Adv. appl. Clifford alg. 18 (2008), 197–236

0188-7009/020197-40, *published online* April 18, 2008
DOI 10.1007/s00006-008-0073-z

Advances in Applied Clifford Algebras

The Graded Structure of Nondegenerate Meson Algebras

Jacques Helmstetter and Artibano Micali

Abstract. Meson algebras are involved in the wave equation of meson particles in the same way as Clifford algebras are involved in the Dirac wave equation of electrons. Here we improve and generalize the information already obtained about their structure and their representations, when the symmetric bilinear form under consideration is nondegenerate. We emphasize their parity grading. We calculate the center of these meson algebras, and the center of their even subalgebra. Finally we show that every nondegenerate meson algebra over a field contains a group isomorphic to the group of automorphisms of the symmetric bilinear form.

Mathematics Subject Classification (2000). 15A66, 15A78, 16S10, 16W50, 81R05, 81R20.

Keywords. Clifford algebras, other algebras built from modules, algebras determined by universal properties, graded algebras and modules, algebras motivated by Physics and their representations, wave equations.

The existence of meson particles (also called π-mesons or pions) was first predicted by Yukawa (see [Y]); he calculated their mass, which amounted to approximatively 200 times the mass of an electron, therefore much less than the mass of a proton. At that time (in 1935) the presence of particles with a similar mass had already been observed in cosmic rays; their average life was about 2 microseconds; but experiments showed that they could not be Yukawa's mesons; they were called mesotrons, and they were supposed to come from the disintegration of mesons; today they are also called muons or Proca particles. The experimental evidence of the existence of Yukawa's mesons was established 12 years later (in 1947) by the "group of Bristol" (so named after the English town where it worked); it was formed by the English physicists Cecil F. Powel and Hugh Muirhead, the Italian Giuseppe Occhialini and the Brazilian Cesar Lattes; but only Powel received the Nobel prize in 1950.

The mathematical treatment of Yukawa's mesons began with articles of Proca, Duffin, Kemmer and Schrödinger (see [P], [D], [K1], [K2], [Sch]). The equations imagined by Proca suggested to Duffin an equation for the wave function ψ of a meson; it looked like the Dirac equation for an electron, and was soon accepted by Kemmer, and later confirmed by Schrödinger; here it is, with somewhat different notations:

$$\beta_1 \frac{\partial\psi}{\partial x} + \beta_2 \frac{\partial\psi}{\partial y} + \beta_3 \frac{\partial\psi}{\partial z} + \beta_4 \frac{1}{ic} \frac{\partial\psi}{\partial t} + \frac{mc}{\hbar} \psi = 0 ;$$

of course x, y, z, t are the usual space and time coordinates, $i = \sqrt{-1}$, and c, m, $\hbar$ are the usual physical constants; yet instead of Dirac's relations, the four matrices β_j satisfy these relations (where the $\delta_{j,k}$ are Kronecker's symbols):

$$\beta_j\beta_k\beta_l + \beta_l\beta_k\beta_j = \delta_{j,k}\beta_l + \delta_{l,k}\beta_j .$$

Let M be the vector space spanned by the four matrices β_j over $\mathbb{C}$ and let $f : M \times M \to \mathbb{C}$ be the symmetric bilinear form such that $f(\beta_j, \beta_k) = \delta_{j,k}$; the above relations are equivalent to this equality involving three arbitrary elements β, β' and β'' of M :

$$\beta\beta'\beta'' + \beta''\beta'\beta = f(\beta, \beta')\,\beta'' + f(\beta'', \beta')\ \beta .$$

It was known that the universal algebra for Dirac's relations was a Clifford algebra of dimension 16, and it was sensible to associate a universal algebra with Duffin's relations too. This new universal algebra had dimension 126, it was the direct sum of three ideals of dimensions respectively 1, 25 and 100, and each ideal was isomorphic to an algebra of matrices of order 1, 5 and 10 respectively.

In the same way as a Clifford algebra is associated with every quadratic form, a (universal) meson algebra is associated with every symmetric bilinear form, as it is now explained. Already Kemmer and Littlewood supplied valuable information about the meson algebras associated with all nondegenerate symmetric bilinear forms over $\mathbb{C}$.

1. Introduction

In this paper we give a precise description of the meson algebra $\mathrm{B}(M, f)$ associated with any nondegenerate symmetric bilinear form $f : M \times M \to K$ over any ring K (associative, commutative, with unit). The word "nondegenerate" must be understood in the strongest sense: M must be a finitely generated projective module (a vector space of finite dimension if K is a field), and f must induce a linear bijection d_f from M onto the dual module M^*.

Nevertheless we can define a meson algebra $\mathrm{B}(M, f)$ for any module M provided with any symmetric bilinear form f ; it is the quotient of the tensor algebra $\mathrm{T}(M)$ by the ideal $\mathrm{J}(M, f)$ generated by all elements $a \otimes b \otimes a - f(a, b)\, a$ with a, $b \in M$. Below it is explained that the canonical mappings $K \to \mathrm{T}(M) \to \mathrm{B}(M, f)$ and $M \to \mathrm{T}(M) \to \mathrm{B}(M, f)$ are injective, and allow us to identify every element of K or M with its image in $\mathrm{B}(M,f)$. Thus we can say that $\mathrm{B}(M,f)$ is the algebra over K generated by M with the only relations $aba = f(a,b)a$ and their consequences. By a well known argument these relations imply

$$abc + cba \;=\; f(a,b)\,c + f(c,b)\,a$$

for all a, b, $c \in M$; and conversely, when 2 is invertible in K, we come back to the initial relations by setting $c = a$.

Of course a universal property can be immediately derived from this definition: any linear mapping φ from M into any algebra A, such that $\varphi(a)\varphi(b)\varphi(a) = f(a,b)\varphi(a)$ for all a, $b \in M$, extends in a unique way to an algebra morphism $\mathrm{B}(M,f) \to A$. As in the case of Clifford algebras, this universal property implies that every meson algebra $\mathrm{B}(M,f)$ is provided with a reversion ρ, that is an involutive linear transformation of $\mathrm{B}(M,f)$ such that $\rho(1) = 1$, $\rho(a) = a$ for all $a \in M$, and $\rho(xy) = \rho(y)\rho(x)$ for all x, $y \in \mathrm{B}(M,f)$.

The assumption that K is any ring (not necessarily a field) does not really complicate the present study, and our text can be read by people that are only concerned with meson algebras over fields; they must simply skip some arguments and translate some expressions; when they read "projective module of constant rank n", they must understand "vector space of dimension n", and so on...

Unless otherwise specified, every grading is a parity grading, in other words, a grading over the group $\mathbb{Z}/2\mathbb{Z}$. For instance the tensor algebra $\mathrm{T}(M)$ is the direct sum of the even subalgebra $\mathrm{T}_0(M) = \bigoplus_k \mathrm{T}^{2k}(M)$ and the odd subspace $\mathrm{T}_1(M) = \bigoplus_k \mathrm{T}^{2k+1}(M)$; and since the ideal $\mathrm{J}(M,f)$ is generated by odd elements, the parity grading of $\mathrm{T}(M)$ is inherited by $\mathrm{B}(M,f)$ which consequently is the direct sum of an even subalgebra $\mathrm{B}_0(M,f)$ and an odd submodule $\mathrm{B}_1(M,f)$. An element $x \in \mathrm{B}(M,f)$ is said to be homogeneous if it belongs to $\mathrm{B}_0(M,f)$ or $\mathrm{B}_1(M,f)$, and its degree, or parity (either 0 or 1), is denoted by ∂x. The parity grading of $\mathrm{B}(M,f)$ is sometimes called the Green decomposition.

The algebra $\mathrm{B}(M,f)$ also inherits an increasing filtration from $\mathrm{T}(M)$; the subspace $\mathrm{B}^{\leq k}(M,f)$ is the image of $\mathrm{T}^{\leq k}(M) = \bigoplus_{j=0}^{k} \mathrm{T}^j(M)$. Thus $\mathrm{B}^{\leq k}(M,f) = 0$ if $k < 0$, $\mathrm{B}^{\leq 0}(M,f) = K$, $\mathrm{B}^{\leq 1}(M,f) = K \oplus M$ and so on... An algebra $\mathrm{Gr}\,\mathrm{B}(M,f)$ graded over $\mathbb{Z}$ (over $\mathbb{N}$ if you prefer it) is associated with this filtration in the usual way; $\mathrm{Gr}^0\mathrm{B}(M,f)$ and $\mathrm{Gr}^1\mathrm{B}(M,f)$ are respectively isomorphic to K and M.

Exceptionally the $\mathbb{Z}$-grading of $\mathrm{T}(M)$ is inherited by the meson algebra when $f = 0$, because the generators $a \otimes b \otimes a$ of the ideal $\mathrm{J}(M)$ are homogeneous of degree 3 for the $\mathbb{Z}$-grading. When $f = 0$, the meson algebra, simply denoted by $\mathrm{B}(M)$, is called the meson algebra of the module M ; among the other meson algebras $\mathrm{B}(M,f)$ it plays the same role as the exterior algebra $\bigwedge(M)$ among the other Clifford algebras $\mathrm{C}\ell(M,q)$. For instance there is a canonical surjective algebra morphism $\mathrm{B}(M) \to \mathrm{Gr}\,\mathrm{B}(M,f)$ analogous to the canonical morphism $\bigwedge(M) \to \mathrm{Gr}\,\mathrm{C}\ell(M,q)$. Indeed, if a' and b' are the images of a and b in $\mathrm{Gr}^1\mathrm{B}(M,f)$, then $a'b'a'$ is the image of aba in $\mathrm{Gr}^3\mathrm{B}(M,f)$ (the quotient of $\mathrm{B}^{\leq 3}(M,f)$ by

$B^{\leq 2}(M,f)$), and this image vanishes because $aba = f(a,b)a \in B^{\leq 1}(M,f)$; thus $a'b'a' = 0$ and the universal property of $B(M)$ implies the existence of the morphism $B(M) \to \mathrm{Gr}\, B(M,f)$. We say that $B(M,f)$ satisfies the PBW-property (the Poincaré-Birkhoff-Witt property) if this morphism is injective (therefore bijective). It follows from Theorem (1.1) below that the PBW-property holds when M is a free module (a module admitting bases).

The following theorem is proved in [AM1,2].

Theorem 1.1. *If M is a free module with a basis $(e_j)_{j\in J}$ indexed by a totally ordered set J, then $B_0(M,f)$ and $B_1(M,f)$ too are free modules. The products*

$$e_{j_1}e_{k_1}e_{j_2}e_{k_2}\dots e_{j_r}e_{k_r} \quad (resp. \quad e_{j_1}e_{k_1}e_{j_2}e_{k_2}\dots e_{j_r}e_{k_r}e_{j_{r+1}})$$

$$\text{where } r \geq 0,\ j_1 < j_2 < \dots < j_r \left(< j_{r+1}\right) \quad \text{and} \quad k_1 < k_2 < \dots < k_r,$$

constitute a basis of $B_0(M,f)$ (resp. $B_1(M,f)$).

Corollary 1.2. *If M is a free module provided with two bases $(e_j)_{j\in J}$ and $(e'_k)_{k\in J'}$ indexed by totally ordered sets, then the products*

$$e_{j_1}e'_{k_1}e_{j_2}e'_{k_2}\dots e_{j_r}e'_{k_r} \quad (resp. \quad e_{j_1}e'_{k_1}e_{j_2}e'_{k_2}\dots e_{j_r}e'_{k_r}e_{j_{r+1}})$$

$$\text{where } r \geq 0\,, j_1 < j_2 < \dots < j_r \left(< j_{r+1}\right) \quad \text{and} \quad k_1 < k_2 < \dots < k_r\ ,$$

constitute a basis of $B_0(M,f)$ (resp. $B_1(M,f)$).

Corollary 1.3. *If M is a free module of rank n, then the ranks of the free modules $B_0(M,f)$, $B_1(M,f)$ and $B(M,f)$ are respectively $\binom{2n}{n}$, $\binom{2n}{n-1}$ and $\binom{2n+1}{n}$.*

The proof of (1.3) involves the identities $\sum_{r=0}^{n}\binom{n}{r}^2 = \binom{2n}{n}$ and $\sum_{r=0}^{n-1}\binom{n}{r}\binom{n}{r+1} = \binom{2n}{n-1}$.
The terms $\binom{n}{r}^2$ and $\binom{n}{r}\binom{n}{r+1}$ are the ranks of the $\mathbb{Z}$-homogeneous components $B^{2r}(M)$ and $B^{2r+1}(M)$ of $B(M)$.

When several basic rings are involved the more precise notation $B_K(M,f)$ may become necessary. If $K \to K'$ is a ring morphism, with the K-module M we can associate the K'-extension $M' = K' \otimes_K M$, and with f its K'-extension $f' : M' \times M' \to K'$. By a classical argument we can prove the bijectiveness of the natural algebra morphism $K' \otimes_K B_K(M,f) \to B_{K'}(M',f')$. Because of this property, all concerned physicists have started their inquiry with the field extension $\mathbb{R} \to \mathbb{C}$, so that the bilinear module (M,f) under consideration admits orthonormal bases. This property also allows us to study $B(M,f)$ by means of localizations when K is not a field, and M not a free module, since every localization of a finitely generated projective module is free. Thus we reach the next corollary.

Corollary 1.4. *If M is a finitely generated projective module of constant rank n, then $B(M,f)$ too is a finitely generated projective module; the ranks of $B_0(M,f)$, $B_1(M,f)$ and $B(M,f)$ are constant and equal to the values given in the previous corollary. Moreover the PBW-property holds true.*

Obviously the quotient mapping $\mathrm{T}(M) \to \mathrm{B}(M)$ induces a bijective mapping $\mathrm{T}^{\leq 2}(M) \to \mathrm{B}^{\leq 2}(M)$; consequently, when the PBW-property holds true, the canonical mappings $K \to \mathrm{B}_0(M,f)$ and $M \to \mathrm{B}_1(M,f)$ are injective; thus their injectiveness is ensured for all modules here under consideration. Besides, another paper (see [H]) proves that the PBW-property holds for *all* meson algebras. It is worth recalling that it does not hold for all Clifford algebras.

The universal property of meson algebras allows us to define a functor B from the category of all K-bilinear modules like (M,f) to the category of all graded K-algebras. A morphism $\varphi : (M,f) \to (M'',f'')$ is a linear mapping $\varphi : M \to M''$ such that $f''(\varphi(a),\varphi(b)) = f(a,b)$ for all $a,\, b \in M$. Such a morphism φ extends to a graded algebra morphism $\mathrm{B}(\varphi) : \mathrm{B}(M,f) \to \mathrm{B}(M'',f'')$. It is clear that $\mathrm{B}(\varphi)$ is surjective when φ is surjective. But it is not always true that $\mathrm{B}(\varphi)$ is injective when φ is injective; because of Theorem (1.1) this assertion is true at least in these two cases: if K is a field, or if M'' is a finitely generated projective module, and $\varphi(M)$ a direct summand of M''.

After this purely mathematical presentation, let us recall how meson algebras were treated in the physicist's works. They always started with a vector space of finite dimension n over $\mathbb{C}$, provided with an orthonormal basis (e_k) (such that $f(e_j,e_k) = \delta_{j,k}$), and it was admitted that the derived meson algebra $\mathrm{B}(M,f)$ was semi-simple. In [K1] (page 104) we read this assertion: *provided an algebra satisfies a certain regularity condition, which has been verified in the present case, the knowledge of the number of independent elements and of the number of elements commuting with all others is sufficient to determine the irreducible representations of the algebra.* This assertion certainly raised critical reactions, because Kemmer added a footnote to explain that the algebra must be "halbeinfach" (or semi-simple) according to v. d. Waerden's definition, and that Pauli found in a work of Artin a general criterion of semi-simplicity, that is the non-vanishing of some determinant. Yet in the case under consideration (when $n = 4$), it is a determinant of order 126, and consequently no serious proof of semi-simplicity has ever been supplied at that time. Thus we may both admire the work achieved by physicists at that time, and claim that it must still be improved. Kemmer gave the dimensions of all irreducible representations of $\mathrm{B}(M,f)$ for all n in [K2]. But for mathematicians the paper [L] is more satisfying; we were informed of its existence after all main results of the present paper had been discovered, and we were surprized to realize that 60 years after Littlewood we followed the same path: first, since Kemmer mentioned the dimension of the center of $\mathrm{B}(M,f)$ without any justification, we wanted a simple and effectice method to calculate this center; secondly the precise description of this center suggested to let $\mathrm{B}(M,f)$ act in the direct sums $\bigwedge^k(M) \oplus \bigwedge^{k-1}(M)$. It is fair to acknowledge that (when $K = \mathbb{C}$) Littlewood reached results almost equivalent to our theorems (3.1), (3.4), (7.1), (7.5). But our theorems (3.2), (3.3), (3.5), (6.6) are quite probably new.

After the physicists, some mathematicians concerned themselves with meson algebras. Jacobson met them because they were associative hulls of some Jordan algebras (see [J]); more details about his work are given at the end of §8. By the instigation of Duffin, Shimpuku wrote a great mathematical work involving meson algebras (see [Sh]); despite of noticeable advances, in his preface he acknowledges not always to respect mathematical rigour: *we sometimes suggest a general result by using an extrapolation or by inference without an exact proof. In this case the result is not given in a form of a theorem.* In [OK] the relations between meson algebras and Physics are developped again.

Our work distinguishes itself by the following four features. It emphasizes the parity grading; for instance the calculation of the center of $\mathrm{B}_0(M, f)$ in §6 is a helpful step before the calculation of the center of $\mathrm{B}(M, f)$ in §7. Like Jacobson's work, it completely ignores Matrix Calculus, but unlike Jacobson's work (motivated by Jordan algebras), it priviledges geometrical methods involving exterior algebras; the little amount of necessary Grassmann Calculus is presented in §2. It respects mathematical rigour, and accepts the most general hypotheses that are compatible with the expected results; by accepting that K is not necessarily a field, we have been led to select the most effective methods. As in the study of Clifford algebras over fields, the automorphisms of (M, f) have been related to twisted inner automorphisms of $\mathrm{B}(M, f)$; with nondegenerate meson algebras over fields, the result is even simpler than with Clifford algebras, since the algebra $\mathrm{B}_0(M, f)$ contains a group $\mathcal{G}$ exactly isomorphic to the group $\mathrm{Aut}(M, f)$.

Our main results can be classified into three subsets. The first subset (in §3 and §4) contains the main theorem, that is Theorem (3.3), which soon reveals the graded structure of nondegenerate meson algebras. The calculation of the centers of $\mathrm{B}_0(M, f)$ and $\mathrm{B}(M, f)$ follows (in §6 and §7); although this calculation was the very beginning of our research (as in Littlewood's work), it is now postponed afer the main theorem; indeed our concern about effective methods has yielded a proof of the main theorem that no longer uses the center of $\mathrm{B}(M, f)$, whereas the main theorem allows an easy and rigorous validation of the results of §6. The group $\mathcal{G}$ which is isomorphic to the group of automorphisms of (M, f) is presented in §9 when K is a field of characteristic $\neq 2$, in §10 when K is a field of characteristic 2.

In an appendix (in §11) we come back to the physical motivation of meson algebras, and we show how geometrical methods (instead of matrix calculations) can solve the problems raised by the meson wave equation.

2. Interior Multiplications with Exterior Algebras

If M' and M are two modules over K, every bilinear mapping $\theta : M' \times M \to K$ induces an interior multiplication $\bigwedge(M') \times \bigwedge(M) \to \bigwedge(M)$ on the left side, and an interior multiplication $\bigwedge(M') \times \bigwedge(M) \to \bigwedge(M')$ on the right side; interior products on the left side (resp. right side) are denoted by $u' \rfloor u$ (resp. $u' \lfloor u$). These interior multiplications are characterized by these four properties which hold for all $a' \in M'$ and all $a \in M$, for all u', $v' \in \bigwedge(M')$ and all u, $v \in \bigwedge(M)$:

(a) $1 \rfloor u = u$ and $u' \lfloor 1 = u'$;

(b) $a' \rfloor a = a' \lfloor a = \theta(a', a)$;

(c) $a' \rfloor (u \wedge v) = (a' \rfloor u) \wedge v + (-1)^{\partial u} u \wedge (a' \rfloor v)$
and $(u' \wedge v') \lfloor a = u' \wedge (v' \lfloor a) + (-1)^{\partial v'} (u' \lfloor a) \wedge v'$;

(d) $u' \rfloor (v' \rfloor u) = (u' \wedge v') \rfloor u$ and $(u' \lfloor u) \lfloor v = u' \lfloor (u \wedge v)$.

Such interior multiplications have appeared in many publications, yet sometimes with other twisting signs; the above formulas have been preferred because they respect this very simple rule: a twisting sign $\pm$ appears whenever two factors are reversed; for instance the twisting sign $(-1)^{\partial u}$ in the first formula (c) comes from the reversion of the factors a' and u, and the twisting sign $(-1)^{\partial v'}$ in the second formula (c) comes from the reversion of a and v'. Whenever we write ∂u or $\partial v'$, we silently assume u or v' to be even or odd.

It is clear that the formulas (a), (b), (c), (d) determine the interior multiplications in a unique way. To prove their existence, we can use the comultiplications $\bigwedge(M') \to \bigwedge(M') \otimes \bigwedge(M')$ and $\bigwedge(M) \to \bigwedge(M) \otimes \bigwedge(M)$ as it is done (with other twisting signs) in [B]. But there is an easier and direct proof which merely verifies that these formulas are compatible with the relations $a' \wedge a' = 0$ and $a \wedge a = 0$ for which $\bigwedge(M')$ and $\bigwedge(M)$ are universal algebras.

Let us suppose that u' and u belong respectively to $\bigwedge^j(M')$ and $\bigwedge^k(M)$. The above formulas imply that $u' \rfloor u$ and $u' \lfloor u$ belong respectively to $\bigwedge^{k-j}(M)$ and $\bigwedge^{j-k}(M')$. This means $u' \rfloor u = 0$ when $j > k$, and $u' \lfloor u = 0$ when $j < k$. When $j = k$, then $u' \rfloor u$ and $u' \lfloor u$ are the same element of $K = \bigwedge^0(M) = \bigwedge^0(M')$; if $u' = a'_k \wedge \dots a'_2 \wedge a'_1$ (with $a'_1, a'_2, \dots, a'_k \in M'$), and if $u = a_1 \wedge a_2 \wedge \dots \wedge a_k$ (with $a_1, a_2, \dots, a_k \in M$), then $u' \rfloor u = u' \lfloor u$ is the determinant of the matrix composed of all $\theta(a'_i, a_h)$ with $i, h = 1, 2, \dots, k$.

Here we only use interior multiplications in this situation: M is a finitely generated projective module of constant rank n, M' is the dual module M^*, and the bilinear mapping θ is the canonical mapping $M^* \times M \to K$ defined by $(\xi, a) \longmapsto \xi(a)$. The assumption that M has a constant rank n is not essential; if its rank were not constant, we had just to replace $\bigwedge^n(M)$ with $\bigwedge^{\max}(M)$ which is by definition the submodule of all $u \in \bigwedge(M)$ such that $a \wedge u = 0$ for all $a \in M$; indeed $\bigwedge^{\max}(M)$ is a projective module of constant rank 1 which coincides with $\bigwedge^n(M)$ when the rank of M is everywhere n ; and similarly we had to replace $\bigwedge^n(M^*)$ with $\bigwedge^{\max}(M^*)$ which can be identified with the dual module of $\bigwedge^{\max}(M)$.

Moreover we assume that there is a bilinear form $f : M \times M \to K$ which determines a linear mapping $\mathrm{d}_f : M \to M^*$ defined by $\mathrm{d}_f(a)(b) = f(a, b)$ for all $a, b \in M$. The extension of d_f to an algebra morphism $\bigwedge(M) \to \bigwedge(M^*)$ is still denoted by d_f ; we are especially interested in its restriction $\bigwedge^n(M) \to \bigwedge^n(M^*)$ because it allows us to give a structure of graded algebra to $\Omega = \bigwedge^0(M) \oplus \bigwedge^n(M)$. By definition the even subalgebra Ω_0 is $\bigwedge^0(M) \oplus 0$ (identified with K), and the odd submodule Ω_1 is $0 \oplus \bigwedge^n(M)$ (whatever the parity of n may be). By definition the product of two elements $(0,\omega)$ and $(0,\omega')$ of Ω_1 is $(\mathrm{d}_f(\omega) \rfloor \omega',\ 0)$. Since $\bigwedge^n(M)$ is a module of constant rank 1, every bilinear mapping $\eta : \bigwedge^n(M) \times \bigwedge^n(M) \to K$ satisfies the equalities

$$\eta(\omega,\omega') = \eta(\omega',\omega) \quad \text{and} \quad \eta(\omega,\omega')\,\omega'' = \eta(\omega'',\omega)\,\omega' = \eta(\omega',\omega'')\,\omega$$

which here imply that Ω is a graded commutative and associative algebra with unit element.

If M is a free module with basis $(e_1, e_2, \dots, e_n)$, then $\bigwedge^n(M)$ is the free module generated by $\omega = e_1 \wedge e_2 \wedge \dots e_n$, and

$$(0,\omega)^2 \;=\; (-1)^{n(n-1)/2} \det\big(\, f(e_i,e_j)\,\big)_{1\le i,\, j\le n} \;.$$

It follows (after localization if M is not a free module) that the multiplication in Ω determines a bijective mapping $\Omega_1 \otimes \Omega_1 \to K$ if and only if f is nondegenerate.

All this is more or less known; only the following two lemmas might require a proof.

Lemma 2.1. *For all $\varphi \in \bigwedge(M^*)$, $u \in \bigwedge(M)$ and $\omega \in \bigwedge^n(M)$ we can write*

$$(\varphi \lfloor u) \rfloor \omega \;=\; (-1)^{(1-\partial\varphi)\partial u}\, u \wedge (\varphi \rfloor \omega) \;.$$

Proof. Let U be the submodule of all $u \in \bigwedge(M)$ such that this formula holds for all φ and ω. It is clear that U contains 1, and we first prove that U contains $u \wedge u'$ whenever it contains u and u'. Indeed

$$(\varphi \lfloor (u \wedge u')) \rfloor \omega \;=\; ((\varphi \lfloor u) \lfloor u') \rfloor \omega \;=\; (-1)^{(1-\partial\varphi-\partial u)\partial u'} u' \wedge ((\varphi \lfloor u) \rfloor \omega)$$

$$=\; (-1)^{(1-\partial\varphi-\partial u)\partial u' + (1-\partial\varphi)\partial u} u' \wedge u \wedge (\varphi \rfloor \omega)$$

$$=\; (-1)^{(1-\partial\varphi)(\partial u+\partial u')} u \wedge u' \wedge (\varphi \rfloor \omega) \;.$$

Thus it remains to prove that U contains every element a of M. We prove by induction on k that $(\varphi \lfloor a) \rfloor \omega = (-1)^{k-1} a \wedge (\varphi \rfloor \omega)$ for all $\varphi \in \bigwedge^k(M^*)$. This is true for $k = 0$ because $1 \lfloor a = 0$ and $a \wedge \omega = 0$. We suppose that the induction has reached k, and we consider $\xi \wedge \varphi$ with $\xi \in M^*$ and $\varphi \in \bigwedge^k(M)$:

$$((\xi \wedge \varphi) \lfloor a) \rfloor \omega$$

$$=\; (\xi \wedge (\varphi \lfloor a)) \rfloor \omega + (-1)^k \xi(a)\, \varphi \rfloor \omega \;=\; \xi \rfloor ((\varphi \lfloor a) \rfloor \omega) + (-1)^k \xi(a)\, \varphi \rfloor \omega$$

$$=\; (-1)^{k-1} \xi \rfloor (a \wedge (\varphi \rfloor \omega)) + (-1)^k \xi(a)\, \varphi \rfloor \omega \;=\; (-1)^k\, a \wedge (\xi \rfloor (\varphi \rfloor \omega)) \;;$$

since $\xi \rfloor (\varphi \rfloor \omega) = (\xi \wedge \varphi) \rfloor \omega$, the proof is ended. □

Lemma 2.2. *If $g : M \times M \to K$ is the bilinear form defined by $g(a,b) = -f(b,a)$, the following formula holds for all $u,\ v \in \bigwedge(M)$:*

$$\mathrm{d}_f(\mathrm{d}_g(u) \rfloor v) \;=\; (-1)^{\partial u \partial v}\, \mathrm{d}_f(v) \lfloor u \;.$$

Proof. Let U be the submodule of all $u \in \bigwedge(M)$ such that this formula holds for all v. It is clear that U contains 1, and we first prove that U contains $u \wedge u'$ whenever it contains u and u'. Indeed

$$\mathrm{d}_f(\mathrm{d}_g(u \wedge u') \rfloor v) = \mathrm{d}_f(\mathrm{d}_g(u) \rfloor (\mathrm{d}_g(u') \rfloor v)) = (-1)^{\partial u(\partial u' + \partial v)} \mathrm{d}_f(\mathrm{d}_g(u') \rfloor v) \lfloor u$$

$$= (-1)^{\partial u(\partial u' + \partial v) + \partial u' \partial v} (\mathrm{d}_f(v) \lfloor u') \lfloor u = (-1)^{(\partial u + \partial u')\partial v} \mathrm{d}_f(v) \lfloor (u \wedge u') .$$

Thus it remains to prove that U contains every element a of M. We prove the equality $\mathrm{d}_f(\mathrm{d}_g(a) \rfloor v) = (-1)^k \, \mathrm{d}_f(v) \lfloor a$ by induction on k for all $v \in \bigwedge^k(M)$. This is true for $k = 0$. We suppose that the induction has reached k, and we consider $v \wedge b$ with $b \in M$ and $v \in \bigwedge^k(M)$:

$$\mathrm{d}_f(\mathrm{d}_g(a) \rfloor (v \wedge b)) = \mathrm{d}_f((\mathrm{d}_g(a) \rfloor v) \wedge \mathrm{d}_f(b) + (-1)^k g(a,b) \, \mathrm{d}_f(v)$$

$$= (-1)^k (\mathrm{d}_f(v) \lfloor a) \wedge \mathrm{d}_f(b) + (-1)^{k-1} f(b,a) \, \mathrm{d}_f(v) = (-1)^{k+1} (\mathrm{d}_f(v) \wedge \mathrm{d}_f(b)) \lfloor a;$$

since $\mathrm{d}_f(v) \wedge \mathrm{d}_f(b) = \mathrm{d}_f(v \wedge b)$, the proof is ended. □

In all following sections f is a symmetric bilinear form; therefore $g = -f$ and $\mathrm{d}_g(u) = (-1)^{\partial u} \mathrm{d}_f(u)$; thus Lemma (2.2) gives the following corollary in which we meet a twisting exponent $(1 - \partial v)\partial u$ quite similar to the twisting exponent $(1 - \partial\varphi)\partial u$ in Lemma (2.1).

Lemma 2.3. *When f is symmetric, for all u, $v \in \bigwedge(M)$ we can write*

$$\mathrm{d}_f(\mathrm{d}_f(u) \rfloor v) = (-1)^{(1-\partial v)\partial u} \, \mathrm{d}_f(v) \lfloor u .$$

3. The Main Theorem

From now on, M is a finitely generated projective module provided with a symmetric bilinear form f. The main theorem, that is Theorem (3.3), requires f to be nondegenerate, and the equivalent theorems (3.4) and (3.5) too. Nevertheless in the preliminary theorems (3.1) and (3.2) the nondegeneracy of f is not necessary. In this section there is no proof of the main theorem, it is only explained why the theorems (3.3), (3.4) and (3.5) are equivalent; then the section §4 begins with the proof of Theorem (3.5), which implies the main theorem (3.3).

Our purpose is to reveal the structure of the meson algebra $\mathrm{B}(M, f)$ by means of a suitable graded representation of $\mathrm{B}(M, f)$ in some graded module $E = E_0 \oplus E_1$; such a representation follows from every linear mapping $a \longmapsto F_a$ which associates an odd endomorphism F_a (such that $F_a(E_i) \subset E_{1-i}$ for $i = 0, 1$) with every $a \in M$, in such a way that $F_a F_b F_a = f(a,b) \, F_a$ for all a, $b \in M$. The following module E and operators F_a have been suggested by the calculation of the primitive central idempotents, which is here presented in §6 although this calculation has been performed before the developments that are exposed in this section.

The module E (or $\mathrm{E}(M)$ to be more precise) is $\bigwedge(M) \oplus \bigwedge(M)$, and its homogeneous components are $E_0 = \bigwedge(M) \oplus 0$ and $E_1 = 0 \oplus \bigwedge(M)$; thus $\bigwedge^k(M) \oplus 0$

and $0 \oplus \bigwedge^k(M)$ are respectively even and odd whatever the parity of k may be. For every $a \in M$, and for all $u, v \in \bigwedge(M)$, we set

$$F_a(u,v) \;=\; (\, a \wedge v\,,\; \mathrm{d}_f(a) \rfloor u\,)\;.$$

Let us calculate

$$F_aF_bF_a(u,v) \;=\; \big(\, a \wedge (\mathrm{d}_f(b) \rfloor (a \wedge v))\,,\; \mathrm{d}_f(a) \rfloor (b \wedge (\mathrm{d}_f(a) \rfloor u))\,\big)$$

$$\text{where } \mathrm{d}_f(b) \rfloor (a \wedge v) \;=\; f(b,a)\, v \,-\, a \wedge (\mathrm{d}_f(b) \rfloor v)\;,$$

$$\mathrm{d}_f(a) \rfloor (b \wedge (\mathrm{d}_f(a) \rfloor u)) \;=\; f(a,b)\, \mathrm{d}_f(a) \rfloor u \,-\, b \wedge (\mathrm{d}_f(a) \rfloor (\mathrm{d}_f(a) \rfloor u))\;;$$

since $a \wedge a = 0$ and $\mathrm{d}_f(a) \wedge \mathrm{d}_f(a) = 0$, we reach the awaited result $F_aF_bF_a = f(a,b)\, F_a$.

Theorem 3.1. *The above mapping $a \longmapsto F_a$ extends to a graded algebra morphism $x \longmapsto F_x$ from the meson algebra $\mathrm{B}(M,f)$ into $\mathrm{End}(E)$.*

Now we set $E^k = \bigwedge^k(M) \oplus \bigwedge^{k-1}(M)$ for every integer k; it is clear that $F_a(E^k) \subset E^k$, and consequently $F_x(E^k) \subset E^k$ for every $x \in \mathrm{B}(M,f)$. To get an easy description of this $\mathrm{B}(M,f)$-module E, we suppose that M has a constant rank n; thus $E^k \neq 0$ only if $0 \le k \le n+1$; the component $E^0 = K \oplus 0$ contains only even elements, and the component $E^{n+1} = 0 \oplus \bigwedge^n(M)$ only odd elements. The other nontrivial components E^k contain both an even component $E^k_0 \cong \bigwedge^k(M)$ and an odd component $E^k_1 \cong \bigwedge^{k-1}(M)$.

It is impossible that the image of the algebra morphism $\mathrm{B}(M,f) \to \mathrm{End}(E)$ be the subalgebra of all endomorphisms of E leaving invariant all components E^k; indeed the rank of $\mathrm{B}(M,f)$ is $\binom{2n+1}{n}$, whereas the rank of this subalgebra of $\mathrm{End}(E)$ is

$$\sum_k \big(\tbinom{n}{k} + \tbinom{n}{k-1} \big)^2 \;=\; \sum_k \tbinom{n+1}{k}^2 \;=\; \tbinom{2n+2}{n+1} \;=\; 2\tbinom{2n+1}{n}\;.$$

The fact that we get twice the rank of $\mathrm{B}(M,f)$, and the examination of the primitive central idempotents (now postponed until §6) suggest the intervention of the commutative algebra $\Omega = \bigwedge^0(M) \oplus \bigwedge^n(M)$ already defined in §2. All people well acquainted with Clifford algebras know that Ω acts in the module $\bigwedge(M)$ at least when 2 is invertible in K; yet the invertibility of 2 is not necessary, and moreover we need a *graded* action of Ω in E. Therefore we consider the multiplication $\Omega \times E \to E$ defined by

$$(\lambda,\omega)\,(u,v) \;=\; (\lambda u + \mathrm{d}_f(v) \rfloor \omega\,,\; \lambda v + \mathrm{d}_f(u) \rfloor \omega\,)$$

for all $\lambda \in K$, all $\omega \in \bigwedge^n(M)$, and all $u, v \in \bigwedge(M)$. Observe that $(0,\omega)$ maps $\bigwedge^k(M) \oplus \bigwedge^j(M)$ into $\bigwedge^{n-j}(M) \oplus \bigwedge^{n-k}(M)$, and consequently E^k into E^{n+1-k}; this agrees with the fact that in Lemma (6.3) a central idempotent $\varepsilon_k + \varepsilon_{n+1-k}$ is associated with every k such that $k \neq n+1-k$. When $k = n+1-k$, (or equivalently when $n = 2k-1$), we get a multiplication $\Omega \times E^k \to E^k$.

Theorem 3.2. *The above multiplication $\Omega \times E \to E$ turns E into a graded Ω-module, and for every $x \in \mathrm{B}(M,f)$ the operation F_x of x in E is Ω-linear.*

Proof. Let ω and ω' be elements of $\bigwedge^n(M)$; to prove that the algebra Ω acts in E, it suffices to verify that the composition of the operations of $(0,\omega)$ and $(0,\omega')$ is the multiplication by the element $\mathrm{d}_f(\omega') \rfloor \omega$ of K. In other words we must verify that $\mathrm{d}_f(\mathrm{d}_f(u) \rfloor \omega') \rfloor \omega$ is equal to $(\mathrm{d}_f(\omega') \rfloor \omega)\, u$ for all $u \in \bigwedge(M)$. This follows from the lemmas (2.3) and (2.1):

$$\mathrm{d}_f(\mathrm{d}_f(u) \rfloor \omega') \rfloor \omega \;=\; (-1)^{(n-1)\partial u}(\mathrm{d}_f(\omega') \lfloor u) \rfloor \omega \;=\; u \wedge (\mathrm{d}_f(\omega') \rfloor \omega)\ .$$

To prove that all F_x are Ω-linear, it suffices to prove that, for every $a \in M$, F_a commutes with the operation of every $(0,\omega) \in \Omega_1$; this means that, for all u, $v \in \bigwedge(M)$,

$$(\, a \wedge (\mathrm{d}_f(u) \rfloor \omega),\ \mathrm{d}_f(a) \rfloor (\mathrm{d}_f(v) \rfloor \omega)\,) \;=\; (\, \mathrm{d}_f(\mathrm{d}_f(a) \rfloor u) \rfloor \omega\,,\ \mathrm{d}_f(a \wedge v) \rfloor \omega\,)\ .$$

Thus we must verify two equalities, one involving u alone, and another involving v alone; the latter is trivial, and the former again follows from the lemmas (2.3) and (2.1):

$$\mathrm{d}_f(\mathrm{d}_f(a) \rfloor u) \rfloor \omega \;=\; (-1)^{1-\partial u}(\mathrm{d}_f(u) \lfloor a) \rfloor \omega \;=\; a \wedge (\mathrm{d}_f(u) \rfloor \omega). \qquad \square$$

Now we can state the main theorem.

Theorem 3.3. *When f is nondegenerate, the algebra morphism $\mathrm{B}(M,f) \to \mathrm{End}(E)$ induces an isomorphism from $\mathrm{B}(M,f)$ onto the subalgebra of all Ω-linear endomorphisms of E that leave E^k invariant for $k = 0,1,2,\ldots,n+1$.*

This theorem is proved in the next section; here we only prove that it is equivalent to the following two theorems, in which f is still assumed to be nondegenerate.

Theorem 3.4. *When n is even (in other words, when $n = 2m$), by restricting all F_x to the direct sum of E^0, E^1, …, E^m, we get a graded algebra isomorphism*

$$\mathrm{B}(M,f) \longrightarrow \prod_{k=0}^{m} \mathrm{End}(E^k)\ .$$

When n is odd (in other words, when $n = 2m-1$), by restricting all F_x to $E^0 \oplus E^1 \oplus \ldots \oplus E^m$, we get a graded algebra isomorphism

$$\mathrm{B}(M,f) \longrightarrow \Big(\prod_{k=0}^{m-1} \mathrm{End}(E^k)\Big) \times \mathrm{End}_\Omega(E^m)\ ;$$

moreover the canonical mapping $\Omega \otimes E_0^m \to E^m$ is bijective and induces a graded algebra isomorphism $\Omega \otimes \mathrm{End}(E_0^m) \to \mathrm{End}_\Omega(E^m)$.

Proof of (3.3) $\Leftrightarrow$(3.4). When f is nondegenerate, the multiplication in Ω determines a bijective mapping $\Omega_1 \otimes \Omega_1 \to K$; in other words, there is a finite sequence $(\mu_1,\nu_1,\mu_2,\nu_2,\ldots,\mu_\ell,\nu_\ell)$ of elements of Ω_1 such that $\sum_{j=1}^{\ell} \mu_j\nu_j = 1$; here ℓ may be the minimal number of elements spanning $\bigwedge^n(M)$ (whence $\ell = 1$ if it is a free module). The action of an element of Ω_1 in E maps each component E^k into E^{n-k+1}; consequently E^m is an Ω-module if $n = 2m-1$. Let B be the subalgebra of all Ω-linear endomorphisms of E that leave invariant all components E^k (with $k = 0,1,\dots,n+1$), and let B' be the algebra of all endomorphisms of $E^0 \oplus E^1 \oplus \dots \oplus E^m$ that leave invariant all E^k (with $k = 0,1,\dots,m$), and that are Ω-linear on E^m when $n = 2m-1$. We must prove that by restriction we get an isomorphism $B \to B'$. If φ' is the restriction of an element φ of B, then for every $w \in E^{m+1} \oplus E^{m+2} \oplus \dots \oplus E^{n+1}$ we can write

$$\varphi(w) \;=\; \sum_j \mu_j \nu_j\, \varphi(w) \;=\; \sum_j \mu_j\, \varphi'(\nu_j w) \;;$$

this proves that φ is determined by its restriction φ', and that the mapping $B \to B'$ is injective. If φ' is any element of B', the equality $\varphi(w) = \sum_j \mu_j \varphi'(\nu_j w)$ is meaningful when w belongs to some E^k with $k > m$, and thus we get an endomorphism φ of E leaving each E^k invariant; it is easy to prove that φ is Ω-linear; therefore the mapping $B \to B'$ is also surjective.

When $n = 2m-1$, the canonical mapping $\Omega_1 \otimes E_0^m \to E_1^m$ (defined by $\lambda \otimes w \longmapsto \lambda w$) is bijective because there is a reciprocal mapping $w \longmapsto \sum_j \mu_j \otimes \nu_j w$; therefore the canonical mapping $\Omega \otimes E_0^m \to E^m$ is bijective. It induces an isomorphism $\Omega \otimes \mathrm{End}(E_0^m) \to \mathrm{End}_\Omega(E^m)$ because the ring extension $K \to \Omega$ is faithfully flat; as a graded algebra, $\mathrm{End}(E_0^m)$ is isomorphic to $\mathrm{End}(\bigwedge^m(M))$ provided with the trivial grading (for which all elements are even). □

Theorem 3.5. *By restricting all F_x associated with an even x to the even component E_0, we get an algebra isomorphism*

$$\mathrm{B}_0(M,f) \longrightarrow \prod_{k=0}^{n} \mathrm{End}(\bigwedge^{k}(M)) \;.$$

And by considering the mappings $E_0 \to E_1$ induced by all F_x associated with an odd x, we get a linear bijection

$$\mathrm{B}_1(M,f) \longrightarrow \prod_{k=1}^{n} \mathrm{Hom}(\bigwedge^{k}(M),\ \bigwedge^{k-1}(M)) \;.$$

Proof of (3.3)⇔(3.5). Let B_0 and B_1 be the even and odd components of the algebra B defined in the previous proof; let C_0 be the algebra of all endomorphisms of E_0 that leave invariant each component E_0^k (with $k = 0,1,\dots,n$ since $E_0^{n+1} = 0$), and C_1 the submodule of all linear mappings $E_0 \to E_1$ that map each E_0^k into E_1^k. We must prove that by restriction to E_0 (at the source) we get two bijections $B_0 \to C_0$ and $B_1 \to C_1$. We write $1 = \sum_j \mu_j \nu_j$ as in the previous proof. If φ is an element of B_0 or B_1, and if ψ is its restriction to E_0, for every $w \in E_1$ we can write

$$\varphi(w) \;=\; \sum_j \mu_j \nu_j\, \varphi(w) \;=\; \sum_j \mu_j\, \psi(\nu_j w) \;;$$

this proves that φ is determined by ψ and that both mappings $B_i \to C_i$ are injective. They are also surjective; indeed when ψ belongs to C_0 or C_1, the formula $\varphi(w) = \sum_j \mu_j \psi(\nu_j w)$ is meaningful for every $w \in E_1$, and thus from ψ we derive an endomorphism φ of E leaving each E^k invariant; it is easy to prove that φ is Ω-linear, and the conclusion follows. □

4. Proof of the Main Theorem, and Complements

We actually prove the theorem (3.5) which is equivalent to (3.3).

Proof of (3.5). It states the bijectiveness of two mappings $\mathrm{B}_i(M,f) \to C_i$ (with $i = 0, 1$); we can identify C_0 with the algebra of all endomorphisms of $\bigwedge(M)$ leaving invariant each component $\bigwedge^k(M)$ (with $k = 0, 1, \dots, n$), and C_1 with the submodule of all endomorphisms of $\bigwedge(M)$ that map each $\bigwedge^k(M)$ into $\bigwedge^{k-1}(M)$. Each target C_i is a finitely generated projective module which has the same rank as $\mathrm{B}_i(M,f)$; this rank is $\sum_{k=0}^n \binom{n}{k}^2 = \binom{2n}{n}$ if $i = 0$, and $\sum_{k=1}^n \binom{n}{k}\binom{n}{k-1} = \binom{2n}{n-1}$ if $i = 1$. Therefore it suffices to prove that both mappings $\mathrm{B}_i(M,f) \to C_i$ are surjective. Indeed if they are surjective onto projective targets, their kernels are direct summands, the ranks of these kernels vanish, and the injectiveness of these mappings follows.

Now we consider the module $M^* \oplus M$ provided with its usual hyperbolic quadratic form $(\xi, a) \longmapsto \xi(a)$, and its Clifford algebra $\mathrm{C}\ell(M^* \oplus M)$. It is well known that there is a graded algebra isomorphism $\mathrm{C}\ell(M^* \oplus M) \to \mathrm{End}(\bigwedge(M))$ that maps every $a \in M$ to the exterior multiplication $u \longmapsto a \wedge u$, and every $\xi \in M^*$ to the interior multiplication $u \longmapsto \xi \rfloor u$. Here an element of $\bigwedge^k(M)$ has the parity of k.

Consequently if an element x of $\mathrm{B}_0(M,f)$ is a product $a_1 b_1 a_2 b_2 \dots a_r b_r$ of $2r$ elements of M, by restricting F_x to E_0 we get the endomorphism of $\bigwedge(M)$ that is also the image of the following element of $\mathrm{C}\ell_0(M^* \oplus M)$: $a_1\, \mathrm{d}_f(b_1)\, a_2\, \mathrm{d}_f(b_2) \dots a_r\, \mathrm{d}_f(b_r)$. Similarly if an element y of $\mathrm{B}_1(M,f)$ is a product $b_0 a_1 b_1 a_2 \dots a_r b_r$ of $2r+1$ elements of M, the mapping $E_0 \to E_1$ induced by F_y gives the endomorphism of $\bigwedge(M)$ that is also the image of this Clifford product:
$\mathrm{d}_f(b_0)\, a_1\, \mathrm{d}_f(b_1)\, a_2 \dots a_r\, \mathrm{d}_f(b_r)$.

Thus we have reduced the problem to the proof of the following statement in which $\mathrm{B}(M,f)$ is no longer involved: by the isomorphism $\mathrm{C}\ell(M^* \oplus M) \to \mathrm{End}(\bigwedge(M)$, the subalgebra C_0 is the image of the submodule C_0'' spanned by all Clifford products $a_1\xi_1 a_2 \xi_2 \dots a_r \xi_r$ (with $r \geq 0$), and the submodule C_1 is the image of the submodule C_1'' spanned by all Clifford products $\xi_0 a_1 \xi_1 a_2 \dots a_r \xi_r$ (with $r \geq 0$).

As a module, $\mathrm{C}\ell(M^* \oplus M)$ is spanned by products of elements of M^* and M. If in such a product the numbers of factors in M and M^* are respectively h and j, the associated endomorphism of $\bigwedge(M)$ maps each component $\bigwedge^k(M)$ into $\bigwedge^{k+h-j}(M)$; when this endomorphism does not vanish, it belongs to C_0 (resp. C_1) if and only if $j = h$ (resp. $j = h + 1$). Thus we must prove that the submodule

A_{2h} (resp. A_{2h+1}) spanned by all products of h elements of M and h (resp. $h+1$) elements of M^* is contained in C_0'' (resp. C_1'').

This can be proved by induction on h. It is trivial if $h = 0$. Let us suppose that all factors $c_1, c_2, \ldots, c_k$ (with $k = 2h$ or $k = 2h+1$) belong to M or M^* and that $c_1c_2\ldots c_k$ belongs to A_k ; if s is any permutation of $\{1, 2, \ldots, k\}$, and if $(-1)^s$ is its signature, then

$$c_1c_2\ldots c_k \ - (-1)^s\, c_{s(1)}c_{s(2)}\ldots c_{s(k)} \in A_{k-2} \ ;$$

it suffices to prove this property when s is the permutation of two consecutive factors; when one factor is an element a of M, and the other an element ξ of M^*, we remember that the equality $a\xi + \xi a = \xi(a)$ holds in $\mathrm{C}\ell(M^* \oplus M)$, and the announced property follows; when both factors belong to M, or both to M^*, they anticommute in $\mathrm{C}\ell(M^*\oplus M)$, and the conclusion is trivial. If we suppose that A_{k-2} is contained in C_0'' or C_1'' (according to the parity of k), by means of a suitable permutation s we realize that every element $c_1c_2\ldots c_k$ of A_k too is contained in it. This completes the proof of Theorem (3.5). □

The main theorem leads us directly to the description of the modules S over $\mathrm{B}(M, f)$ in the next section; yet Quantum Mechanics also needs bilinear forms $g : S \times S \to K$ such that $g(as, t) = \pm g(s, at)$ for all $a \in M$ and all $s, t \in S$. Such bilinear forms can be derived from the next proposition.

First we extend f to a symmetric bilinear form $\bigwedge(M) \times \bigwedge(M) \to K$; several extensions can be proposed; our choice f_ρ is defined below. For every $u \in \bigwedge(M)$ we denote by $\mathrm{Scal}(u)$ its component in $K = \bigwedge^0(M)$. The algebra $\bigwedge(M)$ is provided with a reversion ρ such that $\rho(u) = (-1)^{k(k-1)/2}u$ for all $u \in \bigwedge^k(M)$; we also write $\rho(\lambda, \omega) = (\lambda, \rho(\omega))$ for every $(\lambda, \omega) \in \Omega$. With every parity grading is associated a grade automorphism σ such that $\sigma(t) = (-1)^{\partial t}t$ for every homogeneous t; for instance $\sigma(u) = (-1)^k u$ for every $u \in \bigwedge^k(M)$. But $\sigma(u, v) = (u, -v)$ in the graded module E, and similarly $\sigma(\lambda, \omega) = (\lambda, -\omega)$ in Ω. With these notations we set, for all $u, v \in \bigwedge(M)$,

$$f_\rho(u, v) \ = \ \mathrm{Scal}(\, \mathrm{d}_f(u) \rfloor \rho(v)\,) \ .$$

Obviously $\bigwedge^j(M)$ and $\bigwedge^k(M)$ are orthogonal for f_ρ when $j \neq k$. When u and v are both in $\bigwedge^k(M)$, the equality $f_\rho(u, v) = f_\rho(v, u)$ can be derived from Lemma (2.3):

$$\begin{aligned} f_\rho(u, v) &= (-1)^{k(k-1)/2}\mathrm{d}_f(u) \rfloor v = (-1)^{k(k-1)/2}\mathrm{d}_f(\mathrm{d}_f(u) \rfloor v) \\ &= (-1)^{k(k-1)/2}(-1)^{(1-\partial v)\partial u}\mathrm{d}_f(v) \lfloor u = (-1)^{k(k-1)/2}\mathrm{d}_f(v) \rfloor u = f_\rho(v, u) \ . \end{aligned}$$

All this allows us to derive a symmetric bilinear form $f_E : E \times E \to K$ from f :

$$f_E(\, (u, v),\, (u', v')\,) \ = \ f_\rho(u, u') + f_\rho(v, v') \ .$$

Obviously f_ρ and f_E are nondegenerate if and only if f is nondegenerate; yet the next proposition does not require f to be nondegenerate.

Proposition 4.1. *For all $s,\, t \in E$ we can write $f_E(\sigma(s),\, t) = f_E(s,\, \sigma(t))$. For all $x \in \mathrm{B}(M,f)$, and for all $\mu \in \Omega$, we can also write*

$$f_E(F_x(s),\, t) \;=\; f_E(s,\, F_{\rho(x)}(t)) \quad \textit{and} \quad f_E(\sigma(F_x(s)),\, t) \;=\; f_E(\sigma(s),\, F_{\rho\sigma(x)}(t)) \;;$$

$$f_E(\mu s,\, t) \;=\; f_E(s,\, \rho(\mu)t) \quad \textit{and} \quad f_E(\sigma(\mu s),\, t) \;=\; f_E(\sigma(s),\, \rho\sigma(\mu)t) \;.$$

Proof. The first equality involving σ is trivial. For the equalities involving x, it suffices to prove that

$$f_E(F_a(u,v),\, (u',v')) \;=\; f_E((u,v),\, F_a(u',v'))$$

for all $a \in M$, all $u,\, u' \in \bigwedge^k(M)$, and all $v,\, v' \in \bigwedge^{k-1}(M)$; this can be reduced to an equality involving (u,v'), and another one involving (u',v); but since they are equivalent, it suffices to consider the first one:

$$f_\rho(a \wedge v',\; u) \;=\; f_\rho(v',\; \mathrm{d}_f(a) \rfloor u) \;;$$

it follows from a straightforward calculation:

$$\mathrm{d}_f(\rho(a \wedge v')) \rfloor u \;=\; \mathrm{d}_f(\rho(v') \wedge a) \rfloor u \;=\; \mathrm{d}_f(\rho(v')) \rfloor (\mathrm{d}_f(a) \rfloor u) \;.$$

To prove the equalities involving Ω, it suffices to prove that

$$f_\rho(\mathrm{d}_f(u) \rfloor \omega,\; v') \;=\; (-1)^{n(n-1)/2}\, f_\rho(u,\; \mathrm{d}_f(v') \rfloor \omega)$$

for all $\omega \in \bigwedge^n(M)$, all $u \in \bigwedge^k(M)$ and all $v' \in \bigwedge^{n-k}(M)$; let us notice these two equalities:

$$f_\rho(\mathrm{d}_f(u)) \rfloor \omega,\; v') \;=\; \mathrm{d}_f(\rho(v')) \rfloor (\mathrm{d}_f(u) \rfloor \omega) \;=\; \mathrm{d}_f(\rho(v') \wedge u) \rfloor \omega \;;$$

$$\rho(v') \wedge u \;=\; (-1)^{n(n-1)/2}\, \rho(\rho(v') \wedge u) \;=\; (-1)^{n(n-1)/2}\, \rho(u) \wedge v' \;;$$

they complete the proof of (4.1). □

5. Graded Modules Over $\mathrm{B}(M,f)$

We only consider modules on the left side. Since meson algebras are provided with a reversion ρ, every right module can be turned into a left module.

Let A be a graded algebra $A_0 \oplus A_1$; an A-module S provided with a parity grading $S = S_0 \oplus S_1$ is called a graded module over A if the equality $\partial(xs) = \partial x + \partial s$ holds for all homogeneous $x \in A$ and $s \in S$. An A-linear mapping $S \to S'$ is called a graded morphism (or an even morphism) if it maps S_i into S'_i for $i = 0, 1$. With every graded module S is associated a shifted module $S^\dagger$ which coincides with S as an A-module, but in which the even (resp. odd) elements are the elements of S_1 (resp. S_0); in general S and $S^\dagger$ are not isomorphic as graded modules; nevertheless if A_1 contains an invertible and central element x, the multiplication by x is always an isomorphism $S \to S^\dagger$. Often physicists prefer the word "chirality" to "parity" when they consider modules over graded algebras: the elements of S_0 (resp. S_1) are left-hand (resp. right-hand) elements; but here we maintain the word "parity". By mentioning that a shifted module $S^\dagger$ is associated with every graded module S, we well enough evince that the grading of a module is not of the same nature as the grading of an algebra.

Let us consider the mapping $A_1 \otimes A_1 \to A_0$ defined by $x \otimes y \longmapsto xy$; its image is an ideal of A_0. We say that the grading of A is *regular* if this mapping is surjective onto A_0. The importance of this concept of regularity is explained in the next theorem, which has already been applied to Clifford algebras.

Theorem 5.1. *If the algebra $A = A_0 \oplus A_1$ is provided with a regular parity grading, the category of graded A-modules is equivalent to the category of A_0-modules.*

Indeed if S is a graded A-module, S_0 is an A_0-module, and conversely with every A_0-module S_0 is associated a graded A-module $A \otimes_{A_0} S_0$, the even component of which is canonically isomorphic to S_0 as an A_0-module. Every graded morphism $S \to S'$ induces a morphism $S_0 \to S'_0$ and conversely every (A_0-linear) morphism $\varphi_0 : S_0 \to S'_0$ extends to a graded morphism $\varphi : S \to S'$; if $(\mu_1, \nu_1, \mu_2, \nu_2, \ldots, \mu_k, \nu_k)$ is a sequence of elements of A_1 such that $\sum_j \mu_j \nu_j = 1$, then, for every $s \in S_1$,

$$\varphi(s) \;=\; \sum_j \varphi(\mu_j \nu_j s) \;=\; \sum_j \mu_j\, \varphi_0(\nu_j s)\;;$$

thus we realize that φ is determined by φ_0 in a unique way; and when φ_0 is given, it is easy to prove that the graded extension φ resulting from the above equality is actually A-linear. Moreover φ is injective (resp. surjective) if and only if φ_0 is injective (resp. surjective).

Even without any assumptions on M and f, it is clear that $\mathrm{B}(M, f)$ is the direct sum of K (the subalgebra generated by its unit element 1) and the ideal $M\,\mathrm{B}(M, f) = \mathrm{B}(M, f)\,M$ generated by M; indeed $\mathrm{B}(M, f)$ is the quotient of $\mathrm{T}(M)$ by some ideal $\mathrm{J}(M, f)$ contained in the ideal $\bigoplus_{k>0} \mathrm{T}^k(M)$. Since $\mathrm{B}_1(M, f) \subset M\,\mathrm{B}(M, f)$, there is no invertible elements in $\mathrm{B}_1(M, f)$, and the grading of $\mathrm{B}(M, f)$ is never regular.

Let us suppose that P is a finitely generated and projective K-module. It is known that every module over $\mathrm{End}(P)$ is isomorphic to $P \otimes V$ for some K-module V; you must understand that $\varphi(x \otimes v) = \varphi(x) \otimes v$ for all $\varphi \in \mathrm{End}(P)$, $x \in P$ and $v \in V$. When K is a field, this means that every module over $\mathrm{End}(P)$ is a direct sum of irreducible modules all isomorphic to P. When P is a graded module $P_0 \oplus P_1$, there is a parity grading on $\mathrm{End}(P)$ such that the formula $\partial(\varphi(x)) = \partial\varphi + \partial x$ holds for all homogeneous $\varphi \in \mathrm{End}(P)$ and $x \in P$; thus $\mathrm{End}_0(P)$ is isomorphic to $\mathrm{End}(P_0) \times \mathrm{End}(P_1)$. The grading of $\mathrm{End}(P)$ is regular if and only if the ranks of P_0 and P_1 are both ≥ 1. If it is regular, from Theorem (5.1) we can deduce that every graded module over $\mathrm{End}(P)$ is isomorphic to $(P \otimes U) \oplus (P^\dagger \otimes V)$ for some pair (U, V) of trivially graded K-modules (such that $\partial u = \partial v = 0$ for all $u \in U$ and $v \in V$); we could also write $P \otimes (U \oplus V^\dagger)$. Here P and $P^\dagger$ are not isomorphic as graded modules, and the correspondence $(U, V) \longleftrightarrow (P \otimes U) \oplus (P^\dagger \otimes V)$ is an equivalence between the category of pairs of K-modules and the category of graded $\mathrm{End}(P)$-modules.

After these preliminaries we assume that f is a nondegenerate symmetric bilinear form on a finitely generated projective module M of constant rank n, with $n = 2m$ or $n = 2m-1$ according to its parity. Theorem (3.4) states that $\mathrm{B}(M,f)$ is a direct sum of $m+1$ graded ideals isomorphic to $\mathrm{End}(E^k)$ for $k = 0,1,\ldots,m$ when $n = 2m$, isomorphic to $\mathrm{End}(E^k)$ for $k = 0,1,\ldots,m-1$ and exceptionally to $\Omega \otimes \mathrm{End}(E_0^m)$ when $n = 2m-1$. Since $E^0 = K \oplus 0$, the first ideal is isomorphic to K, and its grading is trivial. But all the other ideals have regular gradings. The bijectiveness of the mapping $\Omega_1 \otimes \Omega_1 \to K$ ensures the regularity of the grading of $\Omega \otimes \mathrm{End}(E_0^m)$, and from Theorem (5.1) we deduce that every graded module over this algebra is isomorphic to $E^m \otimes V$ for some (trivially graded) K-module V. The bijectiveness of $\Omega_1 \otimes \Omega_1 \to K$ shows that the mapping $\Omega_1 \otimes E^k \to E^{n+1-k}$ defined by $\mu \otimes w \longmapsto \mu w$ is bijective; since the action of $\mathrm{B}(M,f)$ in E is Ω-linear, it follows that the graded module $(E^k)^\dagger \otimes \bigwedge^n(M)$ (with a trivially graded factor $\bigwedge^n(M)$) is isomorphic to E^{n+1-k}. When $\bigwedge^n(M)$ is a free module, in particular when K is a field (or a local ring), the graded modules $(E^k)^\dagger$ and E^{n+1-k} are isomorphic; yet E^k and E^{n+1-k} are never isomorphic when $n+1-k \neq k$.

Only one of the $m+1$ ideals mentioned above is not provided with a regular grading; let ε be the projection of 1 in this ideal; this means that F_ε operates on $E^0 \oplus E^{n+1}$ as the identity, but annihilates all E^k with $1 \le k \le n$. This idempotent ε coincides with the idempotent ε_0 that shall be calculated in the next section when M is a free module,

Lemma 5.2. *Among the idempotents of* $\mathrm{B}(M,f)$*, ε is characterized by these two properties: first $\varepsilon a = a\varepsilon = 0$ for all $a \in M$; secondly every equality $\lambda\varepsilon = 0$ with $\lambda \in K$ implies $\lambda = 0$. The complementary idempotent $1-\varepsilon$ generates the ideal $M\,\mathrm{B}(M,f)$; therefore this ideal can be treated as an algebra with unit element $1-\varepsilon$, provided with a regular grading. Every module S over $\mathrm{B}(M,f)$ is the direct sum of the submodules εS and $(1-\varepsilon)S$; εS is the subset of all $s \in S$ such that $as = 0$ for all $a \in M$, and $(1-\varepsilon)S$ is the K-submodule spanned by all as .*

Proof. It is clear that F_a (with $a \in M$) annihilates $E^0 \oplus E^{n+1}$; consequently $F_{a\varepsilon} = F_{\varepsilon a} = 0$, whence $a\varepsilon = \varepsilon a = 0$. It is also clear that the mapping $\lambda \longmapsto \lambda\varepsilon$ is a bijection from K onto the ideal generated by ε. Conversely let ε' be an idempotent such that $a\varepsilon' = \varepsilon' a = 0$ for all $a \in M$. Since $(1-\varepsilon')a = a$, it follows that $M \subset (1-\varepsilon')\mathrm{B}(M,f)$. Since $\mathrm{B}(M,f) = K \oplus M\,\mathrm{B}(M,f)$, we can write $1-\varepsilon' = \lambda + x$ with $\lambda \in K$ and $x \in M\,\mathrm{B}(M,f)$; the equalities $\varepsilon'(1-\varepsilon') = 0$ and $\varepsilon' M = 0$ imply $\lambda\varepsilon' = 0$, and if this last equality implies $\lambda = 0$, we realize that $1-\varepsilon' \in M\,\mathrm{B}(M,f)$. We conclude that $M\,\mathrm{B}(M,f) = (1-\varepsilon')\mathrm{B}(M,f)$. Now the equality $\varepsilon = \varepsilon'$ follows from the fact that $1-\varepsilon$ and $1-\varepsilon'$ are both equal to the unit element of the algebra $M\,\mathrm{B}(M,f)$. The remainder of (5.2) is evident. □

It is clear that every graded module S over $\mathrm{B}(M,f)$ is a direct sum of $m+1$ graded modules over the $m+1$ ideals mentioned above. We already know the modules over the m regularly graded ideals. A graded module over the ideal $K\varepsilon$ is merely a graded K-module; since E^0 (resp. E^{n+1}) is a module of constant rank 1 in which all elements are even (resp. odd), a graded module over $K\varepsilon$ is isomorphic to $(E^0 \otimes U) \oplus (E^{n+1} \otimes V)$ for some pair (U, V) of trivially graded K-modules, which is determined by this graded $K\varepsilon$-module up to isomorphy. Thus we have proved this theorem.

Theorem 5.3. *Every graded module S over $\mathrm{B}(M, f)$ is isomorphic to*

$$\bigoplus_{k=0}^{n+1} (E^k \otimes V_k)$$

for some sequence $(V_0, V_1, \ldots, V_{n+1})$ of trivially graded K-modules. The correspondence $(V_0, V_1, \ldots, V_{n+1}) \longleftrightarrow S$ is an equivalence of categories.

When K is field, this means that every graded module over $\mathrm{B}(M, f)$ is a direct sum of graded irreducible submodules, which are all isomorphic to one (and only one) of the $n+2$ graded modules E^k. In particular $\mathrm{B}(M, f)$ is a graded semi-simple algebra. Nevertheless $\mathrm{B}(M, f)$ is not a semi-simple algebra (in the non-graded sense) when these three conditions are fulfilled: K is a field of characteristic 2, the dimension n is odd, and the determinant of f is any basis of M has a square root in K ; thus Ω_1 contains an element μ such that $\mu^2 = 1$, whence $(1 + \mu)^2 = 0$; this prevents the ideal isomorphic to $\Omega \otimes \mathrm{End}(E_0^m)$ from being semi-simple. Nevertheless since $1 + \mu$ is neither even nor odd, this is not an obstruction to graded semi-simplicity.

Although our methods lead us naturally to graded modules, we can also tackle the description of non-graded modules over $\mathrm{B}(M, f)$. If $n = 2m$, every module is isomorphic to a direct sum $\bigoplus_{k=0}^{m} (E^k \otimes V_k)$ of only $m+1$ direct summands. When $n = 2m - 1$, the last summand (with $k = m$) must be replaced with $E_0^m \otimes W_m$ where W_m is a module over Ω. All these summands can receive a parity grading, except $E_0^m \otimes W_m$ which admits a parity grading if and only if W_m admits one as an Ω-module.

When K is the field $\mathbb{C}$ of complex numbers, Ω is isomorphic to $\mathbb{C} \times \mathbb{C}$ if we forget its grading; consequently there are two isomorphy classes of irreducible modules over the ideal isomorphic to $\Omega \otimes \mathrm{End}(E_0^m)$; these exceptional modules (which cannot be graded, and which only exist when n is odd) were discovered by Schrödinger before Kemmer managed to construct the ordinary irreducible modules.

In the next proposition, K can be again an arbitrary ring.

Proposition 5.4. *When $1 \leq k \leq n$, the algebra morphism $\mathrm{B}(M, f) \to \mathrm{End}(E^k)$ induces an injective mapping $M \to \mathrm{End}(E^k)$.*

Indeed for every $v \in \bigwedge^{k-1}(M)$ we can write $F_a(0, v) = (a \wedge v,\, 0)$; when $a \neq 0$, by means of a basis of M (after localization if necessary) we can prove the existence of some $v \in \bigwedge^{k-1}(M)$ such that $a \wedge v \neq 0$. □

6. The Center of $\mathrm{B}_0(M, f)$

The purpose of the sections §6 and §7 is to calculate the centers of the algebras $\mathrm{B}_0(M, f)$ and $\mathrm{B}(M, f)$ when M is a free module and f a nondegenerate symmetric bilinear form. By localization we can always reduce the problem to the case of a free module, and even to the case of a module provided with orthogonal bases if 2 is invertible, or if n is odd. Nevertheless we begin with an arbitrary basis $(e_1, e_2, \ldots, e_n)$ of M, and we assume it to be orthogonal only at the end (in §7), when we look for the odd component of the center of $\mathrm{B}(M, f)$, since this odd component is $\neq 0$ only if n is odd. The nondegeneracy of f is equivalent to the existence of an *adjoint basis* $(e'_1, e'_2, \ldots, e'_n)$ such that $f(e_i, e'_i) = 1$ for $i = 1, 2, \ldots, n$, and $f(e_i, e'_j) = 0$ whenever $i \neq j$.

Lemma 6.1. *The n products $e_i e'_i$ are idempotents which commute with one another; similarly the n products $e'_i e_i$ are pairwise commuting idempotents. These idempotents (and the complementary idempotents $1 - e_i e'_i$ and $1 - e'_i e_i$) satisfy all these properties:*

$$\text{(a)} \quad e_i\,(e'_i e_i) = (e_i e'_i)\,e_i = e_i\ , \quad e_i\,(1 - e'_i e_i) = (1 - e_i e'_i)\,e_i = 0\ ,$$

$$e'_i\,(e_i e'_i) = (e'_i e_i)\,e'_i = e'_i\ , \quad e'_i\,(1 - e_i e'_i) = (1 - e'_i e_i)\,e'_i = 0\ ,$$

and when $i \neq j$,

$$\text{(b)} \quad e_i\,(e'_j e_j) = (1 - e_j e'_j)\,e_i\ , \quad e_i\,(1 - e'_j e_j) = (e_j e'_j)\,e_i\ ,$$

$$e'_i\,(e_j e'_j) = (1 - e'_j e_j)\,e'_i\ , \quad e'_i\,(1 - e_j e'_j) = (e'_j e_j)\,e'_i\ .$$

Proof. The formulas (a) are direct consequences of $aba = f(a, b)\,a$; they imply that each $e_i e'_i$ or $e'_i e_i$ is an idempotent. The formulas (b) are direct consequences of $abc + cba = f(a, b)c + f(c, b)a$; they imply that the n idempotents $e_i e'_i$ are pairwise commuting:

$$e_i e'_i\ e_j e'_j = e_i\,(1 - e'_j e_j)\,e'_i = e_j e'_j\ e_i e'_i\ ;$$

similarly the n idempotents $e'_i e_i$ commute with one another. □

It follows from Lemma (6.1) that the n idempotents $e_i e'_i$ (resp. $e'_i e_i$) generate a commutative subalgebra $\mathcal{C}$ (resp. $\mathcal{C}'$) in $\mathrm{B}_0(M, f)$; by our definition of the word "subalgebra", the unit element 1 belongs to $\mathcal{C}$ and $\mathcal{C}'$. If the basis $(e_1, \ldots, e_n)$ is orthogonal, then $e'_i = e_i\,f(e_i, e_i)^{-1}$ and consequently $\mathcal{C} = \mathcal{C}'$. When n is even and f is hyperbolic, we can use a hyperbolic basis $(e_1, \ldots, e_n)$ such that $f(e_i, e_{n+1-i}) = 1$ for $i = 1, 2, \ldots, n$, and $f(e_i, e_j) = 0$ if $i + j \neq n + 1$; then $e'_i = e_{n+1-i}$ and again $\mathcal{C} = \mathcal{C}'$. But in general $\mathcal{C} \neq \mathcal{C}'$.

Let us set $N = \{1, 2, 3, \ldots, n\}$; with every subset P of N we associate an idempotent $\varepsilon(P)$ in $\mathcal{C}$ and an idempotent $\varepsilon'(P)$ in $\mathcal{C}'$:

$$\varepsilon(P) = \prod_{i \in P} e_i e'_i \prod_{j \notin P} (1 - e_j e'_j)\ , \qquad \varepsilon'(P) = \prod_{i \in P} e'_i e_i \prod_{j \notin P} (1 - e'_j e_j).$$

Lemma 6.2. *The subalgebra $\mathcal{C}$ (resp. $\mathcal{C}'$) is a free module of rank 2^n which admits the family of all idempotents $\varepsilon(P)$ (resp. $\varepsilon'(P)$) as a basis. If P and Q are different subsets of N, then*

$$\varepsilon(P)\,\varepsilon(Q) \;=\; \varepsilon'(P)\,\varepsilon'(Q) \;=\; 0 \;.$$

Moreover when $i \notin P$, then

$$e_i\,\varepsilon'(P) \;=\; \varepsilon(P)\,e_i \;=\; e'_i\,\varepsilon(P) \;=\; \varepsilon'(P)\,e'_i \;=\; 0 \;;$$

and when $i \in P$, then we consider $Q = (N \setminus P) \cup \{i\}$ (whence conversely $P = (N \setminus Q) \cup \{i\}$) and thus we can write these commutation formulas:

$$e_i\,\varepsilon'(P) \;=\; \varepsilon(Q)\,e_i \qquad \varepsilon(P)\,e_i \;=\; e_i\,\varepsilon'(Q) \;,$$
$$e'_i\,\varepsilon(P) \;=\; \varepsilon'(Q)\,e'_i \qquad \varepsilon'(P)\,e'_i \;=\; e'_i\,\varepsilon(Q) \;.$$

Proof. For each subset P of N, let $\alpha(P)$ be the product of all $e_ie'_i$ with $i \in P$; in particular $\alpha(\emptyset) = 1$ and $\alpha(N) = \varepsilon(N)$; it is clear that $\mathcal{C}$ is spanned (as a submodule) by the 2^n idempotents $\alpha(P)$. Because of Corollary (1.2) the elements $\alpha(P)$ belong to a basis of $\mathrm{B}(M, f)$, and consequently they constitute a basis of $\mathcal{C}$. Each $\varepsilon(P)$ is the sum of $\alpha(P)$ and other terms $\pm\alpha(Q)$ in which Q contains P strictly; this implies that the 2^n elements $\varepsilon(P)$ also constitute a basis of $\mathcal{C}$. Similarly the idempotents $\varepsilon'(P)$ constitute a basis of $\mathcal{C}'$. When $P \neq Q$, there is an integer $i \in N$ that belongs either to P or to Q, but not to P and Q, and the vanishing of $\varepsilon(P)\varepsilon(Q)$ follows from $e_ie'_i(1 - e_ie'_i) = 0$; similarly $\varepsilon'(P)\varepsilon'(Q) = 0$. The other formulas mentioned in Lemma (6.2) are direct consequences of the formulas (a) and (b) of Lemma (6.1). □

For $p = 0, 1, 2, \dots, n$, we define an idempotent ε_p in $\mathcal{C}$ and an idempotent ε'_p in $\mathcal{C}'$:

$$\varepsilon_p \;=\; \sum_{\mathrm{card}(P)=p} \varepsilon(P) \;, \qquad \varepsilon'_p \;=\; \sum_{\mathrm{card}(P)=p} \varepsilon'(P) \;;$$

it is also convenient to set $\varepsilon_{n+1} = \varepsilon'_{n+1} = 0$.

Lemma 6.3. *The idempotents ε_p and ε'_p belong to the center of* $\mathrm{B}_0(M, f)$*, and for all $a \in M$,*

$$a\,\varepsilon'_p \;=\; \varepsilon_{n+1-p}\,a \;, \qquad \varepsilon_p\,a \;=\; a\,\varepsilon'_{n+1-p} \;,$$
$$a\,\varepsilon_p \;=\; \varepsilon'_{n+1-p}\,a \;, \qquad \varepsilon'_p\,a \;=\; a\,\varepsilon_{n+1-p} \;.$$

When $p \neq n+1-p$ (resp. $p = n+1-p$), the idempotent $\varepsilon_p + \varepsilon_{n+1-p}$ (resp. ε_p) belongs to the center of $\mathrm{B}(M, f)$ *provided that $\varepsilon'_p = \varepsilon_p$ and $\varepsilon'_{n+1-p} = \varepsilon_{n+1-p}$.*

When $p = 0$, this last assertion means that ε_0 belongs to the center of $\mathrm{B}(M, f)$ and that $a\varepsilon_0 = \varepsilon_0\,a = 0$; but this already follows from Lemma (6.2) since $\varepsilon_0 = \varepsilon(\emptyset)$.

Proof. Let us first derive the formula $e_i\varepsilon'_p = \varepsilon_{n+1-p}e_i$ from Lemma (6.2). Each product $e_i\varepsilon'(P)$ (with $\mathrm{card}(P) = p$) either vanishes if $i \notin P$, or is equal to some $\varepsilon(Q)e_i$ (with $\mathrm{card}(Q) = n+1-p$) if $i \in P$; and similarly $\varepsilon(Q)e_i$ either vanishes if $i \notin Q$, or is equal to some $e_i\varepsilon'(P)$ if $i \in Q$. The correspondence between the subsets P and Q can be explained in this way: $P \setminus \{i\}$ and $Q \setminus \{i\}$ are complementary subsets of $N \setminus \{i\}$; thus we get a bijection between the subsets P of N that have cardinal p and contain i, and the subsets Q that have cardinal $n+1-p$ and contain i. This proves the first announced formula, and the other three ones are quite similar. Since the algebra $B_0(M,f)$ is generated by all products $e_i e'_j$ (with $i,\ j \in N$), it follows that each ε_p belongs to its center:

$$e_i e'_j \ \varepsilon_p \ = \ e_i \, \varepsilon'_{n+1-p} \, e'_j \ = \ \varepsilon_p \ e_i e'_j \ .$$

The final assertion involving the center of $B(M,f)$ is also clear. □

The previous lemmas have suggested the theorems stated in §3, under the conjecture that these lemmas precisely revealed the center of $B_0(M,f)$ and the even component of the center of $B(M,f)$. When later the main theorem was established, it showed that this conjecture was actually true, by means of the argument that is now presented. In particular it confirmed that ε_p and ε'_p were always equal, and did not depend on the choice of the basis $(e_1, e_2, \ldots, e_n)$.

Now we continue the study of the idempotents under consideration with the help of the algebra morphism $B(M,f) \to \mathrm{End}(E)$ (or $x \longmapsto F_x$) defined in §3. For every subset P of N, we denote by $\hat{e}_P$ (resp. $\hat{e}'_P$) the exterior product of all e_i (resp. e'_i) with $i \in P$; the order of the factors is (for instance) the increasing order of the indices. Let us assume that x belongs to $\mathcal{C}$; when we calculate its operation F_x in $E_0 = \bigwedge(M) \oplus 0$, it is convenient to use the basis of $\bigwedge(M)$ composed of all products $\hat{e}_P$; and when we calculate its operation in $E_1 = 0 \oplus \bigwedge(M)$, it is convenient to use the basis composed of all products $\hat{e}'_P$. Of course if we calculated F_x with $x \in \mathcal{C}'$, we should get the same results after due modifications. The next lemma follows from a straightforward calculation that does not deserve to be written up here; and its corollary too is evident.

Lemma 6.4. *When i belongs to P (a subset of N), the operation of $e_i e'_i$ in E maps $(\hat{e}_P\,, 0)$ to itself, and $(0,\, \hat{e}'_P)$ to 0. When i does not belong to P, it maps $(\hat{e}_P\,, 0)$ to 0, and $(0,\, \hat{e}'_P)$ to itself.*

Corollary 6.5. *The operation of $\varepsilon(P)$ in E maps $(\hat{e}_P\,, 0)$ and $(0,\, \hat{e}'_{N\setminus P})$ to themselves, and annihilates all other $(\hat{e}_Q\,, 0)$ and $(0,\, \hat{e}'_Q)$. The operation of ε_p is the projection onto $E_0^p \oplus E_1^{n+1-p}$ that annihilates all E_0^k such that $k \neq p$, and all E_1^k such that $k \neq n+1-p$. When $p \neq n+1-p$ (resp. $p = n+1-p$), the operation of $\varepsilon_p + \varepsilon_{n+1-p}$ (resp. ε_p) is the projection onto $E^p \oplus E^{n+1-p}$ (resp. E^p) that annihilates all other components E^k.*

From Theorem (3.5) we know that the center of $B_0(M,f)$ is a free module that admits a basis composed of $n+1$ idempotents; by the injective morphism $x \longmapsto F_x$ these idempotents give the operators on E_0 that project E_0 onto its components E_0^k . If we compare this information with Corollary (6.5), we reach the next theorem.

Theorem 6.6. *The idempotents ε_p (with $p = 0, 1, 2, \ldots, n$) do not depend on the choice of the basis $(e_1, e_2, \ldots, e_n)$, and they constitute a basis of the center of $B_0(M,f)$.*

The reversion ρ maps each idempotent $e_i e'_i$ to $e'_i e_i$, and consequently $\rho(\varepsilon_p) = \varepsilon'_p$ for $p = 0, 1, \ldots, n$. Since the bases $(e_1, \ldots, e_n)$ and $(e'_1, \ldots, e'_n)$ give the same idempotent $\varepsilon_p = \varepsilon'_p$, we can add this corollary in which M may be more generally a finitely generated projective module.

Corollary 6.7. *The reversion ρ leaves invariant every element of the center of* $\mathrm{B}_0(M, f)$.

7. The Center of $\mathrm{B}(M, f)$

The even component of the center of $\mathrm{B}(M, f)$ immediately follows from Lemma (6.3) and Theorem (3.4); yet we must separate two cases according to the parity of n. When n is even, we write $n = 2m$, an we already know the center of $\mathrm{B}(M, f)$ since all its elements are even. When n is odd, we write $n = 2m - 1$, and we have still to calculate the odd component of the center of $\mathrm{B}(M, f)$ which is a module of constant rank 1. In all cases, ε_0 is a central idempotent that generates an ideal of dimension 1, and the supplementary ideal $(1 - \varepsilon_0)\mathrm{B}(M, f)$ is the ideal $M\,\mathrm{B}(M, f)$ generated by M.

Theorem 7.1. *When $n = 2m$, the center of* $\mathrm{B}(M, f)$ *is a free module of rank $m+1$ with basis*

$$(\varepsilon_0\,,\ \varepsilon_1 + \varepsilon_n\,,\ \varepsilon_2 + \varepsilon_{n-1}\,,\quad \ldots \quad,\ \varepsilon_m + \varepsilon_{m+1})\ .$$

When $n = 2m - 1$, the even component of the center of $\mathrm{B}(M, f)$ *is a free module of rank $m + 1$ with basis*

$$(\varepsilon_0\,,\ \varepsilon_1 + \varepsilon_n\,,\ \varepsilon_2 + \varepsilon_{n-1}\,,\quad \ldots \quad,\ \varepsilon_{m-1} + \varepsilon_{m+1}\,,\ \varepsilon_m)\ .$$

Before studying the odd central component when $n = 2m - 1$, we add some results that are valid whatever the parity of n may be. We still consider a basis $(e_1, \ldots, e_n)$ of M, the adjoint basis $(e'_1, \ldots, e'_n)$, and the associated elements $\hat{e}_N$ and $\hat{e}'_N$ of $\bigwedge^n(M)$. Let δ be the determinant of the matrix $(\,f(e_i, e_j)_{i,\,j\in N})$; it is clear that $\mathrm{d}_f(\hat{e}_N) \rfloor \rho(\hat{e}_N) = \delta$ and $\mathrm{d}_f(\hat{e}'_N) \rfloor \rho(\hat{e}_N) = 1$; we also know that $\rho(x) = (-1)^{k(k-1)/2} x$ for all $x \in \bigwedge^k(M)$, and consequently:

$$\mathrm{d}_f(\hat{e}_N) \rfloor \hat{e}_N \;=\; (-1)^{n(n-1)/2}\delta \quad \text{and} \quad \hat{e}_N \;=\; \delta\,\hat{e}'_N\ .$$

Since Ω_1 is spanned by $(0, \hat{e}_N)$ or $(0, \hat{e}'_N)$, these formulas describe the algebra Ω. It is also useful to know how these elements of Ω_1 operate in E, and their operations can be deduced from the next lemma. If P is a subset of N, if $(i_1, i_2, \ldots, i_p)$ is the increasing sequence of its elements, and $(j_1, j_2, \ldots, j_{n-p})$ the increasing sequence of the elements of $N \setminus P$, then the signature of P is the signature of the permutation $(i_1, i_2, \ldots, i_p, j_1, j_2, \ldots, j_{n-p})$. It is clear that $\mathrm{sgn}(N \setminus P) = (-1)^{p(n-p)}\mathrm{sgn}(P)$.

Lemma 7.2. *If P is a subset of N, and* $\mathrm{card}(P) = p$, *then*

$$\mathrm{d}_f(\hat{e}'_P) \rfloor \hat{e}_N \;=\; (-1)^{p(p-1)/2}\mathrm{sgn}(P)\,\hat{e}_{N\setminus P} \quad \text{and}$$

$$\mathrm{d}_f(\hat{e}_P) \rfloor \hat{e}'_N \;=\; (-1)^{p(p-1)/2}\mathrm{sgn}(P)\,\hat{e}'_{N\setminus P}\ .$$

This lemma follows from a straightforward calculation. We still recall the lemma that allows us to suppose that the basis $(e_1, \dots, e_n)$ is orthogonal when n is odd (and f always nondegenerate).

Lemma 7.3. *If K is a local ring (or a field), and if M contains an element a such that $f(a, a)$ is invertible, then M admits orthogonal bases. When n is odd, or when the image of 2 in K is invertible, such an element a always exists.*

Proof. If such an element a does not exist, then f induces a symplectic form over the residue field; consequently n is even and 2 is not invertible in K. When such an element a exists, Lemma (7.3) follows from a classical induction on n. Nevertheless when 2 is not invertible (and $n > 2$), it may happen that $f(b, b)$ is never invertible when b is orthogonal to a; in this case there are two elements b and c orthogonal to a such that $f(b, c) = -f(a, a)$, and thus $a + b$ and $a + c$ are orthogonal elements such that $f(a+b,\, a+b)$ and $f(a+c,\, a+c)$ are both invertible; therefore the induction hypothesis holds for the hyperplane orthogonal to $a + b$ (instead of a). □

Now we suppose that $n = 2m-1$ and that $(e_1, \dots, e_n)$ is an orthogonal basis of M.

If $(a_1, a_2, \dots, a_k)$ is a sequence of pairwise orthogonal elements of M, the product $a_1 a_2 \dots a_k$ is changed into the opposite element of $\mathrm{B}(M, f)$ if we permute two factors a_i and a_j such that $j - i$ is even. In particular, when k is odd, then $a_k \dots a_2 a_1 = (-1)^{(k-1)/2} a_1 a_2 \dots a_k$.

Up to the end of this section, P is a subset of N of cardinal m. If $(i_1, i_2, \dots, i_m)$ is the sequence of the elements of P, and $(j_1, j_2, \dots, j_{m-1})$ the sequence of the elements of $N \setminus P$, we set

$$\eta(P) \;=\; \mathrm{sgn}(i_1, j_1, i_2, j_2, \dots, j_{m-1}, i_m)\; e_{i_1} e_{j_1} e_{i_2} e_{j_2} \dots e_{j_{m-1}} e_{i_m} \;;$$

because of the signature of the permutation $(i_1, j_1, \dots, j_{m-1}, i_m)$, this product does not depend on the order of the elements of P or $N \setminus P$ in the sequence $(i_1, \dots, i_m)$ or $(j_1, \dots, j_{m-1})$; when these sequences are both increasing, this signature is equal to $(-1)^{m(m-1)/2}\mathrm{sgn}(P)$, if $\mathrm{sgn}(P)$ is defined as it is before (7.2).

Lemma 7.4. *When i is an element of P, and $Q = (N \setminus P) \cup \{i\}$, then*

$$e_i\, \eta(P) \;=\; \eta(Q)\, e_i \quad \text{and} \quad \eta(P)\, e_i \;=\; e_i\, \eta(Q) \;.$$

When j does not belong to P, then

$$e_j\, \eta(P) \;=\; \eta(P)\, e_j \;=\; 0 \;.$$

The idempotent $\varepsilon(P)$ and $\eta(P)$ span an algebra isomorphic to Ω :

$$\varepsilon(P)\, \eta(P) \;=\; \eta(P)\, \varepsilon(P) \;=\; \eta(P) \quad \textit{and} \quad \eta(P)^2 \;=\; (-1)^{m-1} \delta\; \varepsilon(P) \;.$$

But when Q is any subset of N other than P, then

$$\varepsilon(Q)\, \eta(P) = \eta(P)\, \varepsilon(Q) = 0 \;, \quad \textit{and}$$

$$\eta(Q)\, \eta(P) = \eta(P)\, \eta(Q) = 0 \quad \textit{if} \quad \mathrm{card}(Q) = m \,.$$

Proof. When j does not belong to P, we can assume that j is the element j_1 of $N \setminus P$, and the equality $e_j\,\eta(P) = 0$ follows from $e_j e_{i_1} e_j = 0$. We can also assume that $j = j_{m-1}$; thus we prove that $\eta(P)\,e_j = 0$ because $e_j e_{i_m} e_j = 0$. When i belongs to P, we can assume that $i = i_1$. Let us remember that

$$e_i^2 e_k \;=\; e_k\,(f(e_i,e_i) - e_i^2) \quad \text{and} \quad (f(e_i,e_i) - e_i^2)\,e_k \;=\; e_k e_i^2 \quad \text{if } k \neq i\,;$$

these equalities are the equalities (b) of Lemma (6.1) since $e_i = f(e_i,e_i)\,e'_i$. Thus we realize that

$$e_i^2\; e_{j_1} e_{i_2} e_{j_2} \ldots e_{j_{m-1}} e_{i_m} \;\;=\;\; e_{j_1} e_{i_2} e_{j_2} \ldots e_{j_{m-1}} e_{i_m}\; e_i^2\;,$$

and this is exactly the announced equality $e_i\eta(P) = \eta(Q)e_i$. To prove that $\eta(P)e_i = e_i\eta(Q)$, we would assume that $i = i_m$. The equalities $\varepsilon(P)\eta(P) = \eta(P)$ and $\eta(P)\varepsilon(P) = \eta(P)$ follow from the formulas (a) and (b) in Lemma (6.1). To prove $\eta(P)^2 = (-1)^{m-1}\delta\,\varepsilon(P)$, we notice that δ is the product of the n factors $f(e_i,e_i)$, and we begin in this way:

$$\begin{aligned}
\eta(P)^2 \quad &= \quad (-1)^{m-1}\, e_{i_m} e_{j_{m-1}} \ldots e_{j_2} e_{i_2} e_{j_1} e_{i_1}\; e_{i_1} e_{j_1} e_{i_2} e_{j_2} \ldots e_{j_{m-1}} e_{i_m} \\
&= \quad (-1)^{m-1}\delta\; e_{i_m} e'_{j_{m-1}} \ldots e'_{j_2} e_{i_2} e'_{j_1}\; e_{i_1} e'_{i_1}\; e_{j_1} e'_{i_2} e_{j_2} \ldots e_{j_{m-1}} e'_{i_m} \\
&= \quad (-1)^{m-1}\delta\; (e_{i_1} e'_{i_1})\; e_{i_m} e'_{j_{m-1}} \ldots e'_{j_2} e_{i_2}\; e'_{j_1} e_{j_1}\; e'_{i_2} e_{j_2} \ldots e_{j_{m-1}} e'_{i_m} \\
&= \quad (-1)^{m-1}\delta\; (e_{i_1} e'_{i_1})(1 - e_{j_1} e'_{j_1})\; e_{i_m} e'_{j_{m-1}} \ldots e'_{j_2}\, e_{i_2} e'_{i_2}\, e_{j_2} \ldots e_{j_{m-1}} e'_{i_m}
\end{aligned}$$

and so forth…; by applying many times the formulas (b) in Lemma (6.1) we let the factor $\varepsilon(P)$ appear after $(-1)^{m-1}\delta$. Now the last formulas in (7.4) are evident since $\varepsilon(P)\varepsilon(Q) = 0$ whenever $P \neq Q$; indeed we can always replace $\eta(P)$ with $\varepsilon(P)\eta(P)$ or $\eta(P)\varepsilon(P)$. □

Now it is clear that we get an odd central element if we set

$$\eta \;=\; \sum_{\mathrm{card}(P)=m} \eta(P)\;.$$

Theorem 7.5. *When $n = 2m-1$, the odd component of the center of* $\mathrm{B}(M,f)$ *is spanned by η which belongs to the ideal generated by the idempotent ε_m. This idempotent and η span an algebra isomorphic to Ω since $\eta^2 = (-1)^{m-1}\delta\,\varepsilon_m$. Besides, $\rho(\eta) = (-1)^{m-1}\eta$.*

This theorem, which is an immediate consequence of Lemma (7.4), does not yet satisfy all our curiosity, since we may still wish to know the image of η by the algebra morphism $\mathrm{B}(M,f) \to \mathrm{End}(E)$.

Lemma 7.6. *The endomorphism F_η of E maps all components other that E^m to 0, and it operates in E^m is the same way as the element $(0,\hat{e}_N)$ of Ω.*

Proof. If $\mathrm{card}(P) = m$, from Lemma (7.2) we deduce that the operation of $(0,\hat{e}_N)$ in E maps $(\hat{e}'_P, 0)$ to $(-1)^{m(m-1)/2}\mathrm{sgn}(P)(0,\hat{e}_{N\setminus P})$. Since $\eta(P) = \eta(P)\varepsilon(P)$, we know that $F_{\eta(P)}$ annihilates all $(\hat{e}'_Q, 0)$ such that $Q \neq P$ (remember Corollary

(6.5)). Consequently it suffices to verify that $F_{\eta(P)}$ maps $(\hat{e}'_P, 0)$ to the same element as $(0, \hat{e}_N)$; this proves that the operations of η and $(0, \hat{e}_N)$ coincides on E_0^m and consequently on E^m too, since they are both Ω-linear. If $(i_1, \ldots, i_m)$ and $(j_1, \ldots, j_{m-1})$ are the *increasing* sequences of the elements of P and $N \setminus P$, the signature of the permutation $(i_m, j_{m-1}, \ldots, j_2, i_2, j_1, i_1)$ is $(-1)^{m-1}(-1)^{m(m-1)/2} \cdot \mathrm{sgn}(P)$. Let us set $F_k = F_{e_k}$ for every $k \in N$; thus

$$\begin{aligned}
&(-1)^{m-1}(-1)^{m(m-1)/2}\,\mathrm{sgn}(P)\ F_{\eta(P)}(\hat{e}'_P, 0) \\
&\qquad = F_{i_m} F_{j_{m-1}} \ldots F_{j_2} F_{i_2} F_{j_1} F_{i_1} (e'_{i_1} \wedge e'_{i_2} \wedge \ldots \wedge e'_{i_m}\ ,\ 0) \\
&\qquad = F_{i_m} (e_{j_1} \wedge e_{j_2} \wedge \ldots \wedge e_{j_{m-1}} \wedge e'_{i_m}\ ,\ 0) \\
&\qquad = (-1)^{m-1}\,(0,\ e_{j_1} \wedge e_{j_2} \wedge \ldots \wedge e_{j_{m-1}}).
\end{aligned}$$

between the second and third lines of this calculation, the identity $F_j F_i (e'_i \wedge v,\ 0) = (e_j \wedge v,\ 0)$ (which holds if $\mathrm{d}_f(e_i) \rfloor v = 0$) has been used $(m-1)$ times. The final result confirms (7.6). □

8. Some Applications of the Previous Results

How can we recognize whether an algebra morphism H from $\mathrm{B}(M, f)$ into some algebra A is injective? We still suppose that M is a projective module of constant rank n provided with a nondegenerate symmetric bilinear form f, so that $\mathrm{B}(M, f)$ is the direct sum of $m+1$ ideals isomorphic to $\mathrm{End}(E^k)$ for $k = 0, 1, \ldots, m$ when $n = 2m$, or isomorphic to $\mathrm{End}(E^k)$ for $k = 0, 1, \ldots, m-1$, and to $\Omega \otimes \mathrm{End}(E_0^m)$ when $n = 2m-1$. Thus H is injective if its restriction to the center of $\mathrm{B}(M, f)$ is injective. When n is even, this center contains only even elements, and if M is a free module, it is contained in the algebra $\mathcal{C}$ defined just after Lemma (6.1). When n is odd, we also assume that A is a graded algebra $A_0 \oplus A_1$ and that H is a graded algebra morphism; thus the graded ideals of $\Omega \otimes \mathrm{End}(E_0^m)$ correspond bijectively to the ideals of $\mathrm{End}(E_0^m)$ (see Theorem (5.1)), which correspond bijectively to the ideals of K. Consequently the injectiveness of H can be tested on the even component of the center of $\mathrm{B}(M, f)$, which is still a subalgebra of $\mathcal{C}$.

Proposition 8.1. *When n is even, an algebra morphism $H : \mathrm{B}(M, f) \to A$ is injective if its restriction to the subalgebra $\mathcal{C}$ (defined in §6) is injective. When n is odd, the same assertion holds true if H is a graded algebra morphism.*

Let us apply this criterium of injectiveness to the well-known algebra morphism D from $\mathrm{B}(M, f)$ into $\mathrm{C}\ell(M, f) \otimes \mathrm{C}\ell(M, f)$ which maps every $a \in M$ to $(a \otimes 1 + 1 \otimes a)/2$. The definition of D requires 2 to be invertible in K; the Clifford algebra $\mathrm{C}\ell(M, f)$ is characterized by the relations $a^2 = f(a, a)$ for all $a \in M$, and the target of D is an ordinary tensor product, in which the product of $x \otimes y$ and $x' \otimes y'$ is $xx' \otimes yy'$ even if y and x' are odd. The existence of D follows from the identity

$$(a \otimes 1 + 1 \otimes a)\,(b \otimes 1 + 1 \otimes b)\,(a \otimes 1 + 1 \otimes a) \;=\; 4f(a, b)\,(a \otimes 1 + 1 \otimes a)$$

and the universal property of $\mathrm{B}(M,f)$; besides, D is a graded algebra morphism. These two formulas (with a, $b \in M$) are often useful:

(a) $$D(2a^2 - f(a,a)) \;=\; a \otimes a \;;$$

(b) $$4\,D(ab - ba) \;=\; (ab - ba) \otimes 1 + 1 \otimes (ab - ba)\;.$$

The next theorem does not require f to be nondegenerate.

Theorem 8.2. *If* 2 *is invertible in* K*, and if* M *is a finitely generated projective module, the algebra morphism* D *from* $\mathrm{B}(M,f)$ *into* $\mathrm{C}\ell(M,f) \otimes \mathrm{C}\ell(M,f)$ *is injective.*

Proof. If this theorem is true when f is nondegenerate, it holds for all symmetric bilinear forms f. Indeed if f is degenerate, we can always embed (M,f) in a larger object (M',f') such that f' is nondegenerate, and M' still projective and finitely generated; we can choose for instance $M' = M \oplus M^*$. The injectiveness of D' (defined by means of (M',f')) implies the injectiveness of D. When f is nondegenerate, by localization we can reduce the problem to the case of a module M provided with an orthogonal basis $(e_1, e_2, \dots, e_n)$. Thus $\mathcal{C}$ is the subalgebra generated by all squares e_i^2, or equivalently by all $2e_i^2 - f(e_i,e_i)$; these elements satisfy the relations $(2e_i^2 - f(e_i,e_i))^2 = f(e_i,e_i)^2$, yet even without this piece of information it is already clear that we get a basis of $\mathcal{C}$ if with every subset P of N we associate the product of all $2e_i^2 - f(e_i,e_i)$ with $i \in P$. By means of the above formula (a) it is easy to realize that the elements of this basis of $\mathcal{C}$ are mapped to linearly independent elements of $\mathrm{C}\ell(M,f) \otimes \mathrm{C}\ell(M,f)$. □

Since $\mathrm{C}\ell(M,f) \otimes \mathrm{C}\ell(M,f)$ is the tensor product of two graded algebras, its even subalgebra may receive a parity subgrading for which the sub-even subalgebra and the sub-odd submodule are respectively

$$\mathrm{C}\ell_0(M,f) \otimes \mathrm{C}\ell_0(M,f) \quad \text{and} \quad \mathrm{C}\ell_1(M,f) \otimes \mathrm{C}\ell_1(M,f)\;.$$

As an algebra, $\mathrm{B}_0(M,f)$ is generated by the elements $2a^2 - f(a,a)$ and $ab - ba$; the above formulas (a) and (b) show that their images are respectively in $\mathrm{C}\ell_1(M,f) \otimes \mathrm{C}\ell_1(M,f)$ and $\mathrm{C}\ell_0(M,f) \otimes \mathrm{C}\ell_0(M,f)$. Because of the injectiveness of D, there is also a parity subgrading on $\mathrm{B}_0(M,f)$; it is the direct sum of a subalgebra $\mathrm{B}_{0,0}(M,f)$ that contains all elements $ab - ba$, and a submodule $\mathrm{B}_{0,1}(M,f)$ that contains all elements $2a^2 - f(a,a)$. If an element x of $\mathrm{B}_0(M,f)$ is homogeneous for this parity subgrading, its subparity is denoted by $\wp x$.

Since each element in the image of D is invariant by the automorphism $u \otimes v \longmapsto v \otimes u$, the image of $\mathrm{B}_1(M,f)$ by D is not the direct sum of its intersections with $\mathrm{C}\ell_0(M,f) \otimes \mathrm{C}\ell_1(M,f)$ and $\mathrm{C}\ell_1(M,f) \otimes \mathrm{C}\ell_0(M,f)$. Consequently it is not sensible to look for a parity subgrading in $\mathrm{B}_1(M,f)$.

Comments. (1) The first reliable proof of the injectiveness of D probably stems from Jacobson, who considered the case of a module M provided with a finite orthogonal basis. In his work about meson algebras, the injectiveness of D was the starting point of almost all his main results; therefore his proof was much longer and more arduous. After having proved its injectiveness, Jacobson considered the algebra morphism $\mathrm{B}(M, f) \to \mathrm{C}\ell(M, f) \otimes \mathrm{C}\ell(M, f) \to \mathrm{End}(\mathrm{C}\ell(M, f))$ in which the second arrow maps every $x \otimes y$ to the operator $z \longmapsto xz\,\rho(y)$. Since 2 is invertible in K, there is a canonical linear isomorphism $\bigwedge(M) \to \mathrm{C}\ell(M, f)$ that maps every $a_1 \wedge a_2 \wedge \ldots \wedge a_k$ to $a_1 a_2 \ldots a_k$ when the vectors $a_1, \ldots, a_k$ are pairwise orthogonal; thus we can derive from his method an algebra morphism $\mathrm{B}(M, f) \to \mathrm{End}(\bigwedge(M))$, and then our Theorem (3.4) can be deduced from his results after some manipulations.

(2) The concept of "parity subgrading" in $\mathrm{B}_0(M, f)$ appears here probably for the first time. Despite initial hesitations, its relevance has been confirmed by its usefulness in the next section, and by its usefulness in the subsequent work [H]. Among the first results of this work, it is stated that every $\xi \in M^*$ determines a "twisted semi-derivation" of the algebra $\mathrm{B}(M, f)$, which is denoted by $x \longmapsto \xi \rfloor x$, and which is characterized by these properties: $\xi \rfloor a = \xi(a)$, $\xi \rfloor (ab) = \xi(a)b$ and

$$\forall x \in \mathrm{B}_{0,0}(M, f) \cup \mathrm{B}_{0,1}(M, f),\ \forall y \in \mathrm{B}(M, f), \quad \xi \rfloor (xy) = (\xi \rfloor x)\, y + (-1)^{\wp x}\, x\, (\xi \rfloor y).$$

There is no similar formula involving $\xi \rfloor (xy)$ when x belongs to $\mathrm{B}_1(M, f)$; this is corroborated by the calculation of $\xi \rfloor (abc) = \xi(a)bc + \xi(c)(f(a, b) - ba)$.

9. The Group $\mathrm{Aut}(M, f)$ in Characteristic $\neq 2$

In all cases $\mathrm{Aut}(M, f)$ is the group of all linear transformations φ of M such that $f(\varphi(a), \varphi(b)) = f(a, b)$ for all a and $b \in M$. Here M is a vector space of dimension n over a field K of characteristic $\neq 2$, and f is always nondegenerate; thus $\mathrm{Aut}(M, f)$ is the classical orthogonal group of the quadratic form $a \longmapsto f(a, a)$. The Cartan-Dieudonné Theorem states that this group is generated by the reflections. Each nonisotropic vector $d \in M$ determines a reflection that maps every vector a to $a - 2f(a, d)\, d\, f(d, d)^{-1}$.

Lemma 9.1. *Let d be an element of M such that $f(d, d)$ is invertible, and let us set*

$$z \;=\; \frac{2\,d^2}{f(d, d)} \;-1 \;\in\; \mathrm{B}_{0,1}(M, f)\ ;$$

then $z^2 = 1$, $\rho(z) = z$, and for every $a \in M$,

$$a \;-\frac{2f(a, d)\,d}{f(d, d)} \;=\; -z\,a\,z\ .$$

Proof. The formula (a) in §8 shows that z belongs to $\mathrm{B}_{0,1}(M, f)$. There is an orthogonal basis $(e_1, e_2, \ldots, e_n)$ such that $e_1 = d$, whence $e'_1 = d\, f(d, d)^{-1}$. This implies that $z = e_1 e'_1 - (1 - e_1 e'_1)$. Since $e_1 e'_1$ is an idempotent, it follows that $z^2 = 1$. The equality $\rho(z) = z$ is trivial. From the equalities (a) in Lemma (6.1) we deduce that $zdz = d$, and from the subsequent equalities (b) we deduce that $ze_i z = -e_i$ for $i = 2, 3, \ldots, n$; therefore the mapping $a \longmapsto -zaz$ is actually the reflection with respect to the hyperplane orthogonal to d. □

Theorem 9.2. *Let $\mathcal{G}$ be the multiplicative subgroup of* $\mathrm{B}_0(M,f)$ *generated by the elements* $2d^2 f(d,d)^{-1} - 1$ *mentioned in Lemma* (9.1)*. Each element $x \in \mathcal{G}$ is homogeneous for the parity subgrading of* $\mathrm{B}_0(M,f)$ *defined after Theorem* (8.2)*, and satisfies these properties:*

$$x^{-1} = \rho(x) \quad \textit{and} \quad x - (-1)^{\wp x} \in M\,\mathrm{B}(M,f)\ .$$

Moreover we obtain a group isomorphism $\mathcal{G} \to \mathrm{Aut}(M,f)$ if we map every $x \in \mathcal{G}$ to the linear transformation $a \longmapsto (-1)^{\wp x} x a x^{-1}$.

Proof. Everything in (9.2) is an immediate consequence of (9.1), except the injectiveness of the group morphism $\mathcal{G} \to \mathrm{Aut}(M,f)$. Even its surjectiveness is evident since the orthogonal group $\mathrm{Aut}(M,f)$ is generated by the reflections. The injectiveness of this group morphism follows from the injectiveness of the algebra morphism D in Theorem (8.2). To understand this, we introduce two other groups $\mathcal{G}'$ and $\mathrm{GLip}(M,f)$ and three other group morphisms, which constitute this diagram:

$$\begin{array}{ccc} \mathcal{G} & \longrightarrow & \mathrm{Aut}(M,f) \\ \downarrow & & \uparrow \\ \mathcal{G}' & \longleftarrow & \mathrm{GLip}(M,f) \end{array}$$

The group $\mathcal{G}'$ is the image by D of the group $\mathcal{G}$; according to the formula (a) in §8, it is the multiplicative subgroup of $\mathrm{C}\ell(M,f) \otimes \mathrm{C}\ell(M,f)$ generated by the elements

$$D\left(\frac{2d^2}{f(d,d)} - 1\right) \;=\; \frac{d \otimes d}{f(d,d)}\ .$$

Because of Theorem (8.2), the morphism $\mathcal{G} \to \mathcal{G}'$ induced by D is an isomorphism. Several definitions may be proposed for the Clifford-Lipschitz group $\mathrm{GLip}(M,f)$; here we say that it is the multiplicative subgroup of $\mathrm{C}\ell(M,f)$ generated by the elements $d \in M$ such that $f(d,d)$ is invertible. There is a group morphism from $\mathrm{GLip}(M,f)$ into the group $K^\times$ of invertible scalars that maps every w to $w\rho(w) = \rho(w)w$; therefore there is a group morphism that maps every $w \in \mathrm{GLip}(M,f)$ to the element $w \otimes w\,(w\rho(w))^{-1}$ of $\mathrm{C}\ell(M,f) \otimes \mathrm{C}\ell(M,f)$; since it maps every generator d of $\mathrm{GLip}(M,f)$ to $d \otimes d\, f(d,d)^{-1}$, it induces the surjective group morphism $\mathrm{GLip}(M,f) \to \mathcal{G}'$ mentioned in the diagram above. Finally there is a surjective group morphism $\mathrm{GLip}(M,f) \to \mathrm{Aut}(M,f)$ which maps every $w \in \mathrm{GLip}(M,f)$ to the orthogonal transformation $a \longmapsto (-1)^{\partial w} w a w^{-1}$; the image of every generator d is the reflexion with respect to the hyperplane orthogonal to d, and this piece of information shows that the above diagram becomes a commutative diagram if we replace the morphism $\mathcal{G} \to \mathcal{G}'$ with the reciprocal morphism $\mathcal{G} \leftarrow \mathcal{G}'$. The injectiveness of the surjective morphism $\mathcal{G} \to \mathrm{Aut}(M,f)$ follows from the fact that both surjective morphisms $\mathrm{GLip}(M,f) \to \mathrm{Aut}(M,f)$ and $\mathrm{GLip}(M,f) \to \mathcal{G}'$ have the same kernel, namely $K^\times$; this is a classical result for the first morphism, and an easy assertion for the second morphism. □

It follows immediately from (9.1) and (9.2) that $\mathcal{G}$ is the union of its intersections $\mathcal{G}_0$ and $\mathcal{G}_1$ with $\mathrm{B}_{0,0}(M,f)$ and $\mathrm{B}_{0,1}(M,f)$, and that the elements of $\mathcal{G}_i$

(with $i = 0, 1$) are mapped to orthogonal transformations with determinant $(-1)^i$. It is also clear that every $\varphi \in \mathrm{Aut}(M, f)$ extends to an automorphism of the algebra $\mathrm{B}(M, f)$; if x is the associated element in $\mathcal{G}$, this automorphism of $\mathrm{B}(M, f)$ is precisely $y \longmapsto (-1)^{\wp x\, \partial y}\, xyx^{-1}$.

Let us find the element $\zeta \in \mathcal{G}$ that is mapped to $-\mathrm{id}_M$ by the isomorphism $\mathcal{G} \to \mathrm{Aut}(M, f)$. If $(e_1, e_2, \ldots, e_n)$ is an orthogonal basis of M, then $-\mathrm{id}_M$ is the product of the n reflections determined by the n vectors e_i; therefore ζ is the (commutative) product of the n factors $2e_i^2\, f(e_i, e_i)^{-1} - 1$. Since each factor is equal to $e_i e_i' - (1 - e_i e_i')$, the calculation of this product takes place in the algebra $\mathcal{C}$ defined in §6, and lets soon appear the central idempotents ε_k of $\mathrm{B}_0(M, f)$:

$$\zeta \;=\; \sum_{k=0}^{n} (-1)^{n-k}\, \varepsilon_k \; .$$

When n is even, then $\zeta a \zeta^{-1} = -a$ for all $a \in M$. Consequently, if φ is an orthogonal transformation of determinant -1, there is an element $x' \in \mathcal{G}_1$ (exactly $x' = x\zeta$) such that $\varphi(a) = x' a x'^{-1}$ for all $a \in M$.

When n is odd, ζ belongs to the even component of the center of $\mathrm{B}(M, f)$, which also contains an odd component spanned by the element η calculated in §7. Let x be some element of $\mathcal{G}$, and φ the associated orthogonal transformation. The automorphism $y \longmapsto (-1)^{\wp x\, \partial y} xyx^{-1}$ that extends φ, leaves η invariant when $\wp x = 0$ (whence $\det(\varphi) = +1$), but maps η to $-\eta$ if $\wp x = 1$ (whence $\det(\varphi) = -1$). Therefore φ is not the restriction to M of an inner automorphism $y \longmapsto x' y x'^{-1}$ if $\det(\varphi) = -1$.

Let us consider the Lie algebra $\mathfrak{aut}(M, f)$ derived from the group $\mathrm{Aut}(M, f)$. There is a linear bijection $\bigwedge^2(M) \to \mathfrak{aut}(M, f)$ that maps every bivector $c \wedge d$ to the skew symmetric operator $a \longmapsto f(d, a)c - f(c, a)d$. There is also a linear mapping $\bigwedge^2(M) \to \mathrm{B}(M, f)$ that maps every $c \wedge d$ to $[c, d] = cd - dc$; let $\mathcal{L}$ be its image. The following formula (the double bracket formula) soon appeared in the physicist's works, and to-day it is sometimes attributed to Green:

$$[\,[c, d],\, a] \;=\; f(d, a)\, c - f(c, a)\, d \; ;$$

it easily follows from two applications of the formula $abc + cba = f(a, b)c + f(c, b)a$. It leads to the following two propositions.

Proposition 9.3. *The subspace $\mathcal{L}$ spanned by all $[c, d]$ is a Lie subalgebra of* $\mathrm{B}(M, f)$, *and there is a Lie algebra isomorphism $\mathcal{L} \to \mathfrak{aut}(M, f)$ that maps every $u \in \mathcal{L}$ to the skew symmetric operator $a \longmapsto [u, a]$.*

Proof. The above double bracket formula shows that $[u, a]$ belongs to M for all $u \in \mathcal{L}$ and all $a \in M$. The classical formulas $[u, [c, d]] = [[u, c], d] + [c, [u, d]]$ and $[[u, v], a] = [u, [v, a]] - [v, [u, a]]$ show that $\mathcal{L}$ is a Lie algebra, and that we have defined a Lie algebra morphism from $\mathcal{L}$ into $\mathfrak{aut}(M, f)$. Since the composition of the surjective mappings $\bigwedge^2(M) \to \mathcal{L}$ and $\mathcal{L} \to \mathfrak{aut}(M, f)$ is the classical bijection $\bigwedge^2(M) \to \mathfrak{aut}(M, f)$, both mappings are even bijective. □

Proposition 9.4. *The Lie algebra $\mathcal{L}$ is the Lie algebra derived from the group $\mathcal{G}$ defined in Theorem* (9.2)*, and the Lie algebra isomorphism $\mathcal{L} \to \mathfrak{aut}(M, f)$ is the isomorphism derived from the group isomorphism $\mathcal{G} \to \mathrm{Aut}(M, f)$.*

Proof. Let $K \oplus tK$ the quotient of the polynomial algebra $K[t]$ by the ideal generated by t^2. With every vector space V over K we also associate its extension $V \oplus tV$ over this algebra $K \oplus tK$; an element of $V \oplus tV$ is denoted by $a + tb$ (with $a,\, b \in V$) or even by $a + tb + O(t^2)$ to recall that it is an expansion limited to the order 1. The space $\mathcal{L}$ is spanned by all brackets $[c, d]$ such that $f(d, d)$ is invertible. Let φ be the orthogonal transformation of $M \oplus tM$ that is the product of the reflexions determined by $d + tc$ and d ; it maps every $a \in M$ to

$$a - \frac{2f(a,d)}{f(d,d)}\, d - \frac{2f(a,\, d+tc)}{f(d+tc,\, d+tc)}(d+tc) + \frac{4f(a,d)\, f(d,\, d+tc)}{f(d,d)\, f(d+tc,\, d+tc)}(d+tc)\ .$$

The automorphism of $\mathrm{B}(M, f) \oplus t\, \mathrm{B}(M, f)$ induced by φ is the inner automorphism determined by

$$x \;=\; \left(\frac{2\,(d+tc)^2}{f(d+tc,\, d+tc)} - 1\right)\left(\frac{2\,d^2}{f(d,d)} - 1\right) .$$

To simplify the expressions of $\varphi(a)$ and x, we remember that

$$\frac{1}{f(d+tc,\, d+tc)} \;=\; \frac{1}{f(d,d)} - \frac{2t\, f(d,c)}{f(d,d)^2} + O(t^2)\ ;$$

to simplify the expression of x we also remember that $d^3 = f(d,d)d$ and $dcd = f(d,c)d$; and for the expression of $\varphi(a)$ we also need the double bracket formula written before (9.3). After some calcultations we find that

$$x \;=\; 1 + \frac{2t}{f(d,d)}\,[c,d] + O(t^2) \quad \text{and} \quad \varphi(a) \;=\; a + \frac{2t}{f(d,d)}\,[\,[c,d],\, a] + O(t^2)\ .$$

The conclusions follow. □

10. The Group $\mathrm{Aut}(M, f)$ in Characteristic 2

When K is a field of characteristic 2, the group of automorphisms of a nondegenerate symmetric bilinear form f does not at all look like the orthogonal group of a nondegenerate quadratic form; it seems that such a group $\mathrm{Aut}(M, f)$ has never been seriously studied except when f is a symplectic form (a nondegenerate alternate bilinear form), and when $\mathrm{Aut}(M, f)$ is one of the classical symplectic groups. When f is symplectic, the dimension n of M is even, and the dimension of $\mathrm{Aut}(M, f)$ is $n(n+1)/2$.

In all cases we first look at the quadratic form $a \longmapsto f(a, a)$ which is always degenerate. It is a semi-linear function $M \to K$, in other words,

$$f(a+b,\ a+b) = f(a,a) + f(b,b) \quad \text{and} \quad f(\lambda a,\ \lambda a) = \lambda^2 f(a,a)\ ;$$

consequently the subset M_0 of all isotropic vectors a (such that $f(a, a) = 0$) is a vector subspace of M. Let n_0 be its dimension. When $n_0 = n$, then f is alternate.

When f is not alternate, the equality $n_0 = n-1$ means that $f(a,a)\,f(b,b)$ has a square root in K for all $a,\ b \in M$. When $n_0 = 0$, then f is called anisotropic, the mapping $a \longmapsto f(a,a)$ is injective, and consequently the group $\mathrm{Aut}(M,f)$ is reduced to the only element id_M. The dimension of $\mathrm{Aut}(M,f)$ (which can be defined at least if K is infinite) depends only on n_0 according to this theorem.

Theorem 10.1. *When K is a field of characteristic 2, the group $\mathrm{Aut}(M,f)$ (with a nondegenerate f) is generated by the transformations derived from all couples $(\lambda, d) \in K \times M_0$ in this way:*

$$a \longmapsto a + \lambda f(a,d)\, d\ .$$

The dimension of $\mathrm{Aut}(M,f)$ is $n_0(n_0+1)/2$.

Before proving this theorem, we look at its consequences for the algebra $\mathrm{B}(M,f)$: again this algebra contains a multiplicative group $\mathcal{G}$ isomorphic to $\mathrm{Aut}(M,f)$ by an isomorphism involving the inner automorphisms associated with the elements of $\mathcal{G}$; moreover $x-1$ belongs to the ideal $M\,\mathrm{B}(M,f)$ for every $x \in \mathcal{G}$.

Theorem 10.2. *Let $\mathcal{G}$ be the multiplicative group in $\mathrm{B}_0(M,f)$ generated by all $1+\lambda d^2$ with $d \in M_0$ and $\lambda \in K$. The equality $\rho(x) = x^{-1}$ holds for every $x \in \mathcal{G}$, and we get a group isomorphism $\mathcal{G} \to \mathrm{Aut}(M,f)$ if we map every $x \in \mathcal{G}$ to the linear transformation $a \longmapsto xax^{-1}$.*

Proof. When $f(d,d) = 0$, the equality $d^3 = 0$ implies that the mapping $\lambda \longmapsto 1+\lambda d^2$ is a morphism from the additive group K into the multiplicative group of invertible elements of $\mathrm{B}_0(M,f)$; in particular $(1+\lambda d^2)^{-1} = 1+\lambda d^2 = \rho(1+\lambda d^2)$, and consequently the equality $x^{-1} = \rho(x)$ holds for every $x \in \mathcal{G}$. It is easy to verify that

$$a + \lambda f(a,d)\, d \ = \ (1+\lambda d^2)\, a\, (1+\lambda d^2)\ ;$$

because of Theorem (10.1), it follows that we get a surjective group morphism $\mathcal{G} \to \mathrm{Aut}(M,f)$ if we map every $x \in \mathcal{G}$ to the transformation $a \longmapsto xax^{-1}$. If an element x of $\mathcal{G}$ belongs to the kernel of this morphism, it belongs to the center of $\mathrm{B}_0(M,f)$ which is spanned by the idempotents ε_i (with $i = 0,1,2,\dots,n$) such that $\rho(\varepsilon_i) = \varepsilon_i$. Therefore $x = \sum_{i=0}^{n} \lambda_i \varepsilon_i$ for some $\lambda_i \in K$, and the equality $\rho(x) = x^{-1}$ implies $\lambda_i = \lambda_i^{-1}$ for $i = 0,1,\dots,n$, whence $\lambda_i = 1$ and $x = 1$. □

To prove Theorem (10.1), we consider the subspace $M_0^\perp$ orthogonal to M_0 and we choose a subspace P_4 supplementary to $M_0 + M_0^\perp$ so that M becomes the direct sum of

$$P_1 = M_0 \cap P_4^\perp, \quad P_2 = M_0 \cap M_0^\perp, \quad P_3 = M_0^\perp \cap P_4^\perp \quad \text{and} \quad P_4\ ;$$

thus $M_0 = P_1 \oplus P_2$ and $M_0^\perp = P_2 \oplus P_3$. The subspaces P_1, P_3 and $P_2 \oplus P_4$ are pairwise orthogonal. The restriction of f to P_1 is a symplectic form f_1 which allows us to define a symplectic group $\mathrm{Sp}(P_1, f_1)$. The restriction of f to $P_3 \oplus P_4$ is anisotropic, and its restriction to P_2 is null. Yet f determines a duality between P_2 and P_4.

Proposition 10.3. *If ψ is an automorphism of (M, f), then $\psi(M_0) = M_0$, $\psi(a) = a$ for every $a \in M_0^\perp$ and $\psi(a) - a \in M_0$ for every $a \in M$. Therefore there are four linear mappings*

$$\psi_1 : P_1 \to P_1 \;, \qquad \psi_3 : P_4 \to P_1 \;,$$
$$\psi_2 : P_1 \to P_2 \;, \qquad \psi_4 : P_4 \to P_2$$

such that, for every $(a_1, a_2, a_3, a_4) \in P_1 \times P_2 \times P_3 \times P_4$,

(a) $\psi(a_1 + a_2 + a_3 + a_4) \;=\; (\psi_1(a_1) + \psi_3(a_4)) \;+ (a_2 + \psi_2(a_1) + \psi_4(a_4)) \;+ a_3 + a_4 \;.$

Conversely a quartet of four mappings $(\psi_1, \psi_2, \psi_3, \psi_4)$ like the previous ones determines an automorphism ψ of (M, f) by means of the formula (a) if and only if these three conditions are fulfilled, for all $a_1,\ b_1 \in P_1$ and for all $a_4,\ b_4 \in P_4$:

(b) $$f(\psi_1(a_1),\, \psi_1(b_1)) \;=\; f(a_1, b_1) \;;$$

(c) $$f(\psi_1(a_1),\, \psi_3(b_4)) \;=\; f(\psi_2(a_1),\, b_4) \;;$$

(d) $$f(\psi_3(a_4),\, \psi_3(b_4)) \;=\; f(\psi_4(a_4),\, b_4) + f(a_4,\, \psi_4(b_4)) \;.$$

Proof. If ψ is an automorphism of (M, f), the following assertions are clear: $\psi(M_0) = M_0$, $\psi(M_0^\perp) = M_0^\perp$, $\psi(P_2) = P_2$ and $\psi(a) - a \in M_0$ for every $a \in M$. Since ψ induces the identity transformation on P_4 modulo M_0 and since f determines a duality between P_4 and P_2 (which is orthogonal to M_0), we realize that ψ induces the identity transformation on P_2. Since $\psi(M_0^\perp) = M_0^\perp$, we know that $\psi(a_3) - a_3 \in P_2$ for every $a_3 \in P_3$. Moreover both a_3 and $\psi(a_3)$ are orthogonal to P_4 because $f(\psi(a_3), \psi(a_4)) = f(a_3, a_4) = 0$ (for all $a_4 \in P_4$) and $f(\psi(a_3),\, \psi(a_4) - a_4) = 0$ (indeed $\psi(a_3) \in M_0^\perp$ and $\psi(a_4) - a_4 \in M_0$). Since $\psi(a_3) - a_3$ is an element of P_2 that is orthogonal to P_4, it vanishes. Thus we know that $\psi(a) = a$ for all $a \in P_2 \oplus P_3 = M_0^\perp$. All this proves the first part of (10.3), and the second part follows from straightforward calculations. □

Proof of (10.1). Let us set $n_1 = \dim(P_1)$ and $n_2 = \dim(P_2) = \dim(P_4)$. From the conditions (b), (c), (d) in (10.3) we can immediately deduce the dimension of $\mathrm{Aut}(M, f)$. Indeed (b) means that ψ_1 belongs to the symplectic group $\mathrm{Sp}(P_1, f_1)$; the dimension of this group is $n_1(n_1 + 1)/2$. Then ψ_3 is any element in the space of linear mappings $P_4 \to P_1$; the dimension of this space is $n_1 n_2$. When ψ_1 and ψ_3 are chosen, (c) determines ψ_2 in a unique way. Since f_1 is an alternate form, for every ψ_3 we can find some ψ_4 satisfying (d), but in general this ψ_4 is not unique, since to it we can add any $\psi_4' : P_4 \to P_2$ such that

$$\forall a_4,\ b_4 \in P_4, \quad f(\psi_4'(a_4),\, b_4) + f(a_4,\, \psi_4'(b_4)) \;=\; 0 \;;$$

this condition lets ψ_4' run through a space of dimension $n_2(n_2 + 1)/2$. Thus the dimension of $\mathrm{Aut}(M, f)$ is

$$\frac{1}{2} n_1(n_1 + 1) + \frac{1}{2} n_2(n_2 + 1) + n_1 n_2 = \frac{1}{2} n_0(n_0 + 1) \quad \text{since} \quad n_0 = n_1 + n_2.$$

Now let φ and ψ be two automorphisms of (M,f) which we represent as $(\varphi_1,\varphi_2,\varphi_3,\varphi_4)$ and $(\psi_1,\psi_2,\psi_3,\ \psi_4)$ as it is explained in (10.3). Then $\varphi\circ\psi$ is represented by

$$\varphi\psi \ = \ (\ \varphi_1\psi_1\ ,\ \varphi_2\psi_1+\psi_2\ ,\ \varphi_1\psi_3+\varphi_3\ ,\ \varphi_4+\psi_4+\varphi_2\psi_3\ )\ .$$

With this piece of information we can prove this assertion: the group $\mathrm{Aut}(M,f)$ is generated by the union of three families $(\psi_\alpha)_{\alpha\in A}$, $(\psi_\beta)_{\beta\in B}$ and $(\psi_\gamma)_{\gamma\in\Gamma}$ such that

$$\psi_\alpha=(\psi_{1,\alpha}\,,\,0,\,0,\,0),\quad \psi_\beta=(\mathrm{id}\,,\,0,\,0,\,\psi_{4,\beta}),\quad \psi_\gamma=(\psi_{1,\gamma}\,,\,\psi_{2,\gamma}\,,\,\psi_{3,\gamma}\,,\,\psi_{4,\gamma}),$$

provided that the family $(\psi_{1,\alpha})_{\alpha\in A}$ generates the group $\mathrm{Sp}(P_1,f_1)$, that the family $(\psi_{4,\beta})_{\beta\in B}$ generates the additive group of all $\psi'_4:P_4\to P_2$ satisfying the condition written above, and that the family $(\psi_{3,\gamma})_{\gamma\in\Gamma}$ generates the additive group of all linear mappings $P_4\to P_1$.

All the mappings $a\longmapsto a+\lambda f(a,d)\,d$ are automorphisms of (M,f) if $f(d,d)=0$. First we consider all those automorphisms in which d is an element d_1 of P_1 ; thus we get automorphisms $(\psi_1,0,0,0)$ such that $\psi_1(a_1)=a_1+\lambda f(a_1,d_1)\,d_1$; a classical result of Symplectic Geometry says that all these symplectic transformations of (P_1,f_1) generate its symplectic group. Secondly we suppose that d is an element d_2 of P_2 ; then we get automorphisms $(\mathrm{id},0,0,\psi_4)$ such that $\psi_4(a_4)=\lambda f(a_4,d_2)\,d_2$; it is not difficult to prove that all these ψ_4 generate the additive group of all $\psi'_4:P_4\to P_2$ satisfying the condition written above. Thirdly we suppose that $d=d_1+d_2$, and thus we get automorphisms $(\psi_1,\psi_2,\psi_3,\psi_4)$ such that $\psi_3(a_4)=\lambda f(a_4,d_2)\,d_1$; it is clear that all these ψ_3 generate the additive group of all linear mappings $P_4\to P_1$. □

Remark. Let us assume that $n_0=n-1$, because this assumption is always true when the mapping $\lambda\to\lambda^2$ is surjective (therefore bijective) from K onto itself, and when f is not symplectic. We still use the notations explained just before Proposition (10.3). When n is odd, then $M=P_1\oplus P_3$, and by restriction to $P_1=M_0$ we get an isomorphism from $\mathrm{Aut}(M,f)$ onto the symplectic group $\mathrm{Sp}(P_1,f_1)$. When n is even, the situation is more sophisticated because $M=P_1\oplus P_2\oplus P_4$; here P_2 and P_4 are vector lines spanned by two vectors e_2 and e_4 to which we can impose the relation $f(e_2,e_4)=1$. Let $\psi=(\psi_1,\psi_2,\psi_3,\psi_4)$ be an automorphism of (M,f), and let us set $\psi(e_4)=\beta+\mu e_2+e_4$ with $\beta\in P_1$ and $\mu\in K$; thus ψ is determined by (ψ_1,β,μ) which can be any element of $\mathrm{Sp}(P_1,f_1)\times P_1\times K$; indeed β and μ determine ψ_3 and ψ_4, and then $\psi_2(a_1)=f(\psi_1(a_1),\beta)\,e_2$. If φ is determined in the same way by $(\varphi_1,\alpha,\lambda)$ (another arbitrary element of $\mathrm{Sp}(P_1,f_1)\times P_1\times K$), then $\varphi\psi$ is determined by $\big(\varphi_1\psi_1\,,\ \alpha+\varphi_1(\beta)\,,\ \lambda+\mu+f(\alpha,\,\varphi_1(\beta))\big)$.

11. Some Physical Considerations

Here M is a real vector space of dimension $n=4$, and f is positive definite (resp. negative definite) on some subspaces of dimension 1 (resp. 3). Thus $\bigwedge^4(M)$ contains an element ω such that $\mathrm{d}_f(\omega)\rfloor\omega=-1$, and Ω is isomorphic to $\mathbb{C}$. The canonical bijection $\bigwedge(M) \to \mathrm{C}\ell(M,f)$ allows us to identify ω with an element of $\mathrm{C}\ell(M,f)$, and it is well known that the center of $\mathrm{C}\ell_0(M,f)$ is the subalgebra $\mathbb{R}\oplus\mathbb{R}\omega$ canonically isomorphic to Ω (if we forget the parity grading).

The usual Dirac equation involves functions $\psi : M \to S$ with values in a module S over $\mathrm{C}\ell(M,f)$. If S is a graded module $S_0 \oplus S_1$, then S is determined (up to isomorphy) by the module S_0 over $\mathrm{C}\ell_0(M,f)$ (see Theorem (5.1)). Since the multiplication by ω in S_0 is $\mathrm{C}\ell_0(M,f)$-linear, it extends to a $\mathrm{C}\ell(M,f)$-linear endomorphism i of S such that $i^2 = -1$ (again Theorem (5.1)). Since $\omega\mu = -\mu\omega$ for all $\mu \in \mathrm{C}\ell_1(M,f)$, ω is not in the center of $\mathrm{C}\ell(M,f)$, and to calculate the operation of i in S_1 we use a sequence $(\mu_1,\nu_1,\dots,\mu_k,\nu_k)$ of elements of $\mathrm{C}\ell_1(M,f)$ such that $\sum_j \mu_j\nu_j = 1$; for every $s \in S_1$ we write

$$is \;=\; \sum_j i(\mu_j\nu_j s) \;=\; \sum_j \mu_j(i\nu_j s) \;=\; \sum_j \mu_j(\omega\nu_j s) \;=\; \sum_j -\omega(\mu_j\nu_j s) \;=\; -\omega s \;;$$

thus we come to this formula, valid for all $s \in S_0 \cup S_1$:

$$\text{(a)} \qquad is \;=\; (-1)^{\partial s}\,\omega s \;.$$

Unfortunately the usual presentation by physicists is quite different: they intrude an action of $\mathbb{C}$ in S with the pretext that anyhow such an action shall be necessary; and much later they also pay some attention to Weyl spinors s which are either "left-hand" (when $\omega s = is$) or "right-hand" (when $\omega s = -is$); but they do not realize that this "chirality" is the parity grading of S that determines the intruded complex structure according to the above formula (a). Here we prefer to define S as a graded module over $\mathrm{C}\ell(M,f)$, and to avoid the perplexing intrusion of imaginary numbers.

Now let us consider the wave equation corresponding to meson particles; it involves a function $\psi : M \to S$ with values in a module S over $\mathrm{B}(M,f)$. Since the presence of an electromagnetic potential raises no specific problem, we will forget it. Here (e_1,e_2,e_3,e_4) is any basis of M, (e'_1,e'_2,e'_3,e'_4) is the adjoint basis defined in §6, and ∂_j (with $j = 1,2,3,4$) is the partial derivative along e_j ; since the tensor $\sum_{j=1}^4 e'_j \otimes e_j$ does not depend on the choice of the basis of M, the differential operator $\sum_{j=1}^4 e'_j\partial_j$ (in which the factor e'_j means the operation of e'_j in S) does not depend on it. Duffin proposed a wave equation quite similar to the usual Dirac equation:

$$\text{(b)} \qquad \sum_{j=1}^{4} e'_j\partial_j\psi + \kappa\psi \;=\; 0 \;;$$

the physical constant κ belongs to $\mathbb{R}$ or $i\mathbb{R}$; this point shall be discussed later.

We assume S to be a *graded* module over $\mathrm{B}(M,f)$; anyhow every module over $\mathrm{B}(M,f)$ can be provided with a parity grading (see §5). Besides, when S is provided with a grade automorphism σ (defined at the end of §4), the sign of κ has no importance; indeed if ψ is a solution of the equation (b), then $\sigma(\psi)$ is a solution of the analogous equation in which κ is replaced with $-\kappa$. Theorem (5.3) here means that S is isomorphic to a direct sum of irreducible modules which are each one isomorphic to one of the modules E^k (with $k = 0, 1, 2, 3, 4, 5$). There is no natural action of $\mathbb{C}$ on any one of these irreducible modules; indeed $\mathrm{B}_0(M, f)$ and $\mathrm{B}(M, f)$ are direct sums of ideals in which the center is always isomorphic to $\mathbb{R}$ (see (3.4) and (3.5)). But the action of $\mathrm{B}(M, f)$ in E is Ω-linear (see Theorem (3.2)); thus we obtain modules provided with complex structures if we consider $E^0 \oplus E^5$ or $E^1 \oplus E^4$ or $E^2 \oplus E^3$. Consequently we assume that S is a direct sum of components which are each one isomorphic to one of these three modules. The equation (b) splits into as many equations as there are such components in S; thus we can assume that S is one of these three modules. If $S = E^0 \oplus E^5$, the equation (b) means that $\psi = 0$ (because $\kappa \neq 0$); therefore we can assume that S is either $E^1 \oplus E^4$ or $E^2 \oplus E^3$, with a parity grading that shall soon be discussed. Again no intrusion of imaginary numbers is necessary, since the wanted complex structures appear spontaneously by purely real geometrical constructions; nevertheless here we can identify $\mathbb{C}$ with Ω, whereas such an identification is forbidden by the above formula (a) in the usual Dirac case.

In physical applications every parity grading of S is equivalent to the shifted grading (see §5 for the definition of the shifted module $S^\dagger$). Whereas in the usual Dirac case we can hesitate only between one grading and the shifted grading, in the meson case we can hesitate between two non equivalent gradings, which we call the σ-grading and the $\sigma\tau$-grading by referring to the associated grade automorphisms. Beside σ (the same grade automorphism as in (4.1)) we also consider another involutive transformation τ of E which here is interesting because the dimension n of M is even; it is defined by the equality $\tau(s) = (-1)^k s$ which is valid for all $s \in E^k$. Since σ and τ commute, $\sigma\tau$ is also an involutive transformation.

Proposition 11.1. *The three involutive transformations σ, τ and $\sigma\tau$ satisfy these properties, for all $s \in E$ and for all $a \in M$:*

$$\sigma(is) = -i\sigma(s) , \quad \tau(is) = -i\tau(s) , \quad \sigma\tau(is) = i\sigma\tau(s) ;$$

$$\sigma(as) = -a\sigma(s) , \quad \tau(as) = a\tau(s) , \quad \sigma\tau(as) = -a\sigma\tau(s) .$$

When S is $E^1 \oplus E^4$ or $E^2 \oplus E^3$, every involutive transformation of S that satisfies the same properties as σ (resp. τ, resp. $\sigma\tau$), is equal to $\pm\sigma$ (resp. $\pm\tau$, resp. $\pm\sigma\tau$).

Proof. All the six equalities are evident; the equality $\tau(is) = -i\tau(s)$ follows from $iE^k = E^{n+1-k}$ because here n is even, and the equality $\sigma\tau(as) = -a\sigma\tau(s)$ shows that E is a graded module over $\mathrm{B}(M, f)$ for the $\sigma\tau$-grading too. Now let σ' be an involutive transformation of S that satisfies the same properties as σ (for instance). The equalities $\sigma'(is) = -i\sigma'(s)$ and $\sigma'(as) = -a\sigma'(s)$ are equivalent to $\sigma\sigma'(is) = i\sigma\sigma'(s)$ and $\sigma\sigma'(as) = a\sigma\sigma'(s)$; this means that $\sigma\sigma'$ commutes whith the operation of i in S, and with the operations of all $a \in M$. The subalgebra of $\mathrm{End}_{\mathbb{R}}(S)$ generated by all these operations is equal to $\mathrm{End}_{\mathbb{C}}(S)$. Consequently $\sigma\sigma'$ is the multiplication by some $\lambda \in \mathbb{C}$, and $\sigma'(s) = \lambda\sigma(s)$ (for all s). Since $\sigma'^2 = \mathrm{id}$, we conclude that $\lambda = \pm 1$. □

We continue to use the σ-grading as long as we have no serious reason to prefer the $\sigma\tau$-grading. The following features differentiate these gradings. When S is provided with the σ-grading, it is isomorphic to the shifted module $S^\dagger$; indeed each graded module E^k is isomorphic to $(E^{n+1-k})^\dagger$ by means of the operation of any invertible odd element of Ω. When S is provided with the $\sigma\tau$-grading, it is equal to $(E^1)^\dagger \oplus E^4$ or to $E^2 \oplus (E^3)^\dagger$, therefore isomorphic to $E^4 \oplus E^4$ or to $E^2 \oplus E^2$; moreover the operation of i becomes even as in the usual Dirac case.

For the equation (b) to be sensible for physicists, at least two problems must be immediately settled. The first problem is the invariance of the equation (b) by the action of the orthogonal group $\mathrm{Aut}(M, f)$. Let θ be an element of this group; is there a linear transformation η of S that allows us to claim that a function $\psi : M \to S$ is a solution of the wave equation (b) if and only if $\eta \circ \psi \circ \theta^{-1}$ is a solution? The answer is exactly the same as in the usual Dirac case: if there is an invertible $x \in \mathrm{B}(M, f)$ such that $\theta(a) = xax^{-1}$ for all $a \in M$, this property holds if η is the operation of x in S. From §9 we know that such an x exists for every $\theta \in \mathrm{Aut}(M, f)$. Truly Theorem (9.2) contains a twisting factor $(-1)^{\wp x}$, but after the proof of this theorem it is explained that we can omit it when the dimension of M is even. To prove the invariance of the meson wave equation (b), the physicists undertook infinitesimal calculations which were finally successful after an intervention of the double bracket formula (written just before (9.3)); but such an argument only reaches the neutral connected component of $\mathrm{Aut}(M, f)$, and not the three other ones. Moreover Theorem (9.2) associates exactly one x with every $\theta \in \mathrm{Aut}(M, f)$, whereas in the usual Dirac case the situation is not so easy: with every θ are associated two elements x in $\mathrm{C}\ell(M, f)$ if we impose the extra condition $x^{-1} = \pm\rho(x)$ that confines x to the spin group.

Let us tackle the second problem: from every solution ψ of the equation (b) we must derive a vector field $V : M \to M$ that satisfies this conservation law: its divergence must vanish. This means:

$$\text{(c)} \qquad \sum_{j=1}^{4} \partial_j \, f(e'_j\,, V) \;=\; 0\ .$$

As in the usual Dirac case we derive V from ψ by means of a quadratic mapping $S \to M$ determined by some bilinear form $g : S \times S \to \mathbb{R}$ satisfying these three properties: first it is symmetric; secondly $g(is, t) = -g(s, it)$ for all $s, t \in S$; thirdly for all $a \in M$ (and all s and t) either $g(as, t) = g(s, at)$ (and then we set $g = g_1$) or $g(as, t) = -g(s, at)$ (and then we set $g = g_2$). Although the physicists prefer the sesqui-linear form $S \times S \to \mathbb{C}$ defined by $(s, t) \longmapsto g(s, t) + ig(is, t)$, the real form g is quite sufficient.

Proposition 11.2. *Let g_1 and g_2 be two bilinear forms $S \times S \to \mathbb{R}$ satisfying the three conditions written above, let ψ be a function $M \to S$, and V_1 and V_2 the vector fields $M \to M$ defined in this way:*

$$\forall a \in M, \quad f(a, V_1) = g_1(a\psi, \psi) \quad \textit{and} \quad f(a, V_2) = g_2(a\psi,\, i\psi)\ .$$

If ψ satisfies a wave equation (b) *in which the constant κ is purely imaginary, then V_1 satisfies the conservation equation* (c). *If ψ satisfies a wave equation* (b) *in which the constant κ is real, then V_2 satisfies the conservation equation* (c).

This statement can be verified by straightforward calculations; let us consider the case of V_1 :

$$\sum_j \partial_j f(e'_j, V_1) \;=\; \sum_j \partial_j g_1(e'_j\psi, \psi) \;=\; \sum_j g_1(e'_j\partial_j\psi,\, \psi) + \sum_j g_1(\psi,\, e'_j\partial_j\psi)$$
$$= -g_1(\kappa\psi, \psi) - g_1(\psi, \kappa\psi) \;=\; 0 \quad \text{if} \quad \kappa \in i\mathbb{R}\,.$$

If you object that the vector field V defined by the identity $f(a, V) = g_1(a\psi, i\psi)$ also satisfies (c) when ψ satisfies (b) with $\kappa \in \mathbb{R}$, we will remind you that this definition of V implies $V = -V$, whence $V = 0$. Similarly the definition $f(a, V) = g_2(a\psi, \psi)$ implies $V = 0$.

Now we must find bilinear forms g_1 and g_2 satisfying the required properties. In Proposition (4.1) we immediately recognize that $(s, t) \longmapsto f_E(\sigma(s), t)$ is an example of a bilinear form g_2 ; and from Proposition (11.1) we deduce that $(s, t) \longmapsto f_E(\tau(s), t)$ is an example of a bilinear form g_1 .

Proposition 11.3. *When S is equal to $E^1 \oplus E^4$ or $E^2 \oplus E^3$, the bilinear forms $g_1 : S \times S \to \mathbb{R}$ satisfying the three required properties constitute a vector space of dimension* 1 *over $\mathbb{R}$. The same assertion is true for the bilinear forms g_2 .*

Proof. We have already found a bilinear form g_1 that satisfies the required properties, and that is nondegenerate; therefore if g'_1 is another bilinear form $S \times S \to \mathbb{R}$, there is a $\mathbb{R}$-linear mapping $u : S \to S$ such that $g'_1(s, t) = g_1(u(s), t)$ (for all s and t). The condition $g'_1(is, t) = -g'_1(s, it)$ is equivalent to $u(is) = iu(s)$; the condition $g'_1(as, t) = g'_1(s, at)$ is equivalent to $u(as) = au(s)$; as we did in the proof of (11.1), here we realize that u is the multiplication by some constant $\lambda \in \mathbb{C}$. Finally the condition $g'_1(s, t) = g'_1(t, s)$ implies $\lambda \in \mathbb{R}$. For the bilinear forms g_2 the proof is the same. □

It is not difficult to calculate V_1 and V_2 when g_1 and g_2 are the bilinear forms proposed just before (11.3) When $S = E^1 \oplus E^4$, there are two vector fields b and c ($M \to M$) and two functions λ and μ ($M \to \mathbb{R}$) such that $\psi = (b, \lambda; i\mu, ic)$; in other words, b, λ, $i\mu$, ic are the components of ψ in E^1_0 , E^1_1 , E^4_0 , E^4_1 . From the equalities $f(a, V_1) = f_E(\tau(a\psi), \psi)$ and $f(a, V_2) = f_E(\sigma(a\psi), i\psi)$ we deduce:

$$V_1 \;=\; -2\lambda b - 2\mu c \quad \text{and} \qquad V_2 \;=\; 2\mu b - 2\lambda c\,.$$

When $S = E^2 \oplus E^3$, there are two vector fields b and c and two functions u and v with values in $\bigwedge^2(M)$ such that $\psi = (u, b; ic, iv)$; this implies $a\psi = (a \wedge b\,,\, \mathrm{d}_f(a) \rfloor u\,;\, i(\mathrm{d}_f(a) \rfloor v),\, i(a \wedge c))$, and after some calculations we find:

$$V_1 \;=\; -2\mathrm{d}_f(b) \rfloor u - 2\mathrm{d}_f(c) \rfloor v \quad \text{and} \qquad V_2 \;=\; 2\mathrm{d}_f(b) \rfloor v - 2\mathrm{d}_f(c) \rfloor u\,.$$

In both cases there is no obvious relation between the vector fields V_1 and V_2 ; nevertheless in the usual Dirac case there are important relations between them. Let us remember that the propositions (11.2) and (11.3) hold true in this usual case too; with every (nondegenerate) bilinear form g_1 (satisfying the three required properties) is associated a bilinear form g_2 defined by $g_2(s,t) = g_1(\omega s,\, t)$; indeed the equality $\omega a = -a\omega$ holds in $C\ell(M,f)$, and the operations of ω and i in S commute. From the Fierz relations (which hold even if ψ does not satisfy the Dirac equation) we can extract these relations involving the resulting vector fields V_1 and V_2:

$$f(V_1,V_1) = -f(V_2,V_2) \geq 0 \quad \text{and} \quad f(V_1,V_2) = 0\ .$$

The vector V_1 is a Time-like vector that keeps a constant orientation; for a suitable choice of g_1 it is always oriented toward Future; this explains why the physicists impose the conservation equation (c) to V_1. With every observer is associated a unit vector e oriented toward Future, and with every function $\psi : M \to S$ this observer associates a density function $M \to \mathbb{R}$ that is equal to $f(e,V_1)$; since e and V_1 are both Time-like vectors oriented toward Future, this density is always ≥ 0.

The physicists well realized that in the meson case they could not derive from ψ (a solution of the meson wave equation (b)) a vector field V that both satisfied the equation (c) and allowed the definition of a nonnegative density. Since discussions about the density are beyond our competence, we merely recall this piece of information extracted from [K1] : *"the density ... is, of course, not necessarily positive, but the discussion by Pauli and Weisskopf (1934) has proved that this is in fact not a necessary requirement in the relativistic region"*.

A last question deserves a discussion, especially since Duffin used the bilinear form $-f$ whereas here we prefer f like most physicists to-day: what happens when f is replaced with $-f$? Since we use modules S provided with a parity grading, we use a category that is equivalent to the category of modules over the even subalgebra (see Theorem (5.1)), provided that we refuse trivial modules like E^0 or E^5 in the meson case (see Lemma (5.2)); thus there is no problem since $C\ell_0(M,-f)$ (resp. $B_0(M,-f)$) is canonically isomorphic to $C\ell_0(M,f)$ (resp. $B_0(M,f)$) by an isomorphism that maps every product ab of two vectors to $-ab$. Every module over $C\ell(M,f)$ or $B(M,f)$ that is provided with a complex structure, is naturally a module over $C\ell(M,-f)$ or $B(M,-f)$ too; indeed the operators $s \longmapsto ias$ satisfy the condition that allows them to induce a representation of $C\ell(M,-f)$ or $B(M,-f)$. Now every solution ψ of (b) is also a solution of $\sum_j ie'_j\partial_j\psi + (i\kappa)\psi = 0$; this shows that the constant κ in the wave equation (b) must be multiplied by $\pm i$ when f is replaced with $-f$. Because of Proposition (11.2) the vector field V_1 must be replaced with V_2 (or conversely), and g_1 must be replaced with g_2 (or conversely). This conclusion is corroborated by this observation: since $g_2(ias,\, t) = g_2(s,\, iat)$, the behaviour of g_2 with respect to the operators $s \longmapsto ias$ is the same as the behaviour of g_1 with respect to the operators $s \longmapsto as$. Besides, in the meson case we deduce from Proposition (11.1) that τ behaves with respect to the operators $s \longmapsto ias$ like σ with respect to the operators $s \longmapsto as$, and it is clear that g_1 is related to τ in the same way as g_2 to σ. Without any reference to the operators $s \longmapsto ias$, it is also known that in the usual Dirac case, which requires a

Time-like vector field V oriented toward Future, this property is satisfied by $\pm V_2$ when f is replaced with $-f$.

Duffin used $-f$ and proposed a meson wave equation (b) with a real constant κ, and his choice was based on the physical argument presented by Proca; consequently here we must write a wave equation (b) with a constant κ in $i\mathbb{R}$, and we must choose the vector field V_1 as in the usual Dirac case.

References

[AM1] M. El-Agawany, A. Micali, *Algèbre mésonique d'une forme bilinéaire symétrique.* Rend. Sem. Mat. Univ. Politec. Torino (1) (1980), 1-12.

[AM2] M. El-Agawany, A. Micali, *Algèbre mésonique d'un module.* Communications in Algebra **13** (1985), 399-418.

[BT] S.B. Batdorf, R. Thomas, *Burst production by mesotrons of spin one-half and zero magnetic moment.* Physical Review **59** (1941), 621.

[B] N. Bourbaki, *Algèbre*, chap.III, §11, Hermann, Paris, 1970.

[D] R.J. Duffin, *On the chacarcteristic matrices of covariant systems.* Physical Review **54** (1938), 1114.

[F] H. Frölich, Ch. Terreaux, *Isobaric spin algebra.* Nuclear Physics **42** (1963), 21-26.

[H] J. Helmstetter, *Interior Multiplications and Deformations with Meson Algebras.* Advances in Applied Clifford Algebras **18** (2008), 153-196.

[J] N. Jacobson, *Structure and representations of Jordan algebras.* Amer. Math. Soc., Colloquium Publications **39**, Providence RT, 1968.

[K1] N. Kemmer, *The particle aspect of meson matrices.* Proc. Royal Soc. London, Serie A, **173** (1939), 91-116.

[K2] N. Kemmer, *The algebra of meson matrices.* Proc. Cambridge Phil. Soc. **39** (1943), 189-196.

[L] D.E. Littlewood, *An equation of Quantum Mechanics.* Proc. Cambridge Phil. Soc. **43** (1947), 406-413.

[MR] A. Micali, M. Rachidi, *On meson algebras.* In Proceedings of ICCA7 (Toulouse, May 2005), Advances in Applied Clifford Algebras, (2008).

[OK] Y. Ohnuki, S. Kamefuchi, *Quantum field theory and Parastatistics.* University of Tokyo Press, Springer-Verlag 1982.

[PP] D.N. Poenaru, *Alexandru Proca, the great physicist.* arXiv:physics/0508195v1 [physics.hist-ph] 26 Aug. 2005.

[P] A. Proca, *Sur la théorie ondulatoire des électrons positifs et négatifs.* Le Journal de Physique et le Radium tome VII, série VII (1936), 347-353.

[Sch] E. Schrödinger, *Pentads, tetrads and triads of meson-matrices.* Proc. Royal Irish Acad. A, **48** (1943), 135-146.

[Sh] T. Shimpuku, *Symmetric algebras by direct product of Clifford algebra.* Seikunsha Publishers, Osaka 1988.

[T] Z. Tokuoka, *Generating relations of the sub-algebras in the Duffin-Kemmer algebra.* Nuclear Physics **78** (1966), 681-693.

[Y] H. Yukawa, *On the interaction of elementary particles I.* Proc. Phys.-Math. Soc. Japan **17** (1935), 48.

Jacques Helmstetter
Université Grenoble I
Institut Fourier (Mathématiques)
B.P. 74
38402 Saint-Martin d'Hères
France
e-mail: Jacques.Helmstetter@ujf-grenoble.fr

Artibano Micali
Université Montpellier II
Département des Sciences Mathématiques
Place Eugène Bataillon
34095 Montpellier Cedex 5
France
e-mail: micali@math.univ-montp2.fr

Received: November 22, 2007
Accepted: January 5, 2008