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Degenerate Spin Groups as Semi-Direct Products

Tekin Dereli, Şahin Koçak and Murat Limoncu

Abstract. Let Q be a symmetric bilinear form on $\mathbb{R}^n = \mathbb{R}^{p+q+r}$ with corank r, rank $p+q$ and signature type (p,q), p resp. q denoting positive resp. negative dimensions. We consider the degenerate spin group $Spin(Q) = Spin(p,q,r)$ in the sense of Crumeyrolle and prove that this group is isomorphic to the semi-direct product of the nondegenerate and indefinite spin group $Spin(p,q)$ with the additive matrix group $Mat(p+q,r)$.

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1. Degenerate Clifford Algebra

Let $\langle , \rangle$ be a symmetric bilinear form on $\mathbb{R}^n$ and let us consider the subspace W of $\mathbb{R}^n$

$$W = \{w \in \mathbb{R}^n \mid \langle w, x\rangle = 0 \text{ for all } x \in \mathbb{R}^n\}.$$

W is called the radical and the dimension of W is called the co-rank of $\langle , \rangle$. If co-rank is zero/non-zero, then $\langle , \rangle$ is called non-degenerate/degenerate. If $W' \subset \mathbb{R}^n$ is any complementary subspace to W, then the restriction of $\langle , \rangle$ to W' is non-degenerate. If the co-rank of $\langle , \rangle$ is r and the the restriction of $\langle , \rangle$ to W' has signature (p,q) (in the sense that W' has an orthogonal decomposition $W'' \oplus W'''$, where the dimensions of W'' respectively W''' are p resp. q and the restriction of $\langle , \rangle$ to W'' resp. W''' is positive resp. negative definite), then the pair $(\mathbb{R}^n, \langle , \rangle)$ is said to be of type $\mathbb{R}^{p,q,r}$ (see [1], [2]).

Thus we have a linear basis $\{e_1, \ldots, e_p, e_{p+1}, \ldots, e_{p+q}, f_1, \ldots, f_r\}$ of $\mathbb{R}^{p+q+r}$ such that

$$\begin{aligned} \langle e_i, e_j \rangle &= \delta_{ij} && 1 \leq i, j \leq p \\ \langle e_{p+i}, e_{p+j} \rangle &= -\delta_{ij} && 1 \leq i, j \leq q \\ \langle f_i, f_j \rangle &= 0 && 1 \leq i, j \leq r, \end{aligned}$$

and

$$\begin{aligned} W &= Span\,\{f_1, \ldots, f_r\} \\ W' &= Span\,\{e_1, \ldots, e_p, e_{p+1}, \ldots, e_{p+q}\} \\ W'' &= Span\,\{e_1, \ldots, e_p\} \\ W''' &= Span\,\{e_{p+1}, \ldots, e_{p+q}\}. \end{aligned}$$

In the following, for the degenerate Clifford algebra $\mathcal{C}\ell_{p,q,r} = \mathcal{C}\ell(\mathbb{R}^n, \langle , \rangle)$, we will use the algebra basis $\{e_1, \ldots, e_p, e_{p+1}, \ldots, e_{p+q}, f_1, \ldots, f_r\}$.

We understand $\mathcal{C}\ell_{p,q} \subseteq \mathcal{C}\ell_{p,q,r}$ and recall the (indefinite) spin group:

Definition 1.1. The group $Spin(p, q)$ is defined by

$$Spin(p, q) = \{v_1 \ldots v_m \in \mathcal{C}\ell_{p,q} \mid m \in 2\mathbb{Z}^+, v_i = \sum_{j=1}^{p+q} a_{ij} e_j \;, \langle v_i, v_i \rangle = \mp 1 \;, 1 \leq i \leq m\}.$$

Definition 1.2. The group $SO(p, q)$ is defined by

$$SO(p, q) = \{\sigma \in End(W') \mid \langle \sigma(a), \sigma(b) \rangle = \langle a, b \rangle \;, \forall a, b \in W' \text{ and } det(\sigma) = 1\}.$$

Lemma 1.3. *The map ρ defined by*

$$\begin{aligned} \rho : \quad Spin(p, q) &\longrightarrow SO(p, q) \\ s &\longmapsto \rho(s)(v) := svs^{-1} \qquad (v \in W') \end{aligned}$$

is a $2:1$ group homomorphism.

2. Degenerate Spin Group

Proposition 2.1. *The subset of $\mathcal{C}\ell_{p,q,r}$ defined by*

$$S_{p,q,r} = \{s\gamma_1 \ldots \gamma_{p+q}(1 + G) \mid s \in Spin(p, q),\, \gamma_i = 1 + e_i \sum_{l=1}^{r} c_{il} f_l,\, G \in \Lambda(f)\}$$

is a group under the Clifford multiplication.

Here $1 \leq i \leq p + q$, $c_{il} \in \mathbb{R}$, and $\Lambda(f)$ is defined by

$$\Lambda(f) = Span\{f_{k_1} \ldots f_{k_j} \mid 1 \leq k_1 < k_2 < \ldots < k_j \leq r\}.$$

To prove the Proposition 3.1., we need the following Lemmas.

Lemma 2.2. *Let* $a, b \in Span\{e_1, \dots, e_p, e_{p+1}, \dots, e_{p+q}\} \subset \mathbb{R}^{p,q,r} \subset \mathcal{C}\ell_{p,q,r}$*, and let* $F, F' \in Span\{f_1, \dots, f_r\} \subset \mathbb{R}^{p,q,r} \subset \mathcal{C}\ell_{p,q,r}$ *such that* $\langle a, b\rangle = 0$*. Then we have*

$$\exp(aF) = 1 + aF,$$
$$\exp(aF)\exp(bF') = \exp(aF + bF').$$

Proof. To obtain the first equation, we consider the expression

$$\exp(aF) = 1 + aF + \frac{1}{2}aFaF + \dots,$$

and from this, using the properties $aF = -Fa$ and $F^2 = 0$, we find

$$\exp(aF) = 1 + aF.$$

For the second equation, let us compute both $\exp(aF)\exp(bF')$ and $\exp(aF+bF')$. $\exp(aF)\exp(bF')$ is directly computed as

$$\begin{aligned}\exp(aF)\exp(bF') &= (1 + aF)(1 + bF')\\ &= 1 + aF + bF' + aFbF'.\end{aligned}$$

$\exp(aF + bF')$ is computed as follows:

$$\begin{aligned}\exp(aF + bF') &= 1 + aF + bF' + \frac{1}{2}(aF + bF')(aF + bF') + \dots\\ &= 1 + aF + bF' + \frac{1}{2}(aFaF + aFbF' + bF'aF + bF'bF') + \dots .\end{aligned}$$

Using the facts $aF = -Fa$, $F^2 = 0$, $bF' = -F'b$ and $F'^2 = 0$, we obtain

$$\exp(aF + bF') = 1 + aF + bF' + \frac{1}{2}(aFbF' + bF'aF) + \dots .$$

Since $\langle a, b\rangle = 0$, we have $ab = -ba$. We also have $Fb = -bF$, $F'a = -aF'$ and $FF' = -F'F$. Thus we get

$$\begin{aligned}\exp(aF + bF') &= 1 + aF + bF' + \frac{1}{2}(ab - ba)F'F + \dots\\ \exp(aF + bF') &= 1 + aF + bF' + abF'F + \dots\\ \exp(aF + bF') &= 1 + aF + bF' + aFbF' + \dots .\end{aligned}$$

The terms indicated by dots are equal to zero: In fact, from $F^2 = 0$, $F'^2 = 0$ and $FF' = -F'F$,

$$\begin{aligned}(aF + bF')^3 &= (aF + bF')^2(aF + bF')\\ (aF + bF')^3 &= 2abF'F(aF + bF')\\ (aF + bF')^3 &= 0.\end{aligned}$$

Thus we obtain

$$\exp(aF + bF') = 1 + aF + bF' + aFbF'$$

which gives the second equation. □

One can prove similarly the following

Lemma 2.3. *Let* $a_i \in Span\{e_1, \ldots, e_p, e_{p+1}, \ldots, e_{p+q}\}$ *(*$i = 1, \ldots, N$*),* $\langle a_i, a_j\rangle = 0$ *for* $i \neq j$ *and* $F_i \in Span\{f_1 \ldots f_r\}$ *(*$i = 1, \ldots, N$*). Then*

$$\exp(\sum_{i=1}^{N} a_i F_i) = \exp(a_1 F_1) \ldots \exp(a_N F_N).$$

(we note that, in this case the interesting equation

$$\exp(\sum_{i=1}^{N} a_i F_i) = 1 + \sum_{i=1}^{N} a_i F_i + \sum_{\substack{i<j \\ i,j=1\ldots N}} a_i F_i a_j F_j + \sum_{\substack{i<j<k \\ i,j,k=1\ldots N}} a_i F_i a_j F_j a_k F_k + \cdots + a_1 F_1 a_2 F_2 \ldots a_N F_N$$

holds.)

Proof of Proposition 3.1. First, we will show that $S_{p,q,r}$ is closed with respect to the Clifford multiplication. Let

$$\begin{aligned} \Theta &= s\gamma_1\gamma_2 \ldots \gamma_{p+q}(1 + G) \\ \Theta' &= s'\gamma'_1\gamma'_2 \ldots \gamma'_{p+q}(1 + G') \end{aligned}$$

be two elements of $S_{p,q,r}$. Then

$$\Theta\Theta' = s\gamma_1\gamma_2 \ldots \gamma_{p+q}(1 + G)s'\gamma'_1\gamma'_2 \ldots \gamma'_{p+q}(1 + G')$$

Using the properties $Gs' = s'G$ and $G\gamma'_i = \gamma'_i G$, we have

$$\Theta\Theta' = s\gamma_1\gamma_2 \ldots \gamma_{p+q}s'\gamma'_1\gamma'_2 \ldots \gamma'_{p+q}(1 + G)(1 + G').$$

We use the notation $G'' = G + G' + GG'$:

$$\Theta\Theta' = s\gamma_1\gamma_2 \ldots \gamma_{p+q}s'\gamma'_1\gamma'_2 \ldots \gamma'_{p+q}(1 + G'').$$

Now we set $F_i = \sum_{l=1}^{r} c_{il} f_l$ so that $\gamma_i = 1 + e_i F_i$:

$$\begin{aligned} \Theta\Theta' &= s(1 + e_1F_1) \ldots (1 + e_{p+q}F_{p+q})s'\gamma'_1 \ldots \gamma'_{p+q}(1 + G'') \\ &= s(1 + e_1F_1) \ldots (s' + e_{p+q}F_{p+q}s')\gamma'_1 \ldots \gamma'_{p+q}(1 + G''). \end{aligned}$$

From $F_i s' = s' F_i$, one gets

$$\begin{aligned} \Theta\Theta' &= s(1 + e_1F_1) \ldots (s' + e_{p+q}s'F_{p+q})\gamma'_1 \ldots \gamma'_{p+q}(1 + G'') \\ &= s(1 + e_1F_1) \ldots (s' + s's'^{-1}e_{p+q}s'F_{p+q})\gamma'_1 \ldots \gamma'_{p+q}(1 + G'') \\ &= s(1 + e_1F_1) \ldots s'(1 + s'^{-1}e_{p+q}s'F_{p+q})\gamma'_1 \ldots \gamma'_{p+q}(1 + G''). \end{aligned}$$

By virtue of the $2:1$ group homomorphism

$$\begin{aligned} \rho : \quad Spin(p, q) &\longrightarrow SO(p, q) \\ s &\longmapsto \rho(s)(v) := svs^{-1}, \end{aligned}$$

we find

$$\Theta\Theta' = s(1 + e_1F_1) \ldots s'(1 + \rho(s'^{-1})(e_{p+q})F_{p+q})\gamma'_1 \ldots \gamma'_{p+q}(1 + G'').$$

If this procedure is applied from γ_{p+q-1} to γ_1, then $\Theta\Theta'$ becomes

$$\Theta\Theta' \quad = \quad ss'(1+\rho(s'^{-1})(e_1)F_1)\dots(1+\rho(s'^{-1})(e_{p+q})F_{p+q})\gamma'_1\dots\gamma'_{p+q}(1+G'').$$

With the notation $\gamma'_i = 1+e_iF'_i$ and using the exponential property, one can write:

$$\begin{aligned}\Theta\Theta' &= ss'(1+\rho(s'^{-1})(e_1)F_1)\dots(1+\rho(s'^{-1})(e_{p+q})F_{p+q})(1+e_1F'_1)\\ &\quad\dots(1+e_{p+q}F'_{p+q})(1+G'')\\ \Theta\Theta' &= ss'\exp(\rho(s'^{-1})(e_1)F_1)\dots\exp(\rho(s'^{-1})(e_{p+q})F_{p+q})\exp(e_1F'_1)\\ &\quad\dots\exp(e_{p+q}F'_{p+q})(1+G'').\end{aligned}$$

Because of $\rho(s'^{-1}) \in SO(p,q)$, we have $\langle\rho(s'^{-1})(e_k),\rho(s'^{-1})(e_l)\rangle = \langle e_k, e_l\rangle = 0$. Thus the above expression can be written as

$$\Theta\Theta' \quad = \quad ss'\exp(\sum_{j=1}^{p+q}\rho(s'^{-1})(e_j)F_j)\exp(\sum_{k=1}^{p+q}e_kF'_k)(1+G''). \tag{2.1}$$

If $\rho(s'^{-1})(e_j)$ is expressed as

$$\rho(s'^{-1})(e_j) = \sum_{m=1}^{p+q}\rho'_{mj}e_m \tag{2.2}$$

($\rho'_{mj} \in \mathbb{R}$), then one can write

$$\begin{aligned}\sum_{j=1}^{p+q}\rho(s'^{-1})(e_j)F_j &= \sum_{m,j=1}^{p+q}\rho'_{mj}e_mF_j\\ &= \sum_{m,j=1}^{p+q}e_m\rho'_{mj}F_j\\ &= \sum_{m=1}^{p+q}e_mF''_m.\end{aligned} \tag{2.3}$$

where we set

$$F''_m = \sum_{j=1}^{p+q}\rho'_{mj}F_j. \tag{2.4}$$

Inserting (2.3) into (2.1), we find

$$\begin{aligned}\Theta\Theta' &= ss'\exp(\sum_{m=1}^{p+q}e_mF''_m)\exp(\sum_{k=1}^{p+q}e_kF'_k)(1+G'')\\ &= ss'(1+e_1F''_1)\dots(1+e_{p+q}F''_{p+q})(1+e_1F'_1)\dots(1+e_{p+q}F'_{p+q})(1+G'')\\ &= ss'(1+e_1F''_1)(1+e_1F'_1)\dots(1+e_{p+q}F''_{p+q})(1+e_{p+q}F'_{p+q})(1+G'')\\ &= ss'(1+e_1(F''_1+F'_1)-F''_1F'_1)\dots\\ &\qquad\dots(1+e_{p+q}(F''_{p+q}+F'_{p+q})+F''_{p+q}F'_{p+q})(1+G'')\end{aligned}$$

$$\begin{aligned} &= ss'(1+e_1(F_1''+F_1'))(1-F_1''F_1'))\dots \\ &\qquad \dots(1+e_{p+q}(F_{p+q}''+F_{p+q}'))(1+F_{p+q}''F_{p+q}'))(1+G'') \\ &= ss'(1+e_1(F_1''+F_1'))\dots(1+e_{p+q}(F_{p+q}''+F_{p+q}')). \\ &\qquad (1-F_1''F_1'))\dots(1+F_{p+q}''F_{p+q}'))(1+G'') \\ &= ss'(1+e_1(F_1''+F_1'))\dots(1+e_{p+q}(F_{p+q}''+F_{p+q}'))(1+G''') \end{aligned} \tag{2.5}$$

where we have used $F''F' = -F'F''$, $F''^2 = F'^2 = 0$,

$$(1 \pm F_k''F_k')(1+e_l(F_l''+F_l')) = (1+e_l(F_l''+F_l'))(1 \pm F_k''F_k')$$

and set

$$1+G''' \quad = \quad (1-F_1''F_1'))\dots(1+F_{p+q}''F_{p+q}'))(1+G'').$$

(2.5) implies that $S_{p,q,r}$ is closed with respect to the Clifford multiplication.

Since Clifford algebra is an associative algebra, Clifford multiplication is associative on $S_{p,q,r}$. Unit element of $S_{p,q,r}$ is $1 \in \mathcal{C}\ell_{p,q,r}$.

Finally, the inverse of the element $\Theta = s(1+e_1F_1)\dots(1+e_{p+q}F_{p+q})(1+G) \in S_{p,q,r}$, where $F_k = \sum_{l=1}^{r} c_{kl} f_l$ and $G \in \Lambda(f)$, is

$$\Theta^{-1} = s^{-1}(1-e_1\sum_{k=1}^{p+q}\rho_{1k}F_k)\dots(1-e_{p+q}\sum_{k=1}^{p+q}\rho_{(p+q)k}F_k)(1-G+\dots+(-1)^{\mu-1}G^{\mu-1})$$

where $\rho(s)(e_j) = \sum_{m=1}^{p+q}\rho_{mj}e_m$ $(1 \le i,j \le p+q)$ and the positive integer μ satisfies $G^\mu = 0$ (for any $G \in \Lambda(f)$, there is such an integer). □

The subset of $S_{p,q,r}$ defined by

$$\Delta = \{1+G \mid G \in \Lambda(f)\}$$

is a normal subgroup of $S_{p,q,r}$. We adopt the following definition for the degenerate spin group (see [1]):

Definition 2.4. The quotient group $S_{p,q,r}/\Delta$ is called the degenerate spin group and denoted by $Spin(p,q,r)$.

3. Semi-Direct Products

Now we recall the notion of semi-direct products (see [3]):

Definition 3.1. Let G be a group, let $H \le G$ a subgroup and $N \trianglelefteq G$ a normal subgroup. If $G = HN$ and $H \cap N = 1$, then G is called a semi-direct product of H and N, and denoted by $G = H \ltimes N$.

The map $h \longmapsto \theta(h)(n) := hnh^{-1}$ defines a homomorphism $\theta : H \to Aut(N)$, where $Aut(N)$ denotes the automorphism group of N.

Proposition 3.2. *Let H and N be two groups, and let $\theta : H \longrightarrow Aut(N)$ be a homomorphism. Then, on $H \times N$, the operation*

$$(h, n)(h', n') = (hh', \theta(h'^{-1})(n)n')$$

has the following properties:

1) *$H \times N$ is a group with the above operation,*
2) *$H \simeq H \times 1_N \leq H \times N$,*
3) *$N \simeq 1_H \times N \trianglelefteq H \times N$,*
4) *$H \times N = (H \times 1_N)(1_H \times N)$ and $(H \times 1_N) \cap (1_H \times N) = \{(1_H, 1_N)\}$,*
5) *$H \times N = (H \times 1_N) \ltimes_\theta (1_H \times N) \simeq H \ltimes_\theta N$.*

We omit the straightforward proof.

Definition 3.3. Let $N = \mathbb{M}(p+q, r)$ denote the additive group of $(p+q)\times r$ matrices with real entries and define

$$\begin{array}{rccll} \tilde{\rho}: & Spin(p,q) & \longrightarrow & Aut(\mathbb{M}(p+q,r)) & \\ & s & \longmapsto & \tilde{\rho}(s)(A) := \rho(s)A & (A \in \mathbb{M}(p+q,r)) \end{array}$$

where $\rho(s) \in SO(p, q)$ is understood as a $(p+q) \times (p+q)$ matrix and $\rho(s)A$ is the matrix multiplication.

Now taking $H = Spin(p, q)$, and $\tilde{\rho} : Spin(p, q) \longrightarrow Aut(\mathbb{M}(p + q, r))$, we consider the semi-direct product $Spin(p, q) \ltimes_{\tilde{\rho}} \mathbb{M}(p + q, r)$.

Theorem 3.4. *$Spin(p, q, r) \simeq Spin(p, q) \ltimes_{\tilde{\rho}} \mathbb{M}(p + q, r)$.*

Proof. The isomorphism is given by the following map

$$\begin{array}{rccc} \eta: & Spin(p,q,r) & \longrightarrow & Spin(p,q) \ltimes_{\tilde{\rho}} \mathbb{M}(p+q,r) \\ & [s\gamma_1 \dots \gamma_{p+q}(1+G)] & \longmapsto & (s, (c_{il})) \end{array}$$

where $c_{il} \in \mathbb{M}(p + q, r)$ comes from the expression $\gamma_i = 1 + e_i \sum_{l=1}^{r} c_{il} f_l$.

The map η is well-defined: We must show that images of two equivalent elements in $S_{p,q,r}$ are equal. Let $\Theta, \Theta' \in S_{p,q,r}$ with $[\Theta] = [\Theta'] \in Spin(p, q, r)$ be two equivalent elements. Then we have

$$s\gamma_1 \dots \gamma_{p+q}(1 + G) = s'\gamma'_1 \dots \gamma'_{p+q}(1 + G')(1 + G'').$$

Using the inverse of the $1 + G$, we obtain

$$\begin{aligned} s\gamma_1 \dots \gamma_{p+q} &= s'\gamma'_1 \dots \gamma'_{p+q}(1 + G')(1 + G'')(1 - G + \dots + (-1)^{\mu-1}G^{\mu-1}) \\ &= s'\gamma'_1 \dots \gamma'_{p+q}(1 + G''') \end{aligned}$$

with $1 + G''' = (1 + G')(1 + G'')(1 - G + \dots + (-1)^{\mu-1}G^{\mu-1})$. Moreover, from the last equation, we have

$$(s')^{-1}s\gamma_1 \dots \gamma_{p+q} = \gamma'_1 \dots \gamma'_{p+q}(1 + G''').$$

With the notations $\gamma_1 \dots \gamma_{p+q} = 1 + A$ and $\gamma'_1 \dots \gamma'_{p+q} = 1 + A'$, we can write

$$\begin{aligned} (s')^{-1}s(1+A) &= (1+A')(1+G''') \\ (s')^{-1}s + (s')^{-1}sA &= 1 + G''' + A' + A'G''' \end{aligned} \tag{3.1}$$

where A and A' include terms of the form $e_{i_1}F_{i_1} \dots e_{i_\alpha}F_{i_\alpha}$ and $e_{i_1}F'_{i_1} \dots e_{i_\alpha}F'_{i_\alpha}$, respectively $(1 \le \alpha \le p+q,\ i_1 < i_2 < \dots < i_\alpha)$. Now, equating terms not-containing the f_i's we find $(s')^{-1}s = 1$, i.e., $s = s'$. Thus the equation (3.1) is reduced to

$$A = G''' + A' + A'G''',$$

or

$$G''' = A - A' - A'G'''.$$

G''' must be zero, as there are no terms in the right hand side consisting only of the f_i's. Then we find $A = A'$, i.e.,

$$\gamma_1 \dots \gamma_{p+q} = \gamma'_1 \dots \gamma'_{p+q}. \tag{3.2}$$

From this equation, one can write

$$\gamma_1 \dots \gamma_{p+q-1}(1 + e_{p+q}F_{p+q}) = \gamma'_1 \dots \gamma'_{p+q-1}(1 + e_{p+q}F'_{p+q}).$$

As before, using the notation $\gamma_1 \dots \gamma_{p+q-1} = 1 + B$ and $\gamma'_1 \dots \gamma'_{p+q-1} = 1 + B'$, we find

$$\begin{aligned} (1+B)(1+e_{p+q}F_{p+q}) &= (1+B')(1+e_{p+q}F'_{p+q}) \\ B + e_{p+q}F_{p+q} + Be_{p+q}F_{p+q} &= B' + e_{p+q}F'_{p+q} + B'e_{p+q}F'_{p+q}. \end{aligned}$$

Equating the terms not-containing the e_{p+q} we find $B = B'$. Multiplying then both sides by e_{p+q} we can see $F_{p+q} = F'_{p+q}$ and thus $\gamma_{p+q} = \gamma'_{p+q}$. Proceeding in this way inductively we get $\gamma_{p+q-1} = \gamma'_{p+q-1}, \dots, \gamma_1 = \gamma'_1$. Consequently, the equation

$$[s\gamma_1 \dots \gamma_{p+q}(1+G)] = [s'\gamma'_1 \dots \gamma'_{p+q}(1+G')]$$

implies $s = s'$, $\gamma_1 = \gamma'_1, \dots, \gamma_{p+q} = \gamma'_{p+q}$ whence we obtain $(c_{il}) = (c'_{il})$. Because of $s = s'$ and $(c_{il}) = (c'_{il})$, the map η is well-defined.
Obviously, the map η is both one-to-one and onto.
Homomorphism is obtained as follows:

$$\begin{aligned} [\Theta'] = [\Theta\Theta'] &= [ss'(1 + e_1(F''_1 + F'_1)) \dots (1 + e_{p+q}(F''_{p+q} + F'_{p+q}))(1+G''')] \quad \textit{by } (2.5) \\ &= [ss'(1 + e_1(F''_1 + F'_1)) \dots (1 + e_{p+q}(F''_{p+q} + F'_{p+q}))] \\ &= [ss'(1 + e_1(\sum_{j=1}^{p+q} \rho'_{1j}F_j + F'_1)) \dots \\ &\qquad \dots (1 + e_{p+q}(\sum_{j=1}^{p+q} \rho'_{(p+q)j}F_j + F'_{p+q}))] \qquad \textit{by } (2.4) \end{aligned}$$

$$= [ss'(1 + e_1 \sum_{l=1}^{r}(\sum_{j=1}^{p+q} \rho'_{1j} c_{jl} + c'_{1l}) f_l) \dots$$
$$\dots (1 + e_{p+q} \sum_{l=1}^{r}(\sum_{j=1}^{p+q} \rho'_{(p+q)j} c_{jl} + c'_{(p+q)l}) f_l)].$$

Thus $\eta([\Theta][\Theta'])$ becomes

$$\eta([\Theta][\Theta']) = (ss', (\sum_{j=1}^{p+q} \rho'_{ij} c_{jl} + c'_{il})). \tag{3.3}$$

On the other hand, $\eta([\Theta]) = (s, (c_{il}))$, $\eta([\Theta']) = (s', (c'_{il}))$ and consequently,

$$\begin{aligned} \eta([\Theta])\eta([\Theta']) &= (ss', \rho(s'^{-1})((c_{il})) + (c'_{il}))) \\ &= (ss', (\sum_{j=1}^{p+q} \rho'_{ij} c_{jl} + c'_{il})) \end{aligned}$$

where we have used the matrix of $\rho(s'^{-1})$ given in (2.2). □

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Tekin Dereli
Physics Department
Koc University
Sariyer-Istanbul
Turkey
e-mail: `tdereli@ku.edu.tr`

Şahin Koçak and Murat Limoncu
Mathematics Department
Anadolu University
Eskisehir
Turkey
e-mail: `skocak@anadolu.edu.tr`
`mlimoncu@anadolu.edu.tr`

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