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# Stancu Type Generalization of Dunkl Analogue of Szàsz Operators

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**Abstract.** In this paper, we introduce Stancu type generalization of Dunkl analogue of Szàsz operators. We obtain some direct results, which include asymptotic formula and error estimation in terms of of the modulus continuity. Also, we investigate the convergence of these operators in a weighted space and estimate the rate of convergence.

**Mathematics Subject Classification.** 41A25, 41A36.

**Keywords.** Dunkl Analogue of Szàsz–Stancu Operators, The weighted modulus of continuity, local approximation, weighted space, Voronovskaja type asymptotic formula.

## 1. Introduction

It is well known that the classical Bernstein operators $B_n(f;x)$ converges to $f(x)$ uniformly in $[0,1]$ for every continuous function $f$. In 1968 Stancu [13] showed that the operators

$$B_n^{\alpha,\beta}(f;x)=\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}f\left(\frac{k+\alpha}{n+\beta}\right)$$

converges to continuous function $f(x)$ uniformly in $[0,1]$ for each $\alpha,\beta$ such that $0\leqslant\alpha\leqslant\beta$. This operators are called Bernstein–Stancu operators in literature.

In 1950, Szàsz [15] defined the following operators which are generalized Bernstein operators to the infinite interval

$$S_n(f;x)=e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}f\left(\frac{k}{n}\right),\quad x\in[0,\infty). \tag{1.1}$$

This operators and the generalizations were studied in [1,3,5,8,10,11,16].

Recently, Sucu [14] has introduced Dunkl analogue of Szàsz operators which are generated using generalization of exponential function as follows:

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$$S_n^*(f;x) = \frac{1}{e_\mu(nx)} \sum_{k=0}^{\infty} \frac{(nx)^k}{\gamma_\mu(k)} f\left(\frac{k+2\mu_{\theta_k}}{n}\right), \tag{1.2}$$

where

$$e_\mu(x) = \sum_{k=0}^{\infty} \frac{x^k}{\gamma_\mu(k)}$$

and the coefficients $\gamma_\mu$ are given by for $k \in \mathbb{N}_0$ and $\mu > -1/2$

$$\gamma_\mu(2k) = \frac{2^{2k} k! \Gamma(k+\mu+1/2)}{\Gamma(\mu+1/2)} \quad \gamma_\mu(2k+1) = \frac{2^{2k+1} k! \Gamma(k+\mu+3/2)}{\Gamma(\mu+1/2)}.$$

We note the following recursion relation for the $\gamma_\mu$

$$\gamma_\mu(k+1) = (k+1+2\mu\theta_{k+1})\gamma_\mu(k), \quad k \in \mathbb{N}_0,$$

where $\theta_k$ is defined to be 0 if $k \in 2\mathbb{N}_0$ and 1 if $k \in 2\mathbb{N}_0+1$. He established several approximation properties of these operators. It is clear that the operators defined by (1.2) are positive and linear for $\mu \geqslant 0$, $x \geqslant 0$.

The present paper deals with Stancu type generalization of Dunkl analogue of Szàsz operators. We prove the convergence of our new operator by the help of Korovkin-type theorem on bounded interval. Also, we obtain some local approximation results of these new operators. Further, we investigate the convergence of these operators in a weighted space and estimate the rate of convergence. Finally, we give a Voronovskaja type asymptotic formula for this operators.

## 2. Construction of Operators

For $n \in \mathbb{N}$ and $0 \leqslant \alpha \leqslant \beta$, we introduce Dunkl analogue of Szàsz–Stancu operators as follows:

$$S_n^{\alpha,\beta}(f;x) = \frac{1}{e_\mu(nx)} \sum_{k=0}^{\infty} \frac{(nx)^k}{\gamma_\mu(k)} f\left(\frac{k+2\mu_{\theta_k}+\alpha}{n+\beta}\right). \tag{2.1}$$

It should be noted that if we take $\alpha=\beta=0$, $S_n^{\alpha,\beta}(f;x)$ is reduce to Dunkl analogue of Szàsz operators given by (1.2) and if we take $\mu=\alpha=\beta=0$, $S_n^{\alpha,\beta}(f;x)$ is reduce to the classic Szàsz operators given by (1.1).

To constructed of our approximation theorem for the operators $S_n^{\alpha,\beta}(f;x)$, we need following lemmas.

**Lemma 2.1.** [14, Lemma 1, p. 43] *The following equalities hold for $x \in [0,\infty)$ and $0 \leqslant \alpha \leqslant \beta$:*

$$\begin{aligned}
S_n^*(1;x) &= 1,\\
S_n^*(t;x) &= x,\\
S_n^*(t^2;x) &= x^2 + \left(1+2\mu\frac{e_\mu(-nx)}{e_\mu(nx)}\right)\frac{x}{n},\\
S_n^*(t^3;x) &= x^3 + \left(3-2\mu\frac{e_\mu(-nx)}{e_\mu(nx)}\right)\frac{x^2}{n} + \left(1+4\mu^2+4\mu\frac{e_\mu(-nx)}{e_\mu(nx)}\right)\frac{x}{n^2},
\end{aligned}$$

$$S_n^*(t^4;x) = x^4 + \left(6 + 4\mu\frac{e_\mu(-nx)}{e_\mu(nx)}\right)\frac{x^3}{n} + \left(7 + 4\mu^2 - 8\mu\frac{e_\mu(-nx)}{e_\mu(nx)}\right)\frac{x^2}{n^2} + \left(1 + 12\mu^2 + 2\mu(3 + 4\mu^2)\frac{e_\mu(-nx)}{e_\mu(nx)}\right)\frac{x}{n^3}.$$

**Lemma 2.2.** *For the operators* $S_n^{\alpha,\beta}$, $0 \leqslant \alpha \leqslant \beta$, *and* $x \in [0,\infty)$ *we have*

$$S_n^{\alpha,\beta}(1;x) = 1, \tag{2.2}$$

$$S_n^{\alpha,\beta}(t;x) = \frac{nx}{n+\beta} + \frac{\alpha}{n+\beta}, \tag{2.3}$$

$$S_n^{\alpha,\beta}(t^2;x) = \frac{n^2x^2}{(n+\beta)^2} + \frac{A_0 nx}{(n+\beta)^2} + \frac{\alpha^2}{(n+\beta)^2}, \tag{2.4}$$

$$S_n^{\alpha,\beta}(t^3;x) = \frac{n^3x^3}{(n+\beta)^3} + \frac{A_1 n^2x^2}{(n+\beta)^3} + \frac{A_2 nx}{(n+\beta)^3} + \frac{\alpha^3}{(n+\beta)^3}, \tag{2.5}$$

$$S_n^{\alpha,\beta}(t^4;x) = \frac{n^4x^4}{(n+\beta)^4} + \frac{A_3 n^3x^3}{(n+\beta)^4} + \frac{A_4 n^2x^2}{(n+\beta)^4} + \frac{A_5 nx}{(n+\beta)^4} + \frac{\alpha^4}{(n+\beta)^4}, \tag{2.6}$$

*where*

$$A_0 = 2\alpha + 1 + 2\mu\frac{e_\mu(-nx)}{e_\mu(nx)},$$

$$A_1 = 3\alpha + 3 - 2\mu\frac{e_\mu(-nx)}{e_\mu(nx)},$$

$$A_2 = 1 + 3\alpha^2 + 3\alpha + 4\mu^2 + 2\mu(2+3\alpha)\frac{e_\mu(-nx)}{e_\mu(nx)},$$

$$A_3 = 6 + 4\alpha + 4\mu\frac{e_\mu(-nx)}{e_\mu(nx)},$$

$$A_4 = 7 + 4\mu^2 + 6\alpha^2 + 12\alpha - 8\mu(1+\alpha)\frac{e_\mu(-nx)}{e_\mu(nx)}$$

$$A_5 = 1 + 12\mu^2 + 2\alpha(2 + 8\mu^2 + 2\alpha^2 + 3\alpha) + 2\mu\left(3 + 4\mu^2 + 8\alpha + 6\alpha^2\right)\frac{e_\mu(-nx)}{e_\mu(nx)}.$$

*Proof.* By Lemma 2.1 and definition of $S_n^{\alpha,\beta}(f;x)$, we get

$$S_n^{\alpha,\beta}(1;x) = \frac{1}{e_\mu(nx)}\sum_{k=0}^{\infty}\frac{(nx)^k}{\gamma_\mu(k)} = S_n^*(1;x) = 1,$$

$$\begin{aligned} S_n^{\alpha,\beta}(t;x) &= \frac{1}{e_\mu(nx)}\sum_{k=0}^{\infty}\frac{(nx)^k}{\gamma_\mu(k)}\left(\frac{k + 2\mu_{\theta_k} + \alpha}{n+\beta}\right) \\ &= \frac{1}{(n+\beta)e_\mu(nx)}\left\{\sum_{k=0}^{\infty}\frac{(nx)^k}{\gamma_\mu(k)}(k + 2\mu_{\theta_k}) + \alpha\sum_{k=0}^{\infty}\frac{(nx)^k}{\gamma_\mu(k)}\right\} \\ &= \frac{n}{n+\beta}S_n^*(t;x) + \frac{\alpha}{n+\beta}S_n^*(1;x) \\ &= \frac{nx}{n+\beta} + \frac{\alpha}{n+\beta}. \end{aligned}$$

Now, we consider $S_n^{\alpha,\beta}(t^2;x)$ as follows;

$$
\begin{aligned}
S_n^{\alpha,\beta}(t^2;x) &= \frac{1}{e_\mu(nx)}\sum_{k=0}^{\infty}\frac{(nx)^k}{\gamma_\mu(k)}\left(\frac{k+2\mu_{\theta_k}+\alpha}{n+\beta}\right)^2 \\
&= \frac{1}{(n+\beta)^2 e_\mu(nx)}\left\{\sum_{k=0}^{\infty}\frac{(nx)^k}{\gamma_\mu(k)}\left(k+2\mu_{\theta_k}\right)^2+2\alpha\sum_{k=0}^{\infty}\frac{(nx)^k}{\gamma_\mu(k)}\left(k+2\mu_{\theta_k}\right)\right. \\
&\quad \left.+\alpha^2\sum_{k=0}^{\infty}\frac{(nx)^k}{\gamma_\mu(k)}\right\} \\
&= \frac{n^2}{(n+\beta)^2}S_n^*(t^2;x)+\frac{2\alpha n}{(n+\beta)^2}S_n^*(t;x)+\frac{\alpha^2}{(n+\beta)^2}S_n^*(1;x) \\
&= \frac{n^2x^2}{(n+\beta)^2}+\frac{A_0nx}{(n+\beta)^2}+\frac{\alpha^2}{(n+\beta)^2}.
\end{aligned}
$$

For $S_n^{\alpha,\beta}(t^3;x)$, we get

$$
\begin{aligned}
S_n^{\alpha,\beta}(t^3;x) &= \frac{1}{e_\mu(nx)}\sum_{k=0}^{\infty}\frac{(nx)^k}{\gamma_\mu(k)}\left(\frac{k+2\mu_{\theta_k}+\alpha}{n+\beta}\right)^3 \\
&= \frac{1}{(n+\beta)^3 e_\mu(nx)}\sum_{k=0}^{\infty}\frac{(nx)^k}{\gamma_\mu(k)}\left(k+2\mu_{\theta_k}+\alpha\right)^3 \\
&= \frac{n^3}{(n+\beta)^3}S_n^*(t^3;x)+\frac{3\alpha n^2}{(n+\beta)^3}S_n^*(t^2;x) \\
&\quad +\frac{3\alpha^2 n}{(n+\beta)^3}S_n^*(t;x)+\frac{\alpha^3}{(n+\beta)^3}S_n^*(1;x) \\
&= \frac{n^3x^3}{(n+\beta)^3}+\frac{A_1n^2x^2}{(n+\beta)^3}+\frac{A_2nx}{(n+\beta)^3}+\frac{\alpha^3}{(n+\beta)^3}
\end{aligned}
$$

and

$$
\begin{aligned}
S_n^{\alpha,\beta}(t^4;x) &= \frac{1}{e_\mu(nx)}\sum_{k=0}^{\infty}\frac{(nx)^k}{\gamma_\mu(k)}\left(\frac{k+2\mu_{\theta_k}+\alpha}{n+\beta}\right)^4 \\
&= \frac{1}{(n+\beta)^4 e_\mu(nx)}\sum_{k=0}^{\infty}\frac{(nx)^k}{\gamma_\mu(k)}\left(k+2\mu_{\theta_k}+\alpha\right)^4 \\
&= \frac{n^4}{(n+\beta)^4}S_n^*(t^4;x)+\frac{4\alpha n^3}{(n+\beta)^4}S_n^*(t^3;x)+\frac{6\alpha^2n^2}{(n+\beta)^4}S_n^*(t^2;x) \\
&\quad +\frac{4\alpha^3 n}{(n+\beta)^4}S_n^*(t;x)+\frac{\alpha^4}{(n+\beta)^4}S_n^*(1;x) \\
&= \frac{n^4x^4}{(n+\beta)^4}+\frac{A_3n^3x^3}{(n+\beta)^4}+\frac{A_4n^2x^2}{(n+\beta)^4}+\frac{A_5nx}{(n+\beta)^4}+\frac{\alpha^4}{(n+\beta)^4}.
\end{aligned}
$$

Hence, the proof is completed. □

According to Lemma 2.2 and linearity of $S_n^{\alpha,\beta}(f;x)$, one has the following identities

$$S_n^{\alpha,\beta}((t-x)^2;x) = \frac{\beta^2}{(n+\beta)^2}x^2 + \frac{n\left(1+2\mu\frac{e_\mu(-nx)}{e_\mu(nx)}\right)-2\alpha\beta}{(n+\beta)^2}x + \frac{\alpha^2}{(n+\beta)^2}, \quad (2.7)$$

$$S_n^{\alpha,\beta}((t-x)^4;x) = \frac{x^4\beta^4}{(n+\beta)^4} + \frac{x^3\left(24\mu\frac{e_\mu(-nx)}{e_\mu(nx)}n^3 + 32\beta\mu\frac{e_\mu(-nx)}{e_\mu(nx)}n^2 + C_1 n - 4\alpha\beta^3\right)}{(n+\beta)^4}$$
$$+\frac{x^2\left(C_2n^2 - C_3 n + 6\alpha^2\beta^2\right)}{(n+\beta)^4} + \frac{x(C_4 n - 4\alpha^3\beta)}{(n+\beta)^4} + \frac{\alpha^4}{(n+\beta)^4} \quad (2.8)$$

where

$$C_1 = 6\beta^2\left(1+2\mu\frac{e_\mu(-nx)}{e_\mu(nx)}\right),$$

$$C_2 = 3 - 12\mu^2 - (24+32\alpha)\mu\frac{e_\mu(-nx)}{e_\mu(nx)},$$

$$C_3 = 4\beta\left(1+3\alpha+4\mu^2+(4+6\alpha)\mu\frac{e_\mu(-nx)}{e_\mu(nx)}\right),$$

$$C_4 = 1+2\alpha(2+3\alpha+8\mu^2)+12\mu^2+(6+8\mu^2+16\alpha+12\alpha^2)\mu\frac{e_\mu(-nx)}{e_\mu(nx)}.$$

**Theorem 2.3.** *For $f \in C[0,\infty)\cap E$, the sequence of operators $(S_n^{\alpha,\beta})_{n\geqslant 1}$ converges uniformly to $f$ on $[0,A]$ as $n \longrightarrow \infty$, where*

$$E = \left\{ f : x\in[0,\infty), \frac{f(x)}{x^2+1} \quad \text{is convergent as} \quad x \longrightarrow \infty \right\}$$

*and $A>0$ is absolute constant.*

*Proof.* Using (2.2)–(2.4), we get

$$\lim_{n\to\infty} S_n^{\alpha,\beta}(e_i;x) = e_i(x)$$

uniformly on $[0,A]$, where $e_i(t)=t^i$, $i=0,1,2$. If apply universal Korovkin type theorem [2], we get desired result. □

## 3. Weighted Approximation

In this section, we give approximation properties of the operators $S_n^{\alpha,\beta}$ in weighted space of continuous functions. We consider the following class of functions which are defined on the interval $[0,\infty)$.

Let $B_{x^2}[0,\infty)$ be the set of all functions $f$ defined on $[0,\infty)$ satisfying the condition $|f(x)| \leqslant M_f(1+x^2)$, where $M_f$ is a constant depending on $f$. By $C_{x^2}[0,\infty)$, we denote the subspace of all continuous function belonging to $B_{x^2}[0,\infty)$. Also, $C^*_{x^2}[0,\infty)$ be subspace of all continuous functions $f\in C_{x^2}[0,\infty)$, for which $\lim_{|x|\to\infty}\frac{f(x)}{1+x^2}$ is finite. The norm on $C^*_{x^2}[0,\infty)$ is defined as follows

$$\| f \|_{x^2} = \sup_{x\in[0,\infty)} \frac{|f(x)|}{1+x^2}.$$

The weighted modulus of continuity defined by Ispir and Atakut [6] as follows

$$\Omega(f;\delta) = \sup_{x\in[0,\infty),|h|\leqslant\delta} \frac{|f(x+h)-f(x)|}{(1+h^2)(1+x^2)}.$$

**Lemma 3.1.** *Let $\rho(x) = 1 + x^2$ be a weight function. If $f \in C_{x^2}[0,\infty)$, then*

$$\| S_n^{\alpha,\beta}(\rho;x) \|_{x^2} \leqslant 1 + M,$$

*where $M$ is a positive constant.*

*Proof.* By (2.2) and (2.4), we get

$$S_n^{\alpha,\beta}(\rho;x) = 1 + \frac{n^2x^2}{(n+\beta)^2} + \frac{\left(2\alpha+1+2\mu\frac{e_\mu(-nx)}{e_\mu(nx)}\right)nx}{(n+\beta)^2} + \frac{\alpha^2}{(n+\beta)^2}.$$

Then, we obtain

$$\begin{aligned}\| S_n^{\alpha,\beta}(\rho;x) \|_{x^2} &= \sup_{x\geqslant 0}\left\{\frac{1}{1+x^2} + \frac{n^2x^2}{(n+\beta)^2(1+x^2)} + \frac{\left(2\alpha+1+2\mu\frac{e_\mu(-nx)}{e_\mu(nx)}\right)nx}{(n+\beta)^2(1+x^2)}\right.\\ &\quad \left. + \frac{\alpha^2}{(n+\beta)^2(1+x^2)}\right\}\\ &\leqslant 1 + \frac{n^2}{(n+\beta)^2} + \frac{(\alpha+1/2+\mu)\,n}{(n+\beta)^2} + \frac{\alpha^2}{(n+\beta)^2}.\end{aligned}$$

Since $\lim_{n\to\infty}\frac{n^2}{(n+\beta)^2} = 1$, $\lim_{n\to\infty}\frac{\alpha^2}{(n+\beta)^2} = 0$ and $\lim_{n\to\infty}\frac{(\alpha+1/2+\mu)n}{(n+\beta)^2} = 0$, there exists a positive constant $M$ such that

$$\| S_n^{\alpha,\beta}(\rho;x) \|_{x^2} \leqslant 1 + M.$$

This step concludes the proof. □

By using Lemma 3.1, one can see that the operators $S_n^{\alpha,\beta}$ defined (2.1) act from $C_{x^2}[0,\infty)$ to $B_{x^2}[0,\infty)$.

**Theorem 3.2.** *Let $S_n^{\alpha,\beta}$ be sequence of linear positive operators defined (2.1), then for each $f \in C^*_{x^2}[0,\infty)$*

$$\lim_{n\to\infty} \| S_n^{\alpha,\beta}(f;x) - f(x) \|_{x^2} = 0.$$

*Proof.* By using weighted Korovkin theorem presented by Gadzhiev [7], it is enough to verify the conditions,

$$\lim_{n\to\infty} \| S_n^{\alpha,\beta}(e_v;x) - e_v(x) \|_{x^2} = 0, \quad v = 0,1 \text{ and } 2.$$

By (2.2) and (2.3), we have

$$\| S_n^{\alpha,\beta}(1;x) - 1 \|_{x^2} = 0 \tag{3.1}$$

and

$$\| S_n^{\alpha,\beta}(e_1;x) - e_1(x) \|_{x^2} = \sup_{x\geqslant 0}\left|\frac{\beta+\alpha}{(n+\beta)(x^2+1)}\right| \leqslant \frac{\alpha+\beta}{n+\beta}.$$

which implies that

$$\lim_{n\to\infty} \| S_n^{\alpha,\beta}(e_1;x) - e_1(x) \|_{x^2} = 0. \tag{3.2}$$

By means of (2.4), we obtain that

$$\| S_n^{\alpha,\beta}(e_2;x) - e_2(x) \|_{x^2} = \sup_{x\geqslant 0} \left| \frac{(-2n\beta-\beta^2)x^2}{(n+\beta)^2(1+x^2)} + \frac{\left(2\alpha+1+2\mu\frac{e_\mu(-nx)}{e_\mu(nx)}\right)nx}{(n+\beta)^2(1+x^2)} + \frac{\alpha^2}{(n+\beta)^2(1+x^2)} \right|$$
$$\leqslant \frac{\beta(2n+\beta)}{(n+\beta)^2} + \frac{(\alpha+1/2+\mu)\,n}{(n+\beta)^2} + \frac{\alpha^2}{(n+\beta)^2}.$$

Taking the limit of both sides of above inequality as $n \longrightarrow \infty$, we get

$$\lim_{n\to\infty} \| S_n^{\alpha,\beta}(e_2;x) - e_2(x) \|_{x^2} = 0. \tag{3.3}$$

From (3.1)–(3.3), for $v = 0,1,2$ we have $\lim_{n\to\infty} \| S_n^{\alpha,\beta}(e_v;x) - e_v(x) \|_{x^2} = 0$. □

The proof is completed.

Now, we estimate the weighted modulus of continuity. For this, we need the following lemma.

**Lemma 3.3.** [4,9] *For $f \in C^*_{x^2}[0,\infty)$, then*

(i) *$\Omega(f;\delta)$ is a monotone increasing function of $\delta$,*
(ii) *$\lim_{\delta\to 0}\Omega(f;\delta) = 0$,*
(iii) *for any $\lambda \in [0,\infty)$, $\Omega(f;\lambda\delta) \leqslant (1+\lambda)\Omega(f;\delta)$.*

*By property* (iii), *we get*

$$|f(t)-f(x)| \leqslant 2\left(1+\frac{|t-x|}{\delta}\right)(1+\delta^2)(1+x^2) \times \left(1+(t-x)^2\right)\Omega(f;\lambda\delta), \quad t,x\in[0,\infty). \tag{3.4}$$

**Theorem 3.4.** *Let $f \in C^*_{x^2}[0,\infty)$, then we have*

$$\sup_{x\geqslant 0} \frac{|S_n^{\alpha,\beta}(f;x-f(x))|}{(1+x^2)^3} \leqslant M_{(\alpha,\beta,\mu)}\left(1+\frac{n}{(n+\beta)^2}\right)\Omega\left(f;\sqrt{\frac{n}{(n+\beta)^2}}\right),$$

*where $M_{(\alpha,\beta,\mu)}$ is a constant independent of $n$.*

*Proof.* By (3.4), we have

$$|f(t)-f(x)| \leqslant 2\left(1+\frac{|t-x|}{\delta}\right)(1+\delta^2)(1+x^2)\left(1+(t-x)^2\right)\Omega(f;\lambda\delta).$$

letting $t = \frac{k+2\mu_{\theta_k}+\alpha}{n+\beta}$, we get

$$\left|f\left(\frac{k+2\mu_{\theta_k}+\alpha}{n+\beta}\right)-f(x)\right| \leqslant 2\left(1+\frac{\left|\frac{k+2\mu_{\theta_k}+\alpha}{n+\beta}-x\right|}{\delta}\right)$$
$$\times(1+\delta^2)(1+x^2)\left(1+(t-x)^2\right)\Omega(f;\lambda\delta).$$

Thus,

$$\begin{aligned}S_n^{\alpha,\beta}((f;x)-f(x)) &\leqslant \frac{1}{e_\mu(nx)}\sum_{k=0}^{\infty}\frac{(nx)^k}{\gamma_\mu(k)}\left|f\left(\frac{k+2\mu_{\theta_k}+\alpha}{n+\beta}\right)-f(x)\right| \\ &\leqslant 2(1+\delta^2)(1+x^2)\Omega\left(f;\delta\right)\frac{1}{e_\mu(nx)} \\ &\times\sum_{k=0}^{\infty}\frac{(nx)^k}{\gamma_\mu(k)}\left(1+\frac{\left|\frac{k+2\mu_{\theta_k}+\alpha}{n+\beta}-x\right|}{\delta}\right)\left(1+\left(\frac{k+2\mu_{\theta_k}+\alpha}{n+\beta}-x\right)^2\right) \\ &= 2(1+\delta^2)(1+x^2)\Omega\left(f;\delta\right)\frac{1}{e_\mu(nx)} \\ &\times\left\{\sum_{k=0}^{\infty}\frac{(nx)^k}{\gamma_\mu(k)}+\sum_{k=0}^{\infty}\frac{(nx)^k}{\gamma_\mu(k)}\left(\frac{k+2\mu_{\theta_k}+\alpha}{n+\beta}-x\right)^2\right. \\ &+\frac{1}{\delta}\sum_{k=0}^{\infty}\frac{(nx)^k}{\gamma_\mu(k)}\left|\frac{k+2\mu_{\theta_k}+\alpha}{n+\beta}-x\right|\left(\frac{k+2\mu_{\theta_k}+\alpha}{n+\beta}-x\right) \\ &\left.+\frac{1}{\delta}\sum_{k=0}^{\infty}\frac{(nx)^k}{\gamma_\mu(k)}\left|\frac{k+2\mu_{\theta_k}+\alpha}{n+\beta}-x\right|^2\right\}.\end{aligned}$$

Applying the Cauchy–Schwartz inequality to above last term, we obtain

$$\begin{aligned}S_n^{\alpha,\beta}((f;x)-f(x)) &\leqslant 2(1+\delta^2)(1+x^2)\Omega\left(f;\delta\right) \\ &\times\left(1+S_n^{\alpha,\beta}((t-x)^2;x)+\frac{1}{\delta}\sqrt{S_n^{\alpha,\beta}((t-x)^2;x)}\right. \\ &\left.+\frac{1}{\delta}.\sqrt{S_n^{\alpha,\beta}((t-x)^2;x)S_n^{\alpha,\beta}((t-x)^4;x)}\right).\end{aligned}\tag{3.5}$$

By (2.7) and (2.8), we have the following estimates

$$\begin{aligned}S_n^{\alpha,\beta}((t-x)^2;x) &\leqslant \frac{n}{(n+\beta)^2}\left((\beta+\alpha)^2+1+2\mu\right)(x^2+x+1), \\ S_n^{\alpha,\beta}((t-x)^4;x) &\leqslant \frac{n}{(n+\beta)^2}\Big(\alpha^4+\beta^4+8\mu^3+12\mu^2+24\mu+32\mu(\alpha+\beta) \\ &+6(\alpha^2\beta^2+\alpha^2+\beta^2) \\ &+4\mu(2\mu\alpha+3\alpha+3\beta^2)+4\alpha+3\Big)(x^4+x^3+x^2+x+1).\end{aligned}$$

Combining above inequality with (3.5), one has

$$S_n^{\alpha,\beta}((f;x)-f(x)) \leqslant 2(1+\delta^2)(1+x^2)\Omega(f;\delta) \times\left\{1+A_{(\alpha,\beta,\mu)}(x)+\frac{1}{\delta}\sqrt{\frac{n}{(n+\beta)^2}}\left(\sqrt{A_{(\alpha,\beta,\mu)}(x)} +\sqrt{A_{(\alpha,\beta,\mu)}(x)B_{(\alpha,\beta,\mu)}(x)}\right)\right\},$$

where

$$A_{(\alpha,\beta,\mu)}(x) = \left((\beta+\alpha)^2+1+2\mu\right)(x^2+x+1),$$

$$B_{(\alpha,\beta,\mu)}(x)=\Big(\alpha^4+\beta^4+8\mu^3+12\mu^2+24\mu+32\mu(\alpha+\beta)+6(\alpha^2\beta^2+\alpha^2+\beta^2) + 4\mu(2\mu\alpha+3\alpha+3\beta^2)+4\alpha+3\Big) \times\left(x^4+x^3+x^2+x+1\right).$$

If choose $\delta=\sqrt{\frac{n}{(n+\beta)^2}}$, we get desired the result. □

## 4. Rate of Convergence of the Operators

In this section, we compute the order of approximation of the operator $S_n^{\alpha,\beta}(f;x)$ by means of modulus of continuity which is given by

$$\omega(f;\delta)=\sup_{\substack{|x-y|\leqslant\delta\\ x,y\in[0,a]}}|f(x)-f(y)|.$$

It is well-known that for a function $f\in C[0,a]$,

$$\lim_{\delta\to 0^+}\omega(f;\delta)=0$$

for any $\forall\delta>0$,

$$|f(x)-f(y)|\leqslant\omega(f;\delta)\left(\frac{|x-y|}{\delta}+1\right) \tag{4.1}$$

**Theorem 4.1.** *Let $f\in C[0,\infty)\cap E$ and $x\geqslant 0$. Then, we have*

$$\left|S_n^{\alpha,\beta}(f;x)-f(x)\right|\leqslant 2w(f;\delta_{n,x}),$$

*where*

$$\delta_{n,x}^2=\frac{\beta^2}{(n+\beta)^2}x^2+\frac{n\left(1+2\mu\frac{e_\mu(-nx)}{e_\mu(nx)}\right)}{(n+\beta)^2}x+\frac{\alpha^2}{(n+\beta)^2}.$$

*Proof.* Let $f\in C[0,\infty)\cap E$. By the linearity and positivity properties of the operators $S_n^{\alpha,\beta}(f;x)$, we get

$$\left|S_n^{\alpha,\beta}(f;x)-f(x)\right|\leqslant w(f;\delta)\left\{1+\frac{1}{\delta}\left(S_n^{\alpha,\beta}\left((t-x)^2;x\right)\right)^{\frac{1}{2}}\right\}.$$

By (2.7) and choosing $\delta_{n,x}=\delta$ the proof is completed. □

## 5. Direct Estimate

In this section, we establish the following direct result:

**Theorem 5.1.** *Let $f$ be a bounded function on $[0,\infty)$, and has a second derivative at a point $x \in [0,\infty)$. Then, we have*

$$\lim_{n\to\infty} n\left(S_n^{\alpha,\beta}(f;x) - f(x)\right) = (\alpha - x\beta)f'(x) + \frac{(1+2\mu)x}{2}f''(x).$$

*Proof.* Using Taylor's expansion of $f$, we obtain

$$f(t) = f(x) + f'(x)(t-x) + \frac{1}{2}f''(x)(t-x)^2 + \varepsilon(t,x)(t-x)^2,$$

where $\varepsilon(t,x)$ is the remainder term and $\varepsilon(t,x) \longrightarrow 0$ as $t \longrightarrow x$. By linearity of the operators $S_n^{\alpha,\beta}(f,x)$ we get

$$S_n^{\alpha,\beta}(f;x) - f(x) = f'(x)S_n^{\alpha,\beta}((t-x);x) + \frac{1}{2}f''(x)S_n^{\alpha,\beta}((t-x)^2;x)$$
$$+S_n^{\alpha,\beta}\left(\varepsilon(t,x)(t-x)^2;x\right).$$

Using Cauchy–Schwartz inequality, we get

$$S_n^{\alpha,\beta}\left(\varepsilon(t,x)(t-x)^2;x\right) \leqslant \sqrt{S_n^{\alpha,\beta}(\varepsilon^2(t,x);x)}\sqrt{S_n^{\alpha,\beta}((t-x)^4;x)}. \quad (5.1)$$

Since $\varepsilon^2(x,x) = 0$ and $\varepsilon^2(.,x) \in C_2^*[0,\infty)$, we obtain

$$\lim_{n\to\infty} S_n^{\alpha,\beta}(\varepsilon^2(t,x);x) = \varepsilon^2(x,x) = 0, \quad (5.2)$$

uniformly with respect to $[0,A]$. By (5.1) and (5.2), we have

$$\lim_{n\to\infty} nS_n^{\alpha,\beta}(\varepsilon^2(t,x);x) = 0. \quad (5.3)$$

From Lemma 2.2 and (5.3), one can see that

$$\lim_{n\to\infty} n\left(S_n^{\alpha,\beta}(f;x) - f(x)\right) = \lim_{n\to\infty} n\Big[S_n^{\alpha,\beta}((t-x);x)f'(x)$$
$$+\frac{1}{2}f''(x)S_n^{\alpha,\beta}((t-x)^2;x)$$
$$+ S_n^{\alpha,\beta}(\varepsilon(t,x)(t-x)^2;x)\Big]$$
$$= (\alpha - x\beta)f'(x) + \frac{(1+2\mu)x}{2}f''(x).$$

This step completes the proof. □

Now, we give a local approximation theorem regarding the operators $S_n^{\alpha,\beta}$. Let $C_B[0,\infty)$ denote the space of all real valued bounded and continuous functions $f$ on $[0,\infty)$ with the norm

$$\|f\| = \sup_{x\in[0,\infty)} |f(x)|.$$

For all $\delta > 0$, the Peetre's $K$-functional is given by

$$K_2(f,\delta) = \inf_{h\in C_B^2[0,\infty)}\{\|f-h\| + \delta\|h''\|\}$$

where $C_B^2[0,\infty) = \{h \in C_B[0,\infty) : h', h'' \in C_B[0,\infty)\}$. By [12], $\exists C > 0$ such that

$$K_2(f,\delta) \leqslant C\omega_2(f,\delta)$$

where the second order modulus of continuity of $f \in C_B[0,\infty)$ denoted by

$$\omega_2\left(f,\sqrt{\delta}\right) = \sup_{0<p<\sqrt{\delta}} \sup_{x\in[0,\infty)} |f(x+2p) - 2f(x+p) + f(x)|.$$

Also, let $\omega(f,\delta)$ is denoted the usual modulus of continuity of $f \in C_B[0,\infty)$. Now, we introduce the auxiliary operators

$$\widetilde{S_n^{\alpha,\beta}}(f;x) = S_n^{\alpha,\beta}(f;x) - f\left(\frac{nx+\alpha}{n+\beta}\right) + f(x)$$

where $f \in C_B[0,\infty), x \geqslant 0$.

**Lemma 5.2.** *Let $h \in C_B^2[0,\infty)$. Then, for all $x \geqslant 0$, we get* $\left|\widetilde{S^{\alpha,\beta}}_n(f;x)\right.$ $\left.-h(x)\right| \leqslant \phi_n(x)\|h''\|$ *where*

$$\phi_n(x) = \frac{2\beta^2x^2 + 2\alpha^2 + n(1+2\mu)x}{(n+\beta)^2}.$$

*Proof.* It is clear that $\widetilde{S_n^{\alpha,\beta}}(e_1 - x;x) = 0$. Let $h \in C_B^2[0,\infty)$. From Taylor's expansion of $h$

$$h(t) - h(x) = (t-x)h'(x) + \int_x^t (t-u)h''(u)du$$

where $t \in [0,\infty)$. Applying $\widetilde{S_n^{\alpha,\beta}}$ to the both sides of the above equation, we get

$$\begin{aligned}\widetilde{S_n^{\alpha,\beta}}(h;x) - h(x) &= h'(x)\widetilde{S_n^{\alpha,\beta}}(t-x;x) + \widetilde{S_n^{\alpha,\beta}}\left(\int_x^t (t-u)h''(u)du;x\right)\\ &= \widetilde{S_n^{\alpha,\beta}}\left(\int_x^t (t-u)h''(u)du;x\right)\\ &= S_n^{\alpha,\beta}\left(\int_x^t (t-u)h''(u)du;x\right) - \int_x^{\frac{nx+\alpha}{n+\beta}} \left(\frac{nx+\alpha}{n+\beta} - u\right)h''(u)du\end{aligned}$$

and so

$$\begin{aligned}|\widetilde{S_n^{\alpha,\beta}}(h;x) - h(x)| \leqslant S_n^{\alpha,\beta}&\left(\left|\int_x^t (t-u)h''(u)du\right|;x\right)\\ &+\left|\int_x^{\frac{nx+\alpha}{n+\beta}} \left(\frac{nx+\alpha}{n+\beta} - u\right)h''(u)du\right| \qquad (5.4)\end{aligned}$$

Since

$$\left|\int_x^t (t-u)h''(u)du\right| \leqslant (t-x)^2\|h''\|,$$

we have

$$\left|\int_x^{\frac{nx+\alpha}{n+\beta}} \left(\frac{nx+\alpha}{n+\beta}-u\right) h''(u)du\right| \leqslant \frac{(\alpha-x\beta)^2}{(n+\beta)^2}\|h''\|.$$

By (5.4), we have

$$\left|\widetilde{S_n^{\alpha,\beta}}(h;x)-h(x)\right| \leqslant \left\{S_n^{\alpha,\beta}\left((t-x)^2;x\right)+\frac{(\alpha-x\beta)^2}{(n+\beta)^2}\right\}\|h''\|.$$

By (2.7), one can see that

$$\left|\widetilde{S_n^{\alpha,\beta}}(h;x)-h(x)\right| \leqslant \left\{\frac{2\beta^2x^2+2\alpha^2+n(1+2\mu)x}{(n+\beta)^2}\right\}\|h''\| = \phi_n(x)\|h''\|.$$

This completes the proof of lemma. □

**Theorem 5.3.** *Let $f\in C_B[0,\infty)$. Then, for every $x\geqslant 0$, then exists a constant $C>0$ such that*

$$\left|S_n^{\alpha,\beta}(f;x)-f(x)\right| \leqslant C\omega_2\left(f,\sqrt{\phi_n(x)}\right)+\omega\left(f;\frac{\alpha-x\beta}{n+\beta}\right)$$

*where $\phi_n(x)$ is defined as above.*

*Proof.* For $f\in C_B[0,\infty)$, $h\in C_B^2[0,\infty)$, by definition of $\widetilde{S_n^{\alpha,\beta}}$, we get

$$\left|S_n^{\alpha,\beta}(f;x)-f(x)\right| \leqslant \left|\widetilde{S_n^{\alpha,\beta}}(f-h;x)\right|+|(f-h)(x)|+\left|\widetilde{S_n^{\alpha,\beta}}(h;x)-h(x)\right| + \left|f\left(\frac{nx+\alpha}{n+\beta}\right)-f(x)\right|$$

and

$$\left|\widetilde{S_n^{\alpha,\beta}}(f;x)\right| \leqslant \|f\|\,S_n^{\alpha,\beta}(1;x)+2\|f\| = 3\|f\|.$$

Thus, we get

$$\left|S_n^{\alpha,\beta}(f;x)-f(x)\right| \leqslant 4\|f-h\|+\left|\widetilde{S_n^{\alpha,\beta}}(h;x)-h(x)\right|+\omega\left(f;\frac{\alpha-x\beta}{n+\beta}\right)$$

and using (5.5), it is easy to get

$$|(f;x)-f(x)| \leqslant 4\left(\|f-h\|+\phi_n(x)\,h''\right)+\omega\left(f;\frac{\alpha-x\beta}{n+\beta}\right).$$

Thus, taking the infimum of all $h\in C_B^2[0,\infty)$ on the right hand side of the last inequality, we obtain the desired result. □

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