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Advances in Applied Clifford Algebras

Clifford Algebra Applied to Grover's Algorithm

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Abstract. Grover's algorithm is a quantum algorithm for searching in unstructured databases which provides a quadratic speedup over their classical counterparts. We propose to use Clifford algebra in order to present a new way to describe the operators of Grover's algorithm and to simplify the calculation of its computational complexity.

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1. Introduction

Using classical computers, if we have an unstructured database with N elements and we are looking for a specific one, we test each database element at a time, until we hit the one searched for. This takes, in the worst case, $O(N)$ attempts. Due to the properties of quantum mechanics, Grover's algorithm ([3, 4]) requires only $O(\sqrt{N})$ trials.

In this paper, we propose to use Clifford algebra in order to present a new way to describe the operators of Grover's algorithm and to simplify the calculation of its computational complexity.

In [2], the authors use Clifford algebra for representing quantum algorithms in a new way and apply this technique to generalize the Grover's algorithm. The main points that are considered here and not mentioned in the reference [2] are the following: 1-) we present geometric interpretations of the operators of the Grover's algorithm, motivated by the geometric concepts of Clifford algebra; 2-) we represent the operators of the Grover's algorithm in the language of Clifford algebra using just basic concepts of this language; 3-) we provide an easy way to calculate the computational complexity of the Grover's algorithm (it is important to stress that the calculation of the computational complexity of an algorithm is in general a nontrivial task).

In Section 2, we briefly describe the basic concepts of quantum computing necessary to understand the algorithm. Section 3 presents a review of Grover's algorithm and Section 4 gives geometric interpretations of its operators. In Section 5, we use these geometric interpretations to represent the operators of Grover's algorithm in the language of Clifford algebra in order to simplify the calculation of its computational complexity. Section 6 ends the paper with the final remarks.

2. Basic Concepts of Quantum Computing

In quantum computing, the values 0 and 1 of a bit are replaced by the qubits $|0\rangle$ and $|1\rangle$ (this is the Dirac notation that is standard in quantum mechanics). In addition to the states $|0\rangle$ and $|1\rangle$, a general qubit $|\psi\rangle$ can also be in a linear combination

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle, \tag{2.1}$$

where α and β are complex numbers, called *amplitudes*. $|\psi\rangle$ is a vector in a two-dimensional complex vector space, where $\{|0\rangle, |1\rangle\}$ forms an orthonormal basis, called the *computational basis*. The matrix representations of $|0\rangle$ and $|1\rangle$ are given by

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{2.2}$$

The physical interpretation of $|\psi\rangle$, in (1), is that it coexists in the states $|0\rangle$ and $|1\rangle$. The state $|\psi\rangle$ can store a huge quantity of information in its coefficients α and β, but this information lives in the quantum level. To bring quantum information to the classical level, one must measure the qubit. Quantum mechanics tells us that the measurement process inevitably disturbs a qubit state, yielding a non-deterministic collapse of $|\psi\rangle$ to either $|0\rangle$ or $|1\rangle$: one gets $|0\rangle$ with probability $|\alpha|^2$ or $|1\rangle$ with probability $|\beta|^2$. Thus,

$$|\alpha|^2 + |\beta|^2 = 1. \tag{2.3}$$

If one does not measure a qubit, its evolution is described as the effect of a linear transformation U, the constraint (2.3) being fulfilled throughout the process (that is, U is a unitary transformation).

In order to consider states with more than one qubit, the concept of *tensor product* must be employed. For our purposes, we use the following definition. Given states $|v\rangle \in \mathbb{C}^m$ and $|w\rangle \in \mathbb{C}^n$, the tensor product $|v\rangle \otimes |w\rangle$ is given by

$$|v\rangle \otimes |w\rangle = \begin{bmatrix} v_1|w\rangle \\ \vdots \\ v_m|w\rangle \end{bmatrix}, \tag{2.4}$$

where $v_1, \ldots, v_m$ are the vector $|v\rangle$ coordinates. Either notations $|v\rangle|w\rangle$ or $|vw\rangle$, besides $|v\rangle \otimes |w\rangle$, are used for the tensor product.

In general, the state $|\psi\rangle$ of an n-qubit quantum computer is a superposition of the 2^n states $\{|0\rangle, |1\rangle, ..., |2^n-1\rangle\}$,

$$|\psi\rangle = \sum_{i=0}^{2^n-1} \alpha_i |i\rangle, \tag{2.5}$$

with amplitudes α_i constrained by

$$\sum_{i=0}^{2^n-1} |\alpha_i|^2 = 1. \tag{2.6}$$

As before, a measurement of a generic state $|\psi\rangle$ yields the result $|i_0\rangle$ with probability $|\alpha_{i_0}|^2$, for $0 \le i_0 < 2^n$ (note that the orthonormal basis $\{|0\rangle, \dots, |2^n-1\rangle\}$ is the computational basis in decimal notation). Usually, the measurement is performed qubit by qubit yielding zeroes or ones that are read together to form i_0.

To end this section, we give more two definitions that will be used later. Given two vectors $|\varphi\rangle$ and $|\psi\rangle$ in a vector space V, besides the usual *inner product*, written in the form $\langle\varphi|\psi\rangle$, we will also use the *outer product* $|\psi\rangle\langle\varphi|$, defined as a linear operator on V satisfying

$$(|\psi\rangle\langle\varphi|)|v\rangle = \langle\varphi|v\rangle|\psi\rangle, \;\; \forall|v\rangle \in V. \tag{2.7}$$

More information about quantum computing can be found in Benenti et al. [1], Hirvensalo [5], Kitaev et al. [6], Nielsen and Chuang [9], Pittenger [10], and Preskill [11].

3. Grover's Algorithm

Grover's algorithm uses n qubits in the first register and 1 qubit in the second one (Fig. 1). The first step is to create a superposition of all 2^n computational basis states $\{|0\rangle, ..., |2^n-1\rangle\}$ of the first register which is achieved in the following way. Initialize the first register in the state $|0\rangle \cdots |0\rangle$ and apply the *Hadamard* operator H,

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \tag{3.1}$$

on each qubit $|0\rangle$. For the n-qubit state $|0\rangle \cdots |0\rangle$, the Hadamard operators yield (Fig. 1)

$$|\psi\rangle = \frac{1}{\sqrt{2^n}} \sum_{i=0}^{2^n-1} |i\rangle \,. \tag{3.2}$$

The second register initiates with $|1\rangle$ and, after applying the Hadamard operator, it changes to the state $|-\rangle$ (Fig. 1).

The considered problem deals with the location of a particular element in an unstructured database with N elements, labeled with the integers from 0 to $N-1$.

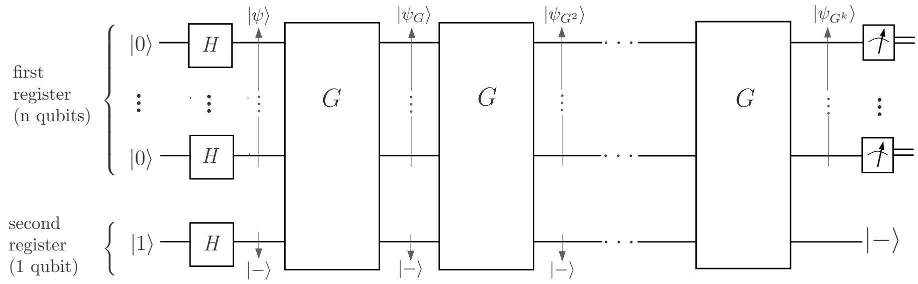

FIGURE 1. Outline of Grover's algorithm.

We assume that $N = 2^n$ for some integer $n \geq 2$ (Grover's algorithm does not work for $n = 1$).

First, Grover's algorithm uses a unitary operator U_f, defined by

$$U_f\left(|i\rangle\,|j\rangle\right) = |i\rangle\,|j \oplus f(i)\rangle\,, \tag{3.3}$$

where $|i\rangle$ stands for a state of the first register ($i \in \{0, ..., N-1\}$), $|j\rangle$ is a state of the second register ($i \in \{0, 1\}$), $\oplus$ is the sum modulo 2, and $f : \{0, ..., N-1\} \to \{0, 1\}$ is a function used to recognize the searched element, given by

$$f(i) = \begin{cases} 1, \text{ if } i \text{ is the label } i_0 \text{ of the searched element} \\ 0, \text{ otherwise.} \end{cases} \tag{3.4}$$

Using the fact that

$$1 \oplus f(i) = \begin{cases} 0 \text{ for } i = i_0 \\ 1 \text{ for } i \neq i_0, \end{cases} \tag{3.5}$$

one may check that

$$U_f\left(|i\rangle\,|-\rangle\right) = (-1)^{f(i)}\,|i\rangle\,|-\rangle\,, \tag{3.6}$$

where

$$|-\rangle = H|1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}. \tag{3.7}$$

Now, applying U_f to the superposition state $|\psi\rangle$, generated at the first step of the algorithm (Fig. 2), we obtain the following result:

$$\begin{aligned} |\psi_1\rangle\,|-\rangle &= U_f\left(|\psi\rangle\,|-\rangle\right) \\ &= \left(\frac{1}{\sqrt{N}}\sum_{i=0}^{N-1}(-1)^{f(i)}|i\rangle\right)|-\rangle, \end{aligned} \tag{3.8}$$

where $|\psi_1\rangle$ is then the resulting state of the first register.

At the quantum level it is possible to identify the searched element, because it is the only one with negative amplitude. However, this quantum information is not fully available at the classical level. A classical information of a quantum state is obtained by practical measurements, and, at this point, it is of no help to

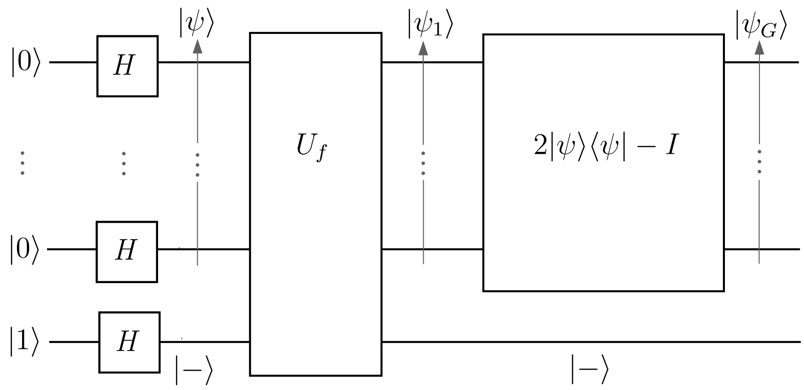

Figure 2. One Grover iteration.

measure the state of the first register, because it is much more likely that we fail to obtain the searched element. Before we can perform a measurement, the next step should be to increase the amplitude of the searched element while decreasing the amplitude of the others. This is obtained using another unitary operator defined by $(2\,|\psi\rangle\,\langle\psi| - I)$ (Fig. 2). We can think that $|\psi\rangle$ is the initial state of the first register and the computing steps are the applications of the unitary operators U_f and $2\,|\psi\rangle\,\langle\psi| - I$ (Fig. 2). The composition of these two operators is called Grover's operator G (Fig. 1) and is defined by

$$G = ((2\,|\psi\rangle\,\langle\psi| - I) \otimes I)\,U_f. \tag{3.9}$$

In [7], it was proved that the resulting action of G^k ($k \in \mathbb{N}$) rotates $|\psi\rangle$ towards $|i_0\rangle$ by $k\theta$ degrees, in the subspace spanned by $|\psi\rangle$ and $|i_0\rangle$, where θ is the angle between $|\psi\rangle$ and $G|\psi\rangle$ (Fig. 3). With this result, it was also proved that the number k of times that the operator G must be applied so that the angle between $|i_0\rangle$ and $G^k|\psi\rangle$ becomes zero is given by

$$k = \frac{\arccos\left(\frac{1}{\sqrt{N}}\right)}{\arccos\left(\frac{N-2}{N}\right)}. \tag{3.10}$$

Comparing k to $\sqrt{N}$, when N is large, it was deduced that

$$\lim_{N\to\infty} \frac{k}{\sqrt{N}} = \frac{\pi}{4}, \tag{3.11}$$

implying that

$$k = O(\sqrt{N}). \tag{3.12}$$

More information about other demonstrations of complexity of Grover's algorithm can be found in Benenti et al. [1], Hirvensalo [5], Kitaev et al. [6], Nielsen and Chuang [9], Pittenger [10], and Preskill [11].

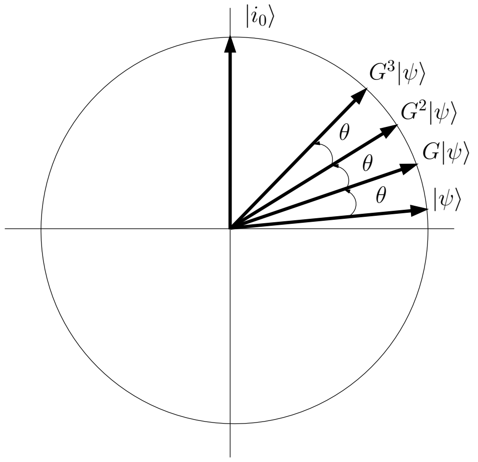

FIGURE 3. Effect of successive applications of G.

4. Geometric Interpretation of the Operators of Grover's Algorithm

Considering a unit vector

$$|v\rangle = \sum_{i=0, i\neq i_0}^{N-1} \alpha_i |i\rangle + \alpha_{i_0}|i_0\rangle, \tag{4.1}$$

we can obtain that

$$U_f\left(|v\rangle|-\rangle\right) = \left(\sum_{i=0, i\neq i_0}^{N-1} \alpha_i |i\rangle - \alpha_{i_0}|i_0\rangle\right)|-\rangle. \tag{4.2}$$

Since the second register does not change, the application of U_f on the vector $|v\rangle$ can be viewed as a reflection of $|v\rangle$ across the orthogonal subspace to $|i_0\rangle$, spanned by all the other elements of the list. To obtain a visualization of this fact, consider first the unit projection of $|v\rangle$ on such orthogonal subspace to $|i_0\rangle$, denoted by the vector $|u\rangle$, given by

$$|u\rangle = \frac{1}{\sqrt{N-1}} \sum_{i=0, i\neq i_0}^{N-1} |i\rangle, \tag{4.3}$$

which can be rewritten as

$$|u\rangle = \frac{\sqrt{N}}{\sqrt{N-1}}|\psi\rangle - \frac{1}{\sqrt{N-1}}|i_0\rangle. \tag{4.4}$$

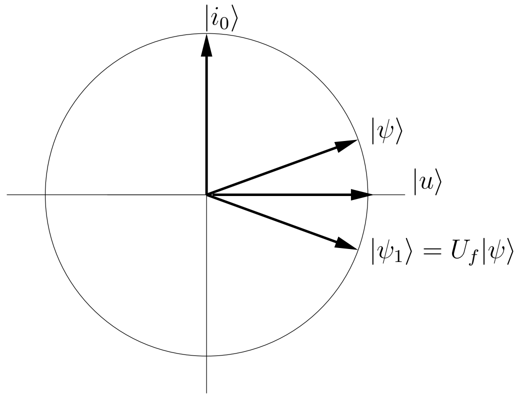

FIGURE 4. Geometric interpretation of the operator U_f.

Calculating

$$\langle\psi|i_0\rangle = \frac{1}{\sqrt{N}}\sum_{i=0}^{N-1}\langle i|i_0\rangle = \frac{1}{\sqrt{N}}\langle i_0|i_0\rangle = \frac{1}{\sqrt{N}} \tag{4.5}$$

and

$$\langle u|i_0\rangle = \frac{1}{\sqrt{N-1}}\sum_{i=0, i\neq i_0}^{N-1}\langle i|i_0\rangle = 0, \tag{4.6}$$

we can deduce that the vectors $|\psi\rangle$ and $|i_0\rangle$ form an angle smaller than $\pi/2$ rad (if N is large, then the angle is nearly $\pi/2$ rad) and the vectors $|u\rangle$ and $|i_0\rangle$ form an angle $\pi/2$ rad. Finally, from (4.4), (4.5), and (4.6), we can obtain a geometric interpretation for the action of U_f, given in Fig. 4.

Now, let us consider the geometric interpretation of the operator $2\,|\psi\rangle\,\langle\psi| - I$. First, calculating the projection of the first register of the state

$$|\psi_1\rangle\,|-\rangle = U_f\,(|\psi\rangle\,|-\rangle) \tag{4.7}$$

on the vector $|\psi\rangle$, denoted by $proj_{|\psi\rangle}(|\psi_1\rangle)$, we obtain that

$$proj_{|\psi\rangle}(|\psi_1\rangle) = \langle\psi|\psi_1\rangle\,|\psi\rangle\,. \tag{4.8}$$

Calculating the reflection of $|\psi_1\rangle$ across $|\psi\rangle$, denoted by $ref_{|\psi\rangle}(|\psi_1\rangle)$, using the Fig. 5, we obtain that

$$ref_{|\psi\rangle}(|\psi_1\rangle) = 2\langle\psi|\psi_1\rangle\,|\psi\rangle - |\psi_1\rangle\,, \tag{4.9}$$

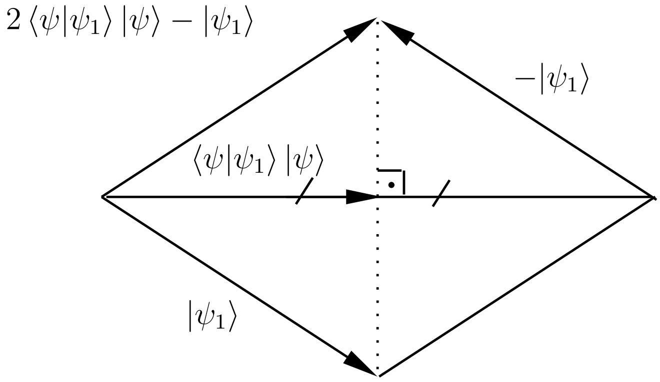

FIGURE 5. Obtaining the operator $2|\psi\rangle\langle\psi| - I$.

which can be rewritten as (see the definition of the outer product, given in (2.7))

$$\begin{aligned} ref_{|\psi\rangle}(|\psi_1\rangle) &= (2\langle\psi|\psi_1\rangle)|\psi\rangle - |\psi_1\rangle \\ &= (2|\psi\rangle\langle\psi|)|\psi_1\rangle - |\psi_1\rangle \\ &= (2|\psi\rangle\langle\psi| - I)|\psi_1\rangle . \end{aligned} \tag{4.10}$$

That is, the action of the operator $2|\psi\rangle\langle\psi| - I$ can be viewed as a reflection of $|\psi_1\rangle$ across $|\psi\rangle$, given in Fig. 6.

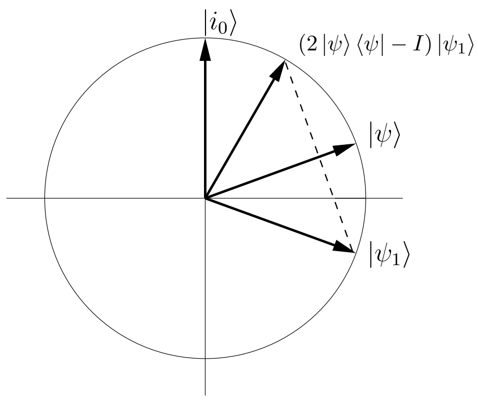

FIGURE 6. Geometric interpretation of the operator $2|\psi\rangle\langle\psi| - I$.

5. Using the Clifford Algebra

Using the standard basis of $\mathbb{R}^N$, $\{e_1, \ldots, e_N\}$, we define

$$\psi = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} e_j \tag{5.1}$$

and

$$u = \frac{1}{\sqrt{N-1}} \sum_{j=1, j \neq j_0}^{N} e_j, \tag{5.2}$$

where $\{e_1, ..., e_N\}$ represents the list in which we have to find the searched element e_{j_0}. From now on, the states of the algorithm will be considered as elements of Cl_N [8].

Based on the geometric interpretation of the operator U_f, given in Section 4, the resulting state of the application of the operator U_f to the ψ, denoted by ψ_1, is then given by (Fig. 4)

$$\psi_1 = u\psi u^{-1}. \tag{5.3}$$

Since u is a unit vector,

$$\psi_1 = u\psi u. \tag{5.4}$$

Again, by the results of the Section 4, the resulting state of the application of the operator $2|\psi\rangle\langle\psi| - I$ to the state ψ_1, denoted by ψ_G, is given by (Fig. 6)

$$\psi_G = \psi\psi_1\psi^{-1}. \tag{5.5}$$

Since ψ is a unit vector and from (5.4), we obtain that

$$\psi_G = \psi\psi_1\psi = \psi u\psi u\psi. \tag{5.6}$$

Defining

$$g = \psi u, \tag{5.7}$$

we can write

$$\psi_G = g^2\psi, \tag{5.8}$$

which implies that the operator G is represented in Cl_N by g^2.

In the following theorem, we obtain a representation in Cl_N for the Grover's operator G, depending on N and the elements of the list.

Theorem 5.1. *The Grover's operator G is represented in Cl_N by the bivector*

$$\frac{N-2}{N} + \frac{2}{N} e_{j_0} \left(\sum_{j=1, j \neq j_0}^{N} e_j \right),$$

where $\{e_1, ..., e_N\}$ is the list in which we have to find the searched element e_{j_0}.

Proof. From (5.1), (5.2), and (5.7), we obtain that

$$
\begin{aligned}
g = \psi u &= \left(\frac{1}{\sqrt{N}}\sum_{j=1}^{N} e_j\right)\left(u = \frac{1}{\sqrt{N-1}}\sum_{j=1, j\neq j_0}^{N} e_j\right) \\
&= \frac{1}{\sqrt{N}\sqrt{N-1}}\left(e_1 + \ldots + e_{j_0} + \ldots + e_N\right)\left(e_1 + \ldots + e_{j_0-1} + e_{j_0+1} + \ldots + e_N\right) \\
&= \frac{1}{\sqrt{N}\sqrt{N-1}}\left(\sum_{j\neq j_0}^{N} e_1 e_j + \ldots + \sum_{j\neq j_0}^{N} e_{j_0} e_j + \ldots + \sum_{j\neq j_0}^{N} e_N e_j\right) \\
&= \frac{1}{\sqrt{N}\sqrt{N-1}}\left(N - 1 + \sum_{j=1, j\neq j_0}^{N} e_{j_0} e_j\right) \\
&= \frac{1}{\sqrt{N}\sqrt{N-1}}\left(N - 1 + e_{j_0}\sum_{j=1, j\neq j_0}^{N} e_j\right).
\end{aligned}
$$

Now, calculating g^2, we obtain

$$
\begin{aligned}
g^2 &= \frac{1}{\sqrt{N}\sqrt{N-1}}\left(N - 1 + e_{j_0}\sum_{j=1, j\neq j_0}^{N} e_j\right)\frac{1}{\sqrt{N}\sqrt{N-1}}\left(N - 1 + e_{j_0}\sum_{j=1, j\neq j_0}^{N} e_j\right) \\
&= \frac{1}{N(N-1)}\left((N-1)^2 + 2(N-1)e_{j_0}\sum_{j=1, j\neq j_0}^{N} e_j + \left(e_{j_0}\sum_{j=1, j\neq j_0}^{N} e_j\right)^2\right) \\
&= \frac{N-1}{N} + \frac{2}{N}e_{j_0}\sum_{j=1, j\neq j_0}^{N} e_j - \frac{1}{N} \\
&= \frac{N-2}{N} + \frac{2}{N}e_{j_0}\left(\sum_{j=1, j\neq j_0}^{N} e_j\right).
\end{aligned}
$$

Since the operator G is represented in Cl_N by g^2, the demonstration is complete. □

It is interesting to note that g can be written, in terms of e_{j_0} and u, as

$$
g = \frac{\sqrt{N-1}}{\sqrt{N}} + \frac{1}{\sqrt{N}} e_{j_0} u,
$$

where $\|u\| = 1$ and $u \perp e_{j_0}$. Let $\phi \in [0, \pi]$ be the angle defined by $\cos(\phi/2) = \frac{\sqrt{N-1}}{\sqrt{N}}$ and $\sin(\phi/2) = \frac{1}{\sqrt{N}}$. Since $(e_{j_0} u)^2 = -1$, we get

$$
g = \exp(\tfrac{\phi}{2}\, e_{j_0} u).
$$

This is readily recognized as the element of Spin(N) (universal covering group of SO(N)) which generates a ϕ-rotation

$$v \mapsto gvg^{-1}$$

on the plane generated by the vectors e_{j_0} and u [8].

An interesting consequence of this result is that it provides a direct way of estimating the complexity of Grover's algorithm. In fact, from the defining relations of ϕ we get $\frac{\phi}{2} = \arctan\left(\frac{1}{\sqrt{N-1}}\right)$, so that $\frac{\phi}{2} \cong \frac{1}{\sqrt{N}}$ for $N \gg 1$. Since we aim a final rotation angle given by $k\phi \cong \frac{\pi}{2}$, the number of times that the algorithm has to be applied is, therefore,

$$k \cong \frac{\pi}{4}\sqrt{N} \quad (\text{for } N \gg 1),$$

which is the well-known expression for the complexity of Grover's algorithm.

6. Final Remarks

Grover's algorithm is a quantum algorithm for searching in unstructured databases, defined by successive applications of the operator G, given by

$$((2\,|\psi\rangle\,\langle\psi| - I) \otimes I)\,U_f, \tag{6.1}$$

where U_f is the operator that identifies the searched element and $2\,|\psi\rangle\,\langle\psi| - I$ is the operator that increases its amplitude, where $|\psi\rangle$ is the superposition of all database elements with equal amplitudes.

In [7], before determining the necessary amount of applications of G to obtain the searched element $|i_0\rangle$, it was necessary to prove that the resulting action of G^k is to rotate $|\psi\rangle$ towards $|i_0\rangle$ by $k\theta$ degrees, in the subspace spanned by $|\psi\rangle$ and $|i_0\rangle$, where θ is the angle between $|\psi\rangle$ and $G|\psi\rangle$.

Using the geometric interpretations of the operators U_f and $2\,|\psi\rangle\,\langle\psi| - I$, we used Clifford algebra to represent the operator G (6.1) as being the bivector given by

$$\frac{N-2}{N} + \frac{2}{N} e_{j_0} \left(\sum_{j=1, j\neq j_0}^{N} e_j \right), \tag{6.2}$$

where $\{e_1, ..., e_N\}$ is the list in which we have to find the searched element e_{j_0}. Based on this new representation, it was easy to calculate the computational complexity of the algorithm.

In addition to the calculation of the computational cost of quantum algorithms, another important point is how to design quantum circuits to implement the operators of a quantum algorithm. Thus, an open question is to investigate how the Clifford algebra could be used in the design of quantum circuits.

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Acknowledgments

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