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(1,1)-Forms Acting on Spinors on Kähler Surfaces

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Abstract. It is known that, for Dirac operators on Riemann surfaces twisted by line bundles with Hermitian-Einstein connections, it is possible to obtain estimates for the first eigenvalue in terms of the topology of the twisting bundle [8]. Attempts to generalize topological estimates for higher rank bundles or higher dimensional manifolds have been so far unsuccessful. In this work, we construct a class of examples, making explicit one problem that must be addressed in attempts to generalize such topological estimates.

Keywords. Spinors, curvature, Kähler surfaces.

1. Introduction

Let (M, g) be a spin Riemannian manifold, $\mathbb{S}$ the spinor bundle of M and D the Dirac operator associated with the Levi-Civita connection of (M, g). There is an extensive literature concerning estimates for the first eigenvalue of this operator, [5]. The main techniques to obtain such estimates are based on the Wietzenböck formula

$$D^2 = \nabla^*\nabla + \frac{1}{4}R.$$

We may also consider the twisted Dirac operator. For this, let $E \to M$ be an Hermitian vector bundle with connection ∇^A, compatible with the Hermitian structure. Then, $\nabla^A \otimes \mathbb{I} + \mathbb{I} \otimes \nabla^{\mathbb{S}}$ is a connection on the tensor product $\mathbb{S} \otimes E$, called the product connection, and we can consider the associated Dirac operator, D_A, in $\mathbb{S} \otimes E$.

For this operator the Witzenböck formula becomes

$$D_A^2 = \nabla^{A^*}\nabla^A + \frac{1}{4}R + F_A,$$

where F_A is the curvature 2-form of the connection ∇^A viewed as an operator on $\mathbb{S} \otimes E$.

In order to obtain estimates for the first eigenvalue of D_A we need to understand how F_A acts on spinors. One special case, where it is possible to obtain a nice characterization of F_A, is that of Hermitian-Einstein connections on Riemann surfaces.

In general, let (M, g, J) be a Kähler manifold of complex dimension n. In this case, M is naturally a $\mathrm{Spin}^{\mathbb{C}}$-manifold and the spinor bundle associated to this structure can be identified with $\wedge^{0,*}M$. If $E \to M$ is a holomorphic Hermitian vector bundle over M then we have[1]

$$\mathbb{S}_{\mathbb{C}} \otimes E \simeq \wedge^{0,*}M \otimes E = \Omega^{0,*}(E).$$

In the particular case of Riemann surfaces, in order to get estimates for the first eigenvalue of D_A, it is sufficient to know how F_A acts on $\wedge^{0,0} \otimes E = \Omega^{0,0}(E)$. Restricted to $\Omega^{0,0}(E)$, the Weitzenböck formula can be written as

$$D_A^2 = \nabla^{A^*}\nabla^A - i\Lambda F_A,$$

where ΛF_A is the contraction of the curvature 2-form F_A by the Kähler form ω.

If we impose that the connection ∇^A is the Hermitian-Einstein connection[2] the above Weitzenböck formula provides the estimate [8]

$$\lambda \geq -\frac{2n}{2n-1}\frac{\pi \deg(E)}{(n-1)!\mathrm{rk}(E)\mathrm{vol}(M)}. \tag{1}$$

involving topological data of the bundle $E \to M$. Using the calculations of [1],it is known that this estimate is sharp for Riemann surfaces.

The key argument for Riemann surfaces is that we can restrict the analysis to $\Omega^{0,0}(E)$. This is possible because on manifolds with even dimension the spinor bundle decomposes as $\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-$ and, if $D_A\psi = \lambda\psi$ is an eigenspinor with non null eigenvalue λ, then $\psi = \psi^+ + \psi^-$ with $\psi^\pm \neq 0$ and we can restrict the analysis to $\mathbb{S}^+$ or $\mathbb{S}^-$. Furthermore, for Riemann surfaces, $\mathbb{S}^+ \simeq \wedge^{0,0}M$, and if the connection ∇^A is Hermitian-Einstein the restriction of F_A to $\mathbb{S}^\pm$ is a definite operator and its norm can be calculated by a topological constant.

For higher dimensional manifolds, this argument does not work because we have $\mathbb{S}^+ = \oplus_{p\ even} \wedge^{0,p} M$ and $\mathbb{S}^- = \oplus_{p\ odd} \wedge^{0,p} M$, and to determine the action of F_A on $\mathbb{S}^\pm$ we need to know the action of F_A on $(0, p)$-forms, which, in general, is not simple.

Despite the efforts to generalize the estimate to Hermitian-Einstein connections on higher dimension Kähler manifolds, the only results in the literature, known to the present author, such as [2, 6], are geometric estimates involving the pointwise eigenvalues of F_A that, in principle, does not carry

[1]When (M, g, J) is a Spin-manifold we have the relation $\mathbb{S}_{\mathbb{C}} = \mathbb{S} \otimes k^{-\frac{1}{2}}$ and we can treat the real case analogously.

[2]Recall that a connection ∇^A is Hermitian-Einstein if $\Lambda F_A = c\mathbb{I}_E$, where c is a topological constant given by

$$\frac{2\pi \deg(E)}{(n-1)!\mathrm{rk}(E)\mathrm{vol}(M)}.$$

topological information. In this work we construct a class of examples showing that, without further hypothesis for $E \to M$ and ∇^A, the argument used on Riemann surfaces to obtain a topological estimate cannot be generalized. The main result is:

Main Theorem. *Let $E \to M$ be a holomorphic line bundle over a Kähler surface*[3] *(M, g, J). Let ∇^A be any connection on E compatible with the holomorphic structure and F_A the curvature 2-form of ∇^A. Then, as an operator acting on spinors, $F_A \mid_{\mathbb{S}^\pm}$ is indefinite for every $p \in M$ such that $F_A \mid_{\mathbb{S}^\pm} (p) \neq 0$.*

This family of examples raises two natural questions:

- Is it possible to find examples where F_A acts as a definite operator on $\mathbb{S}^+$ or $\mathbb{S}^-$? If yes, is it possible to obtain a topological estimate for the first eigenvalue of the twisted Dirac operator?
- Knowing that $D_A\psi = \lambda\psi$ is an eigenspinor is it possible to simplify the action of F_A? If yes, does this lead to a topological estimate?

2. Anti-Selfdual $U(1)$ Connections

Let M be a Kähler manifold with complex dimension 2. All complex manifolds carry a canonical Spin$^{\mathbb{C}}$-structure, and in this structure the spinor bundle is explicitly described in terms of forms:

$$\begin{aligned}
\mathbb{S}_{\mathbb{C}} &\simeq \wedge^{0,*}M = \oplus_{i=0}^{2} \wedge^{0,i} M,\\
\mathbb{S}_{\mathbb{C}}^{+} &\simeq \oplus_{i\ \ even} \wedge^{0,i} M,\\
\mathbb{S}_{\mathbb{C}}^{-} &\simeq \oplus_{i\ \ odd} \wedge^{0,i} M.
\end{aligned}$$

Consequently, the twisted case is described by

$$\mathbb{S}_{\mathbb{C}} \otimes E \simeq \wedge^{0,*}M \otimes E = \Omega^{0,*}(E).$$

This description is very useful, mainly because of two reasons. First, we can explicitly describe the action of $\mathcal{C}\ell(T^*M)$ on $\mathbb{S}_{\mathbb{C}}$. For this, consider an adapted frame $\{\xi^i, \bar{\xi}^i\}$ of $T^*M \otimes \mathbb{C}$. Then, in this frame, the Clifford action is given by:

$$\begin{aligned}
\xi^i\cdot &= -\sqrt{2}\bar{\xi}^i \lrcorner,\\
\bar{\xi}^i\cdot &= \sqrt{2}\bar{\xi}^i \wedge .
\end{aligned} \tag{2}$$

Secondly, the twisted Dirac operator can be described in terms of Cauchy-Riemann operators: if ∇^A is a connection on $E \to M$, the complex structure of M produces the splitting

$$\begin{aligned}
\nabla^A &= \partial_A + \bar{\partial}_A,\\
\partial_A &: \Omega^{p,q}(E) \to \Omega^{p+1,q}(E),\\
\bar{\partial}_A &: \Omega^{p,q}(E) \to \Omega^{p,q+1}(E),
\end{aligned}$$

[3]By a Kähler surface we understand a Kähler manifold of complex dimension 2.

and the twisted Dirac operator is given by

$$D_A = \sqrt{2}\left(\partial_A + \bar{\partial}_A\right).$$

The case of the twisted Dirac operator associated with a Spin-structure can also be described by these identifications; we must only remember that the two spinor spaces are related by $\mathbb{S}_{\mathbb{C}} = \mathbb{S} \otimes k^{-\frac{1}{2}}$, where $k = \wedge^{0,n} M$.

Another important fact about Kähler manifolds with complex dimension 2 is that the 2-forms decompose in self-dual forms, Ω^+, and anti-self-dual forms, Ω^-, and that

$$\begin{aligned} \Omega^+ &= \Omega^{2,0} \oplus \Omega^0\omega \oplus \Omega^{0,2}, \\ \Omega^- &= \Omega_0^{1,1}, \end{aligned} \tag{3}$$

where ω is the Kähler form and $\Omega_0^{1,1}$ is the space of (1,1)-forms orthogonal to ω [4].

Using the adapted frame $\{\xi^i, \bar{\xi}^i\}$ we can explicitly describe the action of elements of $\Omega^{\pm}$ on spinors. First, note that the Kähler form can be written as

$$\omega = i\left(\xi^1 \wedge \bar{\xi}^1 + \xi^2 \wedge \bar{\xi}^2\right),$$

and a basis for $\Omega_0^{1,1}$ is given by $\{\xi^1 \wedge \bar{\xi}^2, \xi^2 \wedge \bar{\xi}^1, \xi^1 \wedge \bar{\xi}^1 - \xi^2 \wedge \bar{\xi}^2\}$. Therefore, if $F_A \in \Omega^-$, locally we can write

$$F_A = a\xi^1 \wedge \bar{\xi}^2 + b\xi^2 \wedge \bar{\xi}^1 + c\left(\xi^1 \wedge \bar{\xi}^1 - \xi^2 \wedge \bar{\xi}^2\right).$$

Proposition 1. *If $F_A \in \Omega^-$, the action of F_A on $\mathbb{S}^-$ is given by*

$$F_A = 2\begin{pmatrix} c & b \\ a & -c \end{pmatrix}.$$

Proof. A 2-form $\alpha \wedge \beta$ acts on spinors through Clifford multiplication by means of the identification

$$\alpha \wedge \beta \simeq \frac{1}{2}\left(\alpha\beta - \beta\alpha\right).$$

Using the action described in (2) we calculate

$$\begin{aligned} \xi^1 \wedge \bar{\xi}^2 \cdot \psi &= \frac{1}{2}\left(\xi^1\bar{\xi}^2 - \bar{\xi}^2\xi^1\right) \cdot \psi \\ &= \frac{1}{2}\left[\xi^1 \cdot \left(\bar{\xi}^2 \cdot \psi\right) - \bar{\xi}^2 \cdot \left(\xi^1 \cdot \psi\right)\right] \\ &= -\bar{\xi}^1 \lrcorner \left(\bar{\xi}^2 \wedge \psi\right) + \bar{\xi}^2 \wedge \left(\bar{\xi}^1 \lrcorner \psi\right). \end{aligned}$$

For 4-manifolds $\mathbb{S}_{\mathbb{C}}^-$ is just $\Omega^{0,1}(E)$; so if $\psi \in \mathbb{S}_{\mathbb{C}}^-$ we have[4]

$$\psi = \psi_1\bar{\xi}^1 + \psi_2\bar{\xi}^2,$$

and the above expression simplifies to

$$(\xi^1 \wedge \bar{\xi}^2) \cdot \psi = \psi_1\bar{\xi}^2 + \psi_1\bar{\xi}^2 = 2\psi_1\bar{\xi}^2$$

[4]Strictly, elements of $\Omega^{(0,1)}(E) \simeq \Gamma(E) \otimes \wedge^{(0,1)} M$ are of the form $\psi = \psi_1 \otimes \bar{\xi}^1 + \psi_2 \otimes \bar{\xi}^2$ and the Clifford action is given by $c(\alpha)(\psi_i \otimes \bar{\xi}^i) = \psi_i \otimes (c(\alpha)\bar{\xi}^i)$. So, to simplify notation, we just write $\psi_i \otimes \bar{\xi}^i \sim \psi_i\bar{\xi}^i$, and the action is as written.

or, in matrix form,

$$(\xi^1 \wedge \bar{\xi}^2) \cdot \psi = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} \cdot \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}.$$

The other terms are calculated in the same manner and are given, in matrix form, by

$$(\xi^2 \wedge \bar{\xi}^1) \cdot \psi = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \cdot \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix},$$

$$(\xi^1 \wedge \bar{\xi}^1 - \xi^2 \wedge \bar{\xi}^2) \cdot \psi = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \cdot \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix},$$

and the result follows. □

To complete the characterization we need to know how F_A acts on $\mathbb{S}^+$.

Proposition 2. *The Clifford action of an* $(1,1)$*-form* α *on sections of* $\wedge^{0,n}(M) \otimes E$*,* $\psi_n \in \Omega^{0,n}(E)$*, is explicit given by*

$$\alpha \cdot \psi_n = i(\Lambda\alpha)\psi_n, \tag{4}$$

where $\Lambda\alpha = \omega \lrcorner \alpha$ *is the contraction of* α *by the Kähler form* ω*.*

Proof. In [8, Proposition1] it was proved that the action of an (1,1)-form, α, on sections of E is given by $-i(\Lambda\alpha)$. The same techniques can be used to explicitly obtain the result. □

With this we can prove:

Theorem 1. *If* $F_A \in \Omega^-$ *then, on points* $p \in M$ *such that* $F_A(p) \neq 0$ *as a 2-form,* $F_A(p)$*, as an operator on* $\mathbb{S}_p$*, is indefinite.*

Proof. By the above proposition, the action of a (1,1)-form α on $\wedge^{0,0}M$ and $\wedge^{0,2}M$ is given by

$$\alpha \cdot \psi = \mp i(\Lambda\alpha)\psi,$$

where the sign is minus on $\wedge^{0,0}M$ and plus on $\wedge^{0,2}M$.

Then, if $F_A \in \Omega^-$, the decomposition (3) implies that $\Lambda F_A = 0$, and F_A acts trivially on $\mathbb{S}^+_{\mathbb{C}}$. The only non trivial part is the action of F_A on $\mathbb{S}^-_{\mathbb{C}}$, which is given by proposition (1).

With the same notation as in the previous proposition, and knowing that the action of F_A on $\mathbb{S}_{\mathbb{C}}$ is Hermitian [3] we find that the eigenvalues of F_A, as an operator on $\mathbb{S}_{\mathbb{C}}$, are

$$\{0, \sqrt{c^2 + ab}, -\sqrt{c^2 + ab}\}.$$

Because the representation of $\mathcal{C}\ell(TM)$ on $\mathbb{S}_{\mathbb{C}}$ is faithful we conclude that for points $p \in M$ where $F_A(p) \neq 0$ the eigenvalues $\pm\sqrt{c^2 + ab}$ cannot be zero, so F_A is indefinite. □

3. Selfdual $U(1)$ Connections

For compatible self-dual connections the decomposition (3) implies that the curvature has the form $F_A = f\omega$, where ω denotes the Kähler form and f is a function on M.

Using proposition (2) and [8, Proposition1] we have:

Proposition 3. *If F_A is of type $(1,1)$, the action of F_A on $\mathbb{S}^+ \simeq \wedge^{0,0}M \oplus \wedge^{0,2}M$ is given by*

$$F_A = i \begin{pmatrix} -\Lambda F_A & 0 \\ 0 & \Lambda F_A \end{pmatrix}.$$

Using this we have:

Theorem 2. *If $F_A \in \Omega^+$ then, on points $p \in M$ such that $F_A(p) \neq 0$ as a 2-form, $F_A(p)$, as an operator on $\mathbb{S}_p$, is indefinite.*

Proof. Using the calculations of proposition (1) we can explicitly verify that ω acts as a null operator on $\mathbb{S}^-$. Thus F_A acts trivially on $\mathbb{S}^-$ and the action of F_A on $\mathbb{S}^+$ is given by the above proposition. Therefore, for $F_A \neq 0$, F_A is indefinite. □

Combining theorems (1) and (2), and decomposition (3), we obtain the main theorem.

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