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Noncommutative Galois Extensions and Ternary Clifford Analysis

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Abstract. In this paper we introduce the idea of Galois extension for a class of associative algebras and discuss binary and ternary Clifford algebras by such an algebraic construction. Nonion algebra is characterized by Galois extensions and a ternary structure is proposed for $\mathfrak{su}(3)$ leading to a duality for certain binary and ternary differential operators.

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A concept of noncommutative Galois extension is proposed for a certain class of associative algebras. Binary Clifford algebras with negative signature are characterized in terms of successive binary Galois extensions. Ternary Clifford algebras are introduced by means of ternary triples and a process of successive ternary Galois extensions is described by means of nonion algebra. Dirac and Laplace operators are discussed in the context of binary and ternary Clifford algebras. A description of $\mathfrak{su}(3)$ is given by means of Gell-Mann matrices and binary extensions. A ternary structure is proposed for $\mathfrak{su}(3)$, leading to a ternary Dirac operator which provides a duality between the associated binary and ternary Laplace operators.

1. Noncommutative Galois Extensions

Here we introduce the idea of noncommutative Galois extension as indicated in [7,8]. For simplicity, we assume that all algebras are defined over $\mathbb{R}$ or $\mathbb{C}$, are associative and unital (not necessarily commutative).

Definition 1.1. Let A and $\tilde{A}$ be algebras such that A is a subalgebra of $\tilde{A}$ and suppose that there are an element $\tau \in \tilde{A}$ and a positive integer n such that $\tau^n = 1$ and $\tau^k \notin A$ for $k = 1, 2, \ldots, n-1$. We denote by $A[\tau]$ the smallest subalgebra of $\tilde{A}$ containing both A and τ and call it *noncommutative Galois extension of A by τ* or, for short, an *extension* of A by τ.

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Noncommutative extensions are usually not easy to control, even when both A and $\tilde{A}$ are free algebras. Hence we restrict ourselves to extensions $A[\tau]$ such that for any $x \in A$ there exists $x' \in A$ satisfying

$$\tau x = x' \tau.$$

In this case $A[\tau]$ is said to be *semi-commutative* and the extension can be written in the most familiar expression as

$$A[\tau] = \left\{a_0 + a_1\tau + \cdots + a_{n-1}\tau^{n-1} | a_0, a_1, \ldots, a_{n-1} \in A\right\}.$$

All extensions we consider here are assumed to be semi-commutative.

We mainly discuss in this paper certain class of binary and ternary Clifford algebras. The algebraic structure of these algebras can be described by means of binary and ternary Galois extensions, defined as follows.

Definition 1.2. An extension $A[\tau]$ is said to be a *binary extension* when $\tau^2 = 1$. In the same way, when $\tau^3 = 1$ we say that $A[\tau]$ a *ternary extension.*

In the following we have examples of a binary and a ternary extension the will apear in the discussions along the paper.

Example. Let $\mathbb{H}$ be the quaternion algebra, generated over $\mathbb{R}$ by the unity 1 and the elements $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ subject to the relations

$$\begin{gathered}\mathbf{i}^2 = -1, \quad \mathbf{j}^2 = -1, \quad \mathbf{k}^2 = -1 \\ \mathbf{ij} + \mathbf{ji} = 0, \quad \mathbf{jk} + \mathbf{kj} = 0, \quad \mathbf{ki} + \mathbf{ik} = 0 \\ \mathbf{ij} = \mathbf{k}, \quad \mathbf{jk} = \mathbf{i}, \quad \mathbf{ki} = \mathbf{j}.\end{gathered}$$

By these relations the elements of $\mathbb{H}$ can be written as

$$a_0 1 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = a_0 1 + a_2\mathbf{j} + \mathbf{i}(a_1 1 + a_3\mathbf{j}).$$

Thus $\mathbb{H} = \mathbb{C}[\mathbf{i}]$, when we consider $\mathbb{C} = \{a_0 1 + a_2\mathbf{j} | a_0, a_2 \in \mathbb{R}\}$, showing that $\mathbb{H}$ is a binary extension.

Example. The 3×3 matrices defined by

$$T_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad T_3 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

satisfy $T_2^2 = T_3$ and $T_2^3 = T_1$. In this case the unity 1 is given by the identity matrix T_1 and the extension $\mathbb{B}$ given by

$$\mathbb{B} = \mathbb{C}[T_2] = \{\theta_1 T_1 + \theta_2 T_2 + \theta_3 T_3 | \theta_1, \theta_2, \theta_3 \in \mathbb{C}\}.$$

is a ternary extension called (complex) *cubic algebra.* Since T_1, T_2, T_3 have only real entries the extension $\mathbb{R}[T_2]$ is also defined, being a ternary counterpart of complex numbers.

If $A[\tau]$ is an extension of A by $\tau \in \tilde{A}$ and $\tau' \in \tilde{A} - A$ we can construct an extension of $A[\tau]$ by τ'. This extension is indicated by $A[\tau, \tau']$ and is called *successive extension of* $A[\tau]$ by τ'. In the first example, given above, observe that $\mathbb{H} = \mathbb{C}[\mathbf{i}]$ and $\mathbb{C} = \mathbb{R}[\mathbf{j}]$. Hence, it follows that $\mathbb{H} = \mathbb{R}[\mathbf{i}, \mathbf{j}]$.

2. Binary and Ternary Clifford Algebras

Binary and ternary algebras have extensively been studied in various domains of theoretical and mathematical physics [4,6]. In particular, ternary algebras plays an important role in the description of quarks by means of a ternary generalization of Pauli's principle [5].

Much of the algebraic structure involved in these constructions can be recovered by Galois extensions where the binary or ternary character is described by particular properties of the extension. This is the case for binary Clifford algebras, that can be obtained by successive binary extensions. By the other side ternary Clifford algebras are characterized by means of ternary triples whose role in the generation of these algebras will be more evident when we consider complex 3×3 matrices.

2.1. Binary Clifford Algebras

Binary Clifford algebras are the usual Clifford algebras defined by a relation on the generators as described for example in [9,10]. We use here the term "binary" to distinguish these algebras from their ternary analogous.

To our purposes it will be enough to consider Clifford algebras defined by a negative signature, since they will be directly related to the positive Laplace operator, as described in [9].

Definition 2.1. An associative algebra generated over $\mathbb{R}$ by a unity 1 and the elements $\{\theta^1, \theta^2, \ldots, \theta^n\}$ subject to the relations

$$\theta^a\theta^b + \theta^b\theta^a = -2\delta_{ab}1,$$

for all $a, b = 1, \ldots, n$ is said to be a *binary Clifford algebra*.

Complex numbers can be obtained as a binary extension of real numbers and quaternions as a binary extension of complex numbers. This reasoning can be improved, characterizing binary Clifford algebras by an inductive process of binary extensions, sometimes called *basic construction* [7,8].

Theorem 2.2. *A binary Clifford algebra can be obtained by a process of successive noncommutative Galois extensions.*

Proof. Let C_m be a binary Clifford algebra generated by a unity 1 and elements $\theta^1, \theta^2, \ldots, \theta^m$, subject to the relations $\theta^a\theta^b + \theta^b\theta^a = -2\delta_{ab}1$. Define new generators as

$$\Theta^a = \theta^a \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \Theta^{m+1} = 1 \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

where $a = 1, 2, \ldots, m$. Hence, it is not difficult to check that the algebra C_{m+1} generated by the elements $\Theta^1, \Theta^2, \ldots, \Theta^m, \Theta^{m+1}$ is a binary Clifford algebra which can be written as $C_{m+1} = C_m[\Theta^{m+1}]$. $\square$

The basic process of extension can also include other algebras, as *binary involutive Clifford algebras*. These are generated over $\mathbb{R}$ by a unity 1 and elements $\{\theta^1, \theta^2, \ldots, \theta^n, \bar{\theta}^1, \bar{\theta}^2, \ldots, \bar{\theta}^n\}$ subject to

$$\theta^a\theta^b + \theta^b\theta^a = -2\delta_{ab}1, \quad \bar{\theta}^a\bar{\theta}^b + \bar{\theta}^b\bar{\theta}^a = -2\delta_{ab}1.$$

for all $a, b = 1, \ldots, m$.

Let C_m be such an involutive algebra and I, J be the matrices

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

We define extended generators by the following block matrices

$$\Theta^a = \theta^a \otimes \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \Theta^{m+1} = 1 \otimes \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$
$$\overline{\Theta}^a = \bar{\theta}^a \otimes \begin{pmatrix} 0 & J \\ -J & 0 \end{pmatrix}, \quad \overline{\Theta}^{m+1} = 1 \otimes \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix}.$$

Since $J^2 = -I$ it is not difficult to check that the algebra C_{m+1}, generated by the elements $\Theta^1, \Theta^2, \ldots, \Theta^m, \Theta^{m+1}, \overline{\Theta}^1, \overline{\Theta}^2, \ldots, \overline{\Theta}^{m+1}$, is a binary involutive Clifford algebra with $C_{m+1} = C_m[\Theta^{m+1}, \overline{\Theta}^{m+1}]$.

Observe that the extended generators may satisfy additional binary relations

$$\Theta^a \overline{\Theta}^b = -\overline{\Theta}^b \Theta^a, \quad \overline{\Theta}^b \Theta^a = -\Theta^a \overline{\Theta}^b$$

once the original generators θ^a, $\bar{\theta}^b$ satisfy similar relations. This is the case when consistence commutation relations on the generators and its conjugate are required to merge both in a bigger algebra [4,5].

2.2. Ternary Clifford Algebra

Ternary Clifford algebras appear in literature as a ternary generalization of binary Clifford algebras. In [2] these algebras are introduced by relations on ternary products of generators while in [1,5] by means of relations on pair of generators, involving the primitive cubic root of 1. Following the lines of [6,7] we present here a definition of ternary Clifford algebras based on ternary triples.

Definition 2.3. Given an associative $\mathbb{C}$-algebra A with unity 1, we say that a triple of generators $\{\theta^1, \theta^2, \theta^3\}$ is a *ternary triple* when the following commutation relations hold

$$\theta^a\theta^b\theta^c + \theta^b\theta^c\theta^a + \theta^c\theta^a\theta^b = 3h^{abc}1$$

where h^{abc} is a totally symmetric tensor whose only non vanishing components are

$$h^{111} = h^{222} = h^{333} = 1, \quad h^{123} = h^{231} = h^{312} = j^2, \quad h^{213} = h^{321} = h^{132} = j,$$

with $j = e^{\frac{2\pi}{3}i}$, the primitive cubic root of 1. The algebra A is said to be a *ternary Clifford algebra* when it is generated by a system of ternary triples.

If A is an involutive $\mathbb{C}$-algebra with unity 1, we say that a triple of generators $\{\theta^1, \theta^2, \theta^3\}$ is a *ternary triple* when

$$\theta^a\theta^b\theta^c + \theta^b\theta^c\theta^a + \theta^c\theta^a\theta^b = 3h^{abc}1, \quad \bar{\theta}^a\bar{\theta}^b\bar{\theta}^c + \bar{\theta}^b\bar{\theta}^c\bar{\theta}^a + \bar{\theta}^c\bar{\theta}^a\bar{\theta}^b = 3h^{abc}1.$$

A is said to be a *ternary involutive Clifford algebra* when it is generated by a system of ternary triples.

If A is endowed with additional relations on ternary triples given by $\theta^a\bar{\theta}^b = k_{ab}\bar{\theta}^b\theta^a$, with $k_{ab} \in \{j, j^2\}$, we say it is a *j-ternary involutive Clifford* *algebra* or, for short, a *j-ternary Clifford algebra*. This is usually the case when binary relations are imposed on "quark" and "anti-quark" [5] to provide constitutive relations between a ternary triple $\{\theta^1, \theta^2, \theta^3\}$ and its associated conjugate triple $\{\bar\theta^1, \bar\theta^2, \bar\theta^3\}$.

Example. Consider the generators Q_1, Q_2, Q_3 given by the complex matrices

$$Q_1 = \begin{pmatrix} 0 & j & 0 \\ 0 & 0 & j^2 \\ 1 & 0 & 0 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 0 & j^2 & 0 \\ 0 & 0 & j \\ 1 & 0 & 0 \end{pmatrix}, \quad Q_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

As the following equalities hold

$$Q_1Q_2 = jQ_2Q_1, \quad Q_2Q_3 = jQ_3Q_2, \quad Q_3Q_1 = jQ_1Q_3,$$

it can be verified that the $\mathbb{C}$-algebra generated by the elements $\{Q_1, Q_2, Q_3\}$ is a ternary Clifford algebra. In addition, by taking hermitian conjugates $\overline{Q}_1 = Q_1^\dagger$, $\overline{Q}_2 = Q_2^\dagger$, $\overline{Q}^3 = Q_3^\dagger$ we also get

$$\overline{Q}_1\overline{Q}_2 = j\overline{Q}_2\overline{Q}_1, \quad \overline{Q}_2\overline{Q}_3 = j\overline{Q}_3\overline{Q}_2, \quad \overline{Q}_3\overline{Q}_1 = j\overline{Q}_1\overline{Q}_3.$$

It is not difficult to check that the $\mathbb{C}$-algebra generated by the elements $\{Q_1, Q_2, Q_3, \overline{Q}_1, \overline{Q}_2, \overline{Q}_3\}$ is a j-ternary Clifford algebra.

A characterization of ternary Clifford algebras by means of successive extensions is not as straightforward as it was done for binary case. This happens because the definition of these algebras by means of a system of ternary triples does not impose additional restrictions on elements of different triples. By the other hand, in [2] a discussion is presented for some algebras where the relation $XY = jYX$ holds for generators X and Y. In these cases it can be checked that $\{X, Y, (jXY)^2\}$ is a ternary triple, and a ternary extension process can be defined [7,8]. In the following we give more detail on these particular algebras.

3. Nonion Algebra and Galois Extensions

Nonion algebra provides the simplest example of a ternary Clifford algebra, being generated by two ternary elements subject to a semi-commutativity condition. In this case ternary triples can be obtained from the generators of nonion algebra, being represented by complex 3×3 matrices.

3.1. Nonion Algebra

Nonion algebra was introduced by Pierce [11] and Sylvester [12] as a ternary analogous to Hamilton quaternions.

Definition 3.1. The associative algebra generated over $\mathbb{C}$ by a unity 1, and elements X and Y subjected to

$$X^3 = Y^3 = 1, \quad XY = jYX$$

with j the primitive cubic root of 1, is called *nonion algebra*.

As pointed before $\{X, Y, (jXY)^2\}$ is a ternary triple, showing that nonion algebra is a ternary Clifford algebra. The structure of this algebra is particularly evident when we consider its lowest dimensional representation, given by 3×3 complex matrices. Sylvester [12] showed that these matrices generate the full algebra $M_3(\mathbb{C})$ where a basis can be written as

$$Q_1 = \begin{pmatrix} 0 & j & 0 \\ 0 & 0 & j^2 \\ 1 & 0 & 0 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 0 & j^2 & 0 \\ 0 & 0 & j \\ 1 & 0 & 0 \end{pmatrix}, \quad Q_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\overline{Q}_1 = \begin{pmatrix} 0 & 0 & 1 \\ j^2 & 0 & 0 \\ 0 & j & 0 \end{pmatrix}, \quad \overline{Q}_2 = \begin{pmatrix} 0 & 0 & 1 \\ j & 0 & 0 \\ 0 & j^2 & 0 \end{pmatrix}, \quad \overline{Q}^3 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$R_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & j & 0 \\ 0 & 0 & j^2 \end{pmatrix}, \quad R_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & j^2 & 0 \\ 0 & 0 & j \end{pmatrix}.$$

A full multiplication table is given in [6,7].

For a nonion algebra generated over $\mathbb{C}$ by elements X and Y the conditions $X^3 = Y^3 = 1$ and $XY = jYX$ allow us to consider ternary extensions, whose structure can be described as follows.

Proposition 3.2. *A nonion algebra N can be obtained by successive ternary extensions and it is isomorphic to the full matrix algebra $M_3(\mathbb{C})$.*

Proof. Suppose that N is a nonion algebra generated over $\mathbb{C}$ by two elements X and Y subject to $X^3 = Y^3 = 1$ and $XY = jYX$. By defining $N_1 = \mathbb{C}[X]$ we have a ternary extension. Besides, it follows that $N = N_1[Y]$, showing that N is also a ternary extension by the adjunction of Y to N_1. Since $1, X, Y, X^2, Y^2, XY, X^2Y, XY^2$ and $(XY)^2$ are linearly independent generators for $N = N_1[Y]$, an algebra isomorphism $\varphi\colon N \to M_3(\mathbb{C})$ is defined. $\square$

The isomorphism indicated in the proposition is not unique. It can be defined on the generators and then extended to all the algebra, for example, as $\varphi(X) = Q_1$ and $\varphi(Y) = Q_2$. Besides, as $\mathbb{C}[X]$ is isomorphic to $\mathbb{B}$, we also refer to $\mathbb{C}[X]$ as a cubic algebra.

Ternary extensions in nonion algebra can also be characterized by the action of adjoint representations defined on the generators.

Definition 3.3. An automorphism $\varphi\colon M_3(\mathbb{C}) \to M_3(\mathbb{C})$ is said to be *inner* if there exists an invertible element $U \in M_3(\mathbb{C})$ such that

$$\varphi(Z) = U^{-1} Z U,$$

for all $Z \in M_3(\mathbb{C})$. In this case φ is called *adjoint representation* defined by U, and indicated by Ad_U.

As an example let us consider the generators R_1, R_2, R_3, Q_1, Q_2, Q_3, $\overline{Q}_1, \overline{Q}_2, \overline{Q}_3$. As they are invertible in $M_3(\mathbb{C})$ it is possible to compute their adjoints. For the triple Q_1, Q_2, Q_3, for example, we have

$$Ad_{Q_s} R_k = j^{k-1} R_k, \quad Ad_{Q_s} Q_k = j^{k-1} Q_k, \quad Ad_{Q_s} \overline{Q}_k = \bar{j}^{k-1} \overline{Q}_k$$

where $s, k = 1, 2, 3$. Other adjoints can be easily calculated by using multiplication table found in [6,7].

The elements $T_1 = R_1$, $T_2 = Q_3$ and $T_3 = \overline{Q}_3$ generate the cubic algebra $\mathbb{B} = \mathbb{C}[T_2]$ which is a commutative ternary sub-algebra of the nonion algebra $M_3(\mathbb{C})$. When $N = \mathbb{B}[X] = M_3(\mathbb{C})$ is an extension over $\mathbb{B}$, with $X \in M_3(\mathbb{C})$, we can calculate the group of all $\mathbb{C}$-automorphisms $\sigma \colon M_3(\mathbb{C}) \to M_3(\mathbb{C})$ such that $\sigma|_{\mathbb{B}} = 1_{\mathbb{B}}$. First notice that every automorphism $\sigma \colon \mathbb{B}[X] \to \mathbb{B}[X]$ is inner, since $\mathbb{B}[X] = M_3(\mathbb{C})$ is a matrix algebra [10]. Thus, there is an invertible element $U \in M_3(\mathbb{C})$ such that $\sigma(Z) = U^{-1}ZU$ for all $Z \in \mathbb{B}[X]$. As $\sigma|_{\mathbb{B}} = 1_{\mathbb{B}}$ we have $\sigma(T_2) = T_2$ and U will be of form

$$U = aT_1 + bT_2 + cT_3,$$

where $a, b, c \in \mathbb{C}$ are not all simultaneously null. As $\{T_1, T_2, T_3\}$ is a cyclic group of order 3 it is isomorphic to $\mathbb{Z}_3$ showing that the group of all such automorphisms is the group $(\mathbb{C}[\mathbb{Z}_3])^*$, of invertible elements in the group algebra $\mathbb{C}[\mathbb{Z}_3]$.

Although we have considered Galois extensions over the cubic algebra, the same reasoning can be performed for general ternary subalgebras of $M_3(\mathbb{C})$ leading to the same invariance group. This fact holds also true for more general algebras showing that noncommutative extensions will have "continuous Galois group".

3.2. Successive Ternary Extensions

Suppose that B is the cubic algebra generated over $\mathbb{C}$ by a ternary element X, i.e., $X^3 = 1$ and $X^k \neq 1$ for $k = 1, 2$. If A is a ternary Clifford algebra and $\{T_1, T_2, T_3\}$ is a ternary triple we can define the triples

$$\{T_1 \otimes X^k, T_2 \otimes X^k, T_3 \otimes X^k\}$$

for $k = 1, 2, 3$. Since X is ternary it is not difficult to see that each of these triples is ternary, endowing $A \otimes B$ with a ternary Clifford algebra structure. By means of the identification $A = A \otimes 1$ we can think $A \otimes B$ as a ternary extension of form $A[X]$, which is called *cubic algebra extension* [7].

By the same reasoning, when N is the nonion algebra, $A \otimes N$ will also be a ternary Clifford algebra. In fact, if X, Y are generators of N, with $B = \mathbb{C}[X]$ and $N = B[Y]$, we can write $A \otimes N = (A \otimes B)[1 \otimes Y] = A[X, Y]$ showing that $A \otimes N$ is a ternary Clifford algebra obtained by successive cubic extensions.

Successive cubic extensions of a ternary Clifford algebra will produce new ternary Clifford algebras. Unfortunately it is not clear that any ternary Clifford algebra can be obtained in this way. This is directly related to the chosen definition for a ternary Clifford algebra. In [1,2], for example, the authors give different definitions for such algebras while here we use the idea of ternary triples. These definitions coincide for nonion algebra although this coincidence is not evident in higher dimensions.

4. Basic Binary and Ternary Differential Operators

Dirac and Laplace operators are well known for real binary Clifford algebras [9]. In ternary case these operators are not uniquely defined, and depend on a choice of a particular ternary triple [6,7].

4.1. Binary Dirac and Laplace Operators

Consider the binary Clifford algebra generated on $\mathbb{R}$ by the unity 1 and the elements $\{T_1, T_2, \ldots, T_m\}$ subject to the relations

$$T_a T_b + T_b T_a = -2\delta_{ab} 1.$$

The *binary Dirac operator* is defined as

$$D = \sum_{k=1}^{m} T_k \frac{\partial}{\partial x_k}.$$

Since the relation

$$(x_1 T_1 + x_2 T_2 + \cdots + x_m T_m)^2 = -(x_1^2 + x_2^2 + \cdots + x_m^2) 1$$

holds for all $x_1, x_2, \ldots, x_m \in \mathbb{R}$, it follows that

$$D^2 = -\left(\sum_{k=1}^{m} \frac{\partial^2}{\partial x_k^2} \right) 1.$$

The operator D^2 is called *binary* (*positive*) *Laplace* operator [9].

4.2. Ternary Dirac and Laplace Operators

In similar way we can introduce ternary Dirac and Laplace operators. To do this let us consider a ternary Clifford algebra over $\mathbb{C}$, with unity 1, and a ternary triple $\{T_1, T_2, T_3\}$ subject to the relations

$$T_a T_b T_c + T_b T_c T_a + T_c T_a T_b = 3h^{abc} 1.$$

We introduce the *ternary Dirac operator*

$$D = T_1 \frac{\partial}{\partial x} + T_2 \frac{\partial}{\partial y} + T_3 \frac{\partial}{\partial z}$$

and its conjugate operators as $D^* = jD$ and $D^{**} = j^2 D$. Since the cubic relation

$$(xT_1 + yT_2 + zT_3)^3 = \left(x^3 + y^3 + z^3 - 3xyz\right) 1$$

holds for all $x, y, z \in \mathbb{C}$ it follows that

$$DD^*D^{**} = \left(\frac{\partial^3}{\partial x^3} + \frac{\partial^3}{\partial y^3} + \frac{\partial^3}{\partial z^3} - 3\frac{\partial^3}{\partial x \partial y \partial z} \right) 1.$$

The operator on the right side is indicated by $\Delta^{(3)}$ and it is called *ternary Laplace* operator [7,8].

There are other choices for defining the conjugate operators. In fact, from the property

$$(x + y + z)(x + j^2 y + jz)(x + jy + j^2 z) = x^3 + y^3 + z^3 - 3xyz$$

we may define

$$D^* = T_1\frac{\partial}{\partial x} + j^2T_2\frac{\partial}{\partial y} + jT_3\frac{\partial}{\partial z}, \quad D^{**} = T_1\frac{\partial}{\partial x} + jT_2\frac{\partial}{\partial y} + j^2T_3\frac{\partial}{\partial z}.$$

Again we have $DD^*D^{**} = \Delta^{(3)}$ showing that the conjugate operators defined in this way lead to the same ternary Laplace operator.

Nonion algebra provides the simplest examples of ternary Dirac operators. Consider, for example, the triple $\{Q_1, Q_2, Q_3\}$ and its conjugate $\{\overline{Q}_1, \overline{Q}_2, \overline{Q}_3\}$ which define two Dirac operators. These operators give raise to the same Laplace operator, indicating an equivalence among solutions. In this direction, the authors in [4,6] define a Dirac operator by means of the triple $\{Q_1, Q_2, Q_3\}$ and, acting three times with this operator, get a ternary Laplace equation. Based on solutions of this third order equation an explanation is proposed on why three quarks can form a freely propagating state while a single quark cannot propagate. Another explanation for this is also given in [5] where quarks are described in terms of colors and a system of coupled equations defined on a binary extension of the nonion algebra generated by Q_3 and R_2.

5. Galois Extension and Clifford Analysis for $\mathfrak{su}(3)$

Here we consider a representation $M(\mathfrak{su}(3))$, of the algebra $\mathfrak{su}(3)$, in terms of 3×3 complex matrices. Then, by means of Gell-Mann basis, we investigate binary and ternary extensions.

5.1. Basic Binary and Ternary extensions in $M(\mathfrak{su}(3))$

First notice that $\mathfrak{su}(2)$ can be represented as a 3-dimensional real algebra generated by $\{i\sigma_1, i\sigma_2, i\sigma_3\}$, where σ_1, σ_2, σ_3 are the Pauli matrices and $i = \sqrt{-1}$. Thus, taking into account the Gell-Mann basis [3] for $M(\mathfrak{su}(3))$, we define the following 3×3 matrices

$$e_1 = \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$e_1' = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad e_2' = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad e_3' = \begin{pmatrix} i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -i \end{pmatrix}$$

$$e_1'' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix}, \quad e_2'' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad e_3'' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix}$$

These nine elements generate $M(\mathfrak{su}(3))$ but are not a basis (in fact we get a basis omitting one of e_3, e_3' or e_3''). They provide the following three copies of $\mathfrak{su}(2)$ inside $M(\mathfrak{su}(3))$:

$$L_1 = \mathrm{span}_{\mathbb{R}}\{e_1, e_2, e_3\}, \quad L_2 = \mathrm{span}_{\mathbb{R}}\{e_1', e_2', e_3'\}, \quad L_3 = \mathrm{span}_{\mathbb{R}}\{e_1'', e_2'', e_3''\}.$$

When we consider the elements

$$e_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_0' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad e_0'' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and the extensions $\widehat{L}_1 = L_1[e_0]$, $\widehat{L}_2 = L_2[e_0']$ and $\widehat{L}_3 = L_3[e_0'']$, we have

Proposition 5.1. *$\widehat{L}_1$, $\widehat{L}_2$ and $\widehat{L}_3$ are binary extensions.*

Proof. The relations

$$e_1^2 = -e_0, \quad e_1e_2 = -e_2e_1 = e_3, \quad e_2e_3 = -e_3e_2 = e_1, \quad e_3e_1 = -e_1e_3 = e_2$$

show that $\widehat{L}_1$ is a quaternion algebra which is a binary extension of $\mathbb{C} = \mathbb{R}[e_0, e_2]$. Similarly we see that $\widehat{L}_2$ and $\widehat{L}_3$ are also binary extensions. □

We now investigate elements $G \in M_3(\mathbb{C})$ that can be adjoined to the generators of $\widehat{L}_1$, $\widehat{L}_2$ and $\widehat{L}_3$ to provide a ternary structure, i.e when $G^3 = I_3$. One simple example is given by $G = \overline{Q}_3$. Unfortunately, we readily see that $\widehat{L}_1[G]$ is not a Galois extension since the semi-commutativity condition does not hold. However it is possible to give a characterization of $M(\mathfrak{su}(3))$ in terms of adjoint representation. In fact we have $\widehat{L}_2 = Ad_G(\widehat{L}_1)$ and $\widehat{L}_3 = Ad_{G^2}(\widehat{L}_1)$, leading to

$$M(\mathfrak{su}(3)) = \widehat{L}_1 + \widehat{L}_2 + \widehat{L}_3 = \widehat{L}_1 + Ad_G(\widehat{L}_1) + Ad_{G^2}(\widehat{L}_1).$$

Hence we see that $M(\mathfrak{su}(3))$ can be obtained by a binary extension and an adjoint extensions given by the ternary element G. In [7,8] the authors already pointed out this fact and suggested that three quarks may be introduced by the three binary extensions $\widehat{L}_1$, $\widehat{L}_2$, $\widehat{L}_3$.

5.2. Ternary Structure on $M(\mathfrak{su}(3))$ and Differential Operators

Despite the generators $\{e_k, e_k', e_k''\}$ are not ternary triples, they satisfy significant ternary conditions. For example, if $X = xe_2 + ye_2' + ze_2''$ then it is possible to verify that $X^3 = -(x^2 + y^2 + z^2)X$. In fact, by performing the product $X^3 = (xe_2 + ye_2' + ze_2'')^3$ we observe that the elements of the triple $\{e_2, e_2', e_2''\}$ satisfy a series of ternary relations. Although our main interest is $M(\mathfrak{su}(3))$, this observation motivates us to introduce the following general definition.

Definition 5.2. The algebra generated over $\mathbb{C}$ by the elements $\{T_1, T_2, T_3\}$ subject to

$$\begin{aligned} &T_\alpha^3 = -T_\alpha \quad (\alpha = 1, 2, 3) \\ &T_1T_2T_3 + T_1T_3T_2 + T_2T_1T_3 + T_2T_3T_1 + T_3T_1T_2 + T_3T_2T_1 = 0 \\ &T_\alpha T_\beta^2 + T_\beta T_\alpha T_\beta + T_\beta^2 T_\alpha = -T_\alpha \quad (\alpha, \beta = 1, 2, 3,\ \alpha \neq \beta) \end{aligned}$$

is called *ternary algebra of quark type* and the triple $\{T_1, T_2, T_3\}$ is said to be *of quark type*.

Thus, if $X = xT_1 + yT_2 + zT_3$ it follows that $X^3 = -(x^2 + y^2 + z^2)X$ for all $x, y, z \in \mathbb{C}$. In this case we say that that X *determines* the ternary algebra generated by $\{T_1, T_2, T_3\}$.

A ternary Dirac operator can be defined for the triple $\{T_1, T_2, T_3\}$ as

$$D = T_1 \frac{\partial}{\partial x} + T_2 \frac{\partial}{\partial y} + T_3 \frac{\partial}{\partial z}.$$

If $\Delta^{(3)} = D^3$ is the corresponding ternary Laplace operator it follows that

$$\Delta^{(3)} = -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) D = \Delta^{(2)} D,$$

where $\Delta^{(2)}$ stands for the binary Laplace operator.

As $\Delta^{(2)} D = D\Delta^{(2)}$, we have a duality between solutions. In fact, if $\psi^{(3)}$ satisfies $\Delta^{(3)}\psi^{(3)} = 0$ we define $\psi^{(2)} = D\psi^{(3)}$. This leads to

$$\Delta^{(2)}\psi^{(2)} = \Delta^{(3)}\psi^{(3)} = 0.$$

By the other hand, when $\psi^{(2)}$ satisfies $\Delta^{(2)}\psi^{(2)} = 0$ and there is $\psi^{(3)}$ such that $D\psi^{(3)} = \psi^{(2)}$, then

$$\Delta^{(3)}\psi^{(3)} = \Delta^{(2)}\psi^{(2)} = 0.$$

The elements of $M(\mathrm{su}(3))$ defined as

$$\begin{aligned} X_1 &= \theta_1 e_1 + \theta_2 e_2' + \theta_3 e_1'' \\ X_2 &= \theta_1' e_2 + \theta_2' e_2' + \theta_3' e_2'' \\ X_3 &= \theta_1'' e_3 + \theta_2'' \tfrac{1}{\sqrt{2}}(e_2' - e_2'') + \theta_3'' \tfrac{1}{\sqrt{2}}(e_1' + e_1'') \end{aligned}$$

with $\theta_k, \theta_k', \theta_k'' \in \mathbb{C}$, determine three distinct ternary algebras of quark type that together generate $M(\mathfrak{su}(3))$. In addition, three corresponding Dirac operators are defined, leading to a duality in each of these ternary algebra. In this context, a question for further investigation is the role played by solutions of these equations in the description of quark model, following the lines of [4–6].

Finally, notice that X_1, X_2 and X_3, as defined above, determine particular ternary algebras of quark type in $M(\mathfrak{su}(3))$ but unfortunately not Ad-invariant by invertible elements of $M(\mathfrak{su}(3))$. Thus, another question for further investigation is whether or not there are elements Y_1, Y_2 an Y_3 determining ternary algebras of quark type, together generating $M(\mathfrak{su}(3))$, that are Ad-invariant by invertible elements $G_1, G_2, G_3 \in M(\mathfrak{su}(3))$ in the sense that

$$Ad_{G_1}(Y_1) = Y_2, \quad Ad_{G_2}(Y_2) = Y_3, \quad Ad_{G_3}(Y_3) = Y_1.$$

An investigation of this kind of invariance in quark models is proposed in [7,8] by means of Galois extensions.

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