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Uniform Approximation by Closed Forms in Several Complex Variables

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Abstract. The first main result in this article provides uniform estimates for solid Bochner-Martinelli-Koppelman transforms of type $(p, n-1)$, $0 \leq p \leq n$, of continuous forms on a compact set Ω in the complex space $\mathbb{C}^n$, in terms of the Euclidean volume of Ω. In the single variable case this result generalizes a classical inequality for the Cauchy kernel due to Ahlfors and Beurling. The second main result is a quantitative Hartogs-Rosenthal theorem which points out that the uniform distance in the space of continuous $(p, n-1)$-forms, $0 \leq p \leq n$, on a compact set Ω in $\mathbb{C}^n$ from a smooth form to the subspace of $\bar{\partial}$-closed forms on Ω is controlled by the Euclidean volume of Ω. This theorem as well as a third main result are generalizations to higher dimensions of an inequality in single variable complex analysis due to Alexander.

Keywords. Differential forms in several complex variables, Bochner-Martinelli-Koppelman integral transforms, quantitative Hartogs-Rosenthal theorems, complex Clifford analysis.

1. Introduction

The primary goal in this article is to formulate and prove a quantitative Hartogs-Rosenthal theorem for differential forms in several complex variables. Though not explicitly indicated in any part of the article, we would like to note that our aim from the outset was essentially motivated by ideas and techniques from complex Clifford analysis, and only as a matter of fact is related to complex analysis in several complex variables. For some significant developments in complex Clifford analysis directly related to the subject matter of this article we refer to the recent research notes by Rocha-Chavez, Shapiro, and Sommen [19]. Other important results addressing related issues in complex Clifford analysis are due to Bernstein [5], Rocha-Chavez, Shapiro, and Sommen [16–18], Ryan [20–24], Shapiro [25], Sommen [26–28], and Vasilevski and Shapiro [29–30], to mention just a few of the contributions in this area.

In the case of a single complex variable, our results provide generalizations of two inequalities due to Alexander [3–4] and Ahlfors and Beurling [1]. Specifically, Alexander's inequality states that if Ω is a compact set in the complex plane $\mathbb{C}$ and $\mathcal{C}(\Omega)$ is the Banach algebra of continuous complex-valued functions on Ω, then

$$\mathrm{dist}_{\mathcal{C}(\Omega)}[\bar{\zeta}, \mathcal{R}(\Omega)] \leq \left[\frac{1}{\pi}\mathrm{area}(\Omega)\right]^{1/2}, \tag{1.1}$$

where $\bar{\zeta}$ is the complex-conjugate coordinate function, and $\mathcal{R}(\Omega)$ stands for the uniform closure in $\mathcal{C}(\Omega)$ of rational functions holomorphic on some open neighborhoods of Ω in $\mathbb{C}$. The classical Hartogs-Rosenthal theorem [7] is just a qualitative consequence of (1.1). Whenever area $(\Omega) = 0$, from (1.1) it follows that $\mathcal{R}(\Omega)$ is a closed unital subalgebra of $\mathcal{C}(\Omega)$ that contains ζ and $\bar{\zeta}$, and by the Stone-Weierstrass theorem one gets $\mathcal{R}(\Omega) = \mathcal{C}(\Omega)$. In its turn, Alexander's distance estimate follows from an inequality discovered by Ahlfors and Beurling which states that for the Cauchy kernel on $\mathbb{C}$ one has the uniform estimate

$$\left|\frac{1}{\pi}\int_{\Omega}\frac{\mathrm{d\ area}(\zeta)}{\zeta - z}\right| \leq \left[\frac{1}{\pi}\mathrm{area}(\Omega)\right]^{1/2}, \qquad z \in \mathbb{C}. \tag{1.2}$$

In our article we are going to generalize both inequalities (1.1) and (1.2) to higher dimensions. As for the generalization of Alexander's inequality (1.1), we will assume that Ω is a compact set in $\mathbb{C}^n$ and replace the Banach algebra $\mathcal{C}(\Omega)$ with the Banach spaces $\mathcal{C}^{p,n-1}(\Omega)$ of all continuous differential forms on Ω of type $(p, n-1)$, $0 \leq p \leq n$. The uniform norm on $\mathcal{C}^{p,n-1}(\Omega)$ will be denoted by $\|\cdot\|_{\Omega,\infty}$. For technical reasons we are also going to use the L^1-norm on $\mathcal{C}^{p,n-1}(\Omega)$ denoted by $\|\cdot\|_{\Omega,1}$. The precise definitions are given in Subsection 2.4. As a substitute for $\mathcal{R}(\Omega)$, we will take the spaces $\mathcal{R}^{p,n-1}(\Omega)$ defined as the uniform closure in $\mathcal{C}^{p,n-1}(\Omega)$ of the subspace consisting of restrictions to Ω of smooth differential $(p, n-1)$-forms that are $\bar{\partial}$-closed on some open neighborhoods of Ω in $\mathbb{C}^n$. Finally, instead of the complex-conjugate coordinate function $\bar{\zeta}$ we will either use the $(0, n-1)$-form γ on $\mathbb{C}^n$ given by

$$\gamma(\zeta) = \sum_{i=1}^{n}(-1)^{i-1}\bar{\zeta}_i d\bar{\zeta}_1 \wedge \cdots \wedge d\bar{\zeta}_{i-1} \wedge d\bar{\zeta}_{i+1} \wedge \cdots \wedge d\bar{\zeta}_n, \qquad \zeta \in \mathbb{C}^n, \tag{1.3}$$

or, more generally, an arbitrary smooth $(p, n-1)$-form ω on $\mathbb{C}^n$, $0 \leq p \leq n$. The distance estimate for γ will be expressed in terms of $\mathrm{vol}(\Omega)$, the Lebesgue measure of Ω as a subset of $\mathbb{C}^n \equiv \mathbb{R}^{2n}$. For an arbitrary $(p, n-1)$-form ω, $0 \leq p \leq n$, the distance estimate will be in terms of $\mathrm{vol}(\Omega)$, $\|\bar{\partial}\omega\|_{\Omega,\infty}$, and $\|\bar{\partial}\omega\|_{\Omega,1}$. The reason we only consider the spaces $\mathcal{C}^{p,n-1}(\Omega)$ and not spaces $\mathcal{C}^{p,q}(\Omega)$ of continuous (p, q)-forms on Ω with $q \neq n-1$ is that the classical Hartogs-Rosenthal theorem, that should result from our estimates whenever $\mathrm{vol}(\Omega) = 0$, is not true when $q \neq n-1$. For more details on this particular issue we refer to the monograph by Aizenberg and Dautov [2; §11], as well as to the articles by Dautov [6] and Weinstock [31].

For the Ahlfors-Beurling's inequality (1.2), in higher dimension we will consider a compact set Ω in $\mathbb{C}^n$ and the solid Bochner-Martinelli-Koppelman transforms $\mathfrak{I}_\Omega^{p,n-1}$ acting on the spaces $\mathcal{C}^{p,n}(\Omega)$, $0 \le p \le n$, defined by

$$\mathfrak{I}_\Omega^{p,n-1}\varphi(z) = \int_{\zeta\in\Omega} \varphi(\zeta) \wedge K_{p,n-1}(\zeta, z), \qquad \varphi \in \mathcal{C}^{p,n}(\Omega),\ z \in \Omega, \tag{1.4}$$

where $K_{p,n-1}(\zeta, z)$ is the component of type $(p, n-1)$ of the Bochner-Martinelli-Koppelman kernel $K(\zeta, z)$ on $\mathbb{C}^n$. More details on this kernel are given in Section 2. The interested reader is referred to Aizenberg and Dautov [2], Henkin and Leiterer [8], Krantz [9], Range [15], or Rocha-Chavez, Shapiro, and Sommen [19].

We are now in a position to state the three main results in this article. The first theorem generalizes the Ahlfors-Beurling's inequality (1.2).

Theorem A. *Suppose $\varphi \in \mathcal{C}^{p,n}(\Omega)$, $0 \le p \le n$, where $\Omega \subset \mathbb{C}^n$ is a compact set, and let $\mathfrak{I}_\Omega^{p,n-1}(\varphi)$ be its solid Bochner-Martinelli-Koppelman transform. Then*

$$\|\mathfrak{I}_\Omega^{p,n-1}(\varphi)\|_{\Omega,\infty} \le \frac{\sqrt{2}n!}{\pi^n}[\mathrm{vol}(\mathbb{B}^{2n})]^{1-1/2n}\|\varphi\|_{\Omega,\infty}^{1-1/2n}\|\varphi\|_{\Omega,1}^{1/2n}, \tag{1.5}$$

where $\mathbb{B}^{2n}$ is the closed unit ball in $\mathbb{C}^n \equiv \mathbb{R}^{2n}$.

The second theorem is a generalization of Alexander's inequality (1.1).

Theorem B. *Suppose $\omega \in \mathcal{E}^{p,n-1}(\mathbb{C}^n)$, $0 \le p \le n$, and $\Omega \subset \mathbb{C}^n$ is a compact set. Then*

$$\mathrm{dist}_{\mathcal{C}^{p,n-1}(\Omega)}[\omega, \mathcal{R}^{p,n-1}(\Omega)] \le \frac{\sqrt{2}n!}{\pi^n}[\mathrm{vol}(\mathbb{B}^{2n})]^{1-1/2n}\|\bar\partial\omega\|_{\Omega,\infty}^{1-1/2n}\|\bar\partial\omega\|_{\Omega,1}^{1/2n}. \tag{1.6}$$

By combining Theorem B with the Stone-Weierstrass theorem we get the next qualitative multivariable Hartogs-Rosenthal theorem. For different proofs of this result we refer to Aizenberg and Dautov [2], Dautov [6], and Weinstock [31].

Corollary. *If Ω is a compact set in $\mathbb{C}^n$ with $\mathrm{vol}(\Omega) = 0$ and $0 \le p \le n$, then $\mathcal{R}^{p,n-1}(\Omega) = \mathcal{C}^{p,n-1}(\Omega)$, that is, every continuous $(p, n-1)$-form on Ω can be uniformly approximated by restrictions to Ω of $(p, n-1)$-forms that are smooth and $\bar\partial$-closed on open neighborhoods of Ω in $\mathbb{C}^n$.*

The constant in estimate (1.6) in Theorem B is no longer the best for the $(0, p-1)$-form γ defined by (1.3). As a direct generalization of Alexander's inequality (1.1), for this form we have the following result that could be regarded as a refined form of Theorem B.

Theorem C. *Suppose $\Omega \subset \mathbb{C}^n$ is a compact set and let γ be the smooth $(0, n-1)$-form on $\mathbb{C}^n$ given by (1.3). Then*

$$\mathrm{dist}_{\mathcal{C}^{0,n-1}(\Omega)}[\gamma, \mathcal{R}^{0,n-1}(\Omega)] \le \frac{2^{(n+1)/2}(n\sqrt{n})(n!)}{\pi^n}[\mathrm{vol}(\mathbb{B}_1^{2n})]^{1-1/2n}[\mathrm{vol}(\Omega)]^{1/2n},$$

where

$$\mathbb{B}_1^{2n} = \{\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_n) \in \mathbb{C}^n : |\zeta|^{2n} \leq \operatorname{Re}\zeta_1\}.$$

The complete proofs of Theorems A, B, and C are given in Section 4. Section 2 introduces some basic prerequisites related to the Bochner-Martinelli-Koppelman kernel. Section 3 presents a technical result that provides sharp pointwise estimates for convolution transforms associated with kernels in the weak Lebesgue spaces $L^\kappa_{\text{weak}}(\mathbb{R}^m)$, where $m \geq 1$ and $\kappa > 1$. It is a refinement of an inequality proved in Martin and Szeptycki [14] for homogeneous kernels on the Euclidean spaces $\mathbb{R}^m$, $m \geq 1$, that better serves our specific purposes.

2. Bochner-Martinelli-Koppelman Transforms

The first part of this section summarizes several basic properties of the Bochner-Martinelli-Koppelman transforms on the complex space $\mathbb{C}^n$. The second part deals with explicit formulas and pointwise estimates for the solid Bochner-Martinelli-Koppelman transforms of type $(p, n-1)$ applied to continuous forms on compact subsets of $\mathbb{C}^n$. For the sake of convenience in what follows we will refer to $\bar{\partial}$-exact or $\bar{\partial}$-closed differential forms on $\mathbb{C}^n$ simply as exact or closed forms.

2.1. Preliminaries

We begin by recalling that the Bochner-Martinelli-Koppelman kernel on $\mathbb{C}^n$ is a double form defined as

$$K(\zeta, z) = \sum_{p,q} K_{p,q}(\zeta, z), \qquad (\zeta, z) \in \mathbb{C}^n \times \mathbb{C}^n,\ \zeta \neq z, \tag{2.1}$$

where the homogeneous components $K_{p,q}(\zeta, z)$ are smooth forms of type (p, q) in z and type $(n-p, n-q-1)$ in ζ, with $0 \leq p \leq n$ and $0 \leq q \leq n-1$.

To the double form K and to each bounded domain Δ in $\mathbb{C}^n$ with a smooth boundary Σ one associates the integral operators

$$\mathfrak{I}^{p,q}_\Sigma : \mathcal{E}^{p,q}(\mathbb{C}^n) \to \mathcal{E}^{p,q}(\mathbb{C}^n \setminus \Sigma) \tag{2.2}$$

and

$$\mathfrak{I}^{p,q}_\Delta : \mathcal{E}^{p,q+1}(\mathbb{C}^n) \to \mathcal{E}^{p,q}(\mathbb{C}^n \setminus \Sigma) \tag{2.3}$$

that act on the space of smooth (p, q)- or $(p, q+1)$-forms in $\mathbb{C}^n$ and take values in the space of (p, q)-forms on $\mathbb{C}^n \setminus \Sigma$. Specifically,

$$\mathfrak{I}^{p,q}_\Sigma(\varphi)(z) = \int_{\zeta\in\Sigma} \varphi(\zeta) \wedge K_{p,q}(\zeta, z), \qquad \varphi \in \mathcal{E}^{p,q}(\mathbb{C}^n),\ z \in \mathbb{C}^n \setminus \Sigma, \tag{2.4}$$

and

$$\mathfrak{I}^{p,q}_\Delta(\varphi)(z) = \int_{\zeta\in\Delta} \varphi(\zeta) \wedge K_{p,q}(\zeta, z), \qquad \varphi \in \mathcal{E}^{p,q+1}(\mathbb{C}^n),\ z \in \mathbb{C}^n \setminus \Sigma. \tag{2.5}$$

Actually, $\mathfrak{I}^{p,q}_\Sigma$ could be applied to forms continuous on Σ and $\mathfrak{I}^{p,q}_\Delta$ can be used for forms continuous or just integrable and essentially bounded on $\overline{\Delta}$, the closure of

Δ in $\mathbb{C}^n$. Moreover, the solid transforms $\mathfrak{I}^{p,q}_{\Delta}$ make sense for more general sets Δ, for instance, compact subsets of $\mathbb{C}^n$.

2.2. Integral Representation Formulas

Suppose next that $\omega \in \mathcal{E}^{p,q}(\mathbb{C}^n)$ is fixed and let $\omega_\Delta \in \mathcal{E}^{p,q}(\mathbb{C}^n \setminus \Sigma)$ be the form that equals ω on Δ and 0 on $\mathbb{C}^n \setminus \overline{\Delta}$. According to the Bochner-Martinelli-Koppelman formula, the so defined truncated form can be represented as

$$\omega_\Delta = \mathfrak{I}^{p,q}_{\Sigma}(\omega) - \mathfrak{I}^{p,q}_{\Delta}(\bar\partial\omega) - \bar\partial\mathfrak{I}^{p,q-1}_{\Delta}(\omega). \tag{2.6}$$

In other words, up to an exact form, the restriction of ω to Δ can be recovered from $\bar\partial\omega$ and the values of ω on the boundary Σ.

When $n = 1$ equation (2.6) reduces to the well-known Cauchy-Pompeiu formula, and the third term is missing for obvious reasons. Moreover, the first term $\mathfrak{I}^{p,q}_{\Sigma}(\omega)$ is a closed form in $\mathbb{C} \setminus \Sigma$. Although this important feature is no longer available for arbitrary values of q when $n \geq 2$, however the forms $\mathfrak{I}^{p,q}_{\Sigma}(\omega)$ turn out to be closed on $\mathbb{C}^n \setminus \Sigma$ provided $q = n-1$. This nice property takes place because the double forms $K_{p,n-1}(\zeta, z)$ are closed as $(p, n-1)$-forms in z. Since the third term in the right-hand side of (2.6) is an exact hence closed form, we conclude that whenever $\omega \in \mathcal{E}^{p,n-1}(\mathbb{C}^n)$, the distance from ω_Δ to the space of closed forms on $\mathbb{C}^n \setminus \Sigma$ is controlled by the form $\mathfrak{I}^{p,n-1}_{\Delta}(\bar\partial\omega)$. Therefore, in order to estimate that distance, we should find uniform estimates for $\mathfrak{I}^{p,n-1}_{\Delta}(\bar\partial\omega)$.

2.3. Explicit Equations

For a later use, we need explicit equations in terms of the standard coordinate functions ζ_i and z_i on $\mathbb{C}^n \times \mathbb{C}^n$ and their complex conjugates $\bar\zeta_i$ and $\bar z_i$, $1 \leq i \leq n$. We are mainly interested in expressing the solid transform (2.5) as a singular integral and in finding uniform estimates of that integral in the case when $q = n-1$. In particular, such estimates will be eventually applied to the second term in the right-hand side of the Bochner-Martinelli-Koppelman formula (2.6).

Each form $\varphi \in \mathcal{E}^{p,q}(\mathbb{C}^n)$ decomposes as

$$\varphi(z) = \sum_{I \in \mathcal{I}_p, J \in \mathcal{I}_q} \varphi_{IJ}(z)\, \mathrm{d}\, z_I \wedge \mathrm{d}\, \bar z_J, \tag{2.7}$$

where $\mathcal{I}_p$ and $\mathcal{I}_q$ are the sets of all ordered multiindices of length p or q, respectively, that is,

$$I = (i_1, i_2, \ldots, i_p), \qquad 1 \leq i_1 < i_2 < \cdots < i_p \leq n,$$

and

$$J = (j_1, j_2, \ldots, j_q), \qquad 1 \leq j_1 < j_2 < \cdots < j_q \leq n,$$

the coefficients φ_{IJ} are smooth complex-valued functions, and

$$\mathrm{d}\, z_I = \bigwedge_{i \in I} \mathrm{d}\, z_i = \mathrm{d}\, z_{i_1} \wedge \mathrm{d}\, z_{i_2} \wedge \cdots \wedge \mathrm{d}\, z_{i_p},$$

$$\mathrm{d}\, z_J = \bigwedge_{j \in J} \mathrm{d}\, \bar z_j = \mathrm{d}\, \bar z_{j_1} \wedge \mathrm{d}\, \bar z_{j_2} \wedge \cdots \wedge \mathrm{d}\, \bar z_{j_q}.$$

The components $K_{p,q}(\zeta, z)$ of the Bochner-Martinelli-Koppelman kernel (2.1) are given by

$$K_{p,q}(\zeta, z) = (-1)^{q(n-p)} \frac{(n-1)!}{(2\pi\sqrt{-1})^n} \sum_{I,J,i} \sigma(I)\sigma(J,i) k_i(\zeta, z) \, \mathrm{d}\,\zeta_I^{\mathrm{c}} \wedge \mathrm{d}\,\bar{\zeta}_{J,i}^{\mathrm{c}} \cdot \mathrm{d}\,z_I \wedge \mathrm{d}\,\bar{z}_J, \tag{2.8}$$

where $\sum_{I,J,i}$ stands for $\sum_{I\in\mathcal{I}_p} \sum_{J\in\mathcal{I}_q} \sum_{i\notin J}$, the functions $k_i(\zeta, z)$ are defined by

$$k_i(\zeta, z) = \frac{\bar{\zeta}_i - \bar{z}_i}{|\zeta - z|^{2n}}, \qquad (\zeta, z) \in \mathbb{C}^n \times \mathbb{C}^n,\ \zeta \neq z, \tag{2.9}$$

the superscript c indicates the use of complementary multiindices, that is,

$$\mathrm{d}\,\zeta_I^{\mathrm{c}} = \bigwedge_{k\notin I} \mathrm{d}\,\zeta_k, \quad \mathrm{d}\,\bar{\zeta}_{J,i}^{\mathrm{c}} = \bigwedge_{k\notin J, k\neq i} \mathrm{d}\,\bar{\zeta}_k,$$

and the coefficients $\sigma(I)$ and $\sigma(J,i)$ equal 1 or -1 acording to the rules

$$\mathrm{d}\,\zeta_I \wedge \mathrm{d}\,\zeta_I^{\mathrm{c}} = \sigma(I)\,\mathrm{d}\,\zeta, \quad \mathrm{d}\,\bar{\zeta}_i \wedge \mathrm{d}\,\bar{\zeta}_J \wedge \mathrm{d}\,\bar{\zeta}_{J,i}^{\mathrm{c}} = \sigma(J,i)\,\mathrm{d}\,\bar{\zeta},$$

with

$$\mathrm{d}\,\zeta = \mathrm{d}\,\zeta_1 \wedge \cdots \wedge \mathrm{d}\,\zeta_n, \quad \mathrm{d}\,\bar{\zeta} = \mathrm{d}\,\bar{\zeta}_1 \wedge \cdots \wedge \mathrm{d}\,\bar{\zeta}_n.$$

2.4. Norm Estimates

Besides smooth forms, we are going to use continuous forms on compact subsets of $\mathbb{C}^n$. If $\Omega \subset \mathbb{C}^n$ is such a compact set, we let $\mathcal{C}^{p,q}(\Omega)$ stand for the space of continuous forms of type (p,q) on Ω. Every form $\varphi \in \mathcal{C}^{p,q}(\Omega)$ is represented as in (2.7), where the coefficients φ_{IJ} are now continuous complex-valued functions on Ω. The Euclidean norm $|\cdot|$ on $\mathbb{C}^n$ can be extended to spaces of exterior forms in a way such that the basis forms $\mathrm{d}\,z_I \wedge \mathrm{d}\,\bar{z}_J$ are orthogonal and

$$|\mathrm{d}\,z_I \wedge \mathrm{d}\,\bar{z}_J|^2 = 2^{p+q}, \qquad I \in \mathcal{I}_p,\ J \in \mathcal{I}_q. \tag{2.10}$$

Consequently, for each $\varphi \in \mathcal{C}^{p,q}(\Omega)$ and $z \in \Omega$ we define the length $|\varphi(z)|$ of $\varphi(z)$ by setting

$$|\varphi(z)|^2 = \sum_{I\in\mathcal{I}_p, J\in\mathcal{I}_q} 2^{p+q} |\varphi_{IJ}(z)|^2. \tag{2.11}$$

The space $\mathcal{C}^{p,q}(\Omega)$ becomes now a Banach space with respect to the uniform norm

$$\|\varphi\|_{\Omega,\infty} = \sup_{\zeta\in\Omega} |\varphi(\zeta)|. \tag{2.12}$$

On $\mathcal{C}^{p,q}(\Omega)$ we also introduce the L^1-norm defined as

$$\|\varphi\|_{\Omega,1} = \int_\Omega |\varphi(\zeta)| \, \mathrm{d}\,\mathrm{vol}(\zeta). \tag{2.13}$$

Our goal is to estimate the uniform norm $\|\mathfrak{I}_\Omega^{p,n-1}(\varphi)\|_{\Omega,\infty}$ for $\varphi \in \mathcal{C}^{p,n}(\Omega)$, where $\mathfrak{I}_\Omega^{p,n-1}(\varphi)$ is given by a formula similar to (2.5), namely,

$$\mathfrak{I}_\Omega^{p,n-1}(\varphi)(z) = \int_{\zeta\in\Omega} \varphi(\zeta) \wedge K_{p,n-1}(\zeta, z), \qquad z \in \Omega. \tag{2.14}$$

By (2.8) we get that the double form $K_{p,n-1}(\zeta, z)$ can be expressed as

$$K_{p,n-1}(\zeta, z) = (-1)^{(n-1)(n-p)} \frac{(n-1)!}{(2\pi\sqrt{-1})^n} \sum_{I,i} (-1)^{i-1} \sigma(I) k_i(\zeta, z)\, \mathrm{d}\,\zeta_I^{\mathrm{c}} \cdot \mathrm{d}\, z_I \wedge \mathrm{d}\,\bar{z}_i^{\mathrm{c}}, \tag{2.15}$$

where $\Sigma_{I,i}$ stands for $\sum_{I\in\mathcal{I}_p}\sum_{i=1}^n$.

Suppose next that $\varphi \in \mathcal{C}^{p,n}(\Omega)$ is represented by

$$\varphi(\zeta) = \sum_{I\in\mathcal{I}_p} \varphi_I(\zeta)\, \mathrm{d}\,\zeta_I \wedge \mathrm{d}\,\bar{\zeta}, \qquad \zeta \in \Omega. \tag{2.16}$$

From (2.15) and (2.16) we get

$$\begin{aligned} &\varphi(\zeta) \wedge K_{p,n-1}(\zeta, z) \\ &\quad = (-1)^{n-p} \frac{(n-1)!}{(2\pi\sqrt{-1})^n} \sum_{I,i} (-1)^{i-1} \varphi_I(\zeta) k_i(\zeta, z)\, \mathrm{d}\,\zeta \wedge \mathrm{d}\,\bar{\zeta} \cdot \mathrm{d}\, z_I \wedge \mathrm{d}\,\bar{z}_i^{\mathrm{c}}. \end{aligned} \tag{2.17}$$

The form $\mathrm{d}\,\zeta \wedge \mathrm{d}\,\bar{\zeta}$ involved in the last equation is a multiple of the Euclidean volume form in $\mathbb{C}^n$ that results when $\mathbb{C}^n$ is identified with $\mathbb{R}^{2n}$, namely,

$$\mathrm{d}\,\zeta \wedge \mathrm{d}\,\bar{\zeta} = (-2\sqrt{-1})^n\, \mathrm{d}\,\mathrm{vol}(\zeta). \tag{2.18}$$

Therefore, based on (2.17) and (2.18) we can express the coefficients of the integral transform (2.14) as principal value integrals of complex-valued functions, that is,

$$\mathfrak{I}_\Omega^{p,n-1}(\varphi)(z) = (-1)^p \frac{(n-1)!}{\pi^n} \sum_{I,i} (-1)^{i-1} \left[\int_\Omega \varphi_I(\zeta) k_i(\zeta, z)\, \mathrm{d}\,\mathrm{vol}(\zeta) \right] \mathrm{d}\, z_I \wedge \mathrm{d}\,\bar{z}_i^{\mathrm{c}}, \tag{2.19}$$

for each $z \in \Omega$. Consequently, by (2.19) and (2.11) we have

$$\left| \mathfrak{I}_\Omega^{p,n-1}(\varphi)(z) \right| = 2^{(p+n-1)/2} \frac{(n-1)!}{\pi^n} \left[\sum_{I,i} \left| \int_\Omega \varphi_I(\zeta) k_i(\zeta, z)\, \mathrm{d}\,\mathrm{vol}(\zeta) \right|^2 \right]^{1/2}, \tag{2.20}$$

for any $z \in \Omega$.

By Minkowski's inequality we get

$$\left[\sum_{I,i} \left| \int_\Omega \varphi_I(\zeta) k_i(\zeta, z)\, \mathrm{d}\,\mathrm{vol}(\zeta) \right|^2 \right]^{1/2} \le \int_\Omega \left[\sum_{I,i} |\varphi_I(\zeta) k_i(\zeta, z)|^2 \right]^{1/2} \mathrm{d}\,\mathrm{vol}(\zeta). \tag{2.21}$$

By (2.16) and (2.11) we now notice that

$$\sum_{I\in\mathcal{I}_p} |\varphi_I(\zeta)|^2 = 2^{-(p+n)} |\varphi(\zeta)|^2, \tag{2.22}$$

and from (2.9) we have

$$\sum_{i=1}^{n} |k_i(\zeta, z)|^2 = \frac{1}{|\zeta - z|^{4n-2}}. \tag{2.23}$$

Finally, combining (2.20)–(2.23) we end up with

$$\left|\mathfrak{I}_\Omega^{p,n-1}(\varphi)(z)\right| \leq \frac{(n-1)!}{\pi^n\sqrt{2}} \int_\Omega \frac{|\varphi(\zeta)|}{|\zeta - z|^{2n-1}} \, \mathrm{d}\,\mathrm{vol}(\zeta). \tag{2.24}$$

The last pointwise estimate will be used in Section 4 as an important step in the proof of Theorem A.

Using (2.8) and (2.7), the previous reasoning and estimate (2.24) can be easily extended to general solid Bochner-Martinelli-Koppelman transforms $\mathfrak{I}_\Omega^{p,q}(\varphi)$, for arbitrary values of p and q. In effect, one gets an extension of Theorem A. The details are left to our reader.

3. Main Technical Result

In this section we prove an inequality that will help in completing the proofs of Theorems A and C.

3.1. The Setting

Suppose $k : \mathbb{R}_0^m \to [0, \infty)$ is a Lebesgue measurable function, where $\mathbb{R}_0^m = \mathbb{R}^m \backslash \{0\}$, and let $\mathfrak{I} = \mathfrak{I}_k$ be the convolution operator associated with k, that is,

$$\mathfrak{I}u(x) = \int_{\mathbb{R}^m} k(y) u(x - y) \, \mathrm{d}\, y, \tag{3.1}$$

where $\mathrm{d}\, y = \mathrm{d}\,\mathrm{vol}(y)$. We are going to use operator $\mathfrak{I}$ for complex-valued functions in the standard Lebesgue spaces $L^1(\mathbb{R}^m)$ and $L^\infty(\mathbb{R}^m)$. The norms of a function u in $L^1(\mathbb{R}^m)$ or $L^\infty(\mathbb{R}^m)$ will be denoted by $\|u\|_1$ and $\|u\|_\infty$, respectively.

To the kernel k we also associate its distribution function $\mu : (0, \infty) \to [0, \infty]$ defined as

$$\mu(t) = \mathrm{vol}\, \Omega[t], \qquad t \in (0, \infty), \tag{3.2}$$

where

$$\Omega[t] = \{y \in \mathbb{R}_0^m : k(y) \geq t\} \cup \{0\}, \qquad t \in (0, \infty). \tag{3.3}$$

3.2. Main Lemma

As expected, the behavior of operator $\mathfrak{I}$ is controlled by the distribution function μ. The specific result that we need is the following:

Main Lemma. *Suppose $1 < \kappa < \infty$. The following two statements are equivalent:*

(i) *There exists a positive constant $A(k)$ such that*

$$|\mathfrak{I}u(x)| \leq A(k) \|u\|_\infty^{1/\kappa} \|u\|_1^{1-1/\kappa}, \qquad x \in \mathbb{R}^m, \tag{3.4}$$

for every $u \in L^1(\mathbb{R}^m) \cap L^\infty(\mathbb{R}^m)$.

(ii) *There exists a positive constant* λ *such that*

$$\mu(t) \leq \lambda t^{-\kappa}, \qquad t \in (0, \infty). \tag{3.5}$$

Whenever (i) *or* (ii) *is true and* λ *is the smallest constant in* (3.5)*, that is,*

$$\lambda = \sup_{t>0} t^{\kappa} \mu(t), \tag{3.6}$$

a possible value of $A(k)$ *in* (3.4) *is given by*

$$A(k) = \frac{\kappa}{\kappa - 1} \cdot \lambda^{1/\kappa}. \tag{3.7}$$

Moreover, if (3.5) *is an equality for all* $t \in (0, \infty)$*, the value of* $A(k)$ *in* (3.7) *is the best constant in inequality* (3.4)*.*

Before continuing with the proof of this result, we should point out that (3.5) is true whenever k is in the Lebesgue space $L^{\kappa}(\mathbb{R}^m)$. In this case, one can take $\lambda = \|k\|_{\kappa}$, the norm of k in $L^{\kappa}(\mathbb{R}^m)$. Condition (3.5) is usually stated by saying that k is in the *weak* Lebesgue space $L^{\kappa}_{\text{weak}}(\mathbb{R}^m)$.

3.3. Proof of Main Lemma—Part 1

We show first that (i) implies (ii). Suppose $x = 0 \in \mathbb{R}^m$ and let χ_τ be the characteristic function of the set $\Omega[\tau]$, where $\tau \in (0, \infty)$ is fixed. Next, define $u : \mathbb{R}^m \to \mathbb{R}$ as

$$u(y) = \chi_\tau(-y), \qquad y \in \mathbb{R}^m, \tag{3.8}$$

and observe that

$$\mathfrak{I}u(0) = \int_{\mathbb{R}^m} k(y) u(-y) \, \mathrm{d}\, y = \int_{\Omega[\tau]} k(y) \, \mathrm{d}\, y \geq \tau \mu(\tau), \tag{3.9}$$

and

$$\|u\|_\infty = 1, \quad \|u\|_1 = \mu(t). \tag{3.10}$$

Using (3.4) in conjunction with (3.9) and (3.10) we get

$$\tau \mu(\tau) \leq A(k) [\mu(\tau)]^{1-1/\kappa},$$

an estimate that amounts to

$$\mu(\tau) \leq [A(k)]^{\kappa} \tau^{-\kappa}.$$

Therefore, (3.5) holds true.

3.4. Proof of Main Lemma—Part 2

We prove now that (ii) implies (i). Suppose $\tau \in (0, \infty)$ is fixed and split the kernel k by setting

$$k = k_{\text{inn}} + k_{\text{out}}, \tag{3.11}$$

where

$$k_{\text{inn}}(y) = \begin{cases} k(y) - \tau, & \text{if } y \in \Omega[\tau] \\ 0, & \text{if } y \in \Omega^{\mathrm{c}}[\tau] \end{cases} \tag{3.12}$$

and

$$k_{\mathrm{out}}(y)=\begin{cases}\tau, & \text{if } y\in\Omega[\tau]\\ k(y), & \text{if } y\in\Omega^{\mathrm{c}}[\tau],\end{cases} \tag{3.13}$$

with $\Omega^{\mathrm{c}}[\tau]=\mathbb{R}^m\setminus\Omega[\tau]$.

By (3.11) we clearly have

$$|\mathfrak{I}u(x)|\le \mathfrak{I}_{\mathrm{inn}}|u|(x)+\mathfrak{I}_{\mathrm{out}}|u|(x), \tag{3.14}$$

where $\mathfrak{I}_{\mathrm{inn}}$ and $\mathfrak{I}_{\mathrm{out}}$ are the convolution operators associated with k_{inn} or k_{out}.

Using now (3.12), the Stieltjes integral associated with the decreasing function μ defined in (3.2), and an integration by parts, we have successively

$$\begin{aligned}\mathfrak{I}_{\mathrm{inn}}|u|(x)&=\int_{\Omega[\tau]}[k(y)-\tau]|u(x-y)|\,\mathrm{d}\,y\\ &\le\|u\|_\infty\int_{\Omega[\tau]}[k(y)-\tau]\,\mathrm{d}\,y\\ &=-\|u\|_\infty\int_\tau^\infty(t-\tau)\,\mathrm{d}\,\mu(t)\\ &=\|u\|_\infty\left[-(t-\tau)\mu(t)|_\tau^\infty+\int_\tau^\infty\mu(t)\,\mathrm{d}\,t\right].\end{aligned}$$

By (3.5) we obtain

$$\lim_{t\to\infty}(t-\tau)\mu(t)=0$$

and

$$\int_\tau^\infty\mu(t)\,\mathrm{d}\,t\le\lambda\int_\tau^\infty t^{-\kappa}\,\mathrm{d}\,t=\frac{\lambda}{\kappa-1}\tau^{1-\kappa},$$

whence

$$\mathfrak{I}_{\mathrm{inn}}|u|(x)\le\frac{\lambda}{\kappa-1}\tau^{1-\kappa}\|u\|_\infty. \tag{3.15}$$

At the same time, from (3.13) we clearly get that $k_{\mathrm{out}}(y)\le\tau$ for all $y\in\mathbb{R}^m$, hence

$$\mathfrak{I}_{\mathrm{out}}|u|(x)\le\tau\|u\|_1. \tag{3.16}$$

Therefore, using (3.14), (3.15), and (3.16) we conclude that

$$|\mathfrak{I}u(x)|\le\frac{\lambda}{\kappa-1}\tau^{1-\kappa}\|u\|_\infty+\tau\|u\|_1. \tag{3.17}$$

To minimize the right-hand side of (3.17) we choose

$$\tau=\left(\lambda\cdot\frac{\|u\|_\infty}{\|u\|_1}\right)^{1/\kappa}.$$

A direct calculation shows that for this value of τ inequality (3.17) reduces to (3.4) with a constant $A(k)$ as in (3.7).

3.5. Proof of Main Lemma—Part 3

To conclude the proof we need to check that (3.7) is the best constant in (3.4) provided

$$\mu(t) = \lambda t^{-\kappa}, \qquad t \in (0, \infty). \tag{3.18}$$

We do that by showing that every function u defined as in (3.8) is extremal. If u is such a function, then by (3.18) we have

$$\begin{aligned}\mathfrak{I}u(0) &= \int_{\Omega[\tau]} k(y)\,\mathrm{d}\,y = -\int_{\tau}^{\infty} t\,\mathrm{d}\,\mu(t) \\ &= \lambda\kappa \int_{\tau}^{\infty} t^{-\kappa}\,\mathrm{d}\,t = \frac{\kappa}{\kappa-1}\cdot\lambda\cdot\tau^{1-\kappa}.\end{aligned}$$

By (3.10) and (3.18), the right-hand side of (3.4) becomes

$$A(k)\|u\|_{\infty}^{1/\kappa}\|u\|_{1}^{1-1/\kappa} = A(k)[\mu(\tau)]^{1-1/\kappa} = A(k)\lambda^{1-1/\kappa}\tau^{1-\kappa},$$

so (3.4) reduces to

$$\frac{\kappa}{\kappa-1}\cdot\lambda \le A(k)\cdot\lambda^{1-1/\kappa},$$

that is,

$$A(k) \ge \frac{\kappa}{\kappa-1}\cdot\lambda^{1/\kappa}. \tag{3.19}$$

Since in Subsection 3.4 above we have already established the opposite inequality for the best value of $A(k)$, the proof is complete.

In addition, we want to point out that in the general case when statement (ii) is true and λ is defined by (3.6), the best value of $A(k)$ in (3.4) must satisfy the double inequality

$$\lambda^{1/\kappa} \le A(k) \le \frac{\kappa}{\kappa-1}\cdot\lambda^{1/\kappa}. \tag{3.20}$$

The lower limit results from the last inequality proved in Subsection 3.3 and the upper limit follows from (3.7).

4. Proofs and Concluding Remarks

Based on Sections 2 and 3, we are now in a position to prove Theorems A, B, and C, as stated in the Introduction to our article.

4.1. Proof of Theorem A

We apply the Main Lemma to the kernel k on $\mathbb{C}^n$, or $\mathbb{R}^{2n}$, defined by

$$k(\zeta) = \frac{1}{|\zeta|^{2n-1}}, \qquad \zeta \in \mathbb{C}^n \setminus \{0\}. \tag{4.1}$$

The corresponding sets $\Omega[t]$ given by (3.3) are closed Euclidean balls in $\mathbb{C}^n$, namely,

$$\Omega[t] = \{\zeta \in \mathbb{C}^n : |\zeta| \le t^{-1/(2n-1)}\}, \qquad t \in (0, \infty). \tag{4.2}$$

Therefore, the distribution function μ of k is

$$\mu(t) = \lambda t^{-\kappa}, \qquad t \in (0, \infty), \tag{4.3}$$

where

$$\lambda = \mu(1) = \mathrm{vol}(\mathbb{B}^{2n}), \tag{4.4}$$

with $\mathbb{B}^{2n}$ the unit closed ball in $\mathbb{R}^{2n}$, and

$$\kappa = 2n/(2n-1). \tag{4.5}$$

Using complex coordinates and (4.4), (4.5), we write inequality (3.4) in the Main Lemma as

$$\left| \int \frac{u(\zeta)}{|\zeta - z|^{2n-1}} \,\mathrm{d}\,\mathrm{vol}(\zeta) \right| \leq 2n [\mathrm{vol}(\mathbb{B}^{2n})]^{1-1/2n} \|u\|_\infty^{1-1/2n} \|u\|_1^{1/2n}, \tag{4.6}$$

for any $u \in L^1(\mathbb{C}^n) \cap L^\infty(\mathbb{C}^n)$ and $z \in \Omega$.

Suppose next that $\Omega \subset \mathbb{C}^n$ is compact, $\varphi \in \mathcal{C}^{p,n}(\Omega)$, and $z \in \Omega$. By (4.6) applied to the function $|\varphi(\cdot)|$, we get

$$\left| \int_\Omega \frac{|\varphi(\zeta)|}{|\zeta - z|^{2n-1}} \,\mathrm{d}\,\mathrm{vol}(\zeta) \right| \leq 2n [\mathrm{vol}(\mathbb{B}^{2n})]^{1-1/2n} \|\varphi\|_{\Omega,\infty}^{1-1/2n} \|\varphi\|_{\Omega,2}^{1/2n}, \tag{4.7}$$

for any $z \in \mathbb{C}^n$.

It remains now to combine estimate (2.24) with (4.7). An easy calculation leads to

$$\left\| \mathfrak{I}_\Omega^{p,n-1}(\varphi) \right\|_{\Omega,\infty} \leq \frac{\sqrt{2}n!}{\pi^n} [\mathrm{vol}(\mathbb{B}^{2n})]^{1-1/2n} \|\varphi\|_{\Omega,\infty}^{1-1/2n} \|\varphi\|_{\Omega,1}^{1/2n}, \tag{4.8}$$

an inequality that concludes the proof of Theorem A.

4.2. Proof of Theorem B

Suppose that $\omega \in \mathcal{E}^{p,n-1}(\mathbb{C}^n)$ and the compact set $\Omega \subset \mathbb{C}^n$ are given. We choose a bounded open neighborhood Δ of Ω in $\mathbb{C}^n$ with a smooth boundary Σ and express the truncated form ω_Δ as in the Bochner-Martinelli-Koppelman formula (2.6). According to comments made in 2.2 above, the first and the third term in that formula result in elements of $\mathcal{R}^{p,n-1}(\Omega)$. Therefore

$$\mathrm{dist}_{\mathcal{C}^{p,n-1}(\Omega)} \left[\omega, \mathcal{R}^{p,n-1}(\Omega) \right] \leq \left\| \mathfrak{I}_\Delta^{p,n-1}(\bar{\partial}\omega) \right\|_{\Omega,\infty}.$$

By letting Δ approach Ω we get

$$\mathrm{dist}_{\mathcal{C}^{p,n-1}(\Omega)} \left[\omega, \mathcal{R}^{p,n-1}(\Omega) \right] \leq \left\| \mathfrak{I}_\Omega^{p,n-1}(\bar{\partial}\omega) \right\|_{\Omega,\infty}. \tag{4.9}$$

Theorem B follows from (4.9) in conjunction with (4.8) applied to $\varphi = \bar{\partial}\omega$.

4.3. Proof of Theorem C

We start the proof by using (4.9) for the $(0, n-1)$-form $\gamma(z) = \sum_{i=1}^{n} (-1)^{i-1} \bar{z}_i \, \mathrm{d}\, \bar{z}_i^{\mathrm{c}}$. Since $\bar{\partial}\gamma = n \, \mathrm{d}\, \bar{z}$, by (2.20) we have

$$\left| \mathfrak{I}_{\Omega}^{0,n-1}(\bar{\partial}\gamma)(z) \right| = \frac{2^{(n-1)/2} n!}{\pi^n} \left[\sum_{i=1}^{n} \left| \int_{\Omega} k_i(\zeta, z) \, \mathrm{d}\, \mathrm{vol}(\zeta) \right|^2 \right]^{1/2}, \tag{4.10}$$

where $k_i(\zeta, z)$ are the functions defined by (2.9).

We claim that

$$\left| \int_{\Omega} k_i(\zeta, z) \, \mathrm{d}\, \mathrm{vol}(\zeta) \right| \leq 2n [\mathrm{vol}(\mathbb{B}_i^{2n})]^{1-1/2n} [\mathrm{vol}(\Omega)]^{1/2n}, \tag{4.11}$$

for all $1 \leq i \leq n$ and $z \in \Omega$, where $\mathbb{B}_i^{2n}$ stand for the compact sets

$$\mathbb{B}_i^{2n} = \left\{ \zeta \in \mathbb{C}^n : |\zeta|^{2n} \leq \mathrm{Re}\, \zeta_i \right\}, \qquad 1 \leq i \leq n. \tag{4.12}$$

The sets $\mathbb{B}_i^{2n}$ have equal volume, that is,

$$\mathrm{vol}(\mathbb{B}_i^{2n}) = \mathrm{vol}(\mathbb{B}_1^{2n}), \qquad 1 \leq i \leq n.$$

Using this remark and taking (4.11) for granted, from (4.10) we get

$$\left\| \mathfrak{I}_{\Omega}^{0,n-1}(\bar{\partial}\gamma) \right\|_{\Omega,\infty} \leq \frac{2^{(n+1)/2} (n\sqrt{n})(n!)}{\pi^n} \left[\mathrm{vol}\left(\mathbb{B}_1^{2n} \right) \right]^{1-1/2n} [\mathrm{vol}(\Omega)]^{1/2n}.$$

Theorem C follows from this estimate and (4.9). Therefore, to complete the proof of Theorem C, we have to show that (4.11) is true.

We assume that $1 \leq i \leq n$ and $z \in \Omega$ in (4.11) are fixed and take a complex number θ with $|\theta| = 1$ such that

$$\left| \int_{\Omega} k_i(\zeta, z) \, \mathrm{d}\, \mathrm{vol}(\zeta) \right| = \theta \cdot \int_{\Omega} k_i(\zeta, z) \, \mathrm{d}\, \mathrm{vol}(\zeta) \geq 0. \tag{4.13}$$

Next, we evaluate the integral over Ω by changing the variables according to

$$\begin{aligned} &\zeta_i - z_i = \theta \zeta_i', \\ &\zeta_j - z_j = \zeta_j', \qquad 1 \leq j \leq n, \ j \neq i, \end{aligned}$$

and letting Ω' stand for the set of the corresponding values of ζ'. In effect we get

$$\int_{\Omega} k_i(\zeta, z) \, \mathrm{d}\, \mathrm{vol}(\zeta) = \bar{\theta} \int_{\Omega'} \frac{\bar{\zeta}_i'}{|\zeta'|^{2n}} \, \mathrm{d}\, \mathrm{vol}(\zeta'),$$

whence, by (4.13) we obtain

$$\left| \int_{\Omega} k_i(\zeta, z) \, \mathrm{d}\, \mathrm{vol}(\zeta) \right| = \int_{\Omega'} \frac{\bar{\zeta}_i'}{|\zeta'|^{2n}} \, \mathrm{d}\, \mathrm{vol}(\zeta'). \tag{4.14}$$

Since the integral in the right-hand side of (4.14) is a nonnegative number we deduce that

$$\int_{\Omega'} \frac{\bar{\zeta}_i'}{|\zeta'|^{2n}} \, \mathrm{d}\, \mathrm{vol}(\zeta') = \int_{\Omega'} \frac{\mathrm{Re}\, \bar{\zeta}_i'}{|\zeta'|^{2n}} \, \mathrm{d}\, \mathrm{vol}(\zeta') \leq \int_{\Omega'} \frac{\left(\mathrm{Re}\, \bar{\zeta}_i' \right)_+}{|\zeta'|^{2n}} \, \mathrm{d}\, \mathrm{vol}(\zeta'),$$

where $(\operatorname{Re}\bar{\zeta}_i')_+ = \max\{\operatorname{Re}\zeta_i', 0\}$, $\zeta \in \Omega'$. Therefore,

$$\left|\int_\Omega k_i(\zeta, z)\,\mathrm{d}\,\mathrm{vol}(\zeta)\right| \leq \int_{\Omega'} \frac{\left(\operatorname{Re}\bar{\zeta}_i'\right)_+}{|\zeta'|^{2n}}\,\mathrm{d}\,\mathrm{vol}(\zeta'). \tag{4.15}$$

We are once more in a position to apply the Main Lemma. This time, the kernel k on $\mathbb{C}^n$ is defined by

$$k(\zeta') = \frac{\left(\operatorname{Re}\bar{\zeta}_i'\right)_+}{|\zeta'|^{2n}}, \qquad \zeta' \in \mathbb{C}^n \setminus \{0\}. \tag{4.16}$$

Since $k(t\zeta') = t^{-(2n-1)}k(\zeta')$ for all $t \in (0, \infty)$, we next get that the distribution function μ of k is given by

$$\mu(t) = \lambda t^{-\kappa}, \qquad t \in (0, \infty), \tag{4.17}$$

where

$$\lambda = \mu(1) = \mathrm{vol}\left(\mathbb{B}_i^{2n}\right), \tag{4.18}$$

and $\kappa = 2n/(2n-1)$. Applying the Main Lemma to the kernel (4.16) with $x = 0$ and u the characteristic function of the set $-\Omega' = \{-\zeta' : \zeta' \in \Omega'\}$, we end up with

$$\int_{\Omega'} \frac{\left(\operatorname{Re}\bar{\zeta}_i'\right)_+}{|\zeta'|^{2n}}\,\mathrm{d}\,\mathrm{vol}(\zeta') \leq 2n \cdot \lambda^{1-1/2n}[\mathrm{vol}(\Omega')]^{1/2n}. \tag{4.19}$$

Since $\mathrm{vol}(\Omega') = \mathrm{vol}(\Omega)$, inequality (4.11) results by combining the last estimate with (4.18) and (4.15).

The proof is complete.

4.4. Remarks

As a final comment, we want to stress that though the three theorems in our article generalize Alexander's and Ahlfors-Beurling's inequalities to several complex variables, however they all should be regarded as results in the framework of complex Clifford analysis and not just multivariable complex analysis. To make the point, we would like to refer to Martin [10–12], where a real Clifford analysis approach addressing similar issues was developed. That approach uses several real variables and provides generalizations of both Alexander's and Ahlfors-Beurling's inequalities similar to Theorems A, B, and C to the setting of Clifford analytic functions, Dirac operators, and the Euclidean Cauchy kernel.

Actually, the standard $\bar{\partial}$ operator, or the Euclidean Dirac operators, should be regarded as prototypes of *non-elliptic*, or, respectively, *elliptic* first-order differential operators for which we have a quantitative Hartogs-Rosenthal theorem, or an Ahlfors-Beurling type inequality. By refining our tools it is possible to establish more general estimates for first-order differential operators with constant coefficients from a real or complex Banach algebra. This specific issue is addressed to a certain extent in Martin [13]. Other recent developments of this research project will be reported elsewhere.

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