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Extended Grassmann and Clifford Algebras

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Abstract. This paper is intended to investigate Grassmann and Clifford algebras over Peano spaces, introducing their respective associated extended algebras, and to explore these concepts also from the counterspace viewpoint. The presented formalism explains how the concept of chirality stems from the bracket, as defined by Rota et all [1]. The exterior (regressive) algebra is shown to share the exterior (progressive) algebra in the direct sum of chiral and achiral subspaces. The duality between scalars and volume elements, respectively under the progressive and the regressive products is shown to have chirality, in the case when the dimension n of the Peano space is even. In other words, the counterspace volume element is shown to be a scalar or a pseudoscalar, depending on the dimension of the vector space to be respectively odd or even. The de Rham cochain associated with the differential operator is constituted by a sequence of exterior algebra homogeneous subspaces subsequently chiral and achiral. Thus we prove that the exterior algebra over the space and the exterior algebra constructed on the counterspace are only *pseudoduals* each other, if we introduce chirality. The extended Clifford algebra is introduced in the light of the periodicity theorem of Clifford algebras context, wherein the Clifford and extended Clifford algebras $C\ell_{p,q}$ can be embedded in $C\ell_{p+1,q+1}$, which is shown to be exactly the extended Clifford algebra. We present the essential character of the Rota's bracket, relating it to the formalism exposed by Conradt [25], introducing the regressive product and subsequently the counterspace. Clifford algebras are constructed over the counterspace, and the duality between progressive and regressive products is presented using the dual Hodge star operator. The differential and codifferential operators are also defined for the extended exterior algebras from the regressive product viewpoint, and it is shown they uniquely tumble right out progressive and regressive exterior products of 1-forms.

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Introduction

Grassmann[1] and Clifford algebras have played an essential role in modern physics (see, e.g., [2, 3, 4]) since their discovery [15]. This work is intended to give a precise mathematical formulation of the concept of chirality associated with these algebras, which is defined as a multiplication by a pseudoscalar[2] ε that satisfies $\varepsilon^2 = 1$. The mathematical description of chirality has fundamental importance, particularly in the context of the extended Clifford algebras to be presented here, and it may bring a deep understanding of Nature. Our viewpoint shed some new light on these old fundamental concepts, and can be applied immediately to physics. In particular the formulation of electromagnetic theory [2, 14] in this formalism is more natural, correct, precise and geometrically sensible if differential forms intrinsically endowed with chirality are used. The metric-free formulation of electrodynamics brings a geometric character and a clear physical interpretation, and the formalism exhibited in some manuscripts [5] motivates the formulation using the Rota's bracket. The (Rota's) bracket, a pseudoscalar that gives chirality to differential forms and multivectors in the Grassmann and Clifford algebras over a Peano space, is presented. Although the extended Grassmann algebra formalism has a didactic explanation for instance in Jancewicz's paper [5], in the light of the Rota's bracket this formalism can be alternatively explored, by introducing the regressive product [15, 25]. After defining chiral differential forms, which changes sign under orientation change, the extended exterior algebra is defined and discussed, constructed as a direct sum of two copies (chiral and achiral) of exterior algebras. We present the quasi-Hodge dual star operators and their chiral partners. After introducing a metric in the Peano space, the extended Grassmann and Clifford algebras are introduced together with the chiral dual Hodge star operators. The regressive product is defined together with the concept of counterspace, preserving the principle of duality [25, 6]. An analogue of the Morgan law to the Grassmann-Cayley algebra, defined to be the Grassmann extended algebra endowed with the regressive product, is also investigated, and the counterspace volume element is shown to be scalar *or* pseudoscalar, depending on the space dimension to be, respectively, odd *or* even. The de Rham cochain, generated by the codifferential operator related to the regressive product, is composed by a sequence of exterior algebra homogeneous subspaces that are subsequently chiral and achiral. This is an astonishing character of the formalism to be presented, since the duality between exterior algebras associated respectively with the space and counterspace is irregular, in the sense that if we take the exterior algebra duality associated with the space, we obtain the exterior algebra associated with the counterspace, but the converse produces the space exterior algebra, which homogeneous even [odd] subspaces are chiral [achiral], depending on the original vector space dimension

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[1] Grassmann algebras are defined as exterior algebras endowed with a metric structure. In this sense exterior algebras are vector spaces (endowed with the wedge product) devoid of a metric.
[2] Pseudoscalars are scalars that change sign under orientation change, and this denomination is not to be confused with pseudoscalars as elements of $\Lambda^n(V)$.

(see eq.(7.6) below). The present formalism explains how the concept of chirality stems from the bracket defined by G.-C. Rota [1].

Denoting $\overset{o}{V}$ the chiral vector space associated with V, the embedding of a vector space V in the vector space $V \oplus \overset{o}{V}$ is necessary in order to be possible to correctly introduce extended Clifford algebras. The units respectively associated with the field $\mathbb{R}$ and $\overset{o}{\mathbb{R}}$, over which V and $\overset{o}{V}$ are constructed, are considered to be distinct, as the metrics in each one of these spaces. Besides, the metric in $V \oplus \overset{o}{V}$ that takes values in distinct subspaces of $V \oplus \overset{o}{V}$ is defined to be identically null, otherwise it can be shown several inconsistencies in the formulation. In a natural manner, the metric in $V \oplus \overset{o}{V}$ is the (direct) sum of the metrics in V and $\overset{o}{V}$. The unit of $V \oplus \overset{o}{V}$ is the sum of the units of V and $\overset{o}{V}$. As $V \simeq \mathbb{R}^{p,q}$, each of the objects acting on $V \oplus \overset{o}{V}$ are shown to be elements of the Clifford algebra $C\ell_{p+1,q+1}$, which is essentially the extended Clifford algebra.

The regressive product is introduced together with the counterspace, providing a formal pre-requisite to define Clifford algebras over the counterspace [25], illustrating in this way the counterspace dual character. Dualities and codualities are defined in space and counterspace, from the use of the dual Hodge star operator. The dual character of contraction operators, defined in space and counterspace, is also established, following Conradt's route [25]. The codifferential operator is uniquely defined in terms of the regressive and progressive exterior products.

This paper is organized as follows: after presenting algebraic preliminaries in Section 1, in Section 2 the Rota's bracket and Peano spaces are introduced. In Section 3 we present the extended exterior algebra and the chiral quasi-Hodge star operators. In Section 4 the extended Grassmann and Clifford algebras, in the context of the Periodicity Theorem, are defined, and in Section 5 the embedding of a vector space in the extended vector space is considered, after which in Section 6 the regressive product is introduced. In Section 7 the chiral counterspace is defined and investigated together with its constituents, the differential coforms. Besides, the counterspace volume element with respect to the regressive product is undefined to be a scalar or a pseudoscalar until we specify whether the dimension of the Peano space is respectively odd or even, by showing that the volume element constructed from a cobasis of the counterspace is a scalar or a pseudoscalar, depending on the vector (Peano) space dimension. In Section 8 Clifford algebras over the counterspace are constructed, and in Section 9 the duality and coduality principles are introduced in space and counterspace, showing a close relation involving the regressive and progressive products, and the dual Hodge star operator. Also, the contraction in counterspace, after defined, is investigated in the light of the duality and coduality. Finally in Section 10 the differential and codifferential operators are introduced in the counterspace context, in an alternative extended formalism.

1. Preliminaries

Let V be a finite n-dimensional real vector space. We consider the tensor algebra $\bigoplus_{i=0}^{\infty} T^i(V)$ from which we restrict our attention to the space $\Lambda(V) = \bigoplus_{k=0}^{n} \Lambda^k(V)$ of multicovectors over V. $\Lambda^k(V)$ denotes the space of the antisymmetric k-tensors, the k-forms. Given $\psi \in \Lambda(V)$, $\tilde{\psi}$ denotes the *reversion*, an algebra antiautomorphism given by $\tilde{\psi} = (-1)^{[k/2]}\psi$ ($[k]$ denotes the integer part of k). $\hat{\psi}$ denotes the *main automorphism or graded involution*, given by $\hat{\psi} = (-1)^k\psi$. The *conjugation* is defined as the reversion followed by the main automorphism. If V is endowed with a non-degenerate, symmetric, bilinear map $g : V \times V \to \mathbb{R}$, it is possible to extend g to $\Lambda(V)$. Given $\psi = \mathbf{u}_1 \wedge \cdots \wedge \mathbf{u}_k$ and $\phi = \mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_l$, $\mathbf{u}_i, \mathbf{v}_j \in V$, one defines $g(\psi, \phi) = \det(g(\mathbf{u}_i, \mathbf{v}_j))$ if $k = l$ and $g(\psi, \phi) = 0$ if $k \neq l$. Finally, the projection of a multivector $\psi = \psi_0 + \psi_1 + \cdots + \psi_n$, $\psi_k \in \Lambda^k(V)$, on its p-vector part is given by $\langle\psi\rangle_p = \psi_p$. The Clifford product between $\mathbf{w} \in V$ and $\psi \in \Lambda(V)$ is given by $\mathbf{w}\psi = \mathbf{w} \wedge \psi + \mathbf{w} \cdot \psi$. The Grassmann algebra $(\Lambda(V), g)$ endowed with this product is denoted by $C\ell(V, g)$ or $C\ell_{p,q}$, the Clifford algebra associated with $V \simeq \mathbb{R}^{p,q}$, $p + q = n$.

2. Peano Spaces

Let V be an n-dimensional vector space over a field[3] $\mathbb{R}$. A basis $\{\mathbf{e}_i\}$ of V is chosen, and V^* denotes the dual space associated with V, which has a basis $\{\mathbf{e}^i\}$ satisfying $\mathbf{e}^i(\mathbf{e}_j) = \delta^i_j$. Since dim V^* = dim V, there exists a non-canonical isomorphism between V and V^*. A Peano space is a pair $(V, [\quad])$, where $[\quad]$ is an alternate n-linear form over $\mathbb{R}$, the *bracket*, defined as the map $[\quad] : \underbrace{V \times V \times \cdots \times V}_{n \text{ times}} \to \mathbb{R}$ with the properties:

1. For all $\mathbf{w}_1, \mathbf{w}_2 \in V$ and $\mu, \nu \in \mathbb{R}$,
$$[\mathbf{v}_1, \ldots, \mathbf{v}_{i-1}, \mu\mathbf{w}_1 + \nu\mathbf{w}_2, \mathbf{v}_{i+1}, \ldots, \mathbf{v}_n] = \mu[\mathbf{v}_1, \ldots, \mathbf{v}_{i-1}, \mathbf{w}_1, \mathbf{v}_{i+1}, \ldots, \mathbf{v}_n] + \nu[\mathbf{v}_1, \ldots, \mathbf{v}_{i-1}, \mathbf{w}_2, \mathbf{v}_{i+1}, \ldots, \mathbf{v}_n];$$
2. $[\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n] = \text{sign}(\sigma)[\mathbf{v}_{\sigma(1)}, \mathbf{v}_{\sigma(2)}, \ldots, \mathbf{v}_{\sigma(n)}]$, where σ is a permutation of the set $\{1, 2, \ldots, n\}$.

Indeed, the bracket is an element of $\Lambda^n(V)$. A Peano space is called *standard* if there exists a basis $\{\mathbf{u}_i\}$ of vectors in V such that $[\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n] \neq 0$ [16]. Unless otherwise stated we assume standard Peano spaces and they shall be denoted uniquely by V. The vectors $\mathbf{w}_1$ and $\mathbf{w}_2$ are linearly independent if there exists $n-2$ vectors $\mathbf{u}_3, \ldots, \mathbf{u}_n$ such that $[\mathbf{w}_1, \mathbf{w}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n] \neq 0$. If another basis $\{\mathbf{v}_i\}$ of V is taken, its bracket in terms of the bracket computed at the basis $\{\mathbf{e}_i\}$ is given by $[\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n] = \det(v_i^j)\ [\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n]$, where $\mathbf{v}_i = v_i^j\mathbf{e}_j$. The number $\det(v_i^j)$ is positive [negative] if the bases $\{\mathbf{e}_i\}$ and $\{\mathbf{v}_i\}$ have the same

[3] Here we shall consider any field $\mathbb{F}$ with characteristic different of two. In particular the complex field $\mathbb{C}$ could be used, but we prefer to use $\mathbb{R}$ in order to clarify the concepts to be introduced.

[opposite] orientation, where the orientation in V is defined as a choice in $\mathbb{Z}_2$ of an equivalence class of a basis in V, i.e., two basis are equivalent if they have the same orientation. Any space V has only two possible orientations, according to the sign of $\det(v_i^j)$ and any even permutation of basis elements induces the same orientation. A basis $\{\mathbf{e}_i\}$ can be transformed as $\mathbf{e}_i \mapsto A\mathbf{e}_i$, where $A \in \mathrm{Hom}(V)$ is a preserving orientation homomorphism, and it still represents the same orientation, since the bracket is non-null. The basis $\{\mathbf{e}_i\}$ is denominated a unimodular basis if $[\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n] = 1$. But as Rota pointed out [1], the orientation can be equivalently defined by an ordered sequence of vectors entering the bracket, and for instance $[\mathbf{e}_2, \mathbf{e}_1, \ldots, \mathbf{e}_n] = -1$ (following from the 2$^{\text{nd}}$ property of bracket definition above). Then the value assumed by the bracket on an ordered sequence of [unit] vectors can assume negative and positive values [± 1], defining two equivalence classes, immediately related to the two possible values for orientation. Hereafter we denote $\varepsilon = [\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n]$. The term ε^2 does not change sign under orientation change and $\varepsilon^2 = 1$. The map $\varepsilon \mapsto -\varepsilon$ corresponds to a orientation change in V and it is clear that $\varepsilon = [\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n] = (-1)^{i-1}[\mathbf{e}_i, \mathbf{e}_1, \mathbf{e}_2, \ldots, \check{\mathbf{e}}_i, \ldots, \mathbf{e}_n]$, where $\check{\mathbf{e}}_i$ means that $\mathbf{e}_i$ is absent from the bracket. Since there is always a natural correspondence between a vector space V and its dual V^*, all considerations above can be asserted *mutatis mutandis* for V^*, and hereon Rota's bracket, now defined as taking values at the dual space, by abuse of notation is also denoted by $[\quad] : \underbrace{V^* \times V^* \times \cdots \times V^*}_{n \text{ times}} \to \mathbb{R}$.

Consider now a canonically isomorphic copy of V^*, denoted by $\overset{o}{V}{}^*$, with a basis $\{\overset{o}{\mathbf{e}}{}^i\}$. This new basis maps vectors in pseudoscalars, according to the definition:

$$\overset{o}{\mathbf{e}}{}^i(\mathbf{e}_j) = (-1)^{i-1}[\mathbf{e}_j, \mathbf{e}_1, \mathbf{e}_2, \ldots, \check{\mathbf{e}}_i, \ldots, \mathbf{e}_n] = \varepsilon\delta^i_j = \varepsilon\mathbf{e}^i(\mathbf{e}_j). \tag{2.1}$$

We can write

$$\overset{o}{\mathbf{e}}{}^i = \varepsilon\mathbf{e}^i. \tag{2.2}$$

Covectors of $\overset{o}{V}{}^*$ change sign under orientation change[4]. Multiplication by ε is clearly an isomorphism between V^* and $\overset{o}{V}{}^*$.

3. The Extended Exterior Algebra

In this section we establish the notion of an extended exterior algebra, using the pseudoscalar ε, and also chiral quasi-Hodge star operators are introduced.

[4]A chiral covector $\overset{o}{\mathbf{e}}{}^i$ is defined directly from its action on a vector of V and $\overset{o}{\mathbf{e}}{}^i(\cdot) = (-1)^{i-1}[\ \cdot, \mathbf{e}_1, \mathbf{e}_2, \ldots, \check{\mathbf{e}}_i, \ldots, \mathbf{e}_n]$.

3.1. The wedge product from the bracket

From a Peano (dual) space V^* an exterior algebra can be constructed, by introducing equivalence classes of ordered vector sequences, using the bracket [16]. Given $\mathbf{a}^i, \mathbf{b}^i \in V^*$, two sequences are said to be equivalent, and denoted by $\mathbf{a}^1, \ldots, \mathbf{a}^k \sim \mathbf{b}^1, \ldots, \mathbf{b}^k$, if for any choice of covectors $\mathbf{v}^{k+1}, \ldots, \mathbf{v}^n \in V^*$ it follows that $[\mathbf{a}^1, \ldots, \mathbf{a}^k, \mathbf{v}^{k+1}, \ldots, \mathbf{v}^n] = [\mathbf{b}^1, \ldots, \mathbf{b}^k, \mathbf{v}^{k+1}, \ldots, \mathbf{v}^n]$. The wedge[5] product between two covectors $\mathbf{e}^i, \mathbf{e}^j \in V^*$ is defined as the elements of the quotient space $T(V^*)/J$, where J denotes the bilateral ideal generated by elements of the form $a \otimes x \otimes x \otimes b$, where $x \in V$ and $a, b \in T(V)$. In what follows we write [16]

$$\mathbf{e}^i \wedge \mathbf{e}^j = \mathbf{e}^i \otimes \mathbf{e}^j \mod \sim \tag{3.1}$$

where $\mathbf{e}^i \otimes \mathbf{e}^j \in V^* \otimes V^*$. For more details see [1, 16]. A k-covector is defined inductively by the wedge product of k covectors, and each k-covector lives in $\Lambda^k(V)$. The exterior algebra is naturally defined as being $\Lambda(V) = \bigoplus_{k=0}^n \Lambda^k(V)$. Analogously chiral k-covectors, elements of $\overset{o}{\Lambda}{}^k(V)$, are defined as the wedge product of elements in V^* and an odd number of elements in $\overset{o}{V}{}^*$. We also define the chiral exterior algebra, which elements change sign under orientation change, as $\overset{o}{\Lambda}(V) := \bigoplus_{k=0}^n \overset{o}{\Lambda}{}^k(V)$. We denote $\Lambda^0(V) = \mathbb{R}$ (scalars), $\overset{o}{\Lambda}{}^0(V) = \varepsilon\mathbb{R}$ (pseudoscalars), $\Lambda^1(V) = V^*$ and $\overset{o}{\Lambda}{}^1(V) = \overset{o}{V}{}^*$. The extended exterior algebra is defined as

$$\breve{\Lambda}(V) = \Lambda(V) \oplus \overset{o}{\Lambda}(V). \tag{3.2}$$

Such an algebra is $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded, where the first $\mathbb{Z}_2$-grading is related to the differential forms[6] chirality and the second one is related to the subspaces $\Lambda^p(V)$, where p is either even or odd. The inclusions

$$\Lambda^k(V) \wedge \Lambda^l(V) \hookrightarrow \Lambda^{k+l}(V), \qquad \overset{o}{\Lambda}{}^k(V) \wedge \overset{o}{\Lambda}{}^l(V) \hookrightarrow \Lambda^{k+l}(V),$$
$$\overset{o}{\Lambda}{}^k(V) \wedge \Lambda^l(V) \hookrightarrow \overset{o}{\Lambda}{}^{k+l}(V)$$

hold. A differential form is said to be *chiral* if it is multiplied by ε, and consequently every chiral form changes sign under orientation change. Obviously the exterior algebra $\overset{o}{\Lambda}(V)$ is chiral by construction. From eq.(2.2), multiplication by the pseudoscalar ε gives a natural isomorphism between $\Lambda(V)$ and $\overset{o}{\Lambda}(V)$. Achirla forms are mapped in chiral forms through multiplication by ε in such a way that $\overset{o}{\Lambda}{}^k(V) = \varepsilon\Lambda^k(V)$. From the relation $\varepsilon^2 = 1$, the set $\{1, \varepsilon\}$ generates the real algebra $\mathbb{D} = \mathbb{R} \oplus \mathbb{R}$ of hyperbolic (or perplex, or pseudocomplex, or Study) numbers [4, 17, 18]. An element of $\mathbb{D}$ can be written as $a + b\varepsilon, a, b \in \mathbb{R}$. So the extended

[5]Grassmann [15] in his original work called this product the progressive product.

[6]Hereon it will be implicit that when we refer to differential forms, there is considered a manifold M and its associated tangent space [cotangent space] $T_xM \simeq V$ $[T_x{}^*M \simeq V^*]$ at a point $x \in M$. Then a differential form is an element of a section $\sec \Lambda(T^*M)$ of the cotangent exterior bundle.

exterior algebra

$$\breve{\Lambda}(V) = \Lambda(V) \oplus \overset{o}{\Lambda}(V) = \Lambda(V) \oplus \varepsilon\Lambda(V)$$

can be written as $\breve{\Lambda}(V) = \mathbb{D} \otimes \Lambda(V)$. If a $2n$-dimensional dual space is considered (n achirla forms and n chiral forms), then only the exterior algebra generated by the n achirla forms is needed, since the other (chiral) forms can be generated from multiplication by ε.

Given an arbitrary basis $\{\mathbf{e}_i\}$ of V, a chiral vector space $\overset{o}{V}$ is defined to be the vector space spanned by vectors $\overset{o}{\mathbf{e}}_i := \varepsilon\mathbf{e}_i$. In this sense, the same formulation is valid both for vector fields *and* for differential forms.

3.2. Dual chiral quasi-Hodge isomorphisms

Consider an n-vector $\Theta = a\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_n$ and an n-covector $\Upsilon = a'\mathbf{e}^1 \wedge \cdots \wedge \mathbf{e}^n$, where a and a' are scalars. Denoting $\lrcorner$ the (left) contraction, we have the relation[7]:

$$\widetilde{\Upsilon}\lrcorner\Theta = aa'(\mathbf{e}^n \wedge \cdots \wedge \mathbf{e}^1)\lrcorner(\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_n) = aa', \tag{3.3}$$

such that

$$0 \neq \widetilde{\Upsilon}\lrcorner\Theta = \begin{cases} > 0 & \text{if } a > 0 \text{ and } a' > 0, \text{ or } a < 0 \text{ and } a' < 0, \\ < 0 & \text{if } a > 0 \text{ and } a' < 0, \text{ or } a < 0 \text{ and } a' > 0. \end{cases}$$

The orientation of V can be related to the orientation of the dual V^*. Both orientations of V and V^*, which are respectively determined by Θ and Υ, are said to be compatible, if $\widetilde{\Upsilon}\lrcorner\Theta > 0$. Assuming the orientations of V and V^* to be compatible, if we choose an orientation for one of these spaces, the orientation of the other one is completely defined. In this case Υ is chosen such that $\widetilde{\Upsilon}\lrcorner\Theta = 1$.

Denoting $\Lambda_k(V) = \Lambda^k(V^*)$ the space of k-vectors, the dual quasi-Hodge star operators are defined as

$$\begin{aligned} \underline{\star} : \Lambda_k(V) &\to \Lambda^{n-k}(V) \\ \psi_k &\mapsto \underline{\star}\psi_k = \widetilde{\psi_k}\lrcorner\Upsilon \end{aligned} \tag{3.4}$$

($\underline{\star}1 = \Upsilon$) and

$$\begin{aligned} \overline{\star} : \Lambda^k(V) &\to \Lambda_{n-k}(V) \\ \psi^k &\mapsto \overline{\star}(\psi^k) = \widetilde{\psi^k}\lrcorner\Theta \end{aligned} \tag{3.5}$$

($\overline{\star}1 = \Theta$). It follows that $\underline{\star}\,\overline{\star} = \overline{\star}\,\underline{\star} = (-1)^{k(n-k)}1$. Analogously we define the dual chiral quasi-Hodge star operators

$$\begin{aligned} \underline{\star}_\varepsilon : \Lambda_k(V) &\to \overset{o}{\Lambda}{}^{n-k}(V) \\ \psi_k &\mapsto \underline{\star}_\varepsilon(\psi_k) = \varepsilon\widetilde{\psi_k}\lrcorner\Upsilon \end{aligned} \tag{3.6}$$

[7]Hereafter we denote $\tilde{\psi}$ the main anti-automorphism of exterior algebras acting on a form ψ.

($\underline{\star}_\varepsilon 1 = \varepsilon\Upsilon$) and

$$\begin{array}{rcl}\overline{\star}_\varepsilon : \overset{o}{\Lambda}{}^k(V) & \to & \Lambda_{n-k}(V)\\ \varepsilon\psi^k & \mapsto & \overline{\star}_\varepsilon(\varepsilon\psi^k) = \widetilde{\psi^k}\lrcorner\Theta\end{array} \tag{3.7}$$

($\overline{\star}_\varepsilon\varepsilon = \Theta$). Obviously $\underline{\star}_\varepsilon = \varepsilon\underline{\star}$ and $\overline{\star}_\varepsilon = \varepsilon\overline{\star}$.

4. The Extended Grassmann Algebra

By considering an isomorphism $V \simeq V^*$ a correlation is defined to be a linear map[8] $\tau : V \to V^*$, which induces a (bilinear, symmetric, non-degenerate) metric $g : V^* \times V^* \to \mathbb{R}$ as $g(\mathbf{e}^i, \mathbf{e}^j) = \tau^{-1}(\mathbf{e}^i)(\mathbf{e}^j) = g^{ik}\mathbf{e}_k(\mathbf{e}^j) = g^{ik}\delta^j_k = g^{ij}$. The extended Grassmann algebra is defined to be the extended exterior algebra endowed with the induced metric. Taking two copies of V^* (V^* and $\overset{o}{V}{}^*$), a correlation for each one of these copies is defined:

$$\begin{array}{rll}\tau: & V & \to V^*\\ & \mathbf{e}_i & \mapsto \tau(\mathbf{e}_i) = g_{ij}\mathbf{e}^j\end{array} \qquad \begin{array}{rll}\overset{o}{\tau} : & V & \to \overset{o}{V}{}^*\\ & \mathbf{e}_i & \mapsto \tau(\mathbf{e}_i) = \overset{o}{g}_{ij}\overset{o}{\mathbf{e}}{}^j = \varepsilon\overset{o}{g}_{ij}\mathbf{e}^j\end{array}$$

and the associated metrics are given by:

$$\begin{array}{rl}g : V^* \times V^* & \to \mathbb{R}\\ (\mathbf{e}^i, \mathbf{e}^j) & \mapsto g(\mathbf{e}^i, \mathbf{e}^j) = g^{ij}\end{array} \qquad \begin{array}{rl}\overset{o}{g} : \overset{o}{V}{}^* \times \overset{o}{V}{}^* & \to \mathbb{R}\\ (\overset{o}{\mathbf{e}}{}^i, \overset{o}{\mathbf{e}}{}^j) & \mapsto \overset{o}{g}(\overset{o}{\mathbf{e}}{}^i, \overset{o}{\mathbf{e}}{}^j) = \overset{o}{g}{}^{ij}\end{array}$$

The metrics $\overset{\epsilon}{g} : \overset{o}{V}{}^* \times V^* \to \mathbb{R}$ and $\overset{\ni}{g} : V^* \times \overset{o}{V}{}^* \to \mathbb{R}$ are defined to be identically null, in such a way that

$$\overset{\epsilon}{g}(\varepsilon\mathbf{e}^i, \mathbf{e}^j) = 0 = \overset{\ni}{g}(\mathbf{e}^i, \varepsilon\mathbf{e}^j). \tag{4.1}$$

Otherwise some inconsistencies arise.

Now, considering $V \simeq \mathbb{R}^{p,q}$ the operators $\Upsilon : \breve{\Lambda}(V) \to \breve{\Lambda}(V)$ admit a fundamental representation

$$\rho(\Upsilon) = \begin{pmatrix}\Upsilon_1 & \Upsilon_2\\ \Upsilon_3 & \Upsilon_4\end{pmatrix}, \tag{4.2}$$

acting on elements $\binom{\psi}{\phi}$, where $\psi \in \Lambda(V)$ and $\phi \in \overset{o}{\Lambda}(V)$. The operators Υ_d ($d = 1, \ldots, 4$) are elements of $C\ell_{p,q}$ defined by the maps

$$\begin{array}{ll}\Upsilon_1 : \Lambda(V) \to \Lambda(V), & \Upsilon_2 : \overset{o}{\Lambda}(V) \to \Lambda(V),\\ \Upsilon_3 : \Lambda(V) \to \overset{o}{\Lambda}(V), & \Upsilon_4 : \Lambda(V) \to \overset{o}{\Lambda}(V).\end{array} \tag{4.3}$$

Using the periodicity theorem of Clifford algebras [3], that asserts $C\ell_{p+1,q+1} \simeq C\ell_{p,q} \otimes C\ell_{1,1} \simeq C\ell_{p,q} \otimes \mathcal{M}(2,\mathbb{C})$, it is immediate that $\Upsilon \in \mathcal{M}(2,\mathbb{R}) \otimes C\ell_{p,q} \simeq C\ell_{1,1} \otimes C\ell_{p,q} \simeq C\ell_{p+1,q+1}$, and since it is well known that algebraic spinors associated with $C\ell_{p+1,q+1}$ define twistors [7, 18], ideals of $\breve{\Lambda}(V)$ are also useful to

[8] Indeed it is a non-canonical isomorphism.

describe twistors, at least when $\dim V = 1, 2, 4$, and consequently to investigate their profound applications in physical theories [7, 8, 18, 9, 10, 11, 12, 13].

We obtain a representation $\rho : \overset{o}{\Lambda}(V) \to \mathrm{End}\,\breve{\Lambda}(V)$ of the pseudoscalar $\varepsilon \in \overset{o}{\Lambda}{}^0(V)$, as

$$\rho(\varepsilon) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{4.4}$$

since ε changes the chirality of differential forms. Indeed, given $\psi \in \Lambda(V)$ and $\phi \in \overset{o}{\Lambda}(V)$,

$$\rho(\varepsilon)\begin{pmatrix} \psi \\ \phi \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \psi \\ \phi \end{pmatrix} = \begin{pmatrix} \phi \\ \psi \end{pmatrix} \in \Lambda(V) \oplus \overset{o}{\Lambda}(V) \simeq \breve{\Lambda}(V). \tag{4.5}$$

Denoting the unit of $\mathbb{R}$ by 1 and the unit of $\overset{o}{\mathbb{R}}$ by $\overset{o}{1}$, they are to be represented respectively by

$$\rho(1) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \rho(\overset{o}{1}) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \tag{4.6}$$

where each matrix is an element of $C\ell_{p+1,q+1}$ with entries in $C\ell_{p,q}$. The unit associated with $\mathbb{R} \oplus \overset{o}{\mathbb{R}}$ (the field over which $V \oplus \overset{o}{V} \simeq \mathbb{D} \otimes V$ is construted) is given by $\breve{1} = 1 + \overset{o}{1}$, and can be represented by $\rho(\breve{1}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Basis elements of V^* and $\overset{o}{V}{}^*$ are respectively represented as[9]

$$\rho(\mathbf{e}^i) = \begin{pmatrix} \mathbf{e}^i & 0 \\ 0 & 0 \end{pmatrix}, \quad \rho(\overset{o}{\mathbf{e}}{}^i) = \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{e}^i \end{pmatrix}. \tag{4.7}$$

When the pseudoscalar ε is represented as in eq.(4.4), some properties defining the extended Clifford algebras are verified. The Clifford product is defined in V^* by

$$\begin{aligned} \mathbf{e}^i\mathbf{e}^j + \mathbf{e}^j\mathbf{e}^i &= \rho^{-1}\left[\begin{pmatrix} \mathbf{e}^i & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \mathbf{e}^j & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \mathbf{e}^j & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \mathbf{e}^i & 0 \\ 0 & 0 \end{pmatrix}\right] \\ &= \rho^{-1}\begin{pmatrix} \mathbf{e}^i\mathbf{e}^j + \mathbf{e}^j\mathbf{e}^i & 0 \\ 0 & 0 \end{pmatrix} = \rho^{-1}\begin{pmatrix} 2g(\mathbf{e}^i, \mathbf{e}^j) & 0 \\ 0 & 0 \end{pmatrix} \\ &= 2g(\mathbf{e}^i, \mathbf{e}^j)\, 1, \end{aligned} \tag{4.8}$$

[9]Here we transit from V $[\overset{o}{V}]$ to V^* $[\overset{o}{V}{}^*]$, since there is a (non-canonical) isomorphism between V and V^*.

and in $\overset{o}{V}$ by

$$\begin{aligned}\overset{o}{\mathbf{e}}{}^i\overset{o}{\mathbf{e}}{}^j+\overset{o}{\mathbf{e}}{}^j\overset{o}{\mathbf{e}}{}^i &= \rho^{-1}\left[\begin{pmatrix}0&0\\0&\mathbf{e}^i\end{pmatrix}\begin{pmatrix}0&0\\0&\mathbf{e}^j\end{pmatrix}+\begin{pmatrix}0&0\\0&\mathbf{e}^j\end{pmatrix}\begin{pmatrix}0&0\\0&\mathbf{e}^i\end{pmatrix}\right]\\ &= \rho^{-1}\begin{pmatrix}0&0\\0&\mathbf{e}^i\mathbf{e}^j+\mathbf{e}^j\mathbf{e}^i\end{pmatrix}=\rho^{-1}\begin{pmatrix}0&0\\0&2g(\mathbf{e}^i,\mathbf{e}^j)\end{pmatrix}\\ &= 2g(\mathbf{e}^i,\mathbf{e}^j)\,\overset{o}{1},\end{aligned} \tag{4.9}$$

where the metrics in V^* and in $\overset{o}{V}{}^*$ can be respectively represented by

$$\rho(g)=\begin{pmatrix}g&0\\0&0\end{pmatrix},\quad \rho(\overset{o}{g})=\begin{pmatrix}0&0\\0&g\end{pmatrix}. \tag{4.10}$$

The representations of g and $\overset{o}{g}$ are related by

$$\rho(g)=\begin{pmatrix}g&0\\0&0\end{pmatrix}=\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0&0\\0&g\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}=\rho(\varepsilon)\rho(\overset{o}{g})\rho(\varepsilon)^{-1}, \tag{4.11}$$

what shows that, analogously to conformal transformations in $\mathbb{R}^{p,q}$ [6, 19, 20, 21, 22], the metrics associated with spaces of different chirality are related by adjoint representations. The extended metric $\breve{g}=g+\overset{o}{g}$ in $V^*\oplus\overset{o}{V}{}^*$ is given by

$$\rho(\breve{g})=\begin{pmatrix}g&0\\0&g\end{pmatrix}. \tag{4.12}$$

Chiral and achiral Clifford algebras are then introduced, in the context of the periodicity theorem of Clifford algebras, from relations (4.8, 4.9) as

$$\mathbf{e}^i\mathbf{e}^j+\mathbf{e}^j\mathbf{e}^i=2g^{ij}\ 1,\qquad \overset{o}{\mathbf{e}}{}^i\overset{o}{\mathbf{e}}{}^j+\overset{o}{\mathbf{e}}{}^j\overset{o}{\mathbf{e}}{}^i=2\overset{o}{g}{}^{ij}\ \overset{o}{1}, \tag{4.13}$$

$$\mathbf{e}^i\overset{o}{\mathbf{e}}{}^j+\overset{o}{\mathbf{e}}{}^j\mathbf{e}^i=0, \tag{4.14}$$

where this last relation denotes definition given by eq.(4.1).

The algebra $\mathbb{D}\otimes C\ell_{p,q}$ can be shown not to be a Clifford algebra. Indeed, considering $C\ell_{1,0}\simeq\mathbb{D}\simeq\mathbb{R}\oplus\mathbb{R}$, then $\mathbb{D}\otimes C\ell_{1,0}\simeq\mathbb{R}\oplus\mathbb{R}\oplus\mathbb{R}\oplus\mathbb{R}$, which is not a Clifford algebra. It can be shown that all subalgebras of a Clifford algebra are either Clifford algebras, or algebras of type $\mathbb{D}\otimes C\ell_{r,s}$, or of type $(\mathbb{D}\otimes\mathbb{D})\otimes C\ell_{r,s}$ [24]. It follows the importance of defining and investigating algebras of type $\mathbb{D}\otimes C\ell_{p,q}$, allowing us completely to classify all subalgebras of a Clifford algebra. Even though $\mathbb{D}\otimes C\ell_{p,q}$ is not a Clifford algebra, as Clifford algebras were also defined in the context given by eqs.(4.8, 4.9), it is possible to define a hyperbolic Clifford algebra, over the vector space $V\oplus\overset{o}{V}\simeq\mathbb{D}\otimes V$, in the light of the formalism presented in Sec. 5. For more details, see, e.g., [23].

4.1. (Chiral) Hodge star operators

The vector spaces $\Lambda^k(V)$ $[\overset{o}{\Lambda}{}^k(V)]$ and $\Lambda^{n-k}(V)$ $[\overset{o}{\Lambda}{}^{n-k}(V)]$ have the same dimension, but it does not exist any canonical isomorphism between these spaces. Let Θ be the volume element in V defined by $\Theta = |\det\tau|^{1/2}\mathbf{e}^1\wedge\cdots\wedge\mathbf{e}^n$, where $\det\tau$ is given[10] implicitly by

$$\tau(\mathbf{e}_1\wedge\mathbf{e}_2\wedge\cdots\wedge\mathbf{e}_n) = \tau(\mathbf{e}_1)\wedge\tau(\mathbf{e}_2)\wedge\cdots\wedge\tau(\mathbf{e}_n) = (\det\tau)\,\mathbf{e}_1\wedge\mathbf{e}_2\wedge\cdots\wedge\mathbf{e}_n. \quad (4.15)$$

The isomorphism given by the dual Hodge star operator $\star : \Lambda^k(V) \to \Lambda^{n-k}(V)$ $[\star : \overset{o}{\Lambda}{}^k(V) \to \overset{o}{\Lambda}{}^{n-k}(V)]$ is defined from the quasi-Hodge star operators. Since a correlation is defined as an isomorphism $\tau : \Lambda_k(V) \to \Lambda^k(V)$ $[\tau : \overset{o}{\Lambda}_k(V) \to \overset{o}{\Lambda}{}^k(V)]$, it follows that $\underline{\star}\circ\tau^{-1} : \Lambda^k(V) \to \Lambda^{n-k}(V)$ $[\underline{\star}\circ\tau^{-1} : \overset{o}{\Lambda}{}^k(V) \to \overset{o}{\Lambda}{}^{n-k}(V)]$ and $\overline{\star}\circ\tau : \Lambda_k(V) \to \Lambda_{n-k}(V)$ $[\overline{\star}\circ\tau : \overset{o}{\Lambda}_k(V) \to \overset{o}{\Lambda}_{n-k}(V)]$. Demand that

$$\underline{\star}\circ\tau^{-1} = \overline{\star}\circ\tau, \quad (4.16)$$

which occurs only when Θ is unitary. The Hodge star operator is defined as

$$\star = \underline{\star}\circ\tau^{-1} = \overline{\star}\circ\tau \quad (4.17)$$

and more explicitly,

$$\star 1 = \eta, \qquad \star\psi = \tau^{-1}(\widetilde{\psi})\lrcorner\eta, \quad (4.18)$$

where $\psi \in \breve{\Lambda}(V)$. The dual Hodge star operator $\star$ does not change the chirality of forms. We define the chiral Hodge star operator as $\star_\varepsilon : \Lambda^k(V) \to \overset{o}{\Lambda}{}^{n-k}(V)$ $[\star_\varepsilon : \overset{o}{\Lambda}{}^k(V) \to \Lambda^{n-k}(V)]$, given by

$$\star_\varepsilon 1 = \varepsilon\eta, \qquad \star_\varepsilon\psi = \varepsilon\tau^{-1}(\widetilde{\psi})\lrcorner\eta. \quad (4.19)$$

We observe that $\star_\varepsilon = \varepsilon\star$ and $\star_\varepsilon$ naturally changes the chirality of forms.

5. Subspaces Embedding and Witt Bases

Consider the metric vector space $(V\oplus\overset{o}{V}, \underline{g})$, and denote $u = \mathbf{u} + \overset{o}{\mathbf{u}}$, $v = \mathbf{v} + \overset{o}{\mathbf{v}} \in V\oplus\overset{o}{V}$. The metric $\underline{g} : (V\oplus\overset{o}{V})\times(V\oplus\overset{o}{V}) \to \mathbb{R}$ is given by

$$\underline{g}(u,v) \;=\; g(\overset{o}{\mathbf{u}},\mathbf{v}) + g(\overset{o}{\mathbf{v}},\mathbf{u}). \quad (5.1)$$

Using the notation introduced in Sec. 4, the metric $\underline{g}$ can be represented as $\begin{pmatrix} 0 & g \\ g & 0 \end{pmatrix}$.

[10]The notation τ is used to describe the map $\tau : V \to V^*$ and its natural extension $\tau : \Lambda_k(V) \to \Lambda^k(V)$, without any distinction. It will be implicit which is each one of them in the text.

Under the inclusion maps

$$i_V: \quad V \to V \oplus \overset{o}{V} \qquad\qquad i_{\overset{o}{V}} : \overset{o}{V} \to V \oplus \overset{o}{V}$$
$$\mathbf{v} \mapsto \mathbf{v} + 0 \qquad\qquad \overset{o}{\mathbf{v}} \mapsto 0 + \overset{o}{\mathbf{v}}$$

since V and $\overset{o}{V}$ are vector subspaces of $V \oplus \overset{o}{V}$, it follows that

$$\underline{g}(\mathbf{u} + 0, 0 + \overset{o}{\mathbf{v}}) = g(\mathbf{u}, \overset{o}{\mathbf{v}}), \quad \underline{g}(0 + \overset{o}{\mathbf{u}}, \mathbf{v} + 0) = g(\overset{o}{\mathbf{u}}, \mathbf{v}),$$

$$\underline{g}(\mathbf{u} + 0, \mathbf{v} + 0) = \underline{g}(0 + \overset{o}{\mathbf{u}}, 0 + \overset{o}{\mathbf{v}}) = 0,$$

and then V and $\overset{o}{V}$ are maximal totally isotropic subspaces of $V \oplus \overset{o}{V}$. There exists a basis $\{\mathbf{e}_i\}_{i=1}^n$ of V and a basis $\{\overset{o}{\mathbf{e}}_j\}_{j=1}^n$ of $\overset{o}{V}$, satisfying

$$\underline{g}(\mathbf{e}_i, \overset{o}{\mathbf{e}}_j) = \delta_{ij}, \quad \underline{g}(\mathbf{e}_i, \mathbf{e}_j) = \underline{g}(\overset{o}{\mathbf{e}}_i, \overset{o}{\mathbf{e}}_j) = 0. \tag{5.2}$$

Motivated by the results in [23], asserting that

$$\xi_i = (\overset{o}{\mathbf{e}}_i + \mathbf{e}_i)/\sqrt{2}, \qquad \xi_{i+n} = (\overset{o}{\mathbf{e}}_i - \mathbf{e}_i)/\sqrt{2}, \tag{5.3}$$

it is easy to see that the vectors $\{\xi_k\}_{k=1}^{2n}$ span $\overset{o}{\mathbb{R}}^{n,n}$, since for $i, j = 1, \ldots, n$ the relations

$$\underline{g}(\xi_i, \xi_j) = -\underline{g}(\xi_{i+n}, \xi_{j+n}) = \delta_{ij}. \tag{5.4}$$

hold. It is worthwhile to emphasize that $\underline{g}(\xi_i, \xi_{k+n}) = 0, \quad 1 \leq i, j \leq n$.

6. The Regressive Product

Given a representation of a k-covector $\psi = \mathbf{a}^1 \wedge \cdots \wedge \mathbf{a}^k$, and $\{h_1, h_2, \ldots, h_r\}$ a set of non-negative integers such that $h_1 + h_2 + \cdots + h_r = k$, a *split of class* $(h_1, h_2, \ldots, h_r)$ *of* ψ is defined as a set of multicovectors $\{\psi^1, \ldots, \psi^r\}$ such that [1]

1. $\psi^i = 1$ if $h_i = 0$ and $\psi^i = \mathbf{a}^{i_1} \wedge \cdots \wedge \mathbf{a}^{i_{h_i}}$, $i_1 < \cdots < i_{h_i}$, if $h_i \neq 0$;
2. $\psi^i \wedge \psi^j \neq 0$;
3. $\psi^1 \wedge \psi^2 \wedge \cdots \wedge \psi^r = \pm\psi$.

(ψ) denotes the finite set of all the possible splits of the k-covector ψ. The regressive product

$$\begin{aligned} \vee : \Lambda^k(V) \times \Lambda^l(V) &\to \Lambda^{k+l-n}(V) \\ (\psi^k, \phi^l) &\mapsto \psi^k \vee \phi^l \end{aligned} \tag{6.1}$$

is defined as [1]

$$\psi^k \vee \phi^l = \sum_{(\psi)} [\psi^k_{(1)}, \phi^l]\, \psi^k_{(2)} = \sum_{(\phi)} [\psi^k, \phi^l_{(2)}]\, \phi^l_{(1)}, \qquad \text{if} \quad k + l \geq n. \tag{6.2}$$

When $k + l < n$ we have the trivial case $\psi^k \vee \phi^l = 0$. The bracket calculated between two k-coforms is defined to be [1]

$$[\mathbf{a}^1 \wedge \cdots \wedge \mathbf{a}^k, \mathbf{b}^1 \wedge \cdots \wedge \mathbf{b}^k] = [\mathbf{a}^1, \ldots, \mathbf{a}^k, \mathbf{b}^1, \ldots, \mathbf{b}^k], \quad \text{if} \quad k + l = n, \tag{6.3}$$

which is identically null when $k + l \neq n$. Given $\psi, \phi, \zeta \in \check{\Lambda}(V)$ the following properties are immediately verified:

1. $(\psi \vee \phi) \vee \zeta = \psi \vee (\phi \vee \zeta)$,
2. $\psi^{[k]} \vee \phi^{[l]} = (-1)^{[k][l]}\phi^{[l]} \vee \psi^{[k]}, \qquad [i] := n - i$,
3. $(\psi + \phi) \vee \zeta = \psi \vee \zeta + \phi \vee \zeta, \qquad \psi \vee (\phi + \zeta) = \psi \vee \phi + \psi \vee \zeta$,
4. $\psi \vee (a\phi) = (a\psi) \vee \phi = a(\psi \vee \phi), \qquad a \in \mathbb{R}$.

The following relations

$$\mathbf{e}^i \vee (\mathbf{e}^1 \wedge \cdots \wedge \mathbf{e}^n) = [1, \mathbf{e}^1 \wedge \cdots \wedge \mathbf{e}^n]\mathbf{e}^i = [\mathbf{e}^1, \ldots, \mathbf{e}^n]\mathbf{e}^i = \varepsilon\mathbf{e}^i, \tag{6.4}$$

$$\begin{aligned}(\mathbf{e}^i \wedge \mathbf{e}^j) \vee (\mathbf{e}^1 \wedge \cdots \wedge \mathbf{e}^n) &= [1, \mathbf{e}^1 \wedge \cdots \wedge \mathbf{e}^n](\mathbf{e}^i \wedge \mathbf{e}^j)\\ &= [\mathbf{e}^1, \ldots, \mathbf{e}^n](\mathbf{e}^i \wedge \mathbf{e}^j) = \varepsilon(\mathbf{e}^i \wedge \mathbf{e}^j),\end{aligned} \tag{6.5}$$

and

$$\begin{aligned}(\mathbf{e}^{i_1} \wedge \mathbf{e}^{i_2} \wedge \cdots \wedge \mathbf{e}^{i_k}) \vee (\mathbf{e}^1 \wedge \cdots \wedge \mathbf{e}^n) &= [1, \mathbf{e}^1 \wedge \cdots \wedge \mathbf{e}^n]\, \mathbf{e}^{i_1} \wedge \mathbf{e}^{i_2} \wedge \cdots \wedge \mathbf{e}^{i_k}\\ &= [\mathbf{e}^1, \ldots, \mathbf{e}^n]\, \mathbf{e}^{i_1} \wedge \mathbf{e}^{i_2} \wedge \cdots \wedge \mathbf{e}^{i_k}\\ &= \varepsilon\mathbf{e}^{i_1} \wedge \mathbf{e}^{i_2} \wedge \cdots \wedge \mathbf{e}^{i_k}.\end{aligned} \tag{6.6}$$

hold. From the above expressions it can be proved, proceeding by induction, and from the linearity of $\psi \in \check{\Lambda}(V)$, that:

$$\psi \vee (\mathbf{e}^1 \wedge \cdots \wedge \mathbf{e}^n) = \varepsilon\psi. \tag{6.7}$$

Using the relation

$$\mathbf{e}^i \vee (\mathbf{e}^1 \wedge \cdots \wedge \check{\mathbf{e}^j} \wedge \cdots \wedge \mathbf{e}^n) = [\mathbf{e}^i, \mathbf{e}^1, \ldots, \check{\mathbf{e}^j}, \ldots, \mathbf{e}^n] = \delta^{ij}(-1)^{i-1}\varepsilon,$$

when $i = j$, the pseudoscalar is represented as ε by

$$\varepsilon = (-1)^{i-1}\, \mathbf{e}^i \vee (\mathbf{e}^1 \wedge \cdots \wedge \check{\mathbf{e}^i} \wedge \cdots \wedge \mathbf{e}^n). \tag{6.8}$$

7. Differential Coforms and the Chiral Counterspace

From the definition of the regressive product, it is immediate that

$$\Lambda^{n-[k]}(V) \vee \Lambda^{n-[l]}(V) \hookrightarrow \Lambda^{n-([k]+[l])}(V), \tag{7.1}$$

$[i] := n - i$. The k-counterspace $\bigvee^k$ is defined [25] as being

$$\bigvee^k = \Lambda^{n-k}(V) \tag{7.2}$$

and under the regressive product it can define a (regressive) exterior algebra. Indeed, it follows from eq.(7.1) that $\bigvee^r \vee \bigvee^s \hookrightarrow \bigvee^{r+s}$. The coexterior algebra is then defined as

$$\bigvee = \overset{0}{\bigvee} \oplus \overset{1}{\bigvee} \oplus \cdots \oplus \overset{n}{\bigvee} = \bigoplus_{j=0}^{n} \overset{j}{\bigvee} \tag{7.3}$$

which is an exterior algebra with respect to the regressive product. From definition given by eq.(7.2) we see that $\bigvee^1 = \Lambda^{n-1}(V)$, and $(n-1)$-forms can be seen as 1-forms associated with the counterspace $\bigvee^1$. A basis for $\bigvee^1$, denominated cobasis, is defined as the set $\{\mathfrak{e}^i\}$ whose elements are defined as [1, 26]

$$\mathfrak{e}^i = (-1)^{i-1}\mathbf{e}^1 \wedge \cdots \wedge \check{\mathbf{e}}^i \wedge \cdots \wedge \mathbf{e}^n \tag{7.4}$$

Elements of $\bigvee^1$ are called 1-coforms. Rota [1] denominated the algebra $(\Lambda(V), \wedge, \vee)$ by dialgebra or double algebra. This concept is extended by considering the algebra $(\breve{\Lambda}(V), \wedge, \vee)$. From the definition above it can be seen that $\mathbf{e}^i \wedge \mathfrak{e}^i = \mathbf{e}^1 \wedge \mathbf{e}^2 \wedge \cdots \wedge \mathbf{e}^n$. Analogously to eq.(2.2), a chiral cobasis $\{\overset{o}{\mathfrak{e}}{}^i\}$, whose elements are defined by the identity $\overset{o}{\mathfrak{e}}{}^i = \varepsilon\mathfrak{e}^i$, can be introduced. The unit of the associative algebra generated by 1-coforms is the volume element $\mathbf{e}^1 \wedge \mathbf{e}^2 \wedge \cdots \wedge \mathbf{e}^n$. The following proposition is a straightforward generalization of the Grassmann-Rota one, yielding information about the chirality of differential forms and coforms.

Proposition 1. ► $\mathfrak{e}^1 \vee \mathfrak{e}^2 \vee \cdots \vee \mathfrak{e}^i = \varepsilon^{i+1}\mathbf{e}^{i+1} \wedge \cdots \wedge \mathbf{e}^n$ ◄.

Proof. Let $\mathfrak{e}^1 = \mathbf{e}^2 \wedge \mathbf{e}^3 \wedge \cdots \wedge \mathbf{e}^n$ and $\mathfrak{e}^2 = -\mathbf{e}^1 \wedge \mathbf{e}^3 \wedge \cdots \wedge \mathbf{e}^n$ be two 1-coforms. It follows that

$$\begin{aligned}
\mathfrak{e}^1 \vee \mathfrak{e}^2 &= -(\mathbf{e}^2 \wedge \mathbf{e}^3 \wedge \cdots \wedge \mathbf{e}^n) \vee (\mathbf{e}^1 \wedge \mathbf{e}^3 \wedge \cdots \wedge \mathbf{e}^n) \\
&= -[\mathbf{e}^2, \mathbf{e}^1, \mathbf{e}^3, \ldots, \mathbf{e}^n]\mathbf{e}^3 \wedge \cdots \wedge \mathbf{e}^n \\
&= \varepsilon\mathbf{e}^3 \wedge \cdots \wedge \mathbf{e}^n.
\end{aligned}$$

We now proceed by induction:

$$\begin{aligned}
&\mathfrak{e}^1 \vee \mathfrak{e}^2 \vee \cdots \vee \mathfrak{e}^i \vee \mathfrak{e}^{i+1} = \\
&\quad = (\varepsilon^{i+1}\mathbf{e}^{i+1} \wedge \cdots \wedge \mathbf{e}^n) \vee \mathfrak{e}^{i+1} \\
&\quad = \varepsilon^{i+1}(\mathbf{e}^{i+1} \wedge \cdots \wedge \mathbf{e}^n) \vee ((-1)^i\mathbf{e}^1 \wedge \cdots \wedge \mathbf{e}^{\check{i+1}} \wedge \cdots \wedge \mathbf{e}^n) \\
&\quad = (-1)^i\varepsilon^{i+1}[\mathbf{e}^{i+1}, \mathbf{e}^1 \wedge \cdots \wedge \check{\mathbf{e}^i} \wedge \cdots \wedge \mathbf{e}^n]\, \mathbf{e}^{i+2} \wedge \cdots \wedge \mathbf{e}^n \\
&\quad = \varepsilon^{i+1}[\mathbf{e}^{i+1}, \mathbf{e}^1, \ldots, \check{\mathbf{e}^i}, \ldots, \mathbf{e}^n]\, \mathbf{e}^{i+2} \wedge \cdots \wedge \mathbf{e}^n \\
&\quad = \varepsilon^{i+1}\varepsilon\mathbf{e}^{i+2} \wedge \cdots \wedge \mathbf{e}^n \\
&\quad = \varepsilon^{i+2}\mathbf{e}^{i+2} \wedge \cdots \wedge \mathbf{e}^n
\end{aligned}$$

□

Depending on the number (i) of elements of the product $\mathfrak{e}^1 \vee \mathfrak{e}^2 \vee \cdots \vee \mathfrak{e}^i$, the RHS of Prop. 1 changes or does not change sign under orientation change. As a corollary, when $i = n$, we obtain

$$\mathfrak{e}^1 \vee \mathfrak{e}^2 \vee \cdots \vee \mathfrak{e}^n = \varepsilon^{n+1} \in \begin{cases} \Lambda^0(V), \text{ if } n = 2k+1, \\ \overset{o}{\Lambda}{}^0(V), \text{ if } n = 2k. \end{cases} \tag{7.5}$$

So the volume element (under the regressive product) $\mathfrak{e}^1 \vee \mathfrak{e}^2 \vee \cdots \vee \mathfrak{e}^n \in \bigvee^n$ is a scalar or a pseudoscalar, depending on the dimension n of V. We conclude from these considerations that

$$\overset{0}{\bigvee} \oplus \overset{1}{\bigvee} \oplus \overset{2}{\bigvee} \oplus \cdots \oplus \overset{n}{\bigvee} = \Lambda^n(V) \oplus \overset{o}{\Lambda}{}^{n-1}(V) \oplus \overset{o}{\Lambda}{}^{n-2} \oplus \cdots \oplus \Lambda^0[\overset{o}{\Lambda}{}^0](V), \tag{7.6}$$

where the last term of the direct sum above denotes the two possibilities, depending on the even or odd value of n. From Prop. 1 and the properties of the dual Hodge star operator, the relation

$$\star(\mathbf{e}^1 \wedge \mathbf{e}^2 \wedge \cdots \wedge \mathbf{e}^k) = \varepsilon^{k+1} \mathfrak{e}^1 \vee \cdots \vee \mathfrak{e}^k \tag{7.7}$$

follows. The Hodge star operator applied on $(2k)$-forms gives chiral $(2k)$-coforms, and when it is applied on $(2k+1)$-forms, its respective $(2k+1)$-coforms have no chirality $(2k+1 \leq n)$.

8. Clifford Algebras Over the Counterspace

In this section the formulation given by Conradt [25] is reinterpreted, and his definitions are slightly modified in order to encompass the present formalism. It is well known that given a volume element $\eta \in \Lambda^n(V)$, the dual Hodge star operator acting on a multivector ψ can be defined as $\star\psi = \tilde{\psi}\eta$ and $\star 1 = \eta$. The Clifford product $* : C\ell_{p,q} \times C\ell_{p,q} \to C\ell_{p,q}$, related to the counterspace is defined as:

$$\psi * \phi := \star^{-1}[(\star\psi)(\star\phi)] \qquad \psi, \phi \in C\ell_{p,q}. \tag{8.1}$$

Such a product is immediately shown to satisfy a Clifford algebra. First of all the associativity is verified [25]. Indeed, given $\psi, \phi, \zeta \in C\ell_{p,q}$, we have:

$$\begin{aligned} (\psi * \phi) * \zeta &= \{\star^{-1}[(\star\psi)(\star\phi)]\} * \zeta \\ &= \star^{-1}\{\star\star^{-1}[(\star\psi)(\star\phi)](\star\zeta)\} \\ &= \star^{-1}\{[(\star\psi)(\star\phi)](\star\zeta)\} \\ &= \star^{-1}\{(\star\psi)[(\star\phi)(\star\zeta)]\} \\ &= \star^{-1}\{(\star\psi) \star [\star^{-1}((\star\phi)(\star\zeta))]\} \\ &= \psi * [\star^{-1}((\star\phi)(\star\zeta))] \\ &= \psi * (\phi * \zeta). \end{aligned} \tag{8.2}$$

The additive distributivity,

$$\begin{aligned} \psi * (\phi + \zeta) &= \psi * \phi + \psi * \zeta && (8.3)\\ (\psi + \phi) * \zeta &= \psi * \zeta + \phi * \zeta, && (8.4) \end{aligned}$$

can be also verified. Indeed:

$$\begin{aligned} \psi * (\phi + \zeta) &= \star[(\star\psi) \star (\phi + \zeta)]\\ &= \star^{-1}[(\star\psi)(\star\phi + \star\zeta)]\\ &= \star^{-1}[(\star\psi)(\star\phi)] + \star^{-1}[(\star\psi)(\star\zeta)]\\ &= \psi * \phi + \psi * \zeta. && (8.5) \end{aligned}$$

Eq.(8.4) is shown in an analogous way.

The volume element η acts as a unit in relation to the product $*$. In fact, η is the left unit

$$\begin{aligned} \eta * \psi &= \star^{-1}[(\star\eta)(\star\psi)]\\ &= \star^{-1}(1 \star \psi) = \star^{-1}(\star\psi)\\ &= \psi && (8.6) \end{aligned}$$

and analogously η is also the right unit related to the product $*$, since

$$\begin{aligned} \psi * \eta &= \star^{-1}[(\star\psi)(\star\eta)]\\ &= \star^{-1}(\star\psi 1) = \star^{-1}(\star\psi)\\ &= \psi. && (8.7) \end{aligned}$$

It is worthwhile to emphasize that both the usual Clifford product, denoted by juxtaposition, and the other Clifford product $* : C\ell_{p,q} \times C\ell_{p,q} \to C\ell_{p,q}$, related itself to the counterspace, act on the underlying vector space of $C\ell_{p,q}$. Since the Clifford algebra, constructed from the usual Clifford product is denoted by $C\ell(\Lambda^1(V), g)$, the 'other' Clifford algebra is denoted by $C\ell(\bigvee^1, g)$ when interpreted as being constructed from the $*$-product. Indeed, the Clifford relation, computed from the product given by eq.(8.1) between two coforms $\mathfrak{e}_i, \mathfrak{e}_j \in \bigvee^1$ is given by

$$\begin{aligned} \mathfrak{e}^i * \mathfrak{e}^j + \mathfrak{e}^j * \mathfrak{e}^i &= \star^{-1}[(\star\mathfrak{e}^i)(\star\mathfrak{e}^j)] + \star^{-1}[(\star\mathfrak{e}^j)(\star\mathfrak{e}^i)]\\ &= \star^{-1}(\mathbf{e}^i\mathbf{e}^j) + \star^{-1}(\mathbf{e}^j\mathbf{e}^i)\\ &= \star^{-1}(\mathbf{e}^i\mathbf{e}^j + \mathbf{e}^j\mathbf{e}^i)\\ &= \star^{-1}(2g(\mathbf{e}^i, \mathbf{e}^j))\\ &= 2g(\mathbf{e}^i, \mathbf{e}^j)\,\eta, && (8.8) \end{aligned}$$

from eq.(4.18). As η is the unit related to the product $*$, the product $*$ indeed defines a Clifford algebra.

Given $\mathbf{v} \in V, \psi \in C\ell_{p,q}$, the regressive product defined in Sec. 6 can now be written in terms of the Clifford product $*$, as:

$$\mathfrak{e}^i \vee \psi = \frac{1}{2}(\mathfrak{e}^i * \psi + \hat{\psi} * \mathfrak{e}^i). \quad (8.9)$$

The right contraction of a vector $\mathbf{v}$ by an element $\psi \in C\ell_{p,q}$, associated to the product $*$ is defined by [25]

$$\mathbf{v}\ulcorner\psi := \frac{1}{2}(\mathbf{v} * \psi - \hat{\psi} * \mathbf{v}), \tag{8.10}$$

while the left contraction is introduced by the expression

$$\psi\urcorner\mathbf{v} := \frac{1}{2}(\psi * \mathbf{v} - \mathbf{v} * \hat{\psi}). \tag{8.11}$$

9. Duality and Coduality

Consider two differential forms $\xi \in \Lambda^i(V)$ and $\omega \in \Lambda^j(V)$. The relations $\overline{\star}(\xi \wedge \omega) = (\overline{\star}\,\xi) \vee (\overline{\star}\,\omega)$, $\overline{\star}(\xi \vee \omega) = (\overline{\star}\,\xi) \wedge (\overline{\star}\,\omega)$, and the same assertions to the operator $\underline{\star}$ are easily shown [1]. If we work with a Grassmann algebra, where a metric is introduced, instead of a Grassmann-Cayley algebra, it is possible to prove that

$$\star(\xi \wedge \omega) = (\star\,\xi) \vee (\star\,\omega), \qquad \star(\xi \vee \omega) = (\star\,\xi) \wedge (\star\,\omega). \tag{9.1}$$

Such relations given by eqs.(9.1), besides being shown inside a formalism devoid of indices and/or components, have origin in the definition of the Clifford product $*$ associated with counterspace. Indeed,

$$\begin{aligned}
\xi \vee \omega &= \langle \xi * \omega \rangle_{n-(i+j)} \\
&= \langle \xi * \omega \eta^{-1} \eta \rangle_{n-(i+j)} \\
&= \langle \star^{-1}[(\star\xi)(\star\omega)]\eta^{-1} \rangle_{i+j}\, \eta \\
&= \langle \star^{-1}\{\star[\widetilde{(\star\xi)(\star\omega)}]\}\eta\eta^{-1} \rangle_{i+j}\, \eta \\
&= \langle \widetilde{(\star\xi)(\star\omega)} \rangle_{i+j}\, \eta \\
&= (\widetilde{\star\omega}) \wedge (\widetilde{\star\xi})\eta \\
&= \widetilde{(\star\xi) \wedge (\star\omega)}\eta \\
&= \star^{-1}[(\star\xi) \wedge (\star\omega)]
\end{aligned} \tag{9.2}$$

and it follows by linearity that

$$\star(\xi \vee \omega) = (\star\xi) \wedge (\star\omega) \quad \forall \xi, \omega \in \Lambda(V), \tag{9.3}$$

which is exactly eq.(9.1). Using the same procedure it is possible to derive eq.(9.1) as an identity involving the Clifford product $*$. Therefore the duality between Clifford algebras over $\bigvee^1$ and $\Lambda^1(V)$ reflects the duality between the spaces $\bigvee^1$ and $\Lambda^1(V)$. Besides there is also a duality related to the contraction between differential forms and differential coforms, where the last was defined by eq.(8.10). This duality is presented by the following proposition:

Proposition 2. ► $\star(\xi \ulcorner \omega) = (\star\xi) \lrcorner (\star\omega) \quad \forall \xi, \omega \in \Lambda(V)$. ◄

Proof. The definition of the product $\xi \ulcorner \omega$ is valid only when ω has degree great or equal to ξ. Then,

$$\begin{aligned}
\xi \ulcorner \omega &= \langle \xi * \omega \rangle_{n-|i-j|} \\
&= \langle \xi * \omega\, (\eta^{-1}\eta) \rangle_{n-|i-j|} \\
&= \langle \star^{-1}[(\star\xi)(\star\omega)]\eta^{-1} \rangle_{|i-j|}\, \eta \\
&= \langle \star^{-1}\{\star[\widetilde{(\star\xi)(\star\omega)}]\}\eta\eta^{-1} \rangle_{|i-j|}\, \eta \\
&= \langle \widetilde{(\star\xi)(\star\omega)} \rangle_{|i-j|}\, \eta \\
&= (\widetilde{\star\omega}) \lrcorner (\widetilde{\star\xi})\eta \\
&= \widetilde{(\star\xi) \lrcorner (\star\omega)}\eta \\
&= \star^{-1}[(\star\xi) \lrcorner (\star\omega)]
\end{aligned} \tag{9.4}$$

Therefore $\star(\xi \ulcorner \omega) = (\star\xi) \lrcorner (\star\omega)$. □

Analogously it can be asserted the following proposition:

Proposition 3. ► $\star(\xi \urcorner \omega) = (\star\xi) \urcorner (\star\omega) \quad \forall \xi, \omega \in \Lambda(V)$ ◄

This proposition is valid in the case when ω has degree less or equal to ξ. The proof is analogous to the Prop. 2. See [25].

10. Differential and Codifferential Operators

In this section the spaces $\Lambda(V)$ are to be viewed as $\Lambda(T_x^*M)$.

10.1. Differential operator

The differential operator $d : \sec \Lambda^k(T^*M) \to \sec \Lambda^{k+1}(T^*M)$ acts on a multivector ψ as $\psi \mapsto d\psi = (\partial_{i_j}\psi_{i_1\cdots i_k})dx^{i_j} \wedge (dx^{i_1} \wedge \cdots \wedge dx^{i_k})$ where M is a manifold which cotangent space T_x^*M, at $x \in M$, is isomorphic to V^*.

We have seen that, in order to map an achiral k-form to a chiral k-form, it is needed the multiplication by ε. Given $\psi \in \sec \Lambda^k(T^*M)$, since d is defined to be "$\mathbb{D}$-linear", i.e., $d(\varepsilon\psi^k) = \varepsilon d\psi^k$, and that $d\psi^k \in \sec \Lambda^{k+1}(T^*M)$, then $\varepsilon d\psi^k \in \sec \overset{o}{\Lambda}{}^{k+1}(T^*M)$. It follows that $d : \sec \overset{o}{\Lambda}{}^k(T^*M) \to \sec \overset{o}{\Lambda}{}^{k+1}(T^*M)$. Motivated by these considerations the exterior derivative is defined as the unique set of operators $d : \sec \Lambda^k(T^*M) \to \sec \Lambda^{k+1}(T^*M)$ and $d : \sec \overset{o}{\Lambda}{}^k(T^*M) \to \sec \overset{o}{\Lambda}{}^{k+1}(T^*M)$ that satisfy the following properties:

1. $d(\zeta + \omega) = d\zeta + d\omega, \quad$ and $\quad d(c\omega) = c\, d\omega,\ \forall \zeta, \omega \in \sec \breve{\Lambda}(T^*M), c \in \mathbb{R},$
2. $d(\omega \wedge \zeta) = d\omega \wedge \zeta + (-1)^k \omega \wedge d\zeta,$
   $\forall \omega \in \sec \Lambda^k(T^*M \oplus \overset{o}{T}{}^*M), \zeta \in \sec \breve{\Lambda}(T^*M),$
3. $d(d\omega) = 0, \qquad \forall \omega \in \sec \breve{\Lambda}(T^*M).$

By linearity it is possible to extend the definition of d to the extended exterior algebra $d : \sec \breve{\Lambda}(T^*M) \to \sec \breve{\Lambda}(T^*M)$.

10.2. Codifferential operator

Considering $l = n - 1$ in eq.(7.1), it can be seen that

$$\sec \Lambda^k(T^*M) \vee \sec \Lambda^{n-1}(T^*M) \hookrightarrow \sec \Lambda^{k-1}(T^*M), \qquad k \geq 1, \tag{10.1}$$

which motivates us to define the codifferential operator from the regressive product as

$$\begin{aligned} \delta : \sec \Lambda^k(T^*M) &\to \sec \Lambda^{k-1}(T^*M) \\ \psi &\mapsto \delta\psi = (g_{i_k i_j} \partial^{i_k} \psi_{i_1 \cdots i_k})(dx^{i_1} \wedge \cdots \wedge dx^{i_k}) \vee \\ &\qquad \left[(dx^1 \wedge \cdots \wedge \check{dx^{i_j}} \wedge \cdots \wedge dx^n)\right]. \end{aligned}$$

From the regressive product associativity it follows that

$$\delta(\psi \vee \phi) = \delta\psi \vee \phi + (-1)^{[\psi]} \psi \vee \delta\phi \tag{10.2}$$

where $\psi, \phi \in \breve{\Lambda}(T^*M)$, and $[\psi] = k$ if $\psi \in \Lambda^k(T^*M)$ or $\psi \in \sec \overset{o}{\Lambda}{}^k(T^*M)$.

The counterspace has the codifferential operator δ acting as its associated differential operator. This can be illustrated by the following de Rham sequences:

$$\Lambda^0(T^*M) \xrightarrow{d} \Lambda^1(T^*M) \xrightarrow{d} \Lambda^2(T^*M) \xrightarrow{d} \cdots \xrightarrow{d} \Lambda^{n-1}(T^*M) \xrightarrow{d} \Lambda^n(T^*M) \xrightarrow{d} 0, \tag{10.3}$$

$$0 \xleftarrow{d} \overset{0}{\bigvee} \xleftarrow{d} \overset{1}{\bigvee} \xleftarrow{d} \overset{2}{\bigvee} \xleftarrow{d} \cdots \xleftarrow{d} \overset{n-1}{\bigvee} \xleftarrow{d} \overset{n}{\bigvee} \tag{10.4}$$

$$\overset{0}{\bigvee} \xrightarrow{\delta} \overset{1}{\bigvee} \xrightarrow{\delta} \overset{2}{\bigvee} \xrightarrow{\delta} \cdots \xrightarrow{\delta} \overset{n-1}{\bigvee} \xrightarrow{\delta} \overset{n}{\bigvee} \xrightarrow{\delta} 0, \tag{10.5}$$

$$0 \xleftarrow{d} \Lambda^0(T^*M) \xleftarrow{\delta} \Lambda^1(T^*M) \xleftarrow{\delta} \Lambda^2(T^*M) \xleftarrow{\delta} \cdots \xleftarrow{\delta} \Lambda^{n-1}(T^*M) \xleftarrow{\delta} \Lambda^n(T^*M). \tag{10.6}$$

10.3. The Hodge-de Rham Laplacian

The Laplacian Δ is naturally defined as

$$\Delta = d\delta + \delta d \tag{10.7}$$

We exhibit a simple example:

Example. *Consider $\psi \in \sec \Lambda^2(T^*\mathbb{R}^3)$, given $\psi = f(x^1, x^2, x^3) dx^1 \wedge dx^2$, where f is a scalar field $f : \mathbb{R}^3 \to \mathbb{R}$. It follows that*

$$d\psi = \frac{\partial f}{\partial x^3} dx^1 \wedge dx^2 \wedge dx^3. \tag{10.8}$$

Therefore,

$$\begin{aligned}\delta d\psi &= \frac{\partial^2 f}{\partial x^1 \partial x^3}(dx^1\wedge dx^2\wedge dx^3)\vee(dx^2\wedge dx^3)\\ &+\frac{\partial^2 f}{\partial x^2 \partial x^3}(dx^1\wedge dx^2\wedge dx^3)\vee(dx^1\wedge dx^3)\\ &+\frac{\partial^2 f}{\partial (x^3)^2}(dx^1\wedge dx^2\wedge dx^3)\vee(dx^1\wedge dx^1)\\ &= \frac{\partial^2 f}{\partial x^1 \partial x^3}(dx^2\wedge dx^3)+\frac{\partial^2 f}{\partial x^2 \partial x^3}(dx^3\wedge dx^1)+\frac{\partial^2 f}{\partial (x^3)^2}(dx^1\wedge dx^2)\end{aligned} \tag{10.9}$$

On the other hand,

$$\begin{aligned}\delta\psi &= \frac{\partial f}{\partial x^1}(dx^1\wedge dx^2)\vee(dx^2\wedge dx^3)+\frac{\partial f}{\partial x^2}(dx^1\wedge dx^2)\vee(dx^1\wedge dx^3)\\ &+\frac{\partial f}{\partial x^3}(dx^1\wedge dx^2)\vee(dx^1\wedge dx^2)\\ &= \frac{\partial f}{\partial x^1}dx^2+\frac{\partial f}{\partial x^2}(-dx^1)+0.\end{aligned} \tag{10.10}$$

It follows that

$$\begin{aligned}d\delta\psi &= \frac{\partial^2 f}{\partial (x^1)^2}(dx^1\wedge dx^2)+\frac{\partial^2 f}{\partial (x^2)^2}(-dx^2\wedge dx^1)+\frac{\partial^2 f}{\partial x^1\partial x^3}dx^3\wedge dx^2\\ &+\frac{\partial^2 f}{\partial x^2\partial x^3}dx^3\wedge dx^1.\end{aligned} \tag{10.11}$$

From eqs.(10.9), (10.11) *we have:*

$$\begin{aligned}(d\delta+\delta d)\psi &= \frac{\partial^2 f}{\partial (x^1)^2}+\frac{\partial^2 f}{\partial (x^2)^2}+\frac{\partial^2 f}{\partial (x^3)^2}\\ &= \Delta\psi.\end{aligned} \tag{10.12}$$

It can be shown by induction that eq. (10.7) *is valid for all* $\psi \in \sec\Lambda(T^*M)$.

Concluding Remarks

Peano spaces are the natural arena to introduce extended exterior, Grassmann, and subsequently Clifford algebras in the light of the regressive product. Besides endowing exterior algebras with chirality, Rota's bracket is suitable to define extended exterior, Grassmann, and Clifford algebras, naturally presenting a $\mathbb{Z}_2\times\mathbb{Z}_2$-graded structure. The introduction of different units providing the construction of respectively achiral and chiral algebras foresees us to use the periodicity theorem of Clifford algebras, asserting that $C\ell_{p+1,q+1}\simeq C\ell_{p,q}\otimes C\ell_{1,1}$, in order to immerge both achiral and chiral Clifford algebras $C\ell_{p,q}$ into $C\ell_{p+1,q+1}$. It gives rise to various possibilities of applications in physical theories, like e.g. twistor theory and conformal field theory. In such embedding, the extended Clifford algebra associated with $C\ell_{p,q}$ is shown to be $C\ell_{p+1,q+1}$, wherein the formulation is more simple and natural. Moreover, Proposition 1 describes a chiral relationship between differential coforms under the regressive product and forms under the progressive product. When the regressive product is used, the dual Hodge star operator acts on k-forms, resulting in a k-coform intrinsically endowed with chirality only if k is an even integer, otherwise its action results in an achiral form. Moreover, the counterspace volume element with respect to the regressive product is scalar or pseudoscalar until we specify whether the dimension of the Peano space is respectively odd or even. The $*$-Clifford product [25] completes the dual characterization of the counterspace. We also introduced pseudoduality between space and counterspace, since the de Rham cochain, generated by the codifferential operator related to the regressive product, is composed by a sequence of exterior algebra homogeneous subspaces that are subsequently chiral and achiral. This is an astonishing character of the formalism to be presented, since the duality between exterior algebras associated respectively with the space and counterspace is irregular, in the sense that if we take the exterior algebra duality associated with the space, we obtain the exterior algebra associated with the counterspace, but the converse produces the space exterior algebra, which homogeneous even [odd] subspaces are chiral [achiral], depending on the original vector space dimension (see eq.(7.6)). Then, duality between space and counterspace is deduced to be a pseudoduality if the exterior algebra is endowed with chirality.

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