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Four Forms Make a Universe

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Abstract. Since Immanuel Kant's Inaugural Dissertation of 1770 we assume that the concepts of space and time are not abstracted from sensations of external things. But outer experience is considered possible at all only through an inner representation of space and time within the cognitive system. In this work we describe a representation which is both inner and outer. We add to the Kantian imagination that "forms of nature, matter, space and time are intelligible, perceivable and comprehensible", the idea that these four are indeed intelligent, perceiving, grasping and clear. They are active systems with their own intelligence. In this paper on the mind-matter interface we create the mathematical prerequisites for an appropriate system representation. We show that there is an oriented logic core within the space–time algebra. This logic core is a commutative subspace from which not only binary logic, but syntax with arbitrary real and complex truth classifiers can be derived. Space–time algebra too is obtained from this inner grammar by two rearrangements of four basic forms of connectives. When we conceive the existence of a few features like polarity between two appearances, identification and rearrangement of the latter as basic and primordial to human cognition and construction, the intelligence of space–time is prior to cognition, as it contains within its representation the basic self-reference necessary for the intelligible de-convolution of space–time. Thus the process of nature extends into the inner space.

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1. Prologue

We are a part of nature. This part, Kant had said, has in itself some representation of the outer space. Why 'the outer space', why not just 'outer space'? We are using the definite article because we are sure about the definite existence of events out there in space moving in time. Space may be conceived

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as both inner and outer, in a definite way. We use to consider space not only outside or surrounding us, but also inside. Space extends into and includes the location where we are. The volume of our body signifies the space we occupy. But things are even more sophisticated. Space extends as a structural event into our inner. Space is an energy invading the mind. Now that we see we are both in nature and in space, we are part of nature and part of space, is it not natural and rational to give to space the same intelligence that nature, by the time, has given to us?

2. A Few Words on Constructivism

We say that the outer is constructed on the basis of inner representations. As soon as we think about these inner systems of representations, they are constructed as well, and so appear as outer objects. Even what we say about the moon is considered as constructed. But radical constructivism cannot allege that the moon would vanish if we all closed our eyes. Post war constructivists stated: “The environment does not contain any information. The environment is as it is”. ([14], p. 189) Here we touch an important argument developed in second order cybernetics (SOC), namely what ([9], p. 418) calls a slippery slope argument on observer-dependencies. We may see it as an observation that discloses the variability of observer-observed-dependencies. We have to realize a specific transfer of attributes within a relation between observer and observed. Foerster considers ‘obscenity’ as an example. Obscenity is not a property of things. Müller points out, the attribute ‘obscene’ has to be transferred away from the image of the observed back to the observer. Processing along the slippery slope, another group of attributes like ‘aesthetic’, ‘beautiful’, ‘satisfying’ “seem to reside within the observer as well”. Foerster made a difference between attributes like the ‘green’ of ‘green spinach’ and ‘good’ or ‘beautiful’ like in ‘good spinach’ which would articulate rather a relation between observer and observed. Actually it seems that even the ‘green’ is brought forth by some specific relation between men and spinach, but moving on the ‘slippery slope’ we may locate the attributes sometimes more at the location of the observed and at other times rather in the observer. At one extreme of the slope we have the environment that does not contain any information. Here the environment is as it is. But at this extreme we can also have the observer/observed both as that which is as it is. Now, with reference to this view, there is something happening when we think about such simple things like arrays of polarities in an unspecified domain of void. Consider for instance such an open sequence like $\ldots, +1, -1, +1, -1, +1, -1, \ldots$, which may separate from another such sequence $\ldots, +1, -1, +1, -1, +1, -1 \ldots$ we see an identity. Or we are aware of a difference, namely only by juxtaposing these two

$$\begin{aligned} &\ldots, +1, -1, +1, -1, +1, -1, \ldots \\ &\ldots, -1, +1, -1, +1, -1, +1, \ldots \end{aligned} \tag{2.1}$$

Actually there might be no difference at all, but a difference can be produced by one step in time. This beautiful construction has been made by ([6,7]).

Now, suppose that such polarities could constitute matter and space–time. The process of separating them in void and juxtaposing them in opposition does not transfer any attribute, because the observer is the observed. The difference is an illusion correlated with time. So this innermost element of a process of polarities which is the most fundamental brick in the cosmic architecture is utterly free of the observer-observed dichotomy. It is not dominated by the inner-outer demarcation line. Now, material and biological things can develop on this basis. If we unfurl the process and remove some confused parts from the muddle of our insight, we can perhaps design the right circuit layout for a cognitive brain, a one that lives integrated within the universal whole. May be we cannot. In that case we shall go on wondering why there is something and not nothing.

3. Algebras and Clifford Algebra

In a series of papers we shall be going into some fundamental unity between logic and geometric structures of physics. These connections have been passed by unrealized since more than hundred years. In fundamental investigation of logic algebraic structures that relate to geometric Clifford algebra of space–time, the following list of algebras is useful. Pondering each of these will promote comprehension of this grand affiliation between logic and geometry.

In this paper we only want to study the relationship between the four algebra structures A2, A4, A5 and A8 shown in Table 1. In order to be able to do so, we must at first give a minimal definition of Clifford algebra. The real Clifford algebra generated by the Minkowski space $\mathbb{R}^{3,1}$ has base unit vectors $\{e_1, e_2, e_3, e_4\}$ and Lorentz Metric $\{+++-\}$ or briefly $\{3, 1\}$. 'Time' is connected with the fourth unit vector e_4. This algebra, denoted by $Cl_{3,1}$, is called the Minkowski algebra. As a linear vectorspace it is spanned by 16 Grassmann monomials $\{Id, e_1, e_2, e_3, e_4, e_{12}, e_{13}, e_{14}, e_{23}, e_{24}, e_{34}, e_{123}, e_{124}, e_{134}, e_{234}, e_{1234}\}$. These have the signature $\{++++---+-++-+++-\}$ or briefly $\{10, 6\}$. Like in every Clifford algebra there can be defined a geometric product in the Minkowski algebra which is called Clifford product. This product can be decomposed into a commutative and a non-commutative component given by the inner, scalar product and the outer or wedge product of vectors $a, b \in Cl_{3,1}$. [1] The Clifford product between vectors a, b is given by

$$ab = a \cdot b + a \wedge b \tag{3.1}$$

where

$$a \cdot b = \frac{1}{2}(ab + ba) = b \cdot a \quad \text{symmetric part} \tag{3.2}$$

$$a \wedge b = \frac{1}{2}(ab - ba) = -b \wedge a \quad \text{anti-symmetric part} \tag{3.3}$$

[1] We are taking for granted some basic knowledge in Clifford Algebra which can be amended by reading ([8], Chapter 3) or ([3], Chapter 1).

Table 1. Fundamental algebraic structures

	Algebra	Carrier	n-ary operations	Unary operation	Features
A1	**2**	B =0,1	$\vee,\wedge$	$\neg$; 'not'	Bivalent
A2	Of binary connectives	16 Terms	$\vee,\wedge$	$\neg$	Symmetries
A3	Free Boolean with k generators	Any B; $\vert \mathrm{B}\vert = 2^k$	$\vee,\wedge$	$\neg$	Free propositional algebra
A4	Of binary sequences $(\mathcal{B},\equiv)$	$\sum^{\mathbb{Z}}$	$\equiv$	Dualising	Self referent intelligence
A5	Dialogic LICO	16 Letters	$\wedge$, d	Complementation	Free linear iconic
A6	Cantor subalgebra $ch \subset cl_{3,1}$	16 Polarity strings	$+,\circ$	–	Color space in Minkowski algebra
A7	LICO vector space	$\mathbb{R}\bigoplus\mathbb{R}\bigoplus\mathbb{R}\bigoplus\mathbb{R}$	Inner, outer, +	–	Commutative
A8	Clifford algebra	Idempotent lattice	Many	Many	Geometric
A9	Boolean	Set of sets	$\bigcup,\bigcap$	Set complement	Poset algebra of inclusion

The exterior or Grassmann product $a \wedge b$ (a wedge b) represents the antisymmetric part of the Clifford product. The unit vectors satisfy the following equations:

$$e_{ij} \stackrel{\text{def}}{=} e_i \wedge e_j = e_i e_j = -e_j \wedge e_i = -e_j e_i = -e_{ji} \tag{3.4}$$

Those Cartan subalgebras, having three (quaternion) generators 'i, j, k' with positive definite signature, can be formed in the standard representation of the Minkowski Algebra $Cl_{3,1}$, they give us six algebras of Cotessarines. The first of these is the Clifford Algebra spanned by the following four magnitudes ([12]).

$$ch_1 \stackrel{\text{def}}{=} \{Id, e_1, e_{24}, e_{124}\} \tag{3.5}$$

We verify

$$e_1^2 = +Id \text{ because of first '+' in signature bracket } \{+++-\}\text{of space } \mathbb{R}^{3,1} \tag{3.6}$$

and also

$$\begin{aligned} e_{24}^2 &= e_{24}e_{24} = -e_{42}e_{24} = -e_4(e_2e_2)e_4 \\ &= -e_4 Id e_4 = -Id e_4 e_4 = -Id(-Id) = +Id \end{aligned} \tag{3.7}$$

because of the second '+' and the '−' in the last entry of the signature bracket $\{+++-\}$. The same sign can be obtained by an analogous calculation for the squared e_{124}. Hence we define the four magnitudes $\{Id, i, j, k\}$ as unit multivectors thusly

$$\begin{array}{ccccc} Id & i = e_1 & j = e_{24} & k = e_{124} & \text{cotessarines} \\ + & + & + & + & \text{signatures} \end{array} \tag{3.8}$$

We have to show that the three cotessarines commute. We have

$$\begin{aligned} e_1 e_{24} &= e_{124} \qquad \text{and further} \\ e_{24} e_1 &= e_{241} = -e_{214} = +e_{124} = e_1 e_{24} \end{aligned} \tag{3.9}$$

We calculate

$$\begin{aligned} e_{24}e_{124} &= e_{24}e_{241} = e_{24}^2 e_1 = Id e_1 = e_1 \quad \text{as well as} \\ e_{124}e_{24} &= e_1 e_{24}^2 = e_1 Id = e_1 \end{aligned} \tag{3.10}$$

Hamilton Quaternions $\mathbb{H}$ are isomorphic with the Clifford Algebra $Cl_{0,2}$. In the Minkowski Algebra $Cl_{3,1}$ they can be represented in standard representation either by the bivectors $\{e_{12}, e_{23}, e_{13}\}$, or by the 'timespace' $\{e_4, e_{123}, e_{1234}\}$. Both have signature $\{---\}$. Connected with Hamilton's Quaternions is the abelian group Q_8, the 'Quaternion Group', the smallest of all Hamiltonian Groups.

4. Minimal Logic Alphabet

To tune in, let us first consider a few statements from several media, to get some examples for statements in binary Boolean logic:

UN-Security Council, 4th September 2013; President Putin is not entirely disclaiming military intervention against the regime in Damascus.

Spiegel Online, 5th September 2013; US-Russia Stalemate: Merkel must take the Initiative on Syria. A Commentary by Frank-Walter Steinmeier. [...] Instead of standing idly on the sidelines, Mrs. Merkel should take advantage of the summit in St. Petersburg and seize the initiative of finding a political solution.

Aljazeera, 6th September 2013; G20 leaders remain divided over Syria action Vladimir Putin leads opposition to possible US unilateral strike, as US envoy accuses Russia of holding UN hostage.

Standard, 8th September 2013; The Gulf Cooperation Council (GCC) urges the International Community to immediate military intervention in Syria.

Yahoo News, 8th September 2013; President Barack Obama braced for a key week in his push to persuade sceptical Americans to back strikes against the Syrian regime.

Spiegel Online, 9th September 2013; (In Petersburg) Merkel did not give Obama allegiance, pointed to Europe and left for Germany. Italians, French and Spanish politicians straightforwardly signed the leaflet of the Americans without the German.

Slate, 9th September 2013; At his press conference on Friday, the president (Obama) explained that Syria's chemical weapons use was not enough of a direct threat to cause him to act without congressional approval. "I put it before Congress because I could not honestly claim that the threat posed by Assad's use of chemical weapons on innocent civilians and women and children posed an imminent, direct threat to the United States." [2]

That strand of statements *A,B,C,...* we would be interested to analyse is built up by sentences like

A. actor, group, institution, nation... favors military intervention against Syria
B. actor, group, institution, nation ... rejects military intervention against Syria
C. actor, group, institution, nation ... favors to delay immediate action and put the case before Congress and so on.

For such statements we can construct a binary logic structure that involves marked and unmarked logic states in the sense of Spencer Brown. In this paper we shall not do this, but we shall use the binary connectives only.

[2] http://www.slate.com/articles/newsandpolitics/politics/2013/09/barackobama'scaseforstrikingsyriathepresident'sargumentsformilitary.html.

TABLE 2. Logic identification of 2 statements

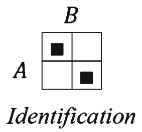


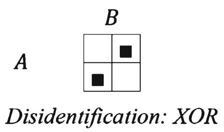
TABLE 3. Truth table for logic equivalence

A	B	$A \equiv B$
1	1	1
1	0	0
0	1	0
0	0	1

TABLE 4. Truth table for logic exclusive disjunction

A	B	$A \neq B$
1	1	0
1	0	1
0	1	1
0	0	0

TABLE 5. Exclusive disjunction of two statements



5. Alphabet of Binary Connectives

Suppose we have (A) American president favors military intervention, and (B) German government favors military intervention. We want to classify the possible relations between (A) and (B). One of 16 possible relations is based on identification. That is, the German government may or could identify its attitude with the attitude of the White House. We represent this event by fourfold Table 2.

This is the same as the binary arrangement in Table 3.

Statements A,B are connected by *identification*. They are equivalent. They may, however, just as well, be connected by *dis-identification*, the *exclusive disjunction* or XOR, Table 4.

This connective can also be represented by a fourfold diagram:

There are more possibilities to relate A with B, since there are 16 'binary connectives'. For example (Tables 5, 6, 7, 8).

Table 6. Truth table for logic conjunction

A	B	$A \wedge B$
1	1	1
1	0	0
0	1	0
0	0	0

Table 7. Conjunction of two statements

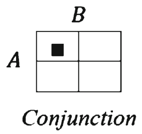


Table 8. Logic NAND and its minimal lettershape

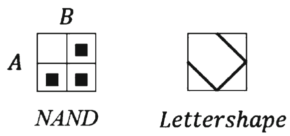
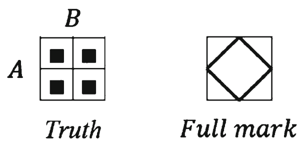


Table 9. Logic truth and full mark

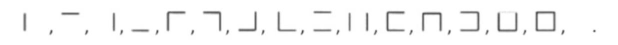


This has a dual, namely the logic 'NAND' meaning that A implies negating B.

When the AND is geometrically superimposed with its dual, we obtain logic truth as in Table 9.

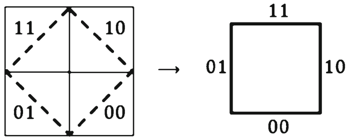
Each of the 16 binary connectives has a dual connective given by its logic opposite, and both superimposed give us the filled up fourfold table often interpreted as 'truth'. From now on we represent this by the square or 'full containment'. To get a geometric alphabet it is possible to still improve the representation a little. We take the four edges of the square as the interior of the four cells. That is, if the inner of a cell is marked by full ■, we draw an edge, if the inner is not marked we do not draw an edge. Hence we obtain the new alphabet.

Table 10. LICO

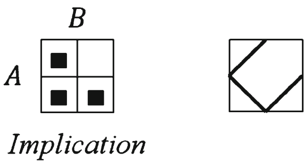


5.1. Letters for the minimal logic alphabet

Note that the last symbol (blank) cannot yet be interpreted as the 'unmarked state' in the sense of Spenser Brown's 'Laws of Form'. To promote understanding of this 'geometric alphabet' we suggest, imagine at first the fourfold table of a connective and draw the figure arising from it:



This procedure provides most conveniently arranged horizontal letter shapes such as $\sqcup$, $\sqcap$, ⌜⌞⌝⌟ $\sqsubset$ $\sqsupset$ $|\,|$ or $|||$ or $\not\equiv$, $=$ or $\equiv$, ■ which, in similar form, are here provided by some symbol packages of TeX. Next we turn fourfold table by $\frac{\pi}{4}$ and carry quarters to edges in a square. We let the upper left hand quarter be the upper edge, and the upper right hand one become the right hand edge, and the lower right quarter becomes the bottom edge, and the lower left hand quarter turns into the left handed edge. Therefore, an 'implication' having table



has an iconic symbol $\sqsubset$. Taking a further look at 'XOR', we find it has letter $|\,|$ whereas its dual is 'identification' and has letter $=$. In this way we obtain a linear iconic alphabet based on squares (Table 10).

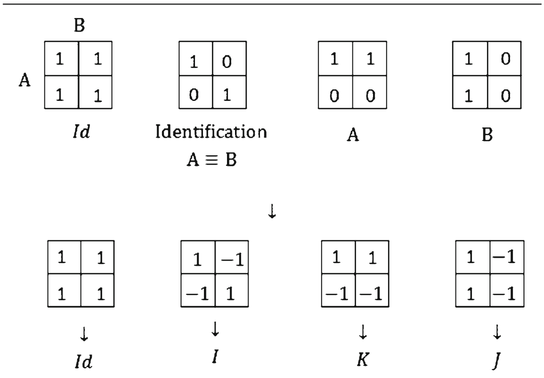
6. Connecting the Algebra Structures

In this section we want to show that we have at least six natural ways to represent logic binary connectives in geometric terms. First there is the familiar manner of writing connectives of any order as well formed terms in Boolean algebra. Then we have the 16 LICO letters capable of 'idemposition', so we can adjoin two LICO letters by geometrically superimposing them. Third we have the Lico vectors formed in the basis of four fundamental LICO letters, fourth we obtain a natural representation within the idempotent lattice of the Clifford algebra $Cl_{3,1}$, from this we obtain in the form of diagonals of matrix representations the fourfold strings formed by four numbers from $\{0, 1\}$, fifth we have them as color vectors, four-vectors in the commutative Cartan subalgebras, the color spaces $ch_{\chi} \subset Cl_{3,1}$ and sixth we can use finite

Table 11. Generators of binary logic space

LICO letter-shape	□	┐	⚌	┌
Logic connective	true	A	A ≡ B	B
Polarity string	[+, +, +, +]	[+, +, −, −]	[+, −, +, −]	[+, −, −, +]
geometric unit	Id	e_1	e_{124}	e_{24}
Vector in $^4\mathbb{R}$	(+1, +1, +1, +1)	(+1, +1, −1, −1)	(+1, −1, +1, −1)	(+1, −1, −1, +1)

Table 12. Logic quad-locations corresponding with four-fold array of quaternions Id, I, J, K



or infinite polarity strings. This construction is natural. For instance we know from physics that the Cartan subalgebras provide the invariants of quantum motion. In this image binary logic presents itself as a 4-dimensional, commutative vector space. This logic can immediately be generalized to real and complex truth classifiers and arbitrarily many generative statements of any order. It is not immediately visible that the linear iconic alphabet is connected with physics. It requires some insight into the whole mathematics of logic and geometric algebra to see this relation. LICO contains a small morphogenetic core or root structure of cognition and space–time equivalent with only four units of a vector space. After neat analysis these base units turn out as graded unit multivectors identical with the generators of a commutative Cartan subalgebra of the geometric Clifford algebra of the Minkowski space. This 'root structure' is essentially given by the four symbols represented in Table 11 and logic quad-locations in Table 12.

These will give us, by transpositions, not only Hamilton's quaternions, but even the 16 unit monomials or basis vectors of the Minkowskian space–time algebra. Recall the truth tables of these four connectives! How important these geometric numbers are in science applications, should be known from the investigation of Quaternion and Clifford calculus by [2], and [12].

Fourfold polarity strings can be put in correspondence with what is denoted as 'quad locations'. They form base units $Id = [+1, +1, +1, +1]$, unit

TABLE 13. Two multiplication tables of Klein four groups for bit strings and multivectors of ch_1

Id	e	f	g	Id	e_1	e_{24}	e_{124}
e	Id	g	f	e_1	Id	e_{124}	e_{24}
f	g	Id	e	e_{24}	e_{124}	Id	e_1
g	f	e	Id	e_{124}	e_{24}	e_1	Id

scalar; $e = [+1, +1, -1, -1] = e_1$, unit vector; $f = [+1, -1, -1, +1] = e_{24}$, directed unit area; $g = [+1, -1, +1, -1] = e_{124}$, directed unit volume in $Cl_{3,1}$. What is so special about these unit vectors? First of all, they form a commutative 4-dimensional vector space. The vectors of this space can be conceived as elements of the fourfold real ring ${}^4\mathbb{R}$ given by real valued quadruples. The four base units commute and satisfy the multiplication table of the Klein 4 group. Multiplication is carried out component-wise, e. g. :

$$ef = e_1 e_{24} = [+1, +1, -1, -1][+1, -1, -1, +1] = [+1, -1, +1, -1] = g \quad (6.1)$$

So we obtain multiplication rules (Table 13).

Every element of this chromatic space, we denote it as ch_1, can be given in two forms, namely either as vector $\phi = u + xe_1 + ye_{24} + ze_{124}$ within the Clifford algebra $Cl_{3,1}$ generated by the units $\{e_1, e_2, e_3, e_4\}$, or as a linear combination of mutually annihilating primitive idempotents. In the Clifford algebra of Minkowski space these primitive idempotents can be given in standard form

$$\begin{aligned} f_1 &= \frac{1}{2}(Id + e_1)\frac{1}{2}(Id + e_{24}) \\ f_2 &= \frac{1}{2}(Id + e_1)\frac{1}{2}(Id - e_{24}) \\ f_3 &= \frac{1}{2}(Id - e_1)\frac{1}{2}(Id - e_{24}) \\ f_4 &= \frac{1}{2}(Id - e_1)\frac{1}{2}(Id + e_{24}). \end{aligned} \quad (6.2)$$

Note the peculiar order of the sign combinations; in physics this order turns out substantial for phenomena of entanglement. These primitive idempotents span a commutative subspace, the 'color-space' ch_1. Let us calculate those primitive idempotents using the above representation in ${}^4\mathbb{R}$ and find out how their corresponding binary sequences and respectively polarity strings may look like

$$\begin{aligned} f_1 &= \frac{1}{2}(Id + e_1)\frac{1}{2}(Id + e_{24}) \\ &= \frac{1}{2}[+2, +2, 0, 0]\frac{1}{2}[+2, 0, 0, +2] = \frac{1}{4}[4, 0, 0, 0] = [1, 0, 0, 0] \\ f_2 &= \frac{1}{2}(Id + e_1)\frac{1}{2}(Id - e_{24}) \\ &= \frac{1}{2}[+2, +2, 0, 0]\frac{1}{2}[0, +2, +2, 0] = \frac{1}{4}[0, 4, 0, 0] = [0, 1, 0, 0] \end{aligned}$$

TABLE 14. Idempotents are bars of square in a LICO letter shape

	$\simeq f_1$
	$\simeq f_2$
	$\simeq f_3$
	$\simeq f_4$

$$\begin{aligned} f_3 &= \frac{1}{2}(Id - e_1)\frac{1}{2}(Id - e_{24}) \\ &= \frac{1}{2}[0, 0, +2, +2]\frac{1}{2}[0, +2, +2, 0] = \frac{1}{4}[0, 0, 4, 0] = [0, 0, 1, 0] \\ f_4 &= \frac{1}{2}(Id - e_1)\frac{1}{2}(Id + e_{24}) \\ &= \frac{1}{2}[0, 0, +2, +2]\frac{1}{2}[+2, 0, 0, +2] = \frac{1}{4}[0, 0, 0, 4] = [0, 0, 0, 1] \end{aligned} \tag{6.3}$$

Every vector $\phi = u + xe_1 + ye_{24} + ze_{124}$ of a commutative color space can be written as a linear combination of the orthogonal primitive idempotents (Table 14)

$$\phi = a_1 f_1 + a_2 f_2 + a_3 f_3 + a_4 f_4 \tag{6.4}$$

with coefficients

$$\begin{aligned} a_1 &= u + x + y + z \\ a_2 &= u + x - y - z \\ a_3 &= u - x - y + z \\ a_4 &= u - x + y - z \end{aligned} \tag{6.5}$$

Theorem 1. *Every letter in the LICO alphabet is a vector.*

Proof. Consider the conjunction. The connective $A \wedge B \simeq [1, 0, 0, 0]$ can be represented by the first primitive idempotent f_1. Further $A \wedge \neg B \simeq [0, 1, 0, 0] = f_2$. $\neg A \wedge \neg B \simeq [0, 0, 1, 0] = f_3$ und finally the $\neg A \wedge B \simeq [0, 0, 0, 1] = f_4$. So we have the simple correspondence between elements in algebras A5 and A8.

But we have a suspicion that we shall later prefer an equivalent representation which allows us to factor in the temporal iteration of a sequence ([7], [11]), that is, sequences in A4. Then we could prefer to represent every zero by a minus one. That is, in iterant algebra we prefer iterant views such as $A \wedge \neg A \simeq [+1, -1, -1, -1]$ over the image $A \wedge \neg A \simeq [+1, 0, 0, 0]$ which both can represent a truth table for conjunction and both allow us to represent that view in the same vector space ch_1. Henceforth, we consider two representations. First as an idempotent element in $Cl_{3,1}$, second in the representation over $\{-1, +1\}$. We have in this peculiar view for example $A \equiv B \simeq_{It} [+1, -1, +1, -1]$ the alternating iterant. In the idempotent representation this is obtained by converting every -1 into zero, that is, we have $A \equiv B \simeq_{Cl} [+1, 0, +1, 0]$. We can test if that is compatible with our previous result, namely that logic identity $A \equiv B$ should be represented by the unit space–time volume e_{124}. Consider the color space $ch_1 = span_{\mathbb{R}}\{Id, e_1, e_{24}, e_{124}\}$. We are recalling the Table 11 of correspondences for coming calculations. Before listing all 16 connectives in a special table, let us carry out two calculations of these geometric representations by hand and show their meaning. Consider at first the Boolean 'false' or contradictory statement $A \wedge \neg A$. We denote it by the LICO letter

which is a fourfold zero in ${}^4\mathbb{R}$. Notice, we consider $\Box$ as a symbol for the unit scalar Id in ch_1. If we work with a display of truth tables over the pair $\{1, -1\}$, we have a representation in iterant algebra as $false \equiv A \wedge \neg A \simeq_{It} [-1, -1, -1, -1]$. Using the explicit representation of primitive idempotents by Eq. 6.2 and summing up, we obtain indeed the negative identity. This can be represented by an iterant or polarity string. This gives the first line in the Table 15. Next we calculate a conjunction, $A \wedge B$ in conventional notation, denoted in iconic notation by the upper stroke. As the LICO letters denote vectors in a linear 4-dimensional space, they can be depicted as a linear combination of four letters, namely the upper bar is equal to or in ch_1 as $(1/2)(-Id + e + f + g)$ which can be thought to be isomorphic with $f_1 - f_2 - f_3 - f_4$. By the aid of the Clifford representation we can calculate this term to be equal to $\frac{1}{2}(-Id + e_1 + e_{24} + e_{124})$. The isomorphic Iterant can be portrayed as a vector over ${}^4\mathbb{R}$ or briefly as a polarity string $A \wedge B \simeq_{It} [+1, -1, -1, -1]$. Thus we obtain the first two lines of a table with 16 lines, namely

$$A \wedge \neg A \simeq_{(cl)} [0, 0, 0, 0] \simeq_{(it)} [-1, -1, -1, -1] \tag{6.6}$$

$$\begin{aligned} A \wedge B &\simeq_{(Li)} \frac{1}{2}(-\Box + {}^{\urcorner} + {}^{\ulcorner} + \equiv) \\ &\simeq_{(cl)} f_1 = \frac{1}{4}(Id + e_1 + e_{24} + e_{124}) \simeq_{(it)} [+1, -1, -1, -1] \end{aligned} \tag{6.7}$$

If we consider the idempotent representation of any 'Lico' binary connective in the Clifford algebra, denote it by L_i with $i = 1 \ldots, 16$, we have a definite representation by one four-vector in ${}^4\mathbb{R}$ and by an idempotent in $Cl_{3,1}$. We can represent $Cl_{3,1}$ by the Majorana algebra $Mat(4, \mathbb{R})$ of 4×4-matrices with real entries. Then the diagonal matrices with unit entries represent the mutually annihilating primitive idempotents. In other words, the fourfold real numbers in ${}^4\mathbb{R}$ are nothing else than the diagonals of diagonal matrices of $Mat(4, \mathbb{R})$. In Table 15 we consider six important representations of binary connectives L_i, namely in terms of

1. Boolean term
2. Lico letter shape
3. Lico vector
4. Idempotent in $Cl_{3,1}$
5. Multivector in color space $ch_1 \subset Cl_{3,1}$
6. Grade 4 iterant view

Table 15. Correspondences in algebra structures A2, A5, A7, A8, A4 in this succession

Boole	LICO letter	LICO vector	Idempotents in $Cl_{3,1}$	Colorspace vector in $Cl_{3,1}$	Polarity string
A ∧ ¬A	⬚	0	0	0	[− − − −]
$A \wedge B$	‾	$\frac{1}{2}$(−□ + ┐+ ┌ + =)	f_1	$\frac{1}{4}(Id + e_1 + e_{24} + e_{124})$	[+ − − −]
¬A ∧ B	|	$\frac{1}{2}$(+□ − ┐+ ┌ − =)	f_4	$\frac{1}{4}(Id + e_1 - e_{24} - e_{124})$	[− − − +]
A ∧ ¬B	|	$\frac{1}{2}$(□ + ┐− ┌ − =)	f_2	$\frac{1}{4}(Id - e_1 + e_{24} - e_{124})$	[− + − −]
¬A ∧ ¬B	_	$\frac{1}{2}$(+□ − ┐− ┌ + =)	f_3	$\frac{1}{4}(Id - e_1 - e_{24} + e_{124})$	[− − + −]
A	┐	┐	$f_1 + f_2$	$\frac{1}{2}(Id + e_1)$	[+ + − −]
¬A	└	−┐	$f_3 + f_4$	$\frac{1}{2}(Id - e_1)$	[− − + +]
A ≡ B	=	=	$f_1 + f_3$	$\frac{1}{2}(Id + e_{124})$	[+ − + −]
A ≢ B	| |	−=	$f_2 + f_4$	$\frac{1}{2}(Id - e_{124})$	[− + − +]
B	┌	┌	$f_1 + f_4$	$\frac{1}{2}(Id + e_{24})$	[+ − − +]
¬B	┘	−┌	$f_2 + f_3$	$\frac{1}{2}(Id - e_{24})$	[− + + −]
$A \vee B$	⊓	$\frac{1}{2}$(□ + ┐+ ┌ − =)	$f_1 + f_2 + f_4$	$\frac{1}{4}(3Id + e_1 + e_{24} - e_{124})$	[+ + − +]
¬A ∨ B	⊏	$\frac{1}{2}$(□ − ┐+ ┌ + =)	$f_1 + f_3 + f_4$	$\frac{1}{4}(3Id - e_1 + e_{24} - e_{124})$	[+ − + +]
A ∨ ¬B	⊐	$\frac{1}{2}$(□ + ┐− ┌ + =)	$f_1 + f_2 + f_3$	$\frac{1}{4}(3Id + e_1 - e_{24} + e_{124})$	[+ + − +]
¬A ∨ ¬B	⊔	$\frac{1}{2}$(□ − ┐− ┌ − =)	$f_2 + f_3 + f_4$	$\frac{1}{4}(3Id - e_1 - e_{24} - e_{124})$	[− + + +]
A ∨ ¬A	□	□	Id	Id	[+ + + +]

These define the six columns of the table. In the third column there appear exactly four LICO letter shapes. These can be adjoined by 'idemposition'. For example, if we idempose

┐ + ┌ + =

we get three upper bars, one left vertical bar, one right vertical bar, one bottom bar, subtracting the square we form the term $\frac{1}{2}(-\square + ^{\neg} + ^{\ulcorner} + \equiv)$. So just one upper bar is left. Therefore we have the isomorphy of $A \wedge B$ with Lico-vector shown in Fig. 1. That is, completing the second line of Table 15, we have the idempotent element f_1 in column 4 and the Clifford number $(1/4)(Id + e_1 + e_{24} + e_{124})$ in column 5. Last we calculate the expression $2f_1 - Id$ and obtain $(1/2)(-Id + e_1 + e_{24} + e_{124})$ which has the diagonal

$$\tfrac{1}{2}(-\square + \urcorner + \ulcorner + \equiv)$$

Figure 1. Upper bar

$$\begin{aligned}
\square &\overset{\text{def}}{=} [+,+,+,+] := +1,+1,+1,+1,+1,+1,+1,+1,\ldots\\
\urcorner &\overset{\text{def}}{=} [+,+,-,-] := +1,+1,-1,-1,+1,+1,-1,-1,\ldots\\
\ulcorner &\overset{\text{def}}{=} [+,-,-,+] := +1,-1,-1,+1,+1,-1,-1,+1,\ldots\\
\equiv &\overset{\text{def}}{=} [+,-,+,-] := +1,-1,+1,-1,+1,-1,+1,-1,\ldots
\end{aligned}$$

Figure 2. Basic geometry of grade four iterant views

representation $[+1,-1,-1,-1]$ and provides us an image for the iterant view of the second Lico lettershape. Proceeding in this way, we obtain the Table 15 showing the six essential representations of the binary connectives.

The first lesson we have learned so far can be compiled as follows: There is a linear iconic alphabet for logic expressions.[3] Binary connectives are simple letter shapes in a square. Further, every connective can be represented in a 4-dimensional commutative vector space. This space can be spanned either by primitive idempotents or by graded commutative base units. The primitive idempotents can and should be identified as those of a Clifford algebra $Cl_{3,1}$ of the Minkowski space or equivalently $Cl_{2,2}$ in neutral signature, as those are isomorphic algebras. The graded commuting base units are identified with the Cartan-subalgebra $ch_1 \subset Cl_{3,1}$ which, in physics, is correlated with the isospin phenomena. Every connective can just as well be depicted as an iterant and respectively as a polarity string built up by numbers $+1$ and -1 only. The correspondence is natural and universal. It has a transdisciplinary significance. It has both natural and cognitive roots. So let us go one step further. The four letters $\square, \ulcorner, \urcorner, \equiv$ can be seen as mathematical symbols for elements of a vector space. These special elements are called polarity strings and read like this $[+,+,+,+],[+,+,-,-],[+,-,-,+],[+,-,+,-]$. These are just abbreviations for vectors with unit entries

$$[+1,+1,+1,+1],[+1,+1,-1,-1],[+1,-1,-1,+],[+1,-1,+1,-1].$$

Sometimes, when we are interested to outline the local temporal iteration of a particle process, we use these tokens as shorthand symbols for iterant sequences of opposite polarity:

These so called iterant views grow out of a careful analysis of quantum motion. In the style of Rowan Hamilton's Algebra as the Science of Pure Time, Kauffman had reformulated the complex numbers and expanded the context of matrix algebra to an interpretation of the unit imaginary i as an oscillatory process. He said, one can regard a wave function such as

[3]There exist two forms of mathematical junction with both algebraic and geometric meaning, the adjunction and the idemposition. These admit two multiplication tables that satisfy the Laws of Form. We shall find out, complex logic expressions can be represented by geometric forms in a window of the plane and/or by linear writing, using the elementary letter shapes.

$\psi(x,t) = exp(i(kx - \omega t))$ as containing a micro-oscillatory system with the special synchronizations of the iterant view $i = [+1, -1]\eta$ which is the product of an iterant, isomorphic with the logic identity connective (Fig. 2),

$$\equiv := [+, -] := +1, -1, +1, -1, +1, -1, +1, -1, \ldots \quad i := \equiv \eta \tag{6.8}$$

with the shift operator η; this has the algebraic property $[a, b]\eta = \eta[b, a]$ and $\eta\eta = Id$. Given an iterant $[a, b] = a, b, a, b, a, b, \ldots$, we can think of $[b, a]$ as the same process with a shift of one time step. It is obvious that the quantity $i = \equiv \eta = [+1, -1]\eta$ has the required algebraic property, namely

$$\begin{aligned} ii &= [+1, -1]\eta[+1, -1]\eta = [+1, -1]\eta\eta[-1, +1] = [+1, -1]Id[-1, +1] \\ &= Id[+1, -1][-1, +1] = Id[-1, -1] = -Id \end{aligned} \tag{6.9}$$

with the two-fold identity operation $Id = [+1, +1]$.

This model has great explanatory power. Kauffman could derive the Schrödinger equation with its Brownian motion and Heisenberg uncertainty, [5] could design the iterant views for a 2-dimensional Dirac equation and analytically reconstructed the Feynman and Hibbs checkerboard model of the Dirac propagator ([1], Problem 2–6, pp. 34–36). In two recent works (2012, 2013) I showed how the iterant model can be extended onto the Clifford algebra of the Minkowski space. In accordance with the idea of primordial observation and the symmetries of the octahedral permutation group S_4 for quad locations we defined time shift operators η and t. These turned out fundamental for the iteration of fourfold locations in a relativistic space–time. □

7. Generalization of Logic Space

We keep polarities as basic events in physics. Consider a basis $\{Id, A, B, \equiv\}$ which provides a 4×4-table of truth values. But we allow for real or even complex entries. That is, algebraically, we consider the Cartan algebra over real and complex numbers, not only binary entries. In this way we obtain the first extension of binary connectives. The four quantities $\{Id, e, f, g\}$ defined as binary, or synonymously as polarity strings, are given by

Lemma 2. *Every compound statement $A \odot B$ constituted by two statements $\{A, B\}$ in logic with arbitrary real or complex truth values can be represented in 4-dimensional commutative vector space.*

The four quantities $\{Id, e, f, g\}$ defined as binary, or synonymously as polarity strings, are given by

$$\begin{aligned} Id &:= [+1, +1, +1, +1] \quad \text{unit scalar;} \\ e &:= [+1, +1, -1, -1] = e_1 \quad \text{unit vector;} \\ f &= [+1, -1, -1, +1] = e_{24} \quad \text{directed unit area;} \\ g &= [+1, -1, +1, -1] = e_{124} \quad \text{directed unit volume in Clifford algebra.} \end{aligned} \tag{7.1}$$

These span a commutative 4-dimensional vector space, with the familiar binary connectives given by linear combinations shown in Table 15. Obviously the logic does not halt here. Because we obtained a vector space ch_1

TABLE 16. Basic geometry of grade 4 iterant views

	B	
A	*a*	*b*
	d	*c*

any connection

over the real numbers, and this has a definite logic interpretation. Since the linear factors of the representation of any vector in that logic (color) space, the logic coordinates, represent a real degree of freedom that can be interpreted as truth value. This is not necessarily equal zero or one.

Example. We want to represent the compound statement $A \odot B$ where both A is 'half true' and the connective 'A implies B' is half true. This can indeed be calculated by linear algebra in color space:

$$\begin{aligned} A \odot B &:= \frac{1}{2} A \wedge \frac{1}{2}(A \rightarrow B) \\ &= \frac{1}{2}(f_1 + f_2) + \frac{1}{2}(f_1 + f_3 + f_4) = \frac{1}{8}(5Id + e_1 + e_{24} + e_{124}) \end{aligned} \quad (7.2)$$

We allow for truth tables of a quite general form
with $a, b, c, d \in \mathbb{C}$.[4] Introducing such truth values as in Table 16, we have a commutative logic color space

$$\begin{aligned} \mathcal{L} &:= \mathcal{L}(2, \mathbb{C}) := \mathbb{C} \otimes ch_1 \\ &= \{uId + xe + yf + zg | u, x, y, z \in \mathbb{C}\} \end{aligned} \quad (7.3)$$

Having introduced the commutative linear color space, it has become possible to match every possible connection between two statements with a real vector which is at the same time a peculiar graded multivector in the Minkowskian space–time algebra.

Definition 3. A logic space for compound statements of finite degree with complex truth values can be defined as a product space having form

$$\mathfrak{L}(\mathbb{C}) := \otimes^n \mathcal{L} \quad (7.4)$$

8. Deriving Space–Time-Algebra from Its Logic Basis

In a quite general sense we would like to conceive the basis of a geometric algebra in terms of a Cartesian product having form $cl = \mathcal{S} \times \mathfrak{J}$ where $\mathcal{S}$ is some small set of symmetries and $\mathfrak{J}$ a small set of binary sequences. Let us go into that problem for the case of the Clifford algebra $Cl_{3,1}$ generated by the

[4]Surprisingly this resembles a superposition of strongly interacting particles in isospin space ch_1.

Minkowski space $\mathbb{R}^{3,1}$. Consider, as before, the small module of self-inverse elements

$$\mathfrak{J}_{(4)} := \{Id, e, f, g\} \tag{8.1}$$

together with the set $\mathcal{T}_{(4)}$ of double transpositions in the symmetric group S_4

$$\begin{aligned} &\mathcal{T}_4 := \{id, \sigma, \phi, \tau\} \qquad \text{with permutations in cycle notation} \\ &id = (1)(2)(3)(4), \phi = (1\ 3)(2\ 4), \sigma = (1\ 2)(3\ 4), \tau = (1\ 4)(2\ 3) \end{aligned} \tag{8.2}$$

It is obvious that $\sigma\sigma = \phi\phi = \tau\tau = id$. Following Kauffman's definition of a temporal shift operator, we should have to shift the symmetry operator to the right side after application, that is, take binary sequences $x, x^{'} \in \mathfrak{J}_{(4)}$, and some transposition $\omega \in \mathcal{T}_{(4)}$, then we demand

$$\omega x = x^{'}\omega \tag{8.3}$$

Investigate for example the multiplication of ϕ with f. We should have

$$\phi f = (1\ 2)(3\ 4)[+1, -1, -1, +1] = [-1, +1, +1, -1](1\ 2)(3\ 4) = -f\phi \tag{8.4}$$

There is something very important that can be learned from this assembly, namely we have

$$(\phi f)^2 = \phi f \phi f = -f\phi\phi f = -fidf = -idff = -idId = -Id \tag{8.5}$$

which tells us that the magnitude ϕf is a 'hypercomplex' number. Following this recipe, we calculate three multiplication tables which we need to formulate the main theorem.

In abstract algebra there is a familiar procedure to represent so called generalized quaternion groups by semi-direct products. A similar thing can be applied here. Namely, consider the group

$$\mathfrak{G} := \mathfrak{J}_{(4)} \cup -\mathfrak{J}_{(4)} \simeq \mathbb{Z}_2^3 \tag{8.6}$$

We define the semi-direct product $\mathfrak{G} \rtimes \mathcal{T}_{(4)}$ by the rule

$$(x, \omega) \cdot (x^{'}, \omega^{'}) = (x\omega(x^{'}), \omega\omega^{'}) \quad \text{with} \quad x, x^{'} \in \mathfrak{G}, \quad \text{and} \quad \omega, \omega^{'} \in \mathcal{T}_{(4)} \tag{8.7}$$

In this notation Kauffman's rule to organize polarity strings takes the form $\omega x = \omega(x)\omega$ for a special case of the Cartesian product 'iterants $\times$ symmetries', namely

$$(Id, \omega) \cdot (x, id) = (\omega(x), \omega) = (\omega(x), id)(Id, \omega) \tag{8.8}$$

It is this peculiar rule that allows us to represent discrete diachronic processes of relativistic quantum motion. As we shall see there is indeed a representation of $\mathfrak{G} \rtimes \mathcal{T}_{(4)}$ in the matrixalgebra $Mat(4, \mathbb{R})$, that is, the identities id and Id turn out to be the same matrix. In order to derive the Clifford Algebra of the Minkowski space from a semi-direct product of polarity strings and temporal transpositions, we need two symmetry operators only, namely the cyclic time of period 4 and the tangle time of period 2. We have iteration time

$$t := (1\ 2\ 3\ 4) \tag{8.9}$$

cyclic quaternion temporal shift, element of the symmetric group S_4 and *tangle-time*

$$\eta := (1\ 2) \tag{8.10}$$

transposition in the symmetric group S_4 which is a *torsion of quadrants.*

9. Derivation of Clifford Algebra of Minkowski Space from Its Logic Basis

Theorem 4. *The iterant algebra with four grades is isomorphic with the Clifford algebra* $Cl_{3,1}$

Proof. Consider the three real iterants e, f, g we are already familiar with

$$e = [+1, +1, -1, -1], f = [+1, -1, -1, +1], g = [+1, -1, +1, -1] \tag{9.1}$$

together with the permutation operators $\sigma = (1\ 2)(34), \phi = (1\ 3)(2\ 4), \tau = (1\ 4)(2\ 3)$. These are generated by iteration time t and tangle time η. Sequences are iterated by iteration time t and by tangle-time η and applied to iterants of degree 4. The iterant time t can be represented by a permutation 4-cycle (1 2 3 4) and the tangle time by a 2-cycle (1 2). These two generate the symmetric group S_4 . We used the three operators $\sigma, \phi, \tau \in S_4$ which shall satisfy the Eq. 8.8.

$$\begin{aligned} \sigma[a, \quad b, \quad c, \quad d] &= [b, \quad a, \quad d, \quad c]\sigma \\ \phi[a, \quad b, \quad c, \quad d] &= [c, \quad d, \quad a, \quad b]\phi \\ \tau[a, \quad b, \quad c, \quad d] &= [d, \quad c, \quad b, \quad a]\tau \end{aligned} \tag{9.2}$$

These can be derived from the generating iterant- and tangle-time operators in the following manner

$$\begin{aligned} &\phi = t^2 = (1\ 2\ 3\ 4)(1\ 2\ 3\ 4) = (1\ 3)(2\ 4), \qquad \text{portrayed as cycles} \\ &\tau = \eta\phi\eta = (2\ 1)((1\ 3)(2\ 4))(2\ 1) = (1\ 4)(2\ 3) \\ &\sigma = \tau\phi \end{aligned}$$

Now there exist nine possibilities to let any permutation operator act on the unit iterants. Among these nine products there are six quaternions. Among those there are the three we already know from the analysis of quad locations. Three of the nine squared give the identity Id. The nine terms are

$$e\sigma, e\phi, e\tau, f\sigma, f\phi, f\tau, g\sigma, g\phi, g\tau$$

The idea to proof theorem 7 is challenged once we understand why among these nine we have six instead of three quaternions. That is, there are indeed two sets of quaternions, and this is also true in the Clifford algebra of Minkowski space. Namely, if we consider the Clifford algebra $Cl_{3,1}$ in a standard basis, we realize that we have a triple of bivectors which represent

Table 17. multiplication tables and commutation relations

Id	e	f	g	id	σ	ϕ	τ	Id	e	f	g
e	Id	g	f	σ	id	τ	ϕ	σ	$e\sigma$	$-f\sigma$	$-g\sigma$
f	g	Id	e	ϕ	τ	id	σ	ϕ	$-e\phi$	$-f\phi$	$g\phi$
g	f	e	Id	τ	ϕ	σ	id	τ	$-e\tau$	$f\tau$	$-g\tau$

quaternions, the bivectors $\{e_{12}, e_{23}, e_{13}\}$ and we have a further triple of time-like quaternions with different grades, the time–space $\{e_4, e_{123}, e_{1234}\}$. If we pose these two sets of quaternions in a proper way, we can see

$$\begin{array}{ccc} e_{12} & e_4 & e_{124} \\ e_{23} & e_{123} \Longrightarrow & e_1 \\ e_{13} & e_{1234} & e_{24} \end{array}$$

how quaternions are carried to the Cartan subalgebra, that is to the color space of logic units. The Clifford product in each row gives a component of the first color space, each of which squared gives the Identity. Therefore it is reasonable to assume that the six quantities □, ⌜, ⌝, ≡, ϕ, τ generate a geometric algebra that includes even more than just two sets of quaternions. This could be the Clifford algebra $Cl_{3,1}$ of the Minkowski space. We can use the above relation to get the base units, one after the other. To abbreviate the proof, let us factor in how the six quantities □, ⌜, ⌝, ≡, ϕ, τ interact. We formulate as

Lemma 5. *The polarity strings e, f, g constitute the commutative algebra of a Klein-4 group; all the same the permutations σ, ϕ, τ satisfy the same algebra. The mixed products of polarity strings and permutations commute or anti-commute according to Table 17.*

Just recall, direct component-wise multiplication gives the first sub-table, the second is well known property of permutation group S_4, the third part can be verified: Use $e = [+1, +1, -1, -1]$, $\sigma = (1\ 2)(3\ 4)$, and rule 8.8, $\sigma[a, b, c, d] = [b, a, d, c]\sigma$ to get $\sigma e = (1\ 2)(3\ 4)[+1, +1, -1, -1] = [+1, +1, -1, -1]\sigma = e\sigma$, the matrix element in first row, first column; further use $f = [+1, -1, -1, +1]$, $\sigma = (1\ 2)(3\ 4)$, and rule 8.8, $\sigma[a, b, c, d] = [b, a, d, c]\sigma$ to get $\sigma f = (1\ 2)(3\ 4)[+1, -1, -1, +1] = [-1, +1, +1, -1]\sigma = -f\sigma$, the matrix element in first row, second column; use $g = [+1, -1, +1, -1]$ and σ to verify $\sigma g = -g\sigma$, the matrix element in first row, third column; and so on until to $\tau g = -g\tau$, the last matrix element in third row, third column. Now it is clear how we place the elements e, f, g at the positions e_1, e_{24}, e_{124}, in the 16 element basis of $Cl_{3,1}$ and if we put $e_2 = \phi$, $e_3 = \tau f$ and $e_4 = \phi f$, we get unit vectors with the appropriate signature $(+ + + -)$ of the Minkowski space in the opposite (Lorentz) metric. These satisfy the commutation relations of this Clifford algebra. The result of exterior multiplication gives us the following representation of the Clifford algebra of Minkowski space $Cl_{3,1}$ (Table 18).

Table 18. Iterant representation of $Cl_{3,1}$

Id	$e_1 := e$	$e_2 := \phi$	$e_3 := \tau f$
$e_4 := \phi f$	$e_{12} := e\phi$	$e_{13} := g\tau$	$e_{14} := g\phi$
$e_{23} := \sigma f$	$e_{24} := f$	$e_{34} := -\sigma$	$e_{123} := \sigma g$
$e_{124} := g$	$e_{134} := -\sigma e$	$e_{234} := -\tau$	$e_{1234} := \tau e$

Verify the signature of the Minkowski space, first its Cartan subalgebra:

$$\begin{aligned} e_1 e_1 &= e^2 = [+1,+1,-1,-1](\equiv)[+1,+1,-1,-1] = \\ &= [+1,+1,+1,+1] = Id \\ f^2 &= [+1,-1,-1,+1](\equiv)[+1,-1,-1,+1] = Id \\ g^2 &= [+1,-1,+1,-1](\equiv)[+1,-1,+1,-1] = Id \end{aligned}$$

we indicate at the same time that component-wise multiplication is brought forth by logical equivalence of sequences in the algebra $(\mathcal{B}, \equiv)$. Also we have by the aid of Table 17

$$\begin{aligned} e_3 e_3 &= f\tau f\tau = f\tau\tau f = fIdf = ffId = IdId = Id \\ e_4 e_4 &= f\phi f\phi = -\phi ff\phi = -\phi Id\phi = -\phi\phi Id = -IdId = -Id \end{aligned}$$

We summarize the first result: $e_1^2 = e_2^2 = e_3^2 = Id$, $e_4^2 = -Id$. Next we verify the (anti)commutation relations for Clifford algebra $Cl_{3,1}$

$$[e_1, e_2] = e\phi - \phi e = e\phi + e\phi = 2e\phi$$

and the anticommutator

$$\{e_1, e_2\} = e_1e_2 + e_2e_1 = e_{12} + e_{21} = e\phi + \phi e = e\phi - e\phi = 0$$

as required, and also

$$e_{13} + e_{31} = g\tau + \tau g = g\tau g\tau = 0$$

$$e_{23} + e_{32} = f\sigma + \sigma f = f\sigma f\sigma = 0$$

We calculate all further signatures of the unit monomials

$$\begin{aligned} e_{14} &= e\phi f = -g\phi; e_{14}^2 = g\phi g\phi = \phi Id\phi = \phi\phi Id = IdId = Id \\ e_{34} &= f\tau\phi f = f\sigma f = -\sigma ff = -\sigma Id = -\sigma; e_{34}^2 = \sigma^2 = Id \\ e_{123} &= e\sigma f = -\sigma ef = -\sigma g = g\sigma; \\ e_{123}^2 &= g\sigma g\sigma = -\sigma gg\sigma = -\sigma Id\sigma = -\sigma\sigma Id = -Id \\ e_{134} &= -e\sigma; e_{134}^2 = e\sigma e\sigma = \sigma ee\sigma = \sigma Id\sigma = \sigma\sigma Id = Id \\ e_{234} &= -\phi\sigma = -\tau; e_{234}^2 = \tau\tau = Id \\ e_{1234} &= \tau e; e_{1234}^2 = \tau e\tau e = -e\tau\tau e = -eIde = -eeId = -IdId = -Id \end{aligned}$$

□

10. Matrix Representations

We can represent the iterants e, f as well as the period 2 permutation operators by $4x4$-matrices as for example by

$$e = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}; \quad f = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}$$

$$\phi = \begin{pmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \\ & 1 & & \end{pmatrix}; \quad \sigma = \begin{pmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{pmatrix}.$$

Taking into account that $g = ef$ and $\tau = \phi\sigma$ we can calculate the matrices of all 16 base units of the Clifford algebra. The generating Minkowski space of the Clifford algebra is then given by the following standard representation

$$e_1 = e = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}; \quad e_2 = \phi = \begin{pmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \\ & 1 & & \end{pmatrix}$$

$$e_3 = \tau f = \begin{pmatrix} & & & 1 \\ & & -1 & \\ & -1 & & \\ 1 & & & \end{pmatrix}; \quad e_4 = \phi f = \begin{pmatrix} & & -1 & \\ & & & 1 \\ 1 & & & \\ & -1 & & \end{pmatrix}.$$

It is this peculiar representation where the four primitive idempotents f_i appear as diagonal matrices with unit entries in raw/column i and the symmetries of $\mathcal{T}_{(4)}$ are positive definite, unitary permutation matrices. This is not necessarily the case in other matrix representations since in $Cl_{3,1}$ there is a manifold of different representations for both $\mathcal{T}_{(4)}$ and $\mathfrak{G}$. The equations 18 and all conditions satisfied by the monomials of $Cl_{3,1}$ have been verified by the computer programs of MAPLE Clifford.

11. Conclusion and Prospect

In this paper we have discovered the most fundamental relation between logic and space–time geometry. The innermost commutative subspace of the Clifford algebra of the Minkowski space is a Boolean logic structure which gives rise to complex multi-valued logic on the one hand and to the isospin spaces of angular 4-momenta of quantum motion on the other. We can denote this as a morphogenetic structure of thought in the sense of the genetic structuralism. Mathematically, this logic core structure is a typical Cartan (sub)algebra of $SU(4) \subset Cl_{3,1}$ having rank three. With four generators $\{Id, A, B, \equiv\}$ the whole universal spook can be brought about. As we know the construction plans of these elements, we can design the right circuit layout for the space-like retinoid system of a cognitive brain ([13]), a one that lives integrated within the universal whole. And we can also get the Lie groups, the motion groups of material bodies and the genetic code from this creature. We can show how icons may lead to linear writing and the latter leads back to iconic notation. These are subjects of prospect.

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