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General two-sided quaternion Fourier transform, convolution and Mustard convolution

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Abstract. In this paper we use the general two-sided quaternion Fourier transform (QFT), and relate the classical convolution of quaternion-valued signals over $\mathbb{R}^2$ with the Mustard convolution. A Mustard convolution can be expressed in the spectral domain as the point wise product of the QFTs of the factor functions. In full generality do we express the classical convolution of quaternion signals in terms of finite linear combinations of Mustard convolutions, and vice versa the Mustard convolution of quaternion signals in terms of finite linear combinations of classical convolutions.

Keywords. Convolution, Mustard convolution, Two-sided quaternion Fourier transform, Quaternion signals, Spatial domain, Frequency domain.

1. Introduction

The two-sided quaternionic Fourier transformation (QFT) was introduced in [9] for the analysis of 2D linear time-invariant partial-differential systems. Subsequently it has been further studied in [4] and applied in many fields, including color image processing [26], edge detection and image filtering [8,25], watermarking [1], pattern recognition [23,28], quaternionic multiresolution analysis (a generalization of discrete wavelets) [2], speech recognition [3], noise removal from video images [21], and efficient and robust image feature detection [11].

In memory of Mrs. Lucy Baker, †31 December 2015, who worked with Seeds Of Hope Foundation, Mumbai, India. The use of this paper is subject to the *Creative Peace License* [16].

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This led to further theoretical investigations [13,14], where a special split of quaternions was introduced, then called ±split. An interesting physical consequence was that this split resulted in a left and right traveling multivector wave packet analysis, when generalizing the QFT to a full spacetime Fourier transform (SFT). Later [15,17,20] this split has been analyzed further, interpreted geometrically and generalized to a freely steerable split of $\mathbb{H}$ into two orthogonal 2D analysis planes, then appropriately named *orthogonal 2D planes split* (OPS).

A key strength of the classical complex Fourier transform is its easy and fast application to filtering problems. The convolution of a signal with its filter function becomes in the spectral domain a point wise product of the respective Fourier transformations. This is not the case for the convolution of two quaternion-valued signals over $\mathbb{R}^2$, due to non-commutativity. Yet it is possible to define from the point wise product of the QFTs of two quaternion signals a new type of convolution, called Mustard convolution [24]. This Mustard convolution can be expressed in terms of sums of classical convolutions and vice versa. For the left-sided QFT this has recently been carried out in [7]. We expand this work in full generality to the two-sided QFT, making significant and efficient use of the two-sided orthogonal planes split of quaternions.

This paper is organized as follows. Section 2 introduces quaternions and reviews the general orthogonal two-dimensional planes split of quaternions. Section 3 introduces to the general form of the two-sided version of the quaternion Fourier transform, and a related mixed exponential-sine transform. Section 4 introduces the Mustard type convolutions based on the QFT and contains the main results of this paper. That is the formulation of the QFT of the classical convolution of quaternion signals in Theorem 4.1, and specialized to the simpler case of only one transform axis in Corollary 4.2. Moreover, the expression of the classical convolution of quaternion signals in terms of the Mustard type convolutions is given in Theorem 4.3, for only one transform axis in Corollary 4.5, and using only standard Mustard convolutions fully general and explicit in Theorem 4.6. Finally the Mustard convolution is fully expanded in terms of classical convolutions in Theorem 4.7.

2. Quaternions and Their Orthogonal Planes Split

2.1. Gauss, Rodrigues and Hamilton's Quaternion Algebra

Gauss, Rodrigues and Hamilton's four-dimensional (4D) quaternion algebra $\mathbb{H}$ is defined over $\mathbb{R}$ with three imaginary units:

$$\begin{aligned} &\boldsymbol{ij} = -\boldsymbol{ji} = \boldsymbol{k},\ \boldsymbol{jk} = -\boldsymbol{kj} = \boldsymbol{i},\ \boldsymbol{ki} = -\boldsymbol{ik} = \boldsymbol{j}, \\ &\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1. \end{aligned} \tag{2.1}$$

The explicit form of a quaternion $q \in \mathbb{H}$ is

$$q = q_r + q_i\boldsymbol{i} + q_j\boldsymbol{j} + q_k\boldsymbol{k} \in \mathbb{H}, \qquad q_r, q_i, q_j, q_k \in \mathbb{R}. \tag{2.2}$$

We have the isomorphisms $Cl(3,0)^+ \cong Cl(0,2) \cong \mathbb{H}$, i.e. $\mathbb{H}$ is isomorphic to the algebra of rotation operators in $Cl(3,0)$. The *quaternion conjugate* (equivalent to Clifford conjugation in $Cl(3,0)^+$ and $Cl(0,2)$) is defined as

$$\overline{q} = q_r - q_i\boldsymbol{i} - q_j\boldsymbol{j} - q_k\boldsymbol{k}, \quad \overline{pq} = \overline{q}\,\overline{p}, \tag{2.3}$$

which leaves the scalar part q_r unchanged. This leads to the *norm* of $q \in \mathbb{H}$

$$|q| = \sqrt{q\overline{q}} = \sqrt{q_r^2 + q_i^2 + q_j^2 + q_k^2}, \qquad |pq| = |p|\,|q|\,. \tag{2.4}$$

The part $\boldsymbol{q} = \mathbf{V}(q) = V(q) = q - q_r = \frac{1}{2}(q - \overline{q}) = q_i\boldsymbol{i} + q_j\boldsymbol{j} + q_k\boldsymbol{k}$ is called a *pure* quaternion (part), or *vector* part. It squares to the *negative* number $-(q_i^2 + q_j^2 + q_k^2)$. Every unit quaternion $\in \mathbb{S}^3$ (i.e. $|q| = 1$) can be written as:

$$\begin{aligned} q &= q_r + q_i\boldsymbol{i} + q_j\boldsymbol{j} + q_k\boldsymbol{k} = q_r + \sqrt{q_i^2 + q_j^2 + q_k^2}\,\widehat{\boldsymbol{q}} \\ &= \cos\alpha + \widehat{\boldsymbol{q}}\sin\alpha = \exp(\alpha\,\widehat{\boldsymbol{q}}), \end{aligned} \tag{2.5}$$

where

$$\begin{aligned} &\cos\alpha = q_r, \quad \sin\alpha = \sqrt{q_i^2 + q_j^2 + q_k^2}, \\ &\widehat{\boldsymbol{q}} = \boldsymbol{q}/\,|\boldsymbol{q}| = (q_i\boldsymbol{i} + q_j\boldsymbol{j} + q_k\boldsymbol{k})/\sqrt{q_i^2 + q_j^2 + q_k^2}, \end{aligned} \tag{2.6}$$

and

$$\widehat{\boldsymbol{q}}^2 = -1, \quad \widehat{\boldsymbol{q}} \in \mathbb{S}^2. \tag{2.7}$$

The left and right *inverse* of a non-zero quaternion is

$$q^{-1} = \overline{q}/\,|q|^2 = \overline{q}/(q\overline{q}). \tag{2.8}$$

The *scalar part* of a quaternion is defined as

$$\mathrm{S}(q) = q_r = \frac{1}{2}(q + \overline{q}). \tag{2.9}$$

with *symmetries* $\forall p, q \in \mathbb{H}$:

$$\mathrm{S}(pq) = \mathrm{S}(qp) = p_r q_r - p_i q_i - p_j q_j - p_k q_k, \quad \mathrm{S}(q) = \mathrm{S}(\overline{q})\,, \tag{2.10}$$

and *linearity*

$$\mathrm{S}(\alpha p + \beta q) = \alpha\,\mathrm{S}(p) + \beta\,\mathrm{S}(q) = \alpha p_r + \beta q_r, \quad \forall p, q \in \mathbb{H}, \;\; \alpha, \beta \in \mathbb{R}, \tag{2.11}$$

The *commutator* of any two quaternions $p, q \in \mathbb{H}$ is a pure quaternion (because $\mathrm{S}(pq) = \mathrm{S}(qp)$) defined as

$$[p, q] = pq - qp, \tag{2.12}$$

For example,

$$[\boldsymbol{i}, \boldsymbol{j}] = \boldsymbol{ij} - \boldsymbol{ji} = 2\boldsymbol{k}. \tag{2.13}$$

The commutator $[f, g]$ of any two pure unit quaternions f, g gives therefore another pure quaternion (or zero for $f = \alpha g$, $\alpha \in \mathbb{R}$). It appears in the commutation of exponentials, e.g.,

$$e^{\alpha f} e^{\beta g} = e^{\beta g} e^{\alpha f} + [f, g]\sin(\alpha)\sin(\beta), \tag{2.14}$$

which (in the same form for general multivector square roots of -1) has been used in [19] in order to derive a general convolution theorem for Clifford Fourier transformations. We furthermore note the useful anticommutation relationships

$$g[f, g] = -[f, g]g, \quad f[f, g] = -[f, g]f, \tag{2.15}$$

and therefore

$$e^{\alpha f}[f,g] = [f,g]e^{-\alpha f}, \quad e^{\beta g}[f,g] = [f,g]e^{-\beta g}. \tag{2.16}$$

The scalar part and the quaternion conjugate allow the definition of the $\mathbb{R}^4$ *inner product* of two quaternions p, q as

$$p \cdot q = \mathrm{S}(p\overline{q}) = p_r q_r + p_i q_i + p_j q_j + p_k q_k \in \mathbb{R}. \tag{2.17}$$

Accordingly we can interpret the four quaternion coefficients as coordinates in $\mathbb{R}^4$. In this interpretation selecting any two-dimensional plane subspace and its orthogonal complement two-dimensional subspace allows to split four-dimensional quaternions $\mathbb{H}$ into pairs of orthogonal two-dimensional planes (compare Theorem 3.5 of [17]).

Definition 2.1. (*Orthogonality of quaternions*) Two quaternions $p, q \in \mathbb{H}$ are orthogonal $p \perp q$, if and only if $\mathrm{S}(p\overline{q}) = 0$.

We will subsequently adopt the following notation for reflecting the argument of functions ([7], Notation 2.4, page 583)

Notation 2.2. (Argument reflection) *For a function* $h : \mathbb{R}^2 \to \mathbb{H}$ *and a multi-index* $\phi = (\phi_1, \phi_2)$ *with* $\phi_1, \phi_2 \in \{0, 1\}$ *we set*

$$h^{\phi} = h^{(\phi_1,\phi_2)}(\boldsymbol{x}) := h((-1)^{\phi_1}x_1, (-1)^{\phi_2}x_2). \tag{2.18}$$

2.2. General Orthogonal Two-Dimensional Planes Split (OPS)

Assume in the following an arbitrary pair of pure unit quaternions f, g, $f^2 = g^2 = -1$. The *orthogonal 2D planes split (OPS)* is then defined with respect to *any two pure unit quaternions* f, g as

Definition 2.3. (*General orthogonal 2D planes split* [17]) Let $f, g \in \mathbb{H}$ be an arbitrary pair of pure quaternions f, g, $f^2 = g^2 = -1$, including the cases $f = \pm g$. The general OPS is then defined with respect to the two pure unit quaternions f, g as

$$q_{\pm} = \frac{1}{2}(q \pm fqg). \tag{2.19}$$

Note that

$$fqg = q_+ - q_-, \tag{2.20}$$

i.e. under the map $f()g$ the q_+ part is invariant, but the q_- part changes sign.

Both parts are two-dimensional, and span two completely orthogonal planes. For $f \neq \pm g$ the q_+ plane is spanned by two orthogonal quaternions $\{f - g, 1 + fg = -f(f - g)\}$, the q_- plane is e.g. spanned by $\{f + g, 1 - fg = -f(f + g)\}$. For $g = f$ a fully *orthonormal* four-dimensional basis of $\mathbb{H}$ is (R acts as rotation operator (rotor))

$$\{1, f, \boldsymbol{j}', \boldsymbol{k}'\} = R^{-1}\{1, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}R, \quad R = \boldsymbol{i}(\boldsymbol{i} + f), \tag{2.21}$$

and the two orthogonal two-dimensional planes basis:

$$q_+\text{-basis: } \{\boldsymbol{j}', \boldsymbol{k}' = f\boldsymbol{j}'\}, \quad q_-\text{-basis: } \{1, f\}. \tag{2.22}$$

Note the notation for normed vectors in [22] $\{q_1, q_2, q_3, q_4\}$ for the resulting total *orthonormal basis of* $\mathbb{H}$. In (2.22), the q_- part commutes with f and is

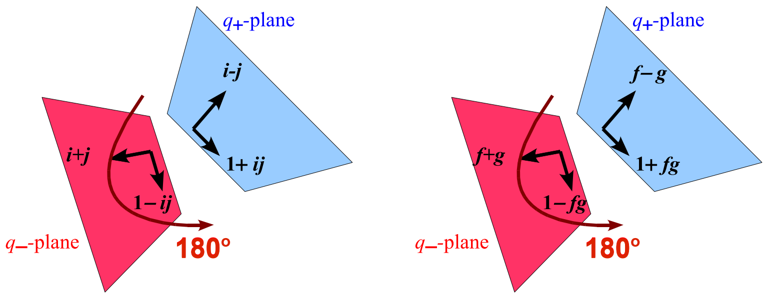

Figure 1. Geometric pictures of the involutions $\boldsymbol{i}(\,)\boldsymbol{j}$ and $f()g$ as half turns

also known as simplex part of q, whereas the q_+ anticommutes with f, and $-q_+\boldsymbol{j}'$ is known as perplex part of q, see [10].

Lemma 2.4. (Orthogonality of two OPS planes [17]) *Given any two quaternions q, p and applying the OPS with respect to any two pure unit quaternions f, g we get zero for the scalar part of the mixed products*

$$Sc(p_+\overline{q}_-) = 0, \qquad Sc(p_-\overline{q}_+) = 0. \tag{2.23}$$

Next we mention the possibility to perform a split along any given set of two (two-dimensional) analysis planes. It has been found, that any two-dimensional plane in $\mathbb{R}^4$ determines in an elementary way an OPS split and vice versa, compare Theorem 3.5 of [17].

Let us turn to the geometric interpretation of the map $f()g$. It *rotates* the q_- plane by 180° around the q_+ axis plane. This is in perfect agreement with Coxeter's notion of *half-turn* [6], see the right side of Fig. 1.

The following *identities* hold

$$e^{\alpha f}q_\pm e^{\beta g} = q_\pm e^{(\beta\mp\alpha)g} = e^{(\alpha\mp\beta)f}q_\pm. \tag{2.24}$$

This leads to a straightforward geometric interpretation of the integrands of the quaternion Fourier transform (QFT or OPS-QFT) with two pure quaternions f, g [17]. Particularly useful cases of (2.24) are $(\alpha, \beta) = (\pi/2, 0)$ and $(0, \pi/2)$:

$$fq_\pm = \mp q_\pm g, \quad q_\pm g = \mp f q_\pm. \tag{2.25}$$

We further note, that with respect to any pure unit quaternion $f \in \mathbb{H}$, $f^2 = -1$, every quaternion $A \in \mathbb{H}$ can be similarly split into *commuting* and *anticommuting* parts [18].

Lemma 2.5. (Commuting and anticommuting with pure unit quaternion [18]) *Every quaternion $A \in \mathbb{H}$ has, with respect to any pure unit quaternion $f \in \mathbb{H}$, $f^2 = -1$, i.e., $f^{-1} = -f$, the unique decomposition denoted by*

$$A_{+f} = \frac{1}{2}(A + f^{-1}Af), \quad A_{-f} = \frac{1}{2}(A - f^{-1}Af)$$
$$A = A_{+f} + A_{-f}, \quad A_{+f}\,f = fA_{+f}, \quad A_{-f}\,f = -fA_{-f}. \tag{2.26}$$

Note, that in Lemma 2.5 the commuting part A_{+f} is also known as simplex part of A, and the anticommuting part is up to a pure quaternion factor the perplex part of A, see [10].

3. The General Two-Sided QFT

Definition 3.1. (*QFT with respect to two pure unit quaternions* f, g [17]) *Let* $f, g \in \mathbb{H}$, $f^2 = g^2 = -1$, *be any two pure unit quaternions. The quaternion Fourier transform (QFT) with respect to* f, g *is*

$$\mathcal{F}\{h\}(\boldsymbol{\omega}) = \mathcal{F}^{f,g}\{h\}(\boldsymbol{\omega}) = \int_{\mathbb{R}^2} e^{-fx_1\omega_1} h(\boldsymbol{x})\, e^{-gx_2\omega_2} d^2\boldsymbol{x}, \tag{3.1}$$

where $h \in L^1(\mathbb{R}^2, \mathbb{H})$, $d^2\boldsymbol{x} = dx_1dx_2$ *and* $\boldsymbol{x}, \boldsymbol{\omega} \in \mathbb{R}^2$.

The QFT can be inverted with

$$h(\boldsymbol{x}) = \mathcal{F}^{-1}\{h\}(\boldsymbol{x}) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{fx_1\omega_1} \mathcal{F}\{h\}(\boldsymbol{\omega})\, e^{gx_2\omega_2} d^2\boldsymbol{\omega}, \tag{3.2}$$

with $d^2\boldsymbol{\omega} = d\omega_1 d\omega_2$.

Remark 3.2. Note, that the general pair of pure unit quaternions f, g in Definition 3.1 includes orthogonal pairs $f \perp g$, non-orthogonal pairs $f \not\perp g$, and parallel pairs $f = \pm g$ (only one transform axis). In the rest of this paper the theorems will be valid for fully general pairs f, g, if not otherwise specified. To avoid clutter we often omit the indication of the pair f, g as in $\mathcal{F} = \mathcal{F}^{f,g}$.

Linearity of the integral (3.1) allows us to use the OPS split $h = h_- + h_+$

$$\begin{aligned}\mathcal{F}^{f,g}\{h\}(\boldsymbol{\omega}) &= \mathcal{F}^{f,g}\{h_-\}(\boldsymbol{\omega}) + \mathcal{F}^{f,g}\{h_+\}(\boldsymbol{\omega}) \\ &= \mathcal{F}_-^{f,g}\{h\}(\boldsymbol{\omega}) + \mathcal{F}_+^{f,g}\{h\}(\boldsymbol{\omega}),\end{aligned} \tag{3.3}$$

since by its construction the operators of the QFT $\mathcal{F}^{f,g}$, and of the OPS with respect to f, g commute. From (2.24) follows

Theorem 3.3. (QFT of $h_\pm$ [17]) *The QFT of the* $h_\pm$ *OPS split parts, with respect to two linearly independent unit quaternions* f, g, *of a quaternion module function* $h \in L^1(\mathbb{R}^2, \mathbb{H})$ *have the quasi-complex forms*

$$\begin{aligned}&\mathcal{F}_\pm^{f,g}\{h\} = \mathcal{F}^{f,g}\{h_\pm\} \\ &= \int_{\mathbb{R}^2} h_\pm e^{-g(x_2\omega_2 \mp x_1\omega_1)} d^2x = \int_{\mathbb{R}^2} e^{-f(x_1\omega_1 \mp x_2\omega_2)} h_\pm d^2x.\end{aligned} \tag{3.4}$$

We further define for later use the following two mixed *exponential-sine* Fourier transforms

$$\mathcal{F}^{f,\pm s}\{h\}(\boldsymbol{\omega}) = \int_{\mathbb{R}^2} e^{-fx_1\omega_1} h(\boldsymbol{x})(\pm 1)\sin(-x_2\omega_2) d^2\boldsymbol{x}, \tag{3.5}$$

$$\mathcal{F}^{\pm s,g}\{h\}(\boldsymbol{\omega}) = \int_{\mathbb{R}^2} (\pm 1)\sin(-x_1\omega_1) h(\boldsymbol{x}) e^{-gx_2\omega_2} d^2\boldsymbol{x}. \tag{3.6}$$

With the help of

$$\sin(-x_1\omega_1) = \frac{f}{2}(e^{-fx_1\omega_1} - e^{fx_1\omega_1}),$$
$$\sin(-x_2\omega_2) = \frac{g}{2}(e^{-gx_2\omega_2} - e^{gx_2\omega_2}), \tag{3.7}$$

we can rewrite the above mixed exponential-sine Fourier transforms in terms of the QFTs of Definition 3.1 as

$$\mathcal{F}^{f,\pm s}\{h\} = \pm\left(\mathcal{F}^{f,g}\left\{h\frac{g}{2}\right\} - \mathcal{F}^{f,-g}\left\{h\frac{g}{2}\right\}\right), \tag{3.8}$$
$$\mathcal{F}^{\pm s,g}\{h\} = \pm\left(\mathcal{F}^{f,g}\left\{\frac{f}{2}h\right\} - \mathcal{F}^{-f,g}\left\{\frac{f}{2}h\right\}\right). \tag{3.9}$$

We further note the following useful relationships using the argument reflection of Notation 2.2

$$\mathcal{F}^{-f,g}\{h\} = \mathcal{F}^{f,g}\{h^{(1,0)}\},$$
$$\mathcal{F}^{f,-g}\{h\} = \mathcal{F}^{f,g}\{h^{(0,1)}\},$$
$$\mathcal{F}^{-f,-g}\{h\} = \mathcal{F}^{f,g}\{h^{(1,1)}\}, \tag{3.10}$$

and similarly

$$\mathcal{F}^{f,-s}\{h\} = \mathcal{F}^{f,s}\{h^{(0,1)}\}, \quad \mathcal{F}^{-s,g}\{h\} = \mathcal{F}^{s,g}\{h^{(1,0)}\}. \tag{3.11}$$

4. Convolution and Mustard Convolution

We define the *convolution* of two quaternion signals $a, b \in L^1(\mathbb{R}^2; \mathbb{H})$ as

$$(a \star b)(\boldsymbol{x}) = \int_{\mathbb{R}^2} a(\boldsymbol{y})b(\boldsymbol{x} - \boldsymbol{y})d^2\boldsymbol{y}, \tag{4.1}$$

provided that the integral exists.

The *Mustard* convolution [5,24] of two quaternion signals $a, b \in L^1(\mathbb{R}^2; \mathbb{H})$ is defined as

$$(a \star_M b)(\boldsymbol{x}) = (\mathcal{F}^{f,g})^{-1}(\mathcal{F}^{f,g}\{a\}\mathcal{F}^{f,g}\{b\}). \tag{4.2}$$

provided that the integral exists. The Mustard convolution has the conceptual and computational advantage to simply yield as spectrum in the QFT Fourier domain the point wise product of the QFTs of the two signals, just as the classical complex Fourier transform.

We additionally define a further type of *exponential-sine* Mustard convolution as

$$(a \star_{Ms} b)(\boldsymbol{x}) = (\mathcal{F}^{f,g})^{-1}(\mathcal{F}^{f,s}\{a\}\mathcal{F}^{s,g}\{b\}). \tag{4.3}$$

In the following two Subsections we will first express the convolution (4.1) in terms of the Mustard convolution (4.2) and vice versa.

4.1. Expressing the Convolution in Terms of the Mustard Convolution

In [7] Theorem 4.1 on page 584 expresses the classical convolution of two quaternion functions with the help of the general left-sided QFT as a sum of 40 Mustard convolutions. In our approach we use Theorem 5.12 on page 327 of [19], which expresses the convolution of two Clifford signal functions (higher dimensional generalizations of quaternion functions) in the Clifford Fourier domain with the help of the general two-sided Clifford Fourier transform (CFT), the latter is in turn a generalization of the QFT. We restate this theorem here again, specialized for quaternion functions and the QFT of Definition 3.1.

Theorem 4.1. (QFT of convolution)
Assuming a general pair of unit pure quaternions f, g, the general two-sided QFT of the convolution (4.1) of two functions $a, b \in L^1(\mathbb{R}^2; \mathbb{H})$ can then be expressed as

$$\begin{aligned}
&\mathcal{F}^{f,g}\{a \star b\} \\
&\quad \mathcal{F}^{f,g}\{a_{+f}\}\mathcal{F}^{f,g}\{b_{+g}\} + \mathcal{F}^{f,-g}\{a_{+f}\}\mathcal{F}^{f,g}\{b_{-g}\} \\
&\qquad + \mathcal{F}^{f,g}\{a_{-f}\}\mathcal{F}^{-f,g}\{b_{+g}\} + \mathcal{F}^{f,-g}\{a_{-f}\}\mathcal{F}^{-f,g}\{b_{-g}\} \\
&\qquad + \mathcal{F}^{f,s}\{a_{+f}\}[f,g]\mathcal{F}^{s,g}\{b_{+g}\} + \mathcal{F}^{f,-s}\{a_{+f}\}[f,g]\mathcal{F}^{s,g}\{b_{-g}\} \\
&\qquad + \mathcal{F}^{f,s}\{a_{-f}\}[f,g]\mathcal{F}^{-s,g}\{b_{+g}\} + \mathcal{F}^{f,-s}\{a_{-f}\}[f,g]\mathcal{F}^{-s,g}\{b_{-g}\}.
\end{aligned} \tag{4.4}$$

Note that due to the commutation properties of (3.5) and (3.6) we can place the commutator $[f, g]$ also inside the exponential-sine transform terms as e.g. in

$$\begin{aligned}
\mathcal{F}^{f,s}\{a_{+f}\}[f,g]\mathcal{F}^{s,g}\{b_{+g}\} &= \mathcal{F}^{f,s}\{a_{+f}[f,g]\}\mathcal{F}^{s,g}\{b_{+g}\} \\
&= \mathcal{F}^{f,s}\{a_{+f}\}\mathcal{F}^{s,g}\{[f,g]b_{+g}\}.
\end{aligned} \tag{4.5}$$

For the special case of parallel unit pure quaternions $f = \pm g$, the commutator vanishes $[f, g] = 0$, and we get the following corollary. Note that in this case $b_{\pm g} = b_{\pm f}$.

Corollary 4.2. (QFT of convolution with $f \parallel g$) *Assuming a parallel pair of unit pure quaternions $f = \pm g$, the general two-sided QFT of the convolution (4.1) of two functions $a, b \in L^1(\mathbb{R}^2; \mathbb{H})$ can be expressed as*

$$\begin{aligned}
&\mathcal{F}^{f,g}\{a \star b\} \\
&\quad = \mathcal{F}^{f,g}\{a_{+f}\}\mathcal{F}^{f,g}\{b_{+f}\} + \mathcal{F}^{f,-g}\{a_{+f}\}\mathcal{F}^{f,g}\{b_{-f}\} \\
&\qquad + \mathcal{F}^{f,g}\{a_{-f}\}\mathcal{F}^{-f,g}\{b_{+f}\} + \mathcal{F}^{f,-g}\{a_{-f}\}\mathcal{F}^{-f,g}\{b_{-f}\}.
\end{aligned} \tag{4.6}$$

By applying the inverse QFT, we can now easily express the convolution of two quaternion signals $a \star b$, in terms of only eight Mustard convolutions (4.2) and (4.3).

Theorem 4.3. (Convolution in terms of two types of Mustard convolution) *Assuming a general pair of unit pure quaternions f, g, the convolution (4.1) of two quaternion functions $a, b \in L^1(\mathbb{R}^2; \mathbb{H})$ can be expressed in terms of four Mustard convolutions (4.2) and four exponential-sine Mustard convolutions (4.3) as*

$$\begin{aligned} a \star b = a_{+f} \star_M b_{+g} + a_{+f}^{(0,1)} \star_M b_{-g} + a_{-f} \star_M b_{+g}^{(1,0)} + a_{-f}^{(0,1)} \star_M b_{-g}^{(1,0)} \\ + a_{+f} \star_{Ms} [f,g] b_{+g} + a_{+f}^{(0,1)} \star_{Ms} [f,g] b_{-g} \\ + a_{-f} \star_{Ms} [f,g] b_{+g}^{(1,0)} + a_{-f}^{(0,1)} \star_{Ms} [f,g] b_{-g}^{(1,0)}. \end{aligned} \tag{4.7}$$

Remark 4.4. We use the convention, that terms such as $a_{+f} \star_{Ms} [f,g]b_{+g}$, should be understood with brackets $a_{+f} \star_{Ms} ([f,g]b_{+g})$, which are omitted to avoid clutter.

Assuming $f \parallel g$, the standard Mustard convolution is sufficient to express the classical convolution.

Corollary 4.5. (Convolution in terms of Mustard convolution with parallel axis) *Assuming a parallel pair of unit pure quaternions $f \parallel g$, the convolution (4.1) of two quaternion functions $a, b \in L^1(\mathbb{R}^2; \mathbb{H})$ can be expressed in terms of four Mustard convolutions (4.2) as*

$$a \star b = a_{+f} \star_M b_{+f} + a_{+f}^{(0,1)} \star_M b_{-f} + a_{-f} \star_M b_{+f}^{(1,0)} + a_{-f}^{(0,1)} \star_M b_{-f}^{(1,0)} \tag{4.8}$$

Furthermore, applying (3.8) and (3.9), we can expand the terms in (4.4) with exponential-sine transforms into sums of products of QFTs. For example, the first term gives

$$\begin{aligned} &\mathcal{F}^{f,s}\{a_{+f}\}[f,g]\mathcal{F}^{s,g}\{b_{+g}\} \\ &= \frac{1}{4}\left(\mathcal{F}^{f,g}\{a_{+f}g\} - \mathcal{F}^{f,-g}\{a_{+f}g\}\right)\left(\mathcal{F}^{f,g}\{f[f,g]b_{+g}\} - \mathcal{F}^{-f,g}\{f[f,g]b_{+g}\}\right) \\ &= \frac{1}{4}\Big(\mathcal{F}\{a_{+f}g\}\mathcal{F}\{f[f,g]b_{+g}\} - \mathcal{F}\{a_{+f}g\}\mathcal{F}\{f[f,g]b_{+g}^{(1,0)}\} \\ &\quad - \mathcal{F}\{a_{+f}^{(0,1)}g\}\mathcal{F}\{f[f,g]b_{+g}\} + \mathcal{F}\{a_{+f}^{(0,1)}g\}\mathcal{F}\{f[f,g]b_{+g}^{(1,0)}\}\Big). \end{aligned} \tag{4.9}$$

This now allows us in turn to express the quaternion signal convolution purely in terms of standard Mustard convolutions

Theorem 4.6. (Convolution in terms of Mustard convolution) *Assuming a general pair of unit pure quaternions f, g, the convolution (4.1) of two quaternion functions $a, b \in L^1(\mathbb{R}^2; \mathbb{H})$ can be expressed in terms of twenty standard Mustard convolutions (4.2) as*

$$\begin{aligned} a \star b = {} & a_{+f} \star_M b_{+g} + a_{+f}^{(0,1)} \star_M b_{-g} + a_{-f} \star_M b_{+g}^{(1,0)} + a_{-f}^{(0,1)} \star_M b_{-g}^{(1,0)} \\ & + \tfrac{1}{4}\,(a_{+f}g \star_M fcb_{+g} - a_{+f}g \star_M fcb_{+g}^{(1,0)} - a_{+f}^{(0,1)}g \star_M fcb_{+g} \\ & + a_{+f}^{(0,1)}g \star_M fcb_{+g}^{(1,0)} + a_{+f}^{(0,1)}g \star_M fcb_{-g} - a_{+f}^{(0,1)}g \star_M fcb_{-g}^{(1,0)} \\ & - a_{+f}g \star_M fcb_{-g} + a_{+f}g \star_M fcb_{-g}^{(1,0)} + a_{-f}g \star_M fcb_{+g}^{(1,0)} \\ & - a_{-f}g \star_M fcb_{+g} - a_{-f}^{(0,1)}g \star_M fcb_{+g}^{(1,0)} + a_{-f}^{(0,1)}g \star_M fcb_{+g} \\ & + a_{-f}^{(0,1)}g \star_M fcb_{-g}^{(1,0)} - a_{-f}^{(0,1)}g \star_M fcb_{-g} \\ & - a_{-f}g \star_M fcb_{-g}^{(1,0)} + a_{-f}g \star_M fcb_{-g}\,), \end{aligned} \tag{4.10}$$

with the abbreviation $c = [f,g]$.

4.2. Expressing the Mustard Convolution in Terms of the Convolution

Now we will simply write out the Mustard convolution (4.2) and simplify it until only standard convolutions (4.1) remain. In this Subsection we will use the general OPS split of Definition 2.3. Our result should be compared with the Theorem 2.5 on page 584 of [7] for the left-sided QFT with 32 classical convolutions for expressing the Mustard convolution of quaternion functions (and for the two-sided QFT in [5], Section 4.4.2, with 16 classical convolutions, stated in a different but apparently equivalent form to Theorem 4.7).

We begin by writing the Mustard convolution (4.2) of two quaternion functions $a, b \in L^2(\mathbb{R}^2, \mathbb{H})$

$$\begin{aligned} a \star_M b(\boldsymbol{x}) &= \tfrac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{f x_1 \omega_1} \mathcal{F}\{a\}(\boldsymbol{\omega}) \mathcal{F}\{b\}(\boldsymbol{\omega}) e^{g x_2 \omega_2} d^2\boldsymbol{\omega} \\ &= \tfrac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{f x_1 \omega_1} \int_{\mathbb{R}^2} e^{-f y_1 \omega_1} a(\boldsymbol{y}) e^{-g y_2 \omega_2} d^2\boldsymbol{y} \\ &\quad \times \int_{\mathbb{R}^2} e^{-f z_1 \omega_1} b(\boldsymbol{z}) e^{-g z_2 \omega_2} d^2\boldsymbol{z} e^{g x_2 \omega_2} d^2\boldsymbol{\omega} \\ &= \tfrac{1}{(2\pi)^2} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{f(x_1-y_1)\omega_1} (a_+(\boldsymbol{y}) + a_-(\boldsymbol{y})) e^{-g y_2 \omega_2} \\ &\quad \times e^{-f z_1 \omega_1} (b_+(\boldsymbol{z}) + b_-(\boldsymbol{z})) e^{g(x_2-z_2)\omega_2} d^2\boldsymbol{y} d^2\boldsymbol{z} d^2\boldsymbol{\omega}. \end{aligned} \tag{4.11}$$

Next, we use the identities (2.24) in order to shift the inner factor $e^{-g y_2 \omega_2}$ to the left and $e^{-f z_1 \omega_1}$ to the right, respectively. We abbreviate $\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}$ to $\iiint$.

$$\begin{aligned} &a \star_M b(\boldsymbol{x}) \\ &= \tfrac{1}{(2\pi)^2} \iiint e^{f(x_1-y_1)\omega_1} e^{f y_2 \omega_2} a_+(\boldsymbol{y}) b_+(\boldsymbol{z}) e^{g z_1 \omega_1} e^{g(x_2-z_2)\omega_2} d^2\boldsymbol{y} d^2\boldsymbol{z} d^2\boldsymbol{\omega} \\ &\quad + \tfrac{1}{(2\pi)^2} \iiint e^{f(x_1-y_1)\omega_1} e^{f y_2 \omega_2} a_+(\boldsymbol{y}) b_-(\boldsymbol{z}) e^{-g z_1 \omega_1} e^{g(x_2-z_2)\omega_2} d^2\boldsymbol{y} d^2\boldsymbol{z} d^2\boldsymbol{\omega} \\ &\quad + \tfrac{1}{(2\pi)^2} \iiint e^{f(x_1-y_1)\omega_1} e^{-f y_2 \omega_2} a_-(\boldsymbol{y}) b_+(\boldsymbol{z}) e^{g z_1 \omega_1} e^{g(x_2-z_2)\omega_2} d^2\boldsymbol{y} d^2\boldsymbol{z} d^2\boldsymbol{\omega} \\ &\quad + \tfrac{1}{(2\pi)^2} \iiint e^{f(x_1-y_1)\omega_1} e^{-f y_2 \omega_2} a_-(\boldsymbol{y}) b_-(\boldsymbol{z}) e^{-g z_1 \omega_1} e^{g(x_2-z_2)\omega_2} d^2\boldsymbol{y} d^2\boldsymbol{z} d^2\boldsymbol{\omega} \end{aligned} \tag{4.12}$$

Furthermore, we abbreviate the inner function products as $ab_{\pm\pm}(\boldsymbol{y}, \boldsymbol{z}) := a_\pm(\boldsymbol{y}) b_\pm(\boldsymbol{z})$, and apply the general OPS split of Definition 2.3 once again to obtain $ab_{\pm\pm}(\boldsymbol{y}, \boldsymbol{z}) = [ab_{\pm\pm}(\boldsymbol{y}, \boldsymbol{z})]_+ + [ab_{\pm\pm}(\boldsymbol{y}, \boldsymbol{z})]_- = ab_{\pm\pm}(\boldsymbol{y}, \boldsymbol{z})_+ + ab_{\pm\pm}(\boldsymbol{y}, \boldsymbol{z})_-$. We omit the square brackets and use the convention that the final OPS split indicated by the final $\pm$ index should be performed last.

This allows to further apply (2.24) again in order to shift the factors $e^{\pm g z_1 \omega_1}$ $e^{g(x_2-z_2)\omega_2}$ to the left. We end up with the following eight terms

$$\begin{aligned}
&a \star_M b(\boldsymbol{x}) \\
&\quad = \tfrac{1}{(2\pi)^2} \iiint e^{f(x_1-y_1-z_1)\omega_1} e^{f(y_2-(x_2-z_2))\omega_2} ab_{++}(\boldsymbol{y},\boldsymbol{z})_+ d^2\boldsymbol{y} d^2\boldsymbol{z} d^2\boldsymbol{\omega} \\
&\qquad + \tfrac{1}{(2\pi)^2} \iiint e^{f(x_1-y_1+z_1)\omega_1} e^{f(y_2+(x_2-z_2))\omega_2} ab_{++}(\boldsymbol{y},\boldsymbol{z})_- d^2\boldsymbol{y} d^2\boldsymbol{z} d^2\boldsymbol{\omega} \\
&\qquad + \tfrac{1}{(2\pi)^2} \iiint e^{f(x_1-y_1+z_1)\omega_1} e^{f(y_2-(x_2-z_2))\omega_2} ab_{+-}(\boldsymbol{y},\boldsymbol{z})_+ d^2\boldsymbol{y} d^2\boldsymbol{z} d^2\boldsymbol{\omega} \\
&\qquad + \tfrac{1}{(2\pi)^2} \iiint e^{f(x_1-y_1-z_1)\omega_1} e^{f(y_2+(x_2-z_2))\omega_2} ab_{+-}(\boldsymbol{y},\boldsymbol{z})_- d^2\boldsymbol{y} d^2\boldsymbol{z} d^2\boldsymbol{\omega} \\
&\qquad + \tfrac{1}{(2\pi)^2} \iiint e^{f(x_1-y_1-z_1)\omega_1} e^{f(-y_2-(x_2-z_2))\omega_2} ab_{-+}(\boldsymbol{y},\boldsymbol{z})_+ d^2\boldsymbol{y} d^2\boldsymbol{z} d^2\boldsymbol{\omega} \\
&\qquad + \tfrac{1}{(2\pi)^2} \iiint e^{f(x_1-y_1+z_1)\omega_1} e^{f(-y_2+(x_2-z_2))\omega_2} ab_{-+}(\boldsymbol{y},\boldsymbol{z})_- d^2\boldsymbol{y} d^2\boldsymbol{z} d^2\boldsymbol{\omega} \\
&\qquad + \tfrac{1}{(2\pi)^2} \iiint e^{f(x_1-y_1+z_1)\omega_1} e^{f(-y_2-(x_2-z_2))\omega_2} ab_{--}(\boldsymbol{y},\boldsymbol{z})_+ d^2\boldsymbol{y} d^2\boldsymbol{z} d^2\boldsymbol{\omega} \\
&\qquad + \tfrac{1}{(2\pi)^2} \iiint e^{f(x_1-y_1-z_1)\omega_1} e^{f(-y_2+(x_2-z_2))\omega_2} ab_{--}(\boldsymbol{y},\boldsymbol{z})_- d^2\boldsymbol{y} d^2\boldsymbol{z} d^2\boldsymbol{\omega}.
\end{aligned} \tag{4.13}$$

We now only show explicitly how to simplify the second triple integral, the others follow the same pattern.

$$\begin{aligned}
&\tfrac{1}{(2\pi)^2} \iiint e^{f(x_1-y_1+z_1)\omega_1} e^{f(y_2+(x_2-z_2))\omega_2} [a_+(\boldsymbol{y}) b_+(\boldsymbol{z})]_- d^2\boldsymbol{y} d^2\boldsymbol{z} d^2\boldsymbol{\omega} \\
&\quad = \tfrac{1}{(2\pi)^2} \iint \int_{\mathbb{R}} e^{f(x_1-y_1+z_1)\omega_1} d\omega_1 \int_{\mathbb{R}} e^{f(y_2+(x_2-z_2))\omega_2} d\omega_2 [a_+(\boldsymbol{y}) b_+(\boldsymbol{z})]_- d^2\boldsymbol{y} d^2\boldsymbol{z} \\
&\quad = \iint \delta(x_1 - y_1 + z_1)\delta(y_2 + (x_2 - z_2))[a_+(\boldsymbol{y}) b_+(z_1, z_2)]_- d^2\boldsymbol{y} d^2\boldsymbol{z} \\
&\quad = \int_{\mathbb{R}^2} [a_+(\boldsymbol{y}) b_+(-(x_1 - y_1), x_2 + y_2)]_- d^2\boldsymbol{y} \\
&\quad = \int_{\mathbb{R}^2} [a_+(\boldsymbol{y}) b_+(-(x_1 - y_1), -(-x_2 - y_2))]_- d^2\boldsymbol{y} \\
&\quad = \int_{\mathbb{R}^2} [a_+(\boldsymbol{y}) b_+^{(1,1)}(x_1 - y_1, -x_2 - y_2)]_- d^2\boldsymbol{y} \\
&\quad = [a_+ \star b_+^{(1,1)}(x_1, -x_2)]_- .
\end{aligned} \tag{4.14}$$

Note that $a_+ \star b_+^{(1,1)}(x_1, -x_2)$ means to first apply the convolution to the pair of functions a_+ and $b_+^{(1,1)}$, and only then to evaluate them with the argument $(x_1, -x_2)$. So in general $a_+ \star b_+^{(1,1)}(x_1, -x_2) \neq a_+ \star b_+(-x_1, x_2)$. Simplifying the other seven triple integrals similarly we finally obtain the desired decomposition of the Mustard convolution (4.2) in terms of the classical convolution.

Theorem 4.7. (Mustard convolution in terms of standard convolution) *The Mustard convolution* (4.2) *of two quaternion functions* $a, b \in L^1(\mathbb{R}^2; \mathbb{H})$ *can be expressed in terms of eight standard convolutions* (4.1) *as*

$$\begin{aligned} a \star_M b(\boldsymbol{x}) = & \\ = & [a_+ \star b_+(\boldsymbol{x})]_+ + [a_+ \star b_+^{(1,1)}(x_1, -x_2)]_- \\ & + [a_+ \star b_-^{(1,0)}(\boldsymbol{x})]_+ + [a_+ \star b_-^{(0,1)}(x_1, -x_2)]_- \\ & + [a_- \star b_+^{(0,1)}(x_1, -x_2)]_+ + [a_- \star b_+^{(1,0)}(\boldsymbol{x})]_- \\ & + [a_- \star b_-^{(1,1)}(x_1, -x_2)]_+ + [a_- \star b_-(\boldsymbol{x})]_- . \end{aligned} \tag{4.15}$$

Remark 4.8. (Levels of computation) Equation (4.15) involves five levels of operation: primary (inner) and secondary (outer) OPS, argument reflections before (pre) and after (post) the actual convolution, and the convolution itself in each of the eight terms.

Remark 4.9. (Efficiency of notation and interpretation) If we would explicitly insert according Definition 2.3 $a_\pm = \frac{1}{2}(a \pm f a g)$ and $b_\pm = \frac{1}{2}(b \pm f b g)$, and similarly explicitly insert the second level OPS split $[\ldots]_\pm$, we would (potentially) obtain up to a maximum of 64 terms[1]. It is therefore obvious how straightforward, significant and efficient the use of the general OPS split is in this context. Efficiency first of all with respect to concise (compact) notation, which in turn may assist geometric (or physical) interpretation in concrete applications.

Regarding *derivation efficiency*, we needed two pages to derive (4.15), but in order to strike the balance with sufficient detail for reasonably self-contained comprehension, this level of detail may be justified. Whether the compact eight term form of (4.15) is advantageous for actual *numerical computations*, is an open question, which requires application to concrete representatives problems, e.g. in the area of quaternionic image processing (see e.g. [1,3,4,8,10,11,21,23,25,26,28]).

And it may depend on the concrete hardware architecture, e.g. how many parallel channels of computation (e.g. number of parallel GPUs) are available, whether quaternion operations are hard coded or need breaking down in elementary real (or matrix) computations; and whether optimized software packages like the precompiler GAALOP [12] or the MATLAB package QFMT [27] would be used, etc.

5. Conclusion

In this paper we have briefly reviewed the algebra of quaternions, their general orthogonal two-dimensional planes split, the general two-sided quaternion

[1] Note that in [5], Section 4.4.2, a fully explicit form is given with only 16 terms.

Fourier transform, and introduced a related mixed quaternionic exponential-sine Fourier transform. We defined the notions of convolution of two quaternion valued functions over $\mathbb{R}^2$, the Mustard convolution (with its QFT as the point wise product of the QFTs of the factor functions), and a special Mustard convolution involving the point wise products of mixed exponential-sine QFTs.

The main results are: An efficient decomposition of the classical convolution of quaternion signals in terms of eight Mustard type convolutions. For the special case of parallel pure unit quaternion axis in the QFT, only four terms of the standard Mustard convolution are sufficient. Even in the case of two general pure unit quaternion axis in the QFT, the classical convolution of two quaternion signals can always be fully expanded in terms of standard Mustard convolutions. Finally we showed how to fully generally expand the Mustard convolution of two quaternion signals in terms of eight classical convolutions.

In view of the many applications of the QFT explained in the introduction, we expect our new results to be of great interest, especially for filter design and feature extraction in signal and color image processing.

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