdist(\pi_1, \pi_2) = sdist(p(\pi_2), \pi_1)$
Angle θ between planes π_1^* and π_2^*	$\cos(\theta) = \frac{2(w_0^* \cdot (w_0 \wedge \pi_1))\cdot(w_0 \cdot (w_0^* \wedge \pi_2^*))}{\lVert w_0^* \cdot (w_0 \wedge \pi_1) \rVert \lVert w_0 \cdot (w_0^* \wedge \pi_2^*) \rVert}$

3. l_1, l_2 intersect $\Leftrightarrow l_1 \wedge l_2 = 0$ and $v(l_1) \wedge v(l_2) \neq 0$.

Theorem 34. *The intersection point of two intersecting lines l_1, l_2 in $\wedge W$ is*

$$p(l_1, l_2) = l_1 \cdot w^*,$$

where $w^* = n^* - 2(n^* \cdot p(l_2))w_0^*$ is the plane in W^* with normal vector $n^* = \big(v(l_2) \times \big(v(l_1) \times v(l_2)\big)\big)^*$ satisfying $p(l_2) \cdot w^* = 0$.

Theorem 35. *Let π be a plane $\wedge W$. Then*

1.

$$n^*({}^*\pi) = 4(2\pi \cdot w_0^*) \cdot (w_1^* \wedge w_2^* \wedge w_3^*)$$

is normal to the plane π;

2.

$$p({}^*\pi) = w_0 + \frac{8\,\pi \cdot (w_1^* \wedge w_2^* \wedge w_3^*)}{\lVert n^*({}^*\pi) \rVert^2} n$$

is a point on the plane π.

Table 5 shows distances and angles between two lines and two planes in 3-dimensions using the algebra of $\mathbb{R}(4,4)$; Gunn derives similar, somewhat simpler formulas for $P(R^*(3,0,1))$ [6,7]. These results are easy to derive. The formula for the distance between two skew lines l_1, l_2 is similar to the formula for the distance between a point and a plane: $p(l_2)$ is the point on the line l_2 closest to w_0 (Proposition 31) and $\pi = p(l_1) \wedge v(l_1) \wedge v(l_2)$ is the plane containing l_1 that is parallel to l_2. The formula for the distance between two parallel lines is similar to the formula for the distance between a point and a line. Similarly, the formula for the distance between two parallel planes is similar to the formula for the distance between a point and a plane. The formula for the angle between two oriented lines l_1 and l_2 is similar to the formula for the angle between two vectors (Table 2): $w_0^* \cdot l_1$ is a vector parallel to line l_1 and $w_0^* \cdot l_2$ is a vector parallel to line l_2 (Proposition 30). Finally the formula for the angle between the two planes is similar to the formula for the angle between the two normal vectors to the planes (Table 2).

Remark 36. In the formula for the distance between parallel planes π_1, π_2, the sign of the distance appears because $p(\pi_2)$ may be above or below the plane π_1 depending on whether the normal to π_1 points towards or away from the plane π_2.

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