



Curvilinear Integral Theorems for Monogenic Functions in Commutative Associative Algebras

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Abstract. We consider an arbitrary finite-dimensional commutative associative algebra, \mathbb{A}_n^m , with unit, over the field of complex number with m idempotents. Let $e_1 = 1, e_2, e_3$ be elements of \mathbb{A}_n^m which are linearly independent over the field of real numbers. We consider monogenic (i.e. continuous and differentiable in the sense of Gateaux) functions of the variable $xe_1 + ye_2 + ze_3$, where x, y, z are real. For mentioned monogenic function we prove curvilinear analogues of the Cauchy integral theorem, the Morera theorem and the Cauchy integral formula.

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1. Introduction

The Cauchy integral theorem and Cauchy integral formula for the holomorphic function of the complex variable are a fundamental result of the classical complex analysis. Analogues of these results are also an important tool in commutative algebras of dimensional more than 2.

In the paper of Lorch [1] for functions differentiable in the sense of Lorch in an arbitrary convex domain of commutative associative Banach algebra, some properties similar to properties of holomorphic functions of complex variable (in particular, the curvilinear integral Cauchy theorem and the integral Cauchy formula, the Taylor expansion and the Morera theorem) are established. Blum [2] withdrew a convexity condition of a domain in the mentioned results from [1].

Let us note that a priori/the differentiability of a function in the sense of Gateaux is a restriction weaker than the differentiability of this function

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in the sense of Lorch. Therefore, we consider a *monogenic* functions defined as a continuous and differentiable in the sense of Gateaux. Also we assume that a monogenic function is given in a domain of three-dimensional subspace of an arbitrary commutative associative algebra with unit over the field of complex numbers. In this situation the results established in the papers [1,2] is not applicable for a mentioned monogenic function, because it deals with an integration along a curve on which the function is not given, generally speaking.

In the papers [3–5] for monogenic function the curvilinear analogues of the Cauchy integral theorem, the Cauchy integral formula and the Morera theorem are obtained in special finite-dimensional commutative associative algebras.

In this paper, we generalize results of the papers [3–5] for an arbitrary commutative associative algebra over the field of complex numbers.

Let us note that some analogues of the curvilinear Cauchy integral theorem and the Cauchy integral formula for another classes of functions in special commutative algebras are established in the papers [6–10].

2. The Algebra \mathbb{A}_n^m

Let \mathbb{N} be the set of natural numbers. We fix the numbers $m, n \in \mathbb{N}$ such that $m \leq n$. Let \mathbb{A}_n^m be an arbitrary commutative associative algebra with unit over the field of complex number \mathbb{C} . Cartan [11, pp. 33–34] proved that in the algebra \mathbb{A}_n^m there exist a basis $\{I_k\}_{k=1}^n$ satisfies the following multiplication rules:

$$\begin{aligned}
 & 1. \forall r, s \in [1, m] \cap \mathbb{N}: \quad I_r I_s = \begin{cases} 0 & \text{if } r \neq s, \\ I_r & \text{if } r = s; \end{cases} \\
 & 2. \forall r, s \in [m + 1, n] \cap \mathbb{N}: \quad I_r I_s = \sum_{k=\max\{r,s\}+1}^n \Upsilon_{r,k}^s I_k; \\
 & 3. \forall s \in [m + 1, n] \cap \mathbb{N} \exists! u_s \in [1, m] \cap \mathbb{N} \forall r \in [1, m] \cap \mathbb{N}: \\
 & \qquad \qquad \qquad I_r I_s = \begin{cases} 0 & \text{if } r \neq u_s, \\ I_s & \text{if } r = u_s. \end{cases} \tag{2.1}
 \end{aligned}$$

Furthermore, the structure constants $\Upsilon_{r,k}^s \in \mathbb{C}$ satisfy the associativity conditions:

- (A1). $(I_r I_s) I_p = I_r (I_s I_p) \quad \forall r, s, p \in [m + 1, n] \cap \mathbb{N}$;
- (A2). $(I_u I_s) I_p = I_u (I_s I_p) \quad \forall u \in [1, m] \cap \mathbb{N} \quad \forall s, p \in [m + 1, n] \cap \mathbb{N}$.

Obviously, the first m basis vectors $\{I_u\}_{u=1}^m$ are the idempotents and, respectively, form a semi-simple subalgebra. Also the vectors $\{I_r\}_{r=m+1}^n$ form a nilpotent subalgebra of algebra \mathbb{A}_n^m . The unit of \mathbb{A}_n^m is the element $1 = \sum_{u=1}^m I_u$. Therefore, we will write that the algebra \mathbb{A}_n^m is a semi-direct sum of the m -dimensional semi-simple subalgebra S and $(n - m)$ -dimensional nilpotent subalgebra N , i.e.

$$\mathbb{A}_n^m = S \oplus_s N.$$

In the cases where \mathbb{A}_n^m has some specific properties, the following propositions are true.

Proposition 2.1. ([15]) *If there exists the unique $u_0 \in [1, m] \cap \mathbb{N}$ such that $I_{u_0} I_s = I_s$ for all $s = m + 1, \dots, n$, then the associativity condition (A2) is satisfied.*

Thus, under the conditions of Proposition 1, the associativity condition (A1) is only required. It means that the nilpotent subalgebra of \mathbb{A}_n^m with the basis $\{I_r\}_{r=m+1}^n$ can be an arbitrary commutative associative nilpotent algebra of dimension $n - m$. We note that such nilpotent algebras are fully described for the dimensions 1, 2, 3 in the paper [12], and some four-dimensional nilpotent algebras can be found in the papers [13, 14].

Proposition 2.2. ([15]) *If all u_r are different in the multiplication rule 3, then $I_s I_p = 0$ for all $s, p = m + 1, \dots, n$.*

Thus, under the conditions of Proposition 2, the multiplication table of the nilpotent subalgebra of \mathbb{A}_n^m with the basis $\{I_r\}_{r=m+1}^n$ consists only of zeros, and all associativity conditions are satisfied.

The algebra \mathbb{A}_n^m contains m maximal ideals

$$\mathcal{I}_u := \left\{ \sum_{k=1, k \neq u}^n \lambda_k I_k : \lambda_k \in \mathbb{C} \right\}, \quad u = 1, 2, \dots, m,$$

the intersection of which is the radical

$$\mathcal{R} := \left\{ \sum_{k=m+1}^n \lambda_k I_k : \lambda_k \in \mathbb{C} \right\}.$$

We define m linear functionals $f_u : \mathbb{A}_n^m \rightarrow \mathbb{C}$ putting

$$f_u(I_u) = 1, \quad f_u(\omega) = 0 \quad \forall \omega \in \mathcal{I}_u, \quad u = 1, 2, \dots, m.$$

Since the kernels of functionals f_u are, respectively, the maximal ideals \mathcal{I}_u , then these functionals are also continuous and multiplicative (see [16, p. 147]).

3. Monogenic Functions

We consider the vectors $e_1 = 1, e_2, e_3$ in \mathbb{A}_n^m which are linearly independent over the field of real number \mathbb{R} . It means that the equality

$$\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 = 0, \quad \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R},$$

holds if and only if $\alpha_1 = \alpha_2 = \alpha_3 = 0$.

Supposing that the vectors $e_1 = 1, e_2, e_3$ have the following decompositions with respect to the basis $\{I_k\}_{k=1}^n$:

$$e_1 = 1, \quad e_2 = \sum_{k=1}^n a_k I_k, \quad e_3 = \sum_{k=1}^n b_k I_k, \tag{3.1}$$

where $a_k, b_k \in \mathbb{C}$.

Let $\zeta := x e_1 + y e_2 + z e_3$, where $x, y, z \in \mathbb{R}$. It is also obvious that $\xi_u := f_u(\zeta) = x + y a_u + z b_u, u = 1, 2, \dots, m$. Let $E_3 := \{\zeta = x e_1 + y e_2 + z e_3 : x, y, z \in \mathbb{R}\}$ be the linear span of vectors e_1, e_2, e_3 over the field of real numbers \mathbb{R} . We note that in the further investigations, it is essential

assumption: $f_u(E_3) = \mathbb{C}$ for all $u = 1, 2, \dots, m$, where $f_u(E_3)$ is the image of E_3 under the mapping f_u . Obviously, it holds if and only if for every fixed $u = 1, 2, \dots, m$ at least one of the numbers a_u or b_u belongs to $\mathbb{C} \setminus \mathbb{R}$.

To a set $Q \subset \mathbb{R}^3$ we associate the set $Q_\zeta := \{\zeta = xe_1 + ye_2 + ze_3 : (x, y, z) \in Q\}$ in E_3 . We also note that the topological properties of a set Q_ζ in E_3 are understood as a corresponding topological properties of a set Q in \mathbb{R}^3 . For example, a homotopicity of a curve $\gamma_\zeta \subset E_3$ to the zero, means a homotopicity of $\gamma \subset \mathbb{R}^3$ to the zero, etc.

Let Ω be a domain in \mathbb{R}^3 . A continuous function $\Phi : \Omega_\zeta \rightarrow \mathbb{A}_n^m$ is *monogenic* in Ω_ζ if Φ is differentiable in the sense of Gateaux in every point of Ω_ζ , i.e. if for every $\zeta \in \Omega_\zeta$ there exists an element $\Phi'(\zeta) \in \mathbb{A}_n^m$ such that

$$\lim_{\varepsilon \rightarrow 0+0} (\Phi(\zeta + \varepsilon h) - \Phi(\zeta))\varepsilon^{-1} = h\Phi'(\zeta) \quad \forall h \in E_3. \tag{3.2}$$

$\Phi'(\zeta)$ is the *Gateaux derivative* of the function Φ in the point ζ .

Consider the decomposition of a function $\Phi : \Omega_\zeta \rightarrow \mathbb{A}_n^m$ with respect to the basis $\{I_k\}_{k=1}^n$:

$$\Phi(\zeta) = \sum_{k=1}^n U_k(x, y, z) I_k. \tag{3.3}$$

In the case where the functions $U_k : \Omega \rightarrow \mathbb{C}$ are \mathbb{R} -differentiable in Ω , i.e. for every $(x, y, z) \in \Omega$,

$$U_k(x + \Delta x, y + \Delta y, z + \Delta z) - U_k(x, y, z) = \frac{\partial U_k}{\partial x} \Delta x + \frac{\partial U_k}{\partial y} \Delta y + \frac{\partial U_k}{\partial z} \Delta z + o\left(\sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}\right), \quad (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 \rightarrow 0,$$

the function Φ is monogenic in the domain Ω_ζ if and only if the following Cauchy–Riemann conditions are satisfied in Ω_ζ :

$$\frac{\partial \Phi}{\partial y} = \frac{\partial \Phi}{\partial x} e_2, \quad \frac{\partial \Phi}{\partial z} = \frac{\partial \Phi}{\partial x} e_3. \tag{3.4}$$

Expansion of the resolvent is of the form

$$(te_1 - \zeta)^{-1} = \sum_{u=1}^m \frac{1}{t - \xi_u} I_u + \sum_{s=m+1}^n \sum_{k=2}^{s-m+1} \frac{Q_{k,s}}{(t - \xi_{u_s})^k} I_s \quad \forall t \in \mathbb{C} : t \neq \xi_u, \quad u = 1, 2, \dots, m, \tag{3.5}$$

where $Q_{k,s}$ are determined by the following recurrence relations:

$$Q_{2,s} := T_s, \quad Q_{k,s} = \sum_{r=k+m-2}^{s-1} Q_{k-1,r} B_{r,s}, \quad k = 3, 4, \dots, s - m + 1. \tag{3.6}$$

with

$$T_s := ya_s + zb_s, \quad B_{r,s} := \sum_{k=m+1}^{s-1} T_k \Upsilon_{r,s}^k, \quad s = m + 2, \dots, n,$$

and natural numbers u_s are defined in the rule 3 of the multiplication table of the algebra \mathbb{A}_n^m .

From relation (3.5), follows that the points $(x, y, z) \in \mathbb{R}^3$ corresponding to the noninvertible elements $\zeta \in \mathbb{A}_n^m$ form the straight lines

$$L_u : \begin{cases} x + y \operatorname{Re} a_u + z \operatorname{Re} b_u = 0, \\ y \operatorname{Im} a_u + z \operatorname{Im} b_u = 0 \end{cases}$$

in the three-dimensional space \mathbb{R}^3 .

Denote by $D_u \subset \mathbb{C}$ the image of Ω_ζ under the mapping f_u , $u = 1, 2, \dots, m$. A constructive description of all monogenic functions in algebra \mathbb{A}_n^m by means of holomorphic functions of the complex variable are obtained in the paper [15]. Namely, it is proved the theorem:

Let a domain $\Omega \subset \mathbb{R}^3$ be convex in the direction of the straight lines L_u and $f_u(E_3) = \mathbb{C}$ for all $u = 1, 2, \dots, m$. Then any monogenic function $\Phi : \Omega_\zeta \rightarrow \mathbb{A}_n^m$ can be expressed in the form

$$\begin{aligned} \Phi(\zeta) &= \sum_{u=1}^m I_u \frac{1}{2\pi i} \int_{\Gamma_u} F_u(t)(te_1 - \zeta)^{-1} dt \\ &+ \sum_{s=m+1}^n I_s \frac{1}{2\pi i} \int_{\Gamma_{u_s}} G_s(t)(te_1 - \zeta)^{-1} dt, \end{aligned} \tag{3.7}$$

where F_u is a certain holomorphic function in a domain D_u ; G_s is a certain holomorphic function in a domain D_{u_s} ; Γ_q is a closed Jordan rectifiable curve lying in the domain D_q surround a point ξ_q and containing no points ξ_ℓ , $\ell, q = 1, 2, \dots, m, \ell \neq q$.

4. Cauchy Integral Theorem for a Curvilinear Integral

Let γ be a Jordan rectifiable curve in \mathbb{R}^3 . For a continuous function $\Psi : \gamma_\zeta \rightarrow \mathbb{A}_n^m$ of the form

$$\Psi(\zeta) = \sum_{k=1}^n U_k(x, y, z) I_k + i \sum_{k=1}^n V_k(x, y, z) I_k, \tag{4.1}$$

where $(x, y, z) \in \gamma$ and $U_k : \gamma \rightarrow \mathbb{R}$, $V_k : \gamma \rightarrow \mathbb{R}$, we define an integral along a Jordan rectifiable curve γ_ζ by the equality:

$$\begin{aligned} \int_{\gamma_\zeta} \Psi(\zeta) d\zeta &:= \sum_{k=1}^n I_k \int_{\gamma} U_k(x, y, z) dx + \sum_{k=1}^n e_2 I_k \int_{\gamma} U_k(x, y, z) dy \\ &+ \sum_{k=1}^n e_3 I_k \int_{\gamma} U_k(x, y, z) dz + i \sum_{k=1}^n I_k \int_{\gamma} V_k(x, y, z) dx \\ &+ i \sum_{k=1}^n e_2 I_k \int_{\gamma} V_k(x, y, z) dy + i \sum_{k=1}^n e_3 I_k \int_{\gamma} V_k(x, y, z) dz, \end{aligned}$$

where $d\zeta := dx + e_2 dy + e_3 dz$.

Also we define a surface integral. Let Σ be a piece-smooth surface in \mathbb{R}^3 . For a continuous function $\Psi : \Sigma_\zeta \rightarrow \mathbb{A}_n^m$ of the form (4.1), where $(x, y, z) \in \Sigma$ and $U_k : \Sigma \rightarrow \mathbb{R}, V_k : \Sigma \rightarrow \mathbb{R}$, we define a surface integral on Σ_ζ with the differential form $dxdy$, by the equality

$$\int_{\Sigma_\zeta} \Psi(\zeta)dxdy := \sum_{k=1}^n I_k \int_{\Sigma} U_k(x, y, z)dxdy + i \sum_{k=1}^n I_k \int_{\Sigma} V_k(x, y, z)dxdy.$$

In the similar way we defined the integrals with the forms $dydz$ and $dzdx$.

If a function $\Phi : \Omega_\zeta \rightarrow \mathbb{A}_n^m$ is continuous together with partial derivatives of the first order in a domain Ω_ζ , and Σ is a piece-smooth surface in Ω , and the edge γ of surface Σ is a rectifiable Jordan curve, then the analogue of the Stokes formula is true, as we can see in the following:

$$\begin{aligned} \int_{\gamma_\zeta} \Psi(\zeta)d\zeta &= \int_{\Sigma_\zeta} \left(\frac{\partial \Psi}{\partial x} e_2 - \frac{\partial \Psi}{\partial y} \right) dxdy + \left(\frac{\partial \Psi}{\partial y} e_3 - \frac{\partial \Psi}{\partial z} e_2 \right) dydz \\ &\quad + \left(\frac{\partial \Psi}{\partial z} - \frac{\partial \Psi}{\partial x} e_3 \right) dzdx. \end{aligned} \tag{4.2}$$

Now, the next theorem is a result of the formula (4.2) and the equalities (3.4).

Theorem 4.1. *Suppose that $\Phi : \Omega_\zeta \rightarrow \mathbb{A}_n^m$ is a monogenic function in a domain Ω_ζ , and Σ is a piece-smooth surface in Ω , and the edge γ of surface Σ is a rectifiable Jordan curve. Then*

$$\int_{\gamma_\zeta} \Phi(\zeta)d\zeta = 0. \tag{4.3}$$

In the case where a domain Ω is convex, then by the usual way (see, e.g., [17]) the equality (4.3) can be proved for an arbitrary closed Jordan rectifiable curve γ_ζ .

In the case where a domain Ω is arbitrary, then similarly to the proof of Theorem 3.2 [2] we can prove the following

Theorem 4.2. *Let $\Phi : \Omega_\zeta \rightarrow \mathbb{A}_n^m$ be a monogenic function in a domain Ω_ζ . Then for every closed Jordan rectifiable curve γ homotopic to a point in Ω , the equality (4.3) is true.*

5. The Morera Theorem

To prove the analogue of Morera theorem in the algebra \mathbb{A}_n^m , we introduce auxiliary notions and prove some auxiliary statements.

Let us consider the algebra $\mathbb{A}_n^m(\mathbb{R})$ with the basis $\{I_k, iI_k\}_{k=1}^n$ over the field \mathbb{R} which is isomorphic to the algebra \mathbb{A}_n^m over the field \mathbb{C} . In the algebra $\mathbb{A}_n^m(\mathbb{R})$ there exist another basis $\{e_k\}_{k=1}^{2n}$, where the vectors e_1, e_2, e_3 are the same as in the Sect. 3.

For the element $a := \sum_{k=1}^{2n} a_k e_k$, $a_k \in \mathbb{R}$ we define the Euclidian norm

$$\|a\| := \sqrt{\sum_{k=1}^{2n} a_k^2}.$$

Accordingly, $\|\zeta\| = \sqrt{x^2 + y^2 + z^2}$ and $\|e_1\| = \|e_2\| = \|e_3\| = 1$.

Using the Theorem on equivalents of norms, for the element $b := \sum_{k=1}^n (b_{1k} + ib_{2k})I_k$, $b_{1k}, b_{2k} \in \mathbb{R}$ we have the following inequalities

$$|b_{1k} + ib_{2k}| \leq \sqrt{\sum_{k=1}^{2n} (b_{1k}^2 + b_{2k}^2)} \leq c\|b\|, \tag{5.1}$$

where c is a positive constant which does not depend on b .

Lemma 5.1. *If γ is a closed Jordan rectifiable curve in \mathbb{R}^3 and function $\Psi : \gamma_\zeta \rightarrow \mathbb{A}_n^m$ is continuous, then*

$$\left\| \int_{\gamma_\zeta} \Psi(\zeta) d\zeta \right\| \leq c \int_{\gamma_\zeta} \|\Psi(\zeta)\| \|d\zeta\|, \tag{5.2}$$

where c is a positive absolutely constant.

Proof. Using the representation of function Ψ in the form (4.1) for $(x, y, z) \in \gamma$, we obtain

$$\begin{aligned} \left\| \int_{\gamma_\zeta} \Psi(\zeta) d\zeta \right\| &\leq \sum_{k=1}^n \|I_k\| \int_{\gamma} |U_k(x, y, z) + iV_k(x, y, z)| dx \\ &\quad + \sum_{k=1}^n \|e_2 I_k\| \int_{\gamma} |U_k(x, y, z) + iV_k(x, y, z)| dy \\ &\quad + \sum_{k=1}^n \|e_3 I_k\| \int_{\gamma} |U_k(x, y, z) + iV_k(x, y, z)| dz. \end{aligned}$$

Now, we can use the inequality (5.1) for $b = \Psi(\zeta)$ and the inequalities $\|e_s I_k\| \leq c_s$, $s = 1, 2, 3$, where c_s are positive absolutely constants, we obtain the relation (5.2). □

Using Lemma 5.1, for functions taking values in the algebra \mathbb{A}_n^m , the following Morera theorem can be established in the usual way.

Theorem 5.2. *If a function $\Phi : \Omega_\zeta \rightarrow \mathbb{A}_n^m$ is continuous in a domain Ω_ζ and satisfies the equality*

$$\int_{\partial\Delta_\zeta} \Phi(\zeta) d\zeta = 0 \tag{5.3}$$

for every triangle Δ_ζ such that closure $\overline{\Delta_\zeta} \subset \Omega_\zeta$, then the function Φ is monogenic in the domain Ω_ζ .

6. Cauchy Integral Formula for a Curvilinear Integral

Let $\zeta_0 := x_0e_1 + y_0e_2 + z_0e_3$ be a point in a domain $\Omega_\zeta \subset E_3$. In a neighborhood of ζ_0 contained in Ω_ζ let us take a circle $C_\zeta(\zeta_0, \varepsilon)$ of radius ε with the center at the point ζ_0 . By $C_u(\xi_u^{(0)}, \varepsilon) \subset \mathbb{C}$ we denote the image of $C_\zeta(\zeta_0, \varepsilon)$ under the mapping f_u , $u = 1, 2, \dots, m$. We assume that the circle $C_\zeta(\zeta_0, \varepsilon)$ embraces the set $\{\zeta - \zeta_0 : (x, y, z) \in \bigcup_{u=1}^m L_u\}$. It means that the curve $C_u(\xi_u^{(0)}, \varepsilon)$ bounds some domain D'_u and $f_u(\zeta_0) = \xi_u^{(0)} \in D'_u$, $u = 1, 2, \dots, m$.

We say that the curve $\gamma_\zeta \subset \Omega_\zeta$ embraces once the set $\{\zeta - \zeta_0 : (x, y, z) \in \bigcup_{u=1}^m L_u\}$, if there exists a circle $C_\zeta(\zeta_0, \varepsilon)$ which embraces the mentioned set and is homotopic to γ_ζ in the domain $\Omega_\zeta \setminus \{\zeta - \zeta_0 : (x, y, z) \in \bigcup_{u=1}^m L_u\}$.

Since the function ζ^{-1} is continuous on the curve $C_\zeta(0, \varepsilon)$, then there exists the integral

$$\lambda := \int_{C_\zeta(0, \varepsilon)} \zeta^{-1} d\zeta. \tag{6.1}$$

The following theorem is an analogue of Cauchy integral theorem for monogenic function $\Phi : \Omega_\zeta \rightarrow \mathbb{A}_n^m$.

Theorem 6.1. *Suppose that a domain $\Omega \subset \mathbb{R}^3$ is convex in the direction of the straight lines L_u and $f_u(E_3) = \mathbb{C}$ for all $u = 1, 2, \dots, m$. Suppose also that $\Phi : \Omega_\zeta \rightarrow \mathbb{A}_n^m$ is a monogenic function in Ω_ζ . Then for every point $\zeta_0 \in \Omega_\zeta$ the following equality is true:*

$$\lambda \Phi(\zeta_0) = \int_{\gamma_\zeta} \Phi(\zeta)(\zeta - \zeta_0)^{-1} d\zeta, \tag{6.2}$$

where γ_ζ is an arbitrary closed Jordan rectifiable curve in Ω_ζ , that embraces once the set $\{\zeta - \zeta_0 : (x, y, z) \in \bigcup_{u=1}^m L_u\}$.

Proof. Inasmuch as γ_ζ is homotopic to $C_\zeta(\zeta_0, \varepsilon)$ in the domain $\Omega_\zeta \setminus \{\zeta - \zeta_0 : (x, y, z) \in \bigcup_{u=1}^m L_u\}$, it follows from Theorem 4.2 that

$$\int_{\gamma_\zeta} \Phi(\zeta)(\zeta - \zeta_0)^{-1} d\zeta = \int_{C_\zeta(\zeta_0, \varepsilon)} \Phi(\zeta)(\zeta - \zeta_0)^{-1} d\zeta. \tag{6.3}$$

Taking into account the equality (6.3) we represent the integral on the right-hand side of equality (6.2) as the sum of the following two integrals:

$$\begin{aligned} \int_{\gamma_\zeta} \Phi(\zeta)(\zeta - \zeta_0)^{-1} d\zeta &= \int_{C_\zeta(\zeta_0, \varepsilon)} (\Phi(\zeta) - \Phi(\zeta_0))(\zeta - \zeta_0)^{-1} d\zeta \\ &\quad + \Phi(\zeta_0) \int_{C_\zeta(\zeta_0, \varepsilon)} (\zeta - \zeta_0)^{-1} d\zeta =: J_1 + J_2. \end{aligned}$$

Let us note that from the relation (6.3) follows that if there exist the integral in the equality (6.1) then it does not depend on ε . As a consequence of the equalities (6.1), (6.3), we have the following relation

$$J_2 = \Phi(\zeta_0) \int_{C_\zeta(0,\varepsilon)} \tau^{-1} d\tau = \lambda \Phi(\zeta_0), \tag{6.4}$$

where $\tau := \zeta - \zeta_0$.

The integrand in the integral J_1 is bounded by a constant which does not depend on ε : when $\varepsilon \rightarrow 0$ the integrand tends to $\Phi'(\zeta_0)$ (see [15, Lemma 5]). Therefore, using the Lemma 5.1, the integral J_1 tends to zero as $\varepsilon \rightarrow 0$. The theorem is proved. \square

In the following, we will prove that the constant λ is an invertible element in \mathbb{A}_n^m .

7. A Constant λ

In some special algebras (see [3-5]) the Cauchy integral formula (6.2) has the form

$$\Phi(\zeta_0) = \frac{1}{2\pi i} \int_{\gamma_\zeta} \Phi(\zeta)(\zeta - \zeta_0)^{-1} d\zeta, \tag{7.1}$$

i.e.

$$\lambda = 2\pi i. \tag{7.2}$$

In this section we indicate a set of algebras \mathbb{A}_n^m for which (7.2) holds. In this a way, we consider first some auxiliary statements.

As a consequence of the expansion (3.5), we obtain the following equality:

$$\zeta^{-1} = \sum_{k=1}^n \tilde{A}_k I_k \tag{7.3}$$

with the coefficients \tilde{A}_k determined by the following relations:

$$\begin{aligned} \tilde{A}_u &= \frac{1}{\xi_u}, \quad u = 1, 2, \dots, m, \\ \tilde{A}_s &= \sum_{k=2}^{s-m+1} \frac{\tilde{Q}_{k,s}}{\xi_{u_s}^k}, \quad s = m + 1, m + 2, \dots, n, \end{aligned} \tag{7.4}$$

where $\tilde{Q}_{k,s}$ are determined by the following recurrence relations:

$$\tilde{Q}_{2,s} := -T_s, \quad \tilde{Q}_{k,s} = - \sum_{r=k+m-2}^{s-1} \tilde{Q}_{k-1,r} B_{r,s}, \quad k = 3, 4, \dots, s - m + 1. \tag{7.5}$$

where T_s and $B_{r,s}$ are the same as in the equalities (3.6), and natural numbers u_s are defined in the rule 3 of the multiplication table of the algebra \mathbb{A}_n^m .

Taking into account the equality (7.3) and the relation

$$d\zeta = dx e_1 + dy e_2 + dz e_3 = \sum_{u=1}^m (dx + dy a_u + dz b_u) I_u + \sum_{r=m+1}^n (dy a_r + dz b_r) I_r = \sum_{u=1}^m d\xi_u I_u + \sum_{r=m+1}^n dT_r I_r,$$

we have the following equality

$$\zeta^{-1} d\zeta = \sum_{u=1}^m \tilde{A}_u d\xi_u I_u + \sum_{r=m+1}^n \tilde{A}_{u_r} dT_r I_r + \sum_{s=m+1}^n \tilde{A}_s d\xi_{u_s} I_s + \sum_{s=m+1}^n \sum_{r=m+1}^n \tilde{A}_s dT_r I_s I_r =: \sum_{k=1}^n \sigma_k I_k. \tag{7.6}$$

Now, taking into account the denotation (7.6) and the equality (7.4), we calculate:

$$\int_{C_\zeta(0,R)} \sum_{u=1}^m \sigma_u I_u = \sum_{u=1}^m I_u \int_{C_u(\xi_u,R)} \frac{d\xi_u}{\xi_u} = 2\pi i \sum_{u=1}^m I_u = 2\pi i.$$

Therefore,

$$\lambda = 2\pi i + \sum_{k=m+1}^n I_k \int_{C_\zeta(0,R)} \sigma_k. \tag{7.7}$$

We note that from the relations (7.7), (7.3), and (7.4) that λ is an invertible element.

Thus, the equality (7.2) holds if and only if

$$\int_{C_\zeta(0,R)} \sigma_k = 0 \quad \forall k = m + 1, \dots, n. \tag{7.8}$$

But, for satisfying the equality (7.8) the differential form σ_k must be a total differential of some function. We note that the property of being a total differential is invariant under admissible transformations of coordinates [18, Theorem 2, p. 328]. In our situation, if we show that σ_k is a total differential of some function depend of the variables $\frac{T_{m+1}}{\xi}, \dots, \frac{T_k}{\xi}$, then it means that σ_k is a total differential of some function depending on x, y, z .

7.1.

In this subsection we indicate a set of algebras in which the vectors (3.1) chosen arbitrarily and the equality (7.2) holds. We remind that an arbitrary commutative associative algebra, \mathbb{A}_n^m , with unit over the field of complex number \mathbb{C} can be represented as $\mathbb{A}_n^m = S \oplus_s N$, where S is m -dimensional semi-simple subalgebra and N is $(n - m)$ -dimensional nilpotent subalgebra (see Sect. 2).

Theorem 7.1. *If $\mathbb{A}_n^m \equiv S$, then the equality (7.2) holds.*

The proof immediately follows from the conditions $\sigma_k \equiv 0$ for $k = m + 1, \dots, n$ and (7.7). This theorem is obtained in the paper [5].

Theorem 7.2. *If $\mathbb{A}_n^m = S \oplus_s N$ and N is a zero nilpotent subalgebra, then the equality (7.2) holds.*

Proof. From the condition of theorem follows that in the relations (7.4) all $B_{k,p} = 0$. Therefore, (7.4) takes the form

$$\tilde{A}_k = -\frac{T_k}{\xi_{u_k}^2}, \quad k = m + 1, \dots, n. \tag{7.9}$$

Since $I_s I_r = 0$ for $r, s = m + 1, \dots, n$, then from the denotation (7.6) and the identity (7.9), we obtain

$$\sigma_k = \frac{dT_k}{\xi_{u_k}} + \tilde{A}_k d\xi_{u_k} = \frac{dT_k}{\xi_{u_k}} - \frac{T_k}{\xi_{u_k}^2} d\xi_{u_k} = d\left(\frac{T_k}{\xi_{u_k}}\right) =: d\tau_k, \quad k = m + 1, \dots, n.$$

Under the transformation $(x, y, z) \rightarrow \tau_k$ the circle $C_\zeta(0, R)$ maps into a closed smooth curve \tilde{C} (Jordan or not) and the singularity $\xi_{u_k} = 0$ maps on $\tau_k = \infty$. Consequently, in an interior of the curve \tilde{C} does not exist singular points. By the Cauchy theorem in complex plane [18, p. 90], we have:

$$\int_{C_\zeta(0,R)} \sigma_k = \int_{\tilde{C}} d\tau_k = 0.$$

So, the equality (7.2) is a consequence of the last relation and (7.7). The theorem is proved. □

The Theorem 7.2 implies the formula (7.1) for monogenic functions in the three-dimensional algebra \mathbb{A}_2 which was investigated in the paper [19].

Further we consider the case when N in non-zero nilpotent subalgebra. For this goal we establish an explicitly form of $\sigma_{m+1}, \sigma_{m+2}, \sigma_{m+3}$ and σ_{m+4} . From the relation (7.6) follows the equalities

$$\begin{aligned} \sigma_{m+1} &= \frac{dT_{m+1}}{\xi_{u_{m+1}}} + \tilde{A}_{m+1} d\xi_{u_{m+1}}, \\ \sigma_k &= \frac{dT_k}{\xi_{u_k}} + \tilde{A}_k d\xi_{u_k} + \sum_{r,s=m+1}^{k-1} \tilde{A}_r dT_s \Upsilon_{r,k}^s, \quad k = m + 2, \dots, n. \end{aligned} \tag{7.10}$$

Now, the equalities (7.4) and (7.5) imply the following equalities:

$$\begin{aligned} \tilde{A}_{m+1} &= -\frac{T_{m+1}}{\xi_{u_{m+1}}^2}, \quad \tilde{A}_{m+2} = -\frac{T_{m+2}}{\xi_{u_{m+2}}^2} + \frac{T_{m+1}^2}{\xi_{u_{m+2}}^3} \Upsilon_{m+1,m+2}^{m+1}, \\ \tilde{A}_{m+3} &= -\frac{T_{m+3}}{\xi_{u_{m+3}}^2} + \frac{T_{m+1}^2}{\xi_{u_{m+3}}^3} \Upsilon_{m+1,m+3}^{m+1} + 2\frac{T_{m+1}T_{m+2}}{\xi_{u_{m+3}}^3} \Upsilon_{m+2,m+3}^{m+1} \\ &\quad - \frac{T_{m+1}^3}{\xi_{u_{m+3}}^4} \Upsilon_{m+1,m+2}^{m+1} \Upsilon_{m+2,m+3}^{m+1} + \frac{T_{m+2}^2}{\xi_{u_{m+3}}^3} \Upsilon_{m+2,m+3}^{m+2} \\ &\quad - \frac{T_{m+1}^2 T_{m+2}}{\xi_{u_{m+3}}^4} \Upsilon_{m+2,m+3}^{m+2} \Upsilon_{m+1,m+2}^{m+1}, \end{aligned}$$

$$\begin{aligned}
\tilde{A}_{m+4} = & -\frac{T_{m+4}}{\xi_{u_{m+4}}^2} + \frac{T_{m+1}^2}{\xi_{u_{m+4}}^3} \Upsilon_{m+1,m+4}^{m+1} + 2\frac{T_{m+1}T_{m+3}}{\xi_{u_{m+4}}^3} \Upsilon_{m+3,m+4}^{m+1} \\
& + 2\frac{T_{m+1}T_{m+2}}{\xi_{u_{m+4}}^3} \Upsilon_{m+2,m+4}^{m+1} + 2\frac{T_{m+2}T_{m+3}}{\xi_{u_{m+4}}^3} \Upsilon_{m+3,m+4}^{m+2} + \frac{T_{m+2}^2}{\xi_{u_{m+4}}^3} \\
& \times \Upsilon_{m+2,m+4}^{m+2} - \frac{T_{m+1}^3}{\xi_{u_{m+4}}^4} \Upsilon_{m+1,m+2}^{m+1} \Upsilon_{m+2,m+4}^{m+1} - \frac{T_{m+1}^2 T_{m+2}}{\xi_{u_{m+4}}^4} \Upsilon_{m+1,m+2}^{m+1} \\
& \times \Upsilon_{m+2,m+4}^{m+2} - \frac{T_{m+1}^2 T_{m+3}}{\xi_{u_{m+4}}^4} \Upsilon_{m+1,m+2}^{m+1} \Upsilon_{m+3,m+4}^{m+2} + \frac{T_{m+3}^2}{\xi_{u_{m+4}}^3} \Upsilon_{m+3,m+4}^{m+3} \\
& - \frac{T_{m+1}^3}{\xi_{u_{m+4}}^4} \Upsilon_{m+1,m+3}^{m+1} \Upsilon_{m+3,m+4}^{m+1} - \frac{T_{m+1}^2 T_{m+2}}{\xi_{u_{m+4}}^4} \Upsilon_{m+1,m+3}^{m+1} \Upsilon_{m+3,m+4}^{m+2} \\
& - \frac{T_{m+1}^2 T_{m+3}}{\xi_{u_{m+4}}^4} \Upsilon_{m+1,m+3}^{m+1} \Upsilon_{m+3,m+4}^{m+3} - 2\frac{T_{m+1}^2 T_{m+2}}{\xi_{u_{m+4}}^4} \Upsilon_{m+2,m+3}^{m+1} \\
& \times \Upsilon_{m+3,m+4}^{m+1} - 2\frac{T_{m+1}T_{m+2}^2}{\xi_{u_{m+4}}^4} \Upsilon_{m+2,m+3}^{m+1} \Upsilon_{m+3,m+4}^{m+2} - 2\frac{T_{m+1}T_{m+2}T_{m+3}}{\xi_{u_{m+4}}^4} \\
& \times \Upsilon_{m+2,m+3}^{m+1} \Upsilon_{m+3,m+4}^{m+3} + \frac{T_{m+1}^4}{\xi_{u_{m+4}}^5} \Upsilon_{m+1,m+2}^{m+1} \Upsilon_{m+2,m+3}^{m+1} \Upsilon_{m+3,m+4}^{m+1} \\
& + \frac{T_{m+1}^3 T_{m+2}}{\xi_{u_{m+4}}^5} \Upsilon_{m+1,m+2}^{m+1} \Upsilon_{m+2,m+3}^{m+1} \Upsilon_{m+3,m+4}^{m+2} \\
& + \frac{T_{m+1}^3 T_{m+3}}{\xi_{u_{m+4}}^5} \Upsilon_{m+1,m+2}^{m+1} \Upsilon_{m+2,m+3}^{m+1} \Upsilon_{m+3,m+4}^{m+3} - \frac{T_{m+1} T_{m+2}^2}{\xi_{u_{m+4}}^4} \\
& \times \Upsilon_{m+2,m+3}^{m+2} \Upsilon_{m+3,m+4}^{m+1} - \frac{T_{m+2}^3}{\xi_{u_{m+4}}^4} \Upsilon_{m+2,m+3}^{m+2} \Upsilon_{m+3,m+4}^{m+2} - \frac{T_{m+2}^2 T_{m+3}}{\xi_{u_{m+4}}^4} \\
& \times \Upsilon_{m+2,m+3}^{m+2} \Upsilon_{m+3,m+4}^{m+3} + \frac{T_{m+1}^3 T_{m+2}}{\xi_{u_{m+4}}^5} \Upsilon_{m+2,m+3}^{m+2} \Upsilon_{m+1,m+2}^{m+1} \Upsilon_{m+3,m+4}^{m+1} \\
& + \frac{T_{m+1}^2 T_{m+2}^2}{\xi_{u_{m+4}}^5} \Upsilon_{m+2,m+3}^{m+2} \Upsilon_{m+1,m+2}^{m+1} \Upsilon_{m+3,m+4}^{m+2} \\
& + \frac{T_{m+1}^3 T_{m+2} T_{m+3}}{\xi_{u_{m+4}}^5} \Upsilon_{m+2,m+3}^{m+2} \Upsilon_{m+1,m+2}^{m+1} \Upsilon_{m+3,m+4}^{m+3}.
\end{aligned}$$

Finally, a consequence of the previous equalities and the relations (7.10) is the following differential representation of σ_{m+1} , σ_{m+2} , σ_{m+3} and σ_{m+4} :

$$\sigma_{m+1} = d\left(\frac{T_{m+1}}{\xi_{u_{m+1}}}\right), \quad \sigma_{m+2} = d\left(\frac{T_{m+2}}{\xi_{u_{m+2}}} - \frac{1}{2}\Upsilon_{m+1,m+2}^{m+1} \frac{T_{m+1}^2}{\xi_{u_{m+2}}^2}\right), \quad (7.11)$$

$$\begin{aligned}
\sigma_{m+3} = & d\left(\frac{T_{m+3}}{\xi_{u_{m+3}}} - \frac{1}{2}\Upsilon_{m+1,m+3}^{m+1} \frac{T_{m+1}^2}{\xi_{u_{m+3}}^2} - \Upsilon_{m+2,m+3}^{m+1} \frac{T_{m+1}T_{m+2}}{\xi_{u_{m+3}}^2}\right. \\
& \left. - \frac{1}{2}\Upsilon_{m+2,m+3}^{m+2} \frac{T_{m+2}}{\xi_{u_{m+3}}^2} + \frac{1}{3}\Upsilon_{m+1,m+2}^{m+1} \Upsilon_{m+2,m+3}^{m+1} \frac{T_{m+1}^3}{\xi_{u_{m+3}}^3}\right) \\
& + \Upsilon_{m+1,m+2}^{m+1} \Upsilon_{m+2,m+3}^{m+2} \sigma_{m+3}^{(1)}, \quad (7.12)
\end{aligned}$$

$$\begin{aligned}
 \sigma_{m+4} = & d \left(\frac{T_{m+4}}{\xi u_{m+4}} - \frac{1}{2} \Upsilon_{m+1, m+4} \frac{T_{m+1}^2}{\xi^2 u_{m+4}} - \Upsilon_{m+3, m+4} \frac{T_{m+1} T_{m+3}}{\xi^2 u_{m+4}} \right. \\
 & - \Upsilon_{m+2, m+4}^{m+1} \frac{T_{m+1} T_{m+2}}{\xi^2 u_{m+4}} - \frac{1}{2} \Upsilon_{m+2, m+4}^{m+2} \frac{T_{m+2}^2}{\xi^2 u_{m+4}} - \Upsilon_{m+3, m+4}^{m+2} \\
 & \times \frac{T_{m+2} T_{m+3}}{\xi^2 u_{m+3}} + \frac{1}{3} \Upsilon_{m+1, m+2}^{m+1} \Upsilon_{m+2, m+4}^{m+1} \frac{T_{m+1}^3}{\xi^3 u_{m+4}} - \frac{1}{2} \Upsilon_{m+3, m+4}^{m+3} \frac{T_{m+3}^2}{\xi^2 u_{m+4}} \\
 & + \frac{1}{3} \Upsilon_{m+1, m+3}^{m+1} \Upsilon_{m+3, m+4}^{m+1} \frac{T_{m+1}^3}{\xi^3 u_{m+4}} - \frac{1}{4} \Upsilon_{m+1, m+2}^{m+1} \Upsilon_{m+2, m+3}^{m+1} \Upsilon_{m+3, m+4}^{m+1} \\
 & \times \left. \frac{T_{m+1}^4}{\xi^4 u_{m+4}} + \frac{1}{3} \Upsilon_{m+2, m+3}^{m+2} \Upsilon_{m+3, m+4}^{m+2} \frac{T_{m+2}^3}{\xi^3 u_{m+4}} \right) \Upsilon_{m+1, m+2}^{m+1} \Upsilon_{m+2, m+4}^{m+2} \sigma_{m+4}^{(1,2)} \\
 & + \Upsilon_{m+1, m+2}^{m+1} \Upsilon_{m+3, m+4}^{m+2} \sigma_{m+4}^{(2,2)} + \Upsilon_{m+1, m+3}^{m+1} \Upsilon_{m+3, m+4}^{m+2} \sigma_{m+4}^{(3,2)} \\
 & + \Upsilon_{m+3, m+4}^{m+3} \Upsilon_{m+1, m+3}^{m+1} \sigma_{m+4}^{(4,3)} + \Upsilon_{m+2, m+3}^{m+1} \Upsilon_{m+3, m+4}^{m+1} \sigma_{m+4}^{(5,1)} \\
 & + \Upsilon_{m+2, m+3}^{m+1} \Upsilon_{m+3, m+4}^{m+2} \sigma_{m+4}^{(6,2)} + \Upsilon_{m+2, m+3}^{m+1} \Upsilon_{m+3, m+4}^{m+3} \sigma_{m+4}^{(7,3)} \\
 & - \Upsilon_{m+1, m+2}^{m+1} \Upsilon_{m+2, m+3}^{m+1} \Upsilon_{m+3, m+4}^{m+2} \sigma_{m+4}^{(8,2)} - \Upsilon_{m+1, m+2}^{m+1} \Upsilon_{m+2, m+3}^{m+1} \\
 & \times \Upsilon_{m+3, m+4}^{m+3} \sigma_{m+4}^{(9,3)} + \Upsilon_{m+2, m+3}^{m+2} \Upsilon_{m+3, m+4}^{m+1} \sigma_{m+4}^{(10,1)} + \Upsilon_{m+2, m+3}^{m+2} \\
 & \times \Upsilon_{m+3, m+4}^{m+3} \sigma_{m+4}^{(11,3)} - \Upsilon_{m+2, m+3}^{m+2} \Upsilon_{m+1, m+2}^{m+1} \Upsilon_{m+3, m+4}^{m+1} \sigma_{m+4}^{(12,1)} \\
 & - \Upsilon_{m+2, m+3}^{m+2} \Upsilon_{m+1, m+2}^{m+1} \Upsilon_{m+3, m+4}^{m+2} \sigma_{m+4}^{(13,2)} - \Upsilon_{m+2, m+3}^{m+2} \Upsilon_{m+1, m+2}^{m+1} \\
 & \times \Upsilon_{m+3, m+4}^{m+3} \sigma_{m+4}^{(14,3)}, \tag{7.13}
 \end{aligned}$$

where

$$\sigma_{m+3}^{(1)} := \frac{T_{m+1}^2}{\xi^3 u_{m+3}} \left(dT_{m+2} - \frac{T_{m+2}}{\xi u_{m+3}} d\xi u_{m+3} \right), \tag{7.14}$$

and $\sigma_{m+4}^{(\ell, r)}$, $\ell = 1, 2, \dots, 14$ are determined by the following relations:

$$\sigma_{m+4}^{(\ell, r)} := \begin{cases} \frac{T_{m+1}^2}{\xi^3 u_{m+4}} g(r) & \text{for } \ell = 1, 2, 3, 4, \\ \frac{2 T_{m+1} T_{m+2}}{\xi^3 u_{m+4}} g(r) & \text{for } \ell = 5, 6, 7, \\ \frac{T_{m+1}^3}{\xi^4 u_{m+4}} g(r) & \text{for } \ell = 8, 9, \\ \frac{T_{m+2}^2}{\xi^3 u_{m+4}} g(r) & \text{for } \ell = 10, 11, \\ \frac{T_{m+1} T_{m+2}}{\xi^4 u_{m+4}} g(r) & \text{for } \ell = 12, 13, 14, \end{cases} \tag{7.15}$$

where $g(r) := dT_{m+r} - \frac{T_{m+r}}{\xi u_{m+4}} d\xi u_{m+4}$.

Theorem 7.3. *If $\mathbb{A}_n^m = S \oplus_s N$ and $\dim_{\mathbb{C}} N \leq 3$, then the equality (7.2) holds.*

Proof. From the equality (7.11) for σ_{m+1} , we have

$$\sigma_{m+1} = d \left(\frac{T_{m+1}}{\xi_{u_{m+1}}} \right) =: d\tau_{m+1}.$$

Now, the identity $\int_{C_\zeta(0,R)} \sigma_{m+1} = 0$ is proved as in Theorem 7.2.

Consider σ_{m+2} from the equality (7.11), which is a total differential of the certain function depending on the variables $\frac{T_{m+1}}{\xi_{u_{m+2}}}, \frac{T_{m+2}}{\xi_{u_{m+2}}}$. Under the transformation $(x, y, z) \rightarrow (\frac{T_{m+1}}{\xi_{u_{m+2}}}, \frac{T_{m+2}}{\xi_{u_{m+2}}})$ the circle $C_\zeta(0, R)$ maps into a closed smooth curve $\tilde{\tilde{C}}$ (Jordan or not) and the singularity $\xi_{u_{m+2}} = 0$ maps on ∞ . Consequently, in an interior of the curve $\tilde{\tilde{C}}$ does not exist singular points. Then by the Cauchy theorem in the space \mathbb{C}^2 [18, p. 334], we have:

$$\int_{C_\zeta(0,R)} \sigma_{m+2}(x, y, z) = \int_{\tilde{\tilde{C}}} \sigma_{m+2} \left(\frac{T_{m+1}}{\xi_{u_{m+2}}}, \frac{T_{m+2}}{\xi_{u_{m+2}}} \right) = 0.$$

Finally, we prove the equality (7.8) for $k = m + 3$. In the paper [12], is described all commutative associative nilpotent algebras over the field \mathbb{C} of dimension 1, 2, 3. From results of the paper [12, (Table 1)], immediately follows that for all mentioned algebras the relation $\Upsilon_{m+1,m+2}^{m+1} \Upsilon_{m+2,m+3}^{m+2} = 0$ is always satisfied. Therefore, the equality (7.12) implies that under the conditions of theorem, σ_{m+3} is always a total differential of the certain function depending on the variables $\frac{T_{m+1}}{\xi_{u_{m+3}}}, \frac{T_{m+2}}{\xi_{u_{m+3}}}, \frac{T_{m+3}}{\xi_{u_{m+3}}}$.

Now as before, under the transformation $(x, y, z) \rightarrow (\frac{T_{m+1}}{\xi_{u_{m+3}}}, \frac{T_{m+2}}{\xi_{u_{m+3}}}, \frac{T_{m+3}}{\xi_{u_{m+3}}})$ the circle $C_\zeta(0, R)$ maps into a closed smooth curve \hat{C} (Jordan or not) and the singularity $\xi_{u_{m+3}} = 0$ maps on ∞ . Hence, in an interior of the curve \hat{C} does not exist singular points. Then by the Cauchy theorem in the space \mathbb{C}^3 [18, p. 334], we have:

$$\int_{C_\zeta(0,R)} \sigma_{m+3}(x, y, z) = \int_{\hat{C}} \sigma_{m+3} \left(\frac{T_{m+1}}{\xi_{u_{m+3}}}, \frac{T_{m+2}}{\xi_{u_{m+3}}}, \frac{T_{m+3}}{\xi_{u_{m+3}}} \right) = 0.$$

So, the equality (7.2) is a consequence of the last relation and (7.7). The theorem is proved. □

Let us note that from the Theorem 7.3 follows the formula (7.1), for monogenic functions in the three-dimensional algebra \mathbb{A}_3 (see [3]) and in the three-dimensional algebra \mathbb{A}_2 , which was considered in the paper [19].

Theorem 7.4. *Let $\mathbb{A}_n^m = S \oplus_s N$ and $\dim_{\mathbb{C}} N = 4$. Then the equality (7.2) holds if the following relations are satisfied*

$$\begin{aligned} \Upsilon_{m+1,m+2}^{m+1} \Upsilon_{m+2,m+3}^{m+2} &= \Upsilon_{m+1,m+2}^{m+1} \Upsilon_{m+2,m+4}^{m+2} = \Upsilon_{m+1,m+3}^{m+1} \Upsilon_{m+3,m+4}^{m+2} \\ &= \Upsilon_{m+3,m+4}^{m+3} \Upsilon_{m+1,m+3}^{m+1} = \Upsilon_{m+2,m+3}^{m+1} \Upsilon_{m+3,m+4}^{m+1} = \Upsilon_{m+2,m+3}^{m+1} \Upsilon_{m+3,m+4}^{m+2} \\ &= \Upsilon_{m+2,m+3}^{m+1} \Upsilon_{m+3,m+4}^{m+3} = \Upsilon_{m+1,m+2}^{m+1} \Upsilon_{m+2,m+3}^{m+1} \Upsilon_{m+3,m+4}^{m+2} \end{aligned}$$

$$\begin{aligned}
 &= \Upsilon_{m+1,m+2}^{m+1} \Upsilon_{m+2,m+3}^{m+1} \Upsilon_{m+3,m+4}^{m+3} = \Upsilon_{m+2,m+3}^{m+2} \Upsilon_{m+3,m+4}^{m+1} \\
 &= \Upsilon_{m+2,m+3}^{m+2} \Upsilon_{m+3,m+4}^{m+3} = \Upsilon_{m+2,m+3}^{m+2} \Upsilon_{m+1,m+2}^{m+1} \Upsilon_{m+3,m+4}^{m+1} \\
 &= \Upsilon_{m+2,m+3}^{m+2} \Upsilon_{m+1,m+2}^{m+1} \Upsilon_{m+3,m+4}^{m+2} = \Upsilon_{m+2,m+3}^{m+2} \Upsilon_{m+1,m+2}^{m+1} \Upsilon_{m+3,m+4}^{m+3} = 0.
 \end{aligned}
 \tag{7.16}$$

Proof. From the equalities (7.12) and (7.13), it is obvious that under conditions (7.16) expressions for σ_{m+3} and σ_{m+4} are total differentials. Further proof is similar to proof of the Theorem 7.3. \square

Further we consider some examples of algebras, which satisfy the relations (7.16).

Examples.

- Consider the algebra with the basis $\{I_1 := 1, I_2, I_3, I_4, I_5\}$ and multiplication rules:

$$I_2^2 = I_3, \quad I_2 I_4 = I_5$$

and other products are zeros (for nilpotent subalgebra see [14, Table 21], algebra \mathcal{J}_{69} and [13, page 590], algebra $A_{1,4}$).

- Consider the algebra with the basis $\{I_1 := 1, I_2, I_3, I_4, I_5\}$ and multiplication rules:

$$I_2^2 = I_3$$

and other products are zeros (for nilpotent subalgebra see [13, page 590], algebra $A_{1,2} \oplus A_{0,1}^2$).

- The algebra with the basis $\{I_1 := 1, I_2, I_3, I_4, I_5\}$ and multiplication rules:

$$I_2^2 = I_3, \quad I_4^2 = I_5$$

and other products are zeros (for nilpotent subalgebra see [13, page 590], algebra $A_{1,2} \oplus A_{1,2}$).

- The algebra with the basis $\{I_1 := 1, I_2, I_3, I_4, I_5\}$ and multiplication rules:

$$I_2^2 = I_3, \quad I_2 I_3 = I_4$$

and other products are zeros (for nilpotent subalgebra see [14, Table 21], algebra \mathcal{J}_{71}).

Now we consider an example of algebra, which does not satisfy the relations (7.16). Moreover, we choose the vectors e_1, e_2, e_3 of the form (3.1) such that the equality (7.2) is not true.

Example. Consider the algebra \mathbb{A}_5 with the basis $\{1, \rho, \rho^2, \rho^3, \rho^4\}$, where $\rho^5 = 0$ (see [4] and [10, par. 11]). Here $n = 5, m = 1$. It is obvious that $\Upsilon_{2,3}^2 \Upsilon_{3,4}^3 = 1$ and the relations (7.16) are not true. Consider the vectors:

$$e_1 = 1, \quad e_2 = i + \rho^2 + \rho^4, \quad e_3 = (1 - i)\rho + \left(\frac{1}{4} - \frac{3}{4}i\right)\rho^3,$$

which are linearly independent over \mathbb{R} and satisfy the equality

$$e_1^2 + e_2^2 + e_3^2 = 0.$$

Let $\zeta = xe_1 + ye_2 + ze_3$. In the algebra \mathbb{A}_5 for given ζ , we have

$$\xi_{u_2} = \xi_{u_3} = \xi_{u_4} = \xi_{u_5} = x + iy =: \xi.$$

The inverse element ζ^{-1} is of the form (7.3), where

$$\begin{aligned} \tilde{A}_0 &= \frac{1}{\xi}, & \tilde{A}_1 &= \frac{z(i-1)}{\xi^2}, & \tilde{A}_2 &= -\frac{y}{\xi^2} + \frac{z^2(1-i)^2}{\xi^3}, \\ \tilde{A}_3 &= \frac{1}{4} \frac{z(3i-1)}{\xi^2} + \frac{2yz(1-i)}{\xi^3} - \frac{z^3(1-i)^3}{\xi^4}, \\ \tilde{A}_4 &= -\frac{y}{\xi^2} + \frac{y^2 + \frac{1}{2}z^2(1-i)(1-3i)}{\xi^3} - \frac{3yz^2(1-i)^2}{\xi^4} + \frac{z^4(1-i)^4}{\xi^5}. \end{aligned}$$

Let us set

$$C_\zeta(0, R) := \{\zeta = xe_1 + ye_2 \in E_3 : x^2 + y^2 = R^2\}. \tag{7.17}$$

On the circle of integration (7.17), we obtain:

$$\tilde{A}_0 = \frac{1}{\xi}, \quad \tilde{A}_1 = \tilde{A}_3 = 0, \quad \tilde{A}_2 = -\frac{y}{\xi^2}, \quad \tilde{A}_4 = -\frac{y}{\xi^2} + \frac{y^2}{\xi^3}. \tag{7.18}$$

As a consequence of the equations (7.10), (7.18) on the circle (7.17) we obtain the following expression

$$\sigma_5 = \left(\frac{1}{\xi} - \frac{y}{\xi^2}\right) dy + \left(-\frac{y}{\xi^2} + \frac{y^2}{\xi^3}\right) d\xi.$$

It is easy to calculate that

$$\int_{C_\zeta(0,R)} \sigma_5 = \frac{\pi i}{2}$$

and

$$\int_{C_\zeta(0,R)} \sigma_1 = \int_{|\xi|=R} \frac{d\xi}{\xi} = 2\pi i, \quad \int_{C_\zeta(0,R)} \sigma_k = 0, \quad k = 2, 3, 4.$$

Hence, in this example

$$\lambda = \int_{C_\zeta(0,R)} \zeta^{-1} d\zeta = 2\pi i + \frac{\pi i}{2} \rho^4.$$

7.2.

In this subsection we indicate sufficient conditions on a choose of the vectors (3.1) for which the equality (7.2) is true. Let the algebra \mathbb{A}_n^m be represented as $\mathbb{A}_n^m = S \oplus_s N$. Let us note that the condition $\zeta \in E_3 \subset S$ means that in the decomposition (3.1) $a_k = b_k = 0$ for all $k = m + 1, \dots, n$.

Theorem 7.5. *If $\mathbb{A}_n^m = S \oplus_s N$ and $\zeta \in E_3 \subset S$, then the equality (7.2) holds.*

Proof. Since $\zeta \in S$, then $T_k = 0$ for $k = m + 1, \dots, n$ [see denotation (3.6)]. From (7.5), (7.4) follows that $\tilde{A}_k = 0$, and now from (7.10) follows that $\sigma_k = 0$ for $k = m + 1, \dots, n$. The equality (7.2) is a consequence of the equality $\sigma_k = 0$ and the relation (7.7). □

Let us note that by essentially the Theorem 7.5 generalizes the Theorem 3 of the paper [20].

Now we consider a case where $\zeta \notin S$. If $\mathbb{A}_n^m = S \oplus_s N$ and $\dim_{\mathbb{C}} N \leq 3$, then by Theorem 7.3 the equality (7.2) holds for any $\zeta \in E_3$.

Theorem 7.6. *Let $\mathbb{A}_n^m = S \oplus_s N$ and $\dim_{\mathbb{C}} N = 4$. Then the equality (7.2) holds if the following two conditions are satisfied:*

1. $a_{m+1} = b_{m+1} = 0$;
2. *at least one of the relations $a_{m+2} = b_{m+2} = 0$ or $a_{m+3} = b_{m+3} = 0$ are true.*

Proof. It follows from the condition of theorem that $T_{m+1} = 0$ and at least one of the equalities $T_{m+2} = 0$ or $T_{m+3} = 0$ are true. To prove (7.2) it is need to prove the equality (7.8) for $k = m + 1, \dots, m + 4$. The equality (7.8) is proved in Theorem 7.3 for $k = m + 1, m + 2$. Under the condition $T_{m+1} = 0$ from (7.14), we have $\sigma_{m+3}^{(1)} = 0$. Since now σ_{m+3} is a total differential, then similar to proof of Theorem 7.3 can be proved the equality (7.8) for $k = m + 3$.

Moreover, under the conditions of theorem from the denotation (7.15) follows the equalities $\sigma_{m+4}^{(\ell, r)} = 0$ for all $\ell = 1, \dots, 14$. Therefore, σ_{m+4} is a total differential, then similar to proof of Theorem 7.3 can be proved the equality (7.8) for $k = m + 4$. \square

Conflict of interest

The author declare that he have no conflict of interest or commercial relationship.

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