Adv. Appl. Clifford Algebras 26 (2016), 969–984

0188-7009/030969-16
published online November 2, 2015
DOI 10.1007/s00006-015-0620-3

Advances in Applied Clifford Algebras

The Quaternion Domain Fourier Transform and its Properties

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Abstract. So far quaternion Fourier transforms have been mainly defined over $\mathbb{R}^2$ as signal domain space. But it seems natural to define a quaternion Fourier transform for quaternion valued signals over *quaternion domains*. This *quaternion domain Fourier transform* (QDFT) transforms quaternion valued signals (for example electromagnetic scalar-vector potentials, color data, space-time data, etc.) defined over a quaternion domain (space-time or other 4D domains) from a quaternion position space to a quaternion frequency space. The QDFT uses the full potential provided by hypercomplex algebra in higher dimensions and may moreover be useful for solving quaternion partial differential equations or functional equations, and in crystallographic texture analysis. We define the QDFT and analyze its *main properties*, including quaternion dilation, modulation and shift properties, Plancherel and Parseval identities, covariance under orthogonal transformations, transformations of coordinate polynomials and differential operator polynomials, transformations of derivative and Dirac derivative operators, as well as signal width related to band width uncertainty relationships.

Mathematics Subject Classification. Primary 11R52; Secondary 42A38, 15A66.

Keywords. Quaternions, Fourier transform, Signal and image processing, Uncertainty, Orthogonal transformations in three and four dimensions, Quaternionic differential operators.

1. Introduction

The electromagnetic field equations were originally formulated by Maxwell [27] in the language of Hamilton's quaternions [18]. Later, among many other

This paper is dedicated to Wu Xiangdong, who died 4th June 1989 on Tienanmen Square in Beijing [1]. The use of this paper is subject to the *Creative Peace License* [19].
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applications, quaternions began to play an important role in aerospace engineering [25], color signal processing [15], artificial intelligence [3–5], and in material science for texture analysis (understood as the distribution of crystallographic orientations of a polycrystalline sample [34]) [2,28].

Quaternion Fourier transforms (QFT) are since over 20 years a mathematically well researched and frequently applied subject [9]. Yet interesting enough most publications on QFTs concentrate on transformations for signals with domain $\mathbb{R}^2$. Motivated by private communication with T. L. Saaty related to *quaternion valued functions over the domain of quaternions*, we establish here a genuine Fourier transform with a quaternionic kernel operating on such functions.

This paper begins by introducing quaternions and their relevant properties, including quaternion domain functions in Sect. 2. The quaternion (algebra) domain Fourier transform (QDFT) is defined in Sect. 3. Many fundamental properties of the QDFT are investigated in Sect. 4. The conclusions in Sect. 5 give an outlook into the wide area of possible applications and the rich possibilities of studying related transforms for quaternion domain signals.

2. Definition and Properties of Quaternions $\mathbb{H}$

2.1. Basic Facts About Quaternions

Gauss, Rodrigues and Hamilton's four-dimensional (4D) *quaternion algebra* $\mathbb{H}$ is defined over $\mathbb{R}$ with three imaginary units:

$$\begin{aligned} &\boldsymbol{ij} = -\boldsymbol{ji} = \boldsymbol{k}, \quad \boldsymbol{jk} = -\boldsymbol{kj} = \boldsymbol{i}, \quad \boldsymbol{ki} = -\boldsymbol{ik} = \boldsymbol{j}, \\ &\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1. \end{aligned} \tag{2.1}$$

Every quaternion can be written explicitly as

$$q = q_r + q_i\boldsymbol{i} + q_j\boldsymbol{j} + q_k\boldsymbol{k} \in \mathbb{H}, \quad q_r, q_i, q_j, q_k \in \mathbb{R}, \tag{2.2}$$

and has a *quaternion conjugate* (equivalent to reversion in $Cl^+_{3,0}$)

$$\tilde{q} = q_r - q_i\boldsymbol{i} - q_j\boldsymbol{j} - q_k\boldsymbol{k}, \quad \widetilde{pq} = \tilde{q}\tilde{p}. \tag{2.3}$$

This leads to the *norm* of $q \in \mathbb{H}$

$$|q| = \sqrt{q\tilde{q}} = \sqrt{q_r^2 + q_i^2 + q_j^2 + q_k^2}, \quad |pq| = |p||q|. \tag{2.4}$$

The *inverse* of a non-zero quaternion $q \in \mathbb{H}$ is

$$q^{-1} = \frac{\tilde{q}}{|q|^2}. \tag{2.5}$$

The (*symmetric*) *scalar part* of a quaternion is defined as

$$\langle q\rangle_0 = Sc(q) = q_r = \frac{1}{2}(q + \tilde{q}), \quad Sc(pq) = Sc(qp) = Sc(\tilde{p}\tilde{q}), \tag{2.6}$$

$$Sc(pqr) = Sc(qrp) = Sc(rpq). \tag{2.7}$$

Every quaternion $a \in \mathbb{H}$, $a \neq 0$, can be written as scalar part plus (*pure*) *vector part*

$$a = a_r + a_i\boldsymbol{i} + a_j\boldsymbol{j} + a_k\boldsymbol{k} = a_r + \mathbf{a} = |a|\left(\cos\alpha + \frac{\mathbf{a}}{|a|}\sin\alpha\right) = |a|e^{\hat{\mathbf{a}}\alpha}, \tag{2.8}$$

with $\hat{\mathbf{a}} = \mathbf{a}/|a|$, $\cos\alpha = a_r/|a|$, $\alpha \in [0,\pi)$. A *scalar product* of quaternions can be defined for $x, y \in \mathbb{H}$ as

$$x \cdot y = Sc(\tilde{x}y) = x_r y_r + x_i y_i + x_j y_j + x_k y_k. \tag{2.9}$$

Two quaternions interpreted as elements of $\mathbb{R}^4$ are defined to be *orthogonal*, if and only if their scalar product is zero

$$x \perp y \quad \Leftrightarrow \quad x \cdot y = 0. \tag{2.10}$$

Pure quaternions have zero scalar part. A (*normed*) *unit pure quaternion* $\mathbf{q}$ squares to -1

$$\mathbf{q}^2 = -(q_i^2 + q_j^2 + q_k^2) = -1. \tag{2.11}$$

The set of unit pure quaternions is isomorphic to the unit sphere $S^2 \subset \mathbb{R}^3$. Quaternion multiplication pq can be alternatively represented by the following *matrix vector multiplication* [35]

$$\begin{pmatrix} Sc(pq) \\ (pq)_i \\ (pq)_j \\ (pq)_k \end{pmatrix} = \begin{pmatrix} p_r & -p_i & -p_j & -p_k \\ p_i & p_r & -p_k & p_j \\ p_j & p_k & p_r & -p_i \\ p_k & -p_j & p_i & p_r \end{pmatrix} \begin{pmatrix} q_r \\ q_i \\ q_j \\ q_k \end{pmatrix}. \tag{2.12}$$

The *determinant* of the above matrix $P(p)$ is simply

$$\det P(p) = |p|^4. \tag{2.13}$$

If we interpret the four real coefficients of $x \in \mathbb{H}$, $x_r, x_i, x_j, x_k \in \mathbb{R}$ as *coordinates* in $\mathbb{R}^4$, with *infinitesimal volume element* $d^4x = dx_r dx_i dx_j dx_k$, then the substitution $z = ax$, $a \in \mathbb{H}$, yields

$$z = ax \quad \Rightarrow \quad d^4z = |a|^4 d^4x, \qquad d^4x = |a|^{-4} d^4z, \tag{2.14}$$

assuming $a \neq 0$ for the last identity.

For the transformation $z = axb$, $a, b, x \in \mathbb{H}$, we set $y = xb$ and then

$$d^4z = |a|^4 d^4y. \tag{2.15}$$

Quaternion *conjugation* leads to

$$\tilde{y} = \tilde{b}\tilde{x}, \tag{2.16}$$

such that

$$-d^4y = d^4\tilde{y} = |\tilde{b}|^4 d^4\tilde{x} = -|b|^4 d^4x, \tag{2.17}$$

because $|\tilde{b}| = |b|$, $d^4\tilde{x} = dx_r(-dx_i)(-dx_j)(-dx_k) = -d^4x$, and similarly $d^4\tilde{y} = -d^4y$. Hence

$$d^4z = |a|^4 d^4y = |a|^4|b|^4 d^4x = |ab|^4 d^4x. \tag{2.18}$$

As expected the *rotation* (2.25) does not change the infinitesimal volume element

$$z = axa^{-1} \quad \Rightarrow \quad d^4x = |aa^{-1}|^4 d^4x = d^4x. \tag{2.19}$$

We follow [17] in defining the following *derivative operators*

$$\tilde{\partial} = \partial_{x_r} + \partial_{x_i}\boldsymbol{i} + \partial_{x_j}\boldsymbol{j} + \partial_{x_k}\boldsymbol{k}, \tag{2.20}$$

$$\partial = \partial_{x_r} - \partial_{x_i}\boldsymbol{i} - \partial_{x_j}\boldsymbol{j} - \partial_{x_k}\boldsymbol{k}, \tag{2.21}$$

where $\partial_{x_r} = \partial/\partial x_r$, etc. We further define the three-dimensional *Dirac operator*

$$D = \tilde{\partial} - \partial_{x_r} = \partial_{x_i}\boldsymbol{i} + \partial_{x_j}\boldsymbol{j} + \partial_{x_k}\boldsymbol{k}, \quad \tilde{\partial} = \partial_{x_r} + D. \tag{2.22}$$

The *orthogonal planes split* of $q \in \mathbb{H}$ with (one) pure unit quaternion $f = g = I$, $I \in \mathbb{H}$, $I^2 = -1$, [20,22,24] is defined[1] as

$$q_{\pm} = \frac{1}{2}(q \pm IqI), \quad q_- = q_r + q_I I,$$
$$q_+ = q_J J + q_K K = (q_J + q_K I)J, \tag{2.23}$$

with *rotation operator* $R = (\boldsymbol{i}+I)\boldsymbol{i}$, $J = R\boldsymbol{j}R^{-1}$ and $K = R\boldsymbol{k}R^{-1}$, $J^2 = K^2 = -1$, $q_r, q_I, q_J, q_K \in \mathbb{R}$, similar to [24]. Note, that there is a *gauge freedom* in this split by changing $R \to R\exp(I\varphi/2)$, $\varphi \in [0, 2\pi)$, i.e. a rotation freedom in the q_+-plane. The units $\{1, I, J, K\}$ form another *equivalent representation* of quaternions $\mathbb{H}$. Note further, that the q_- part *commutes* with I, whereas the q_+ part *anticommutes*

$$q_- I = I q_-, \quad q_+ I = -I q_+. \tag{2.24}$$

2.2. Quaternions and Reflections and Rotations in Three and Four Dimensions

The *geometry of reflections and rotations* in three and four dimensions, expressed in the language of quaternions is discussed in [12,23,24,28]. We give an overview of how important *orthogonal transformations* in three-dimensional and four-dimensional Euclidean space can be expressed by means of quaternions.

A three-dimensional *rotation* of the vector part $\mathbf{x}$ of the quaternion $x \in \mathbb{H}$ by the angle 2α around the axis $\hat{\mathbf{a}}$, leaving the scalar part x_r invariant, is given by

$$x' = axa^{-1}, \quad a = e^{\alpha\hat{\mathbf{a}}}, \quad \hat{\mathbf{a}}^2 = -1. \tag{2.25}$$

For example $a = \cos\alpha + \boldsymbol{k}\sin\alpha = \exp(\boldsymbol{k}\alpha)$ rotates $x = \boldsymbol{i}$ to

$$x' = a\boldsymbol{i}a^{-1} = e^{\boldsymbol{k}\alpha}\boldsymbol{i}e^{-\boldsymbol{k}\alpha} = e^{2\boldsymbol{k}\alpha}\boldsymbol{i} = (\cos 2\alpha + \boldsymbol{k}\sin 2\alpha)\boldsymbol{i}$$
$$= \cos(2\alpha)\boldsymbol{i} + \sin(2\alpha)\boldsymbol{j}. \tag{2.26}$$

We further note, that the transformation

$$x' = axb, \quad a = e^{\alpha\hat{\mathbf{a}}}, \quad b = e^{\beta\hat{\mathbf{a}}}, \tag{2.27}$$

[1] The references [20,22,24] contain examples with values of ($f = \boldsymbol{i}$, $g = \boldsymbol{j}$), ($f = \boldsymbol{i}$, $g = \boldsymbol{i}$), etc. In the case that $f = g = (\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k})/\sqrt{3}$ we obtain the split into luminosity and chromaticity of a color image [15].

rotates the x_--part by the angle $\alpha+\beta$ in the q_--plane (determined by (2.23), setting $I = \hat{\mathbf{a}}$), and rotates the x_+-part by $\alpha - \beta$ in the q_+-plane.

The *4D reflection* at the *real line* is given by quaternion conjugation $x \to \tilde{x}$, leaving the real line pointwise invariant.

The *4D reflection* at the *3D hyperplane* of pure quaternions is therefore given by $x \to -\tilde{x}$, leaving the 3D hyperplane of pure quaternions pointwise invariant.

A *reflection* at a (pointwise invariant) *general line* in $\mathbb{R}^4$ in the direction of the unit quaternion $a \in \mathbb{H}$, $|a| = 1$, is given by $x \to a\tilde{x}a$.

A *reflection* at the (pointwise invariant) *three-dimensional hyperplane* orthogonal to the direction in four dimensions given by the unit quaternion a, $|a| = 1$, is given by $x \to -a\tilde{x}a$.

A *general rotation* in $\mathbb{R}^4$ is given by

$$x \to axb, \quad a, b \in \mathbb{H}, \quad |a| = |b| = 1. \tag{2.28}$$

To understand the *geometry of this rotation* [24], we rewrite the unit quaternions a, b as

$$a = e^{\alpha\hat{\mathbf{a}}}, \quad b = e^{\beta\hat{\mathbf{b}}}. \tag{2.29}$$

The pure unit quaternions $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ define two orthogonal two-dimensional rotation planes in $\mathbb{R}^4$, where without restriction of generality we assume $\hat{\mathbf{a}} \neq \hat{\mathbf{b}}$, because the case $\hat{\mathbf{a}} = \hat{\mathbf{b}}$ has already been discussed in (2.27). The $q_+^{a,b}$ plane with orthogonal basis and projection

$$q_+^{a,b} \text{ basis} : \{\hat{\mathbf{a}} - \hat{\mathbf{b}}, 1 + \hat{\mathbf{a}}\hat{\mathbf{b}}\}, \quad q_+^{a,b} = \frac{1}{2}(q + \hat{\mathbf{a}}q\hat{\mathbf{b}}), \tag{2.30}$$

and the orthogonal $q_-^{a,b}$ plane orthogonal basis and projection

$$q_-^{a,b} \text{ basis} : \{\hat{\mathbf{a}} + \hat{\mathbf{b}}, 1 - \hat{\mathbf{a}}\hat{\mathbf{b}}\}, \quad q_-^{a,b} = \frac{1}{2}(q - \hat{\mathbf{a}}q\hat{\mathbf{b}}), \tag{2.31}$$

such that $q = q_+^{a,b} + q_-^{a,b}$, for all $q \in \mathbb{H}$. The transformation $x \to axb$ of (2.28) then means geometrically a rotation by the angle $\alpha - \beta$ in the $q_+^{a,b}$ plane (around the $q_-^{a,b}$ plane as axis) and a rotation by the angle $\alpha + \beta$ in the $q_-^{a,b}$ plane (around the $q_+^{a,b}$ plane as axis). This also tells us, that for $\alpha = \beta$ the rotation degenerates to a single two-dimensional rotation by 2α in the $q_-^{a,b}$ plane, and for $\alpha = -\beta$ it degenerates to a single two-dimensional rotation by 2α in the $q_+^{a,b}$ plane.

A *general rotary reflection* (rotation reflection) in $\mathbb{R}^4$ is given by

$$x \to a\tilde{x}b, \quad a, b \in \mathbb{H}, \quad |a| = |b| = 1. \tag{2.32}$$

This rotary reflection has the pointwise invariant line through $a + b$. In the remaining three-dimensional hyperplane, orthogonal to the $a+b$ line, the axis of the rotary reflection is the line in the direction $a - b$, because $a(a - b)b = -(a - b)$. The rotation plane of the rotary reflection is spanned by the two orthogonal quaternions $v_{1,2} = [a, b](1 \pm \tilde{a}b)$, $[a, b] = ab - ba$, and the angle of rotation is $\Gamma = \pi - \text{acrccos}(Sc(\tilde{a}b))$ [24].

2.3. Quaternion Domain Functions

Every real valued quaternion domain function f maps $\mathbb{H} \to \mathbb{R}$:

$$f : x \mapsto f(x) \in \mathbb{R}, \quad \forall x \in \mathbb{H}. \tag{2.33}$$

Every *quaternion valued quaternion domain function* f maps $\mathbb{H} \to \mathbb{H}$, its four coefficient functions f_r, f_i, f_j, f_k, are in turn real valued quaternion domain functions:

$$f : x \mapsto f(x) = f_r(x) + f_i(x)\boldsymbol{i} + f_j(x)\boldsymbol{j} + f_k(x)\boldsymbol{k} \in \mathbb{H}. \tag{2.34}$$

Quaternion valued quaternion domain functions have been historically studied in [16,30,32,33], and applications are described in [17].

We define for two functions $f, g : \mathbb{H} \to \mathbb{H}$ the following *quaternion valued inner product*

$$(f, g) = \int_{\mathbb{H}} f(x)\tilde{g}(x)\, d^4x \tag{2.35}$$

with $d^4x = dx_r dx_i dx_j dx_k \in \mathbb{R}$. Note that quaternion conjugation yields

$$\widetilde{(f, g)} = (g, f). \tag{2.36}$$

This means that the real scalar part of the *inner product* (f, g) is *symmetric*

$$\langle f, g\rangle = \frac{1}{2}[(f, g) + (g, f)] = \int_{\mathbb{H}} \langle f(x)\tilde{g}(x)\rangle_0 d^4x \in \mathbb{R},$$
$$\langle f, g\rangle = \langle g, f\rangle. \tag{2.37}$$

We further define the $L^2(\mathbb{H}; \mathbb{H})$-*norm*[2] as

$$\|f\| = \sqrt{(f, f)} = \sqrt{\langle f, f\rangle} = \sqrt{\int_{\mathbb{H}} |f(x)|^2\, d^4x} \geq 0. \tag{2.38}$$

The *quaternion domain module* $L^2(\mathbb{H}; \mathbb{H})$ is the set of all finite $L^2(\mathbb{H}; \mathbb{H})$-norm functions

$$L^2(\mathbb{H}; \mathbb{H}) = \{f | f : \mathbb{H} \to \mathbb{H}, \|f\| \leq \infty\}. \tag{2.39}$$

The *convolution* of two functions $f, g \in L^2(\mathbb{H}; \mathbb{H})$ is defined as

$$(f * g)(x) = \int_{\mathbb{H}} f(y)g(x - y)\, d^4y. \tag{2.40}$$

For unit norm signals $f \in L^2(\mathbb{H}; \mathbb{H})$, $\|f\| = 1$, we define the *effective spatial width* or spatial uncertainty (or signal width) of f *in the direction* of the unit quaternion $a \in \mathbb{H}$, $|a| = 1$, as the square root of the variance of the energy distribution of f along the a-axis

$$\Delta x_a = \|(x \cdot a)f\| = \sqrt{\int_{\mathbb{H}} (x \cdot a)^2 |f(x)|^2\, d^4x}. \tag{2.41}$$

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[2] Note that in equation (13) of [20] the square root is missing over the integral in the definition of the $L^2(\mathbb{R}^2; \mathbb{H})$-norm.

Also for unit norm signals f, we define the *effective spatial width* (spatial uncertainty) as the square root of the variance of the energy distribution of f

$$\Delta x = \|xf\| = \sqrt{\int_{\mathbb{H}} |x|^2 |f(x)|^2 \, d^4x}. \tag{2.42}$$

3. The Quaternion Domain Fourier Transform

Since the traditional quaternion Fourier transform (QFT) [10,14,20] is only defined for real or quaternion valued signals over the domain $\mathbb{R}^2$, we newly define the *quaternion domain Fourier transform* (QDFT) for $h \in L^1(\mathbb{H};\mathbb{H})$ as

$$\mathcal{F}\{h\}(\omega) = \hat{h}(\omega) = \frac{1}{(2\pi)^2} \int_{\mathbb{H}} h(x) e^{-Ix\cdot\omega} \, d^4x, \tag{3.1}$$

with $x, \omega \in \mathbb{H}$, $d^4x = dx_r dx_i dx_j dx_k \in \mathbb{R}$, and some constant $I \in \mathbb{H}$, $I^2 = -1$. The constant unit pure quaternion I can be chosen specific for each problem.

Note that the QDFT of (3.1) is *steerable* due to the free choice of the unit pure quaternion unit $I \in S^2$.

This QDFT definition is *left linear*

$$\mathcal{F}\{\alpha h + \beta g\}(\omega) = \alpha \hat{h}(\omega) + \beta \hat{g}(\omega), \tag{3.2}$$

for $g, h \in L^1(\mathbb{H};\mathbb{H})$ and constants $\alpha, \beta \in \mathbb{H}$. See (4.4) for linear combinations of signals with constant quaternion coefficients multiplied from the right.

Applying the orthogonal planes split (2.23) to the signal function $h = h_+ + h_-$ and to the QDFT $\hat{h}$ we find

$$\hat{h}(\omega) = \hat{h}_+(\omega) + \hat{h}_-(\omega), \tag{3.3}$$

$$\hat{h}_+(\omega) = \frac{1}{(2\pi)^2} \int_{\mathbb{H}} h_+(x) e^{-Ix\cdot\omega} \, d^4x = \frac{1}{(2\pi)^2} \int_{\mathbb{H}} e^{+Ix\cdot\omega} h_+(x) \, d^4x, \tag{3.4}$$

$$\hat{h}_-(\omega) = \frac{1}{(2\pi)^2} \int_{\mathbb{H}} h_-(x) e^{-Ix\cdot\omega} \, d^4x = \frac{1}{(2\pi)^2} \int_{\mathbb{H}} e^{-Ix\cdot\omega} h_-(x) \, d^4x. \tag{3.5}$$

Example Following the suggestion of T. L. Saaty, we apply the QDFT transform to the *functional quaternion equation*[3]

$$h(ax) = bh(x), \quad h : \mathbb{H} \to \mathbb{H}, \tag{3.6}$$

[3] The simplest solutions of this equation take the form $h(x) = cxd$, $bc = ca$, with quaternion constants $c, d \in \mathbb{H}$. (I thank the anonymous reviewer for this hint.) In the complex domain T. L. Saaty has developed interesting solutions [31].

with quaternion constants $a, b \in \mathbb{H}$. We define the auxiliary function $h_a(x) = h(ax)$ and compute

$$\begin{aligned}\hat{h}_a(\omega) &= \frac{1}{(2\pi)^2}\int_{\mathbb{H}} h(ax)e^{-Ix\cdot\omega}\,d^4x \\ &= \frac{1}{(2\pi)^2}\int_{\mathbb{H}} h(ax)e^{-ISc((ax)^{\sim}\tilde{a}^{-1}\omega)}\,d^4x \\ &\overset{z=ax}{=} \frac{1}{(2\pi)^2}\int_{\mathbb{H}} h(z)e^{-ISc(\tilde{z}\tilde{a}^{-1}\omega)}|a|^{-4}\,d^4z \\ &= |a|^{-4}\hat{h}(\tilde{a}^{-1}\omega),\end{aligned} \tag{3.7}$$

where we used that

$$x\cdot\omega = Sc(\tilde{x}\omega) = Sc(\tilde{x}\tilde{a}\tilde{a}^{-1}\omega) = Sc((ax)^{\sim}\tilde{a}^{-1}\omega) = (ax)\cdot(\tilde{a}^{-1}\omega). \tag{3.8}$$

For $a = \alpha \in \mathbb{R}$ we get

$$\hat{h}_\alpha(\omega) = \frac{1}{|\alpha|^4}\hat{h}\left(\frac{\omega}{\alpha}\right), \tag{3.9}$$

and for $a \in \mathbb{H}$, $|a| = 1$, we get

$$\hat{h}_a(\omega) = \hat{h}(a\omega), \tag{3.10}$$

because for $a \in \mathbb{H}$, $|a| = 1$, we have $\tilde{a}^{-1} = a$. Using relationship (3.7) and left linearity we arrive at the QDFT of (3.6)

$$|a|^{-4}\hat{h}(\tilde{a}^{-1}\omega) = b\hat{h}(\omega), \tag{3.11}$$

or equivalently

$$\hat{h}(\tilde{a}^{-1}\omega) = |a|^4 b\hat{h}(\omega), \tag{3.12}$$

which seems neither less nor more complicated to solve than the original Eq. (3.6). But note, that for $a \in \mathbb{H}$, $|a| = 1$, Eqs. (3.6) and (3.12) become identical, because (3.12) then reads

$$\hat{h}(a\omega) = b\hat{h}(\omega), \tag{3.13}$$

i.e. then (3.6) is invariant under the QDFT operator.

An application of (3.7) is the four-dimensional inversion at the origin $x \to -x$ results in

$$\hat{h}_{-1}(\omega) = \hat{h}(-\omega). \tag{3.14}$$

The QDFT can *separate* the two components of a "complex" signal $f : \mathbb{H} \to \mathbb{R} + \boldsymbol{i}\mathbb{R}$, $f(x) = f_r(x) + \boldsymbol{i}f_i(x)$, into *even* and *odd components* with respect to the inversion $x \to -x$. Let

$$f(x) = f_r(x) + \boldsymbol{i}f_i(x) = f_r^e(x) + f_r^o(x) + \boldsymbol{i}(f_i^e(x) + f_i^o(x)), \tag{3.15}$$

with

$$f_r^e(-x) = f_r^e(x) = \frac{1}{2}(f_r(x) + f_r(-x)),$$
$$f_r^o(-x) = -f_r^o(x) = \frac{1}{2}(f_r(x) - f_r(-x)),$$
$$f_i^e(x) = f_i^e(-x), \quad f_i^o(x) = -f_i^o(x). \tag{3.16}$$

Using the steerability of the QDFT (3.1) we select for $I = \boldsymbol{j}$ (we could also set $I = \boldsymbol{k}$ or any other pure quaternion $\perp \boldsymbol{i}$) and have by linearity

$$\begin{aligned}\hat{f}(\omega) &= \hat{f}_r^e(\omega) + \hat{f}_r^o(\omega) + \boldsymbol{i}(\hat{f}_i^e(\omega) + \hat{f}_i^o(\omega)) \\ &= \int_{\mathbb{H}} f_r^e(x)\cos(x\cdot\omega)\,d^4\omega + \boldsymbol{j}\int_{\mathbb{H}} f_r^o(x)\sin(x\cdot\omega)\,d^4\omega \\ &\quad +\boldsymbol{i}\int_{\mathbb{H}} f_i^e(x)\cos(x\cdot\omega)\,d^4\omega + \boldsymbol{k}\int_{\mathbb{H}} f_i^o(x)\sin(x\cdot\omega)\,d^4\omega. \end{aligned} \tag{3.17}$$

Compare [26] for a similar approach to the symmetry analysis of signals $f : \mathbb{R} \to \mathbb{C}$.

4. Properties of the QDFT

Properties of the QDFT that can easily be established are *inversion*

$$h(x) = \frac{1}{(2\pi)^2}\int_{\mathbb{H}} \hat{h}(x)e^{+Ix\cdot\omega}\,d^4\omega, \tag{4.1}$$

a *shift theorem* for $g(x) = h(x-a)$, constant $a \in \mathbb{H}$,

$$\hat{g}(\omega) = \hat{h}(\omega)e^{-Ia\cdot\omega}, \tag{4.2}$$

and a *modulation theorem* for $m(x) = h(x)e^{Ix\cdot\omega_0}$, constant $\omega_0 \in \mathbb{H}$,

$$\hat{m}(\omega) = \hat{h}(\omega - \omega_0). \tag{4.3}$$

Linear combinations with constant quaternion coefficients $\alpha, \beta \in \mathbb{H}$ from the *right* lead due to (2.24) to

$$\mathcal{F}\{h\alpha + g\beta\} = \hat{h}(\omega)\alpha_+ + \hat{h}(-\omega)\alpha_- + \hat{g}(\omega)\beta_+ + \hat{g}(-\omega)\beta_-. \tag{4.4}$$

We define $g_l(x) = \partial_{x_l}h(x)$, $l \in \{r,i,j,k\}$ for the *partial derivative* of the signal function h and obtain its QDFT as

$$\widehat{g_l}(\omega) = \hat{h}(\omega)I\omega_l. \tag{4.5}$$

For example for $l = r$ we obtain

$$\widehat{\partial_{x_r}h}(\omega) = \omega_r\hat{h}(\omega)I. \tag{4.6}$$

This leads to the QDFT of the *derivative operators* (2.20) and (2.21)

$$\widehat{\tilde{\partial}^m h}(\omega) = \omega^m\hat{h}(\omega)I^m, \quad \widehat{\partial^m h}(\omega) = \widetilde{\omega}^m\hat{h}(\omega)I^m, \qquad m \in \mathbb{N}. \tag{4.7}$$

Applying the *derivative operators from the right* to the signal function h we further obtain

$$\widehat{h\tilde{\partial}^m}(\omega) = \widehat{h\omega^m}(\omega)I^m, \quad \widehat{h\partial^m}(\omega) = \widehat{h\tilde{\omega}^m}(\omega)I^m, \qquad m \in \mathbb{N}. \tag{4.8}$$

QDFT transformations of the *Dirac operator* D applied from the left and right, respectively, give

$$\widehat{D^m h}(\omega) = \boldsymbol{\omega}^m \hat{h}(\omega)I^m, \quad \widehat{hD^m}(\omega) = \widehat{h\boldsymbol{\omega}^m}(\omega)I^m, \qquad m \in \mathbb{N}. \tag{4.9}$$

where the pure quaternion part of the quaternion frequency ω is $\boldsymbol{\omega} = \omega - \omega_r$.

The QDFT of m-fold *powers of coordinates* x_l, $l \in \{r, i, j, k\}$, $m \in \mathbb{N}$, times the signal function h leads to (dual to (4.5))

$$\widehat{x_l^m h}(\omega) = \partial_{\omega_l}^m \hat{h}(\omega)I^m. \tag{4.10}$$

For example for $l = r$ we obtain

$$\widehat{x_r h}(\omega) = \partial_{\omega_r} \hat{h}(\omega)I. \tag{4.11}$$

If $P(x_r, x_i, x_j, x_k) = \sum_{m_r,m_i,m_j,m_k} \alpha_{m_r,m_i,m_j,m_k} x_r^{m_r} x_i^{m_i} x_j^{m_j} x_k^{m_k}$, with quaternion coefficients $\alpha_{m_r,m_i,m_j,m_k} \in \mathbb{H}$, is a *polynomial* of the four *coordinates* $\{x_r, x_i, x_j, x_k\}$, then the QDFT yields

$$\mathcal{F}\{P(x_r, x_i, x_j, x_k)h\}(\omega) = \sum_{m_r,m_i,m_j,m_k} \alpha_{m_r,m_i,m_j,m_k} \partial_{\omega_r}^{m_r} \partial_{\omega_i}^{m_i} \partial_{\omega_j}^{m_j} \partial_{\omega_k}^{m_k} \hat{h}(\omega) I^{m_r+m_i+m_j+m_k}. \tag{4.12}$$

For example for $P(x) = a \cdot x = a_r x_r + a_i x_i + a_j x_j + a_k x_k$ we obtain

$$\mathcal{F}\{(a \cdot x)h\}(\omega) = (a \cdot \tilde{\partial}_\omega)\hat{h}(\omega)I, \tag{4.13}$$

with $\tilde{\partial}_\omega = \partial_{\omega_r} + \partial_{\omega_i}\boldsymbol{i} + \partial_{\omega_j}\boldsymbol{j} + \partial_{\omega_k}\boldsymbol{k}$ and $a \cdot \tilde{\partial}_\omega = a_r\partial_{\omega_r} + a_i\partial_{\omega_i} + a_j\partial_{\omega_j} + a_k\partial_{\omega_k}$. We have the *dual* to (4.12) result that

$$\mathcal{F}\{P(\partial_{x_r}, \partial_{x_i}, \partial_{x_j}, \partial_{x_k})h\}(\omega) = \sum_{m_r,m_i,m_j,m_k} \alpha_{m_r,m_i,m_j,m_k} {\omega_r}^{m_r} {\omega_i}^{m_i} {\omega_j}^{m_j} {\omega_k}^{m_k} \hat{h}(\omega) I^{m_r+m_i+m_j+m_k}, \tag{4.14}$$

with the special case (dual to (4.13))

$$\mathcal{F}\{(a \cdot \tilde{\partial})h\}(\omega) = (a \cdot \omega)\hat{h}(\omega)I. \tag{4.15}$$

Note that (4.14) shows how the QDFT (with $t = x_0$, $x_1 = x_i$, $x_2 = x_j$, $x_3 = x_k$) can be used to *treat* important *partial differential equations in physics*, e.g. the heat equation, wave equation, Klein–Gordon equation, the Maxwell equations in vacuum, free particle Schrödinger and Dirac equations [36–39].

Equation (4.12) leads further (dual to left side of (4.7)) to,

$$\widehat{xh}(\omega) = \tilde{\partial}\hat{h}(\omega)I, \quad \widehat{x^m h}(\omega) = \tilde{\partial}^m \hat{h}(\omega)I^m, \qquad m \in \mathbb{N}. \tag{4.16}$$

Multiplying instead with the *quaternion conjugate* $\tilde{x}$ we obtain (*dual* to right side of (4.7))

$$\widehat{\tilde{x}h}(\omega) = \partial\hat{h}(\omega)I, \quad \widehat{\tilde{x}^m h}(\omega) = \partial^m\hat{h}(\omega)I^m, \qquad m \in \mathbb{N}. \tag{4.17}$$

Taking only the *pure vector part* of x, $\mathbf{x} = x - x_r$ we obtain (*dual* to (4.9))

$$\widehat{\mathbf{x}h}(\omega) = D_\omega\hat{h}(\omega)I, \quad \widehat{\mathbf{x}^m h}(\omega) = D_\omega^m\hat{h}(\omega)I^m, \qquad m \in \mathbb{N}, \tag{4.18}$$

where $D_\omega = \partial_{\omega_i}\boldsymbol{i} + \partial_{\omega_j}\boldsymbol{j} + \partial_{\omega_k}\boldsymbol{k}$.

We further obtain the following *QDFT Plancherel identity*, which expresses, that the quaternion valued *inner product* (2.35) of two quaternion domain module functions $f, g \in L^2(\mathbb{H};\mathbb{H})$ is given by the quaternion valued inner product of the corresponding QDFTs $\hat{f}$ and $\hat{g}$

$$(f, g) = (\hat{f}, \hat{g}). \tag{4.19}$$

As corollaries we get the corresponding *QDFT Plancherel identity* for the *scalar inner product* of Eq. (2.37)

$$\langle f, g\rangle = \langle\hat{f}, \hat{g}\rangle, \tag{4.20}$$

as well as the *QDFT Parseval identity*

$$\|f\| = \|\hat{f}\|. \tag{4.21}$$

The QDFT Parseval identity means, that the QDFT preserves the signal energy when applied in signal processing.

We now define analogous to (2.41) for unit norm signals $f \in L^2(\mathbb{H};\mathbb{H})$, $\|f\| = 1$, the *effective spectral width* (or band width) of f *in the direction* of the unit quaternion $a \in \mathbb{H}$, $|a| = 1$, as the square root of the variance of the frequency spectrum of f along the a-axis

$$\Delta\omega_a = \|(\omega\cdot a)\hat{f}\| = \sqrt{\int_{\mathbb{H}} (\omega\cdot a)^2|\hat{f}(\omega)|^2\, d^4\omega}. \tag{4.22}$$

We further define the *effective spectral width* (frequency uncertainty) as the square root of the variance of the energy distribution of $\hat{f}$

$$\Delta\omega = \|\omega\hat{f}\| = \sqrt{\int_{\mathbb{H}} |\omega|^2|\hat{f}(\omega)|^2\, d^4\omega}. \tag{4.23}$$

We can now state the *directional uncertainty principle* for the QDFT of unit norm signals $f \in L^2(\mathbb{H};\mathbb{H})$, $\|f\| = 1$ as

$$\Delta x_a \Delta\omega_b \geq \frac{|a\cdot b|}{2}. \tag{4.24}$$

The *uncertainty principle* takes the form

$$\Delta x\Delta\omega \geq 1. \tag{4.25}$$

Equality holds in (4.24) and (4.25) for *Gaussian signals*

$$f(x) = Ce^{-c|x|^2}, \quad 0 < c \in \mathbb{R}, \tag{4.26}$$

with constant factor $C \in \mathbb{H}$.

The QDFT of the *convolution* (2.40) of two functions $f, g \in L^2(\mathbb{H}; \mathbb{H})$ results in

$$\widehat{(f * g)}(\omega) = (2\pi)^2[\hat{f}(\omega)\hat{g}_-(\omega) + \hat{f}(-\omega)\hat{g}_+(\omega)]. \tag{4.27}$$

Note that for $\hat{g}_+(\omega) = 0$ or if $\hat{f}(\omega) = \hat{f}(-\omega)$ we obtain

$$\widehat{(f * g)}(\omega) = (2\pi)^2\hat{f}(\omega)\hat{g}(\omega). \tag{4.28}$$

The QDFT of the convolution of the two functions $f, g \in L^2(\mathbb{H}; \mathbb{H})$ in opposite order results in

$$\widehat{(g * f)}(\omega) = (2\pi)^2[\hat{g}(\omega)\hat{f}_-(\omega) + \hat{g}(-\omega)\hat{f}_+(\omega)], \tag{4.29}$$

and is usually different from (4.27), because of the general non-commutativity of $f, g \in L^2(\mathbb{H}; \mathbb{H})$.

An application of the QDFT convolution (4.27) is, e.g., the *fast convolution* (via simple multiplication of the QDFTs in the Fourier domain) of a *quaternion domain signal* $f : \mathbb{H} \to \mathbb{H}$ with a *pair of complex filters* $g_1(x) = g_{1,r}(x) + g_{1,i}(x)\boldsymbol{i} = g_-(x)$, $g_2(x) = g_{2,r}(x) + g_{2,i}(x)\boldsymbol{i} = g_+(x)(-\boldsymbol{j})$, choosing $I = \boldsymbol{i}$ in (3.1).

Next, we study the *covariance properties* of the QDFT under *orthogonal transformations*. We find that a *three-dimensional rotation* (2.25) of the argument $g(x) = h(a^{-1}xa)$ leads to

$$\hat{g}(\omega) = \hat{h}(a^{-1}\omega a). \tag{4.30}$$

The *reflection at* the pointwise invariant *real scalar line* $x \to \tilde{x}$, $g(x) = h(\tilde{x})$ gives

$$\hat{g}(\omega) = -\hat{h}(\widetilde{\omega}). \tag{4.31}$$

The *reflection at* the three-dimensional *hyperplane of pure quaternions* $x \to -\tilde{x}$, $g(x) = h(-\tilde{x})$ results in

$$\hat{g}(\omega) = -\hat{h}(-\widetilde{\omega}). \tag{4.32}$$

The *reflection at* the pointwise invariant *line through* $a \in \mathbb{H}$, $|a| = 1$, $x \to a\tilde{x}a$, $g(x) = h(a\tilde{x}a)$ gives

$$\hat{g}(\omega) = -\hat{h}(\tilde{a}^{-1}\widetilde{\omega}\tilde{a}^{-1}) = -\hat{h}(a\widetilde{\omega}a), \tag{4.33}$$

because $\tilde{a}^{-1} = a$ for $|a| = 1$. The *reflection at* the three-dimensional *hyperplane orthogonal to* the line through $a \in \mathbb{H}$, $|a| = 1$, $x \to -a\tilde{x}a$, $g(x) = h(-a\tilde{x}a)$ results in

$$\hat{g}(\omega) = -\hat{h}(-a\widetilde{\omega}a). \tag{4.34}$$

A general *four-dimensional rotation* in $\mathbb{R}^4$, $x \to axb$, $a, b \in \mathbb{H}$, $|a| = |b| = 1$, $g(x) = h(axb)$ leads to

$$\hat{g}(\omega) = \hat{h}(a\omega b). \tag{4.35}$$

We have thus studied the behavior of the QDFT under *all point group transformations* in three and four dimensions (reflections, rotations, rotary reflections, inversions), which are of importance in crystallography. We note, that quaternions have already been employed for the description of crystallographic symmetry in [2] and for the description of root systems of finite groups in three and four dimensions in [13].

5. Conclusion

We first *reviewed quaternion algebra*, its expression in terms of matrices and vectors, quaternion derivatives and the Dirac derivative, the orthogonal planes split of quaternions with respect to a single unit pure quaternion, and the description of three-dimensional and four-dimensional orthogonal transformations. Then we defined *quaternion domain functions*, the quaternion module $L^2(\mathbb{H};\mathbb{H})$, convolution of quaternion domain functions, and their effective spatial width (uncertainty).

We established the *steerable quaternion domain Fourier transform* (QDFT) with a free choice a single constant pure unit quaternion in the kernel. We examined the *properties* of left and right linearity, orthogonal plane split property, and gave an example of the QDFT applied to a functional equation. Further properties studied are the inverse QDFT, shift and modulation theorems, the QDFT of quaternion coordinate polynomials[4] multiplied with quaternion domain signals, as well as products with powers of the signal argument x, and the corresponding dual properties (polynomials of partial differential operators, quaternion derivatives and Dirac derivatives). We found that the QDFT can separate the symmetry components of complex signals, and can be applied to many partial differential equations in physics. Quaternion *non-commutativity* means, that multiplication from the right and left need to be distinguished carefully. Next we established Parseval and Plancherel identities, uncertainty principles and convolution properties for the QDFT. The convolution allows e.g. fast filtering with pairs of complex filters. Finally we studied the covariance properties of the QDFT under orthogonal transformations of the signal arguments, which may a.o. be of importance for applications in crystallography.

We expect that this new quaternionic Fourier transformation[5] may find rich *applications* in mathematics (e.g. higher dimensional holomorphic functions [17]) and physics, including relativity and spacetime physics, in three-dimensional color field processing, neural signal processing, space color video processing, crystallography, quaternion analysis, and for the solution of many types of quaternionic differential equations. We further expect that the QDFT can be successfully *extended* to *localized transforms*, e.g., quaternion domain window Fourier transforms, and continuous quaternionic wavelets and quater-

[4] Note that real and complex polynomial generated moment invariants have recently been successfully used for translation, rotation and scale invariant normalized moment description of vector field features, including flow fields [6–8].

[5] A review of previous types of quaternionic Fourier transforms and their applications can be found in [9].

nionic ridgelets [11][6]. Further research should be done into *operator versions* of the QDFT, and its related *linear canonical transforms*, which may open up many further areas of interesting applications, including quantum physics. Especially for applications, *discretization* and *fast implementation* with pairs of complex fast Fourier transforms will be of great interest.

Acknowledgements

I thank *God* for his wonderful creation: "They speak of the glorious splendor of your majesty – and I will meditate on your wonderful works." [29]. I thank my *dear family* for always supporting my research activities. I thank Prof. *T. L. Saaty* for very inspiring private communication. I finally thank the organizers of the ICCA10 conference: Prof. *Viktor Abramov, Olga Liivapuu, Priit Lätt, Md. Raknuzzaman, Sergei Shitov, Jaan Vajakas*, and Prof. *Wolfgang Sprössig* for successfully taking for many years the overall responsibility for the ICCA conference series.

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Received: December 13, 2014.
Accepted: October 16, 2015.