Nexus Network Journal
https://doi.org/10.1007/s00004-024-00794-9

RESEARCH

Monge Surfaces. Generation, Discretisation and Application in Architecture

Francisco Gonzalez-Quintial[1] · Andrés Martin-Pastor[2]

Accepted: 11 July 2024

Abstract

This paper describes a revision of so-called Monge surfaces through using digital graphic tools, based on the synthetic conception exposed by Gaspard Monge in the 18th and 19th centuries, through their geometric relation with polar surface. As a starting point, we propose a graphical system that integrally solves the generation of these kind of surfaces, their rationalization and discretization, with the objective of their specific application in architecture. The geometric system described allows the generation of a type of surface in which the pair of curves -generatrix and directrix- give rise to the network of principal lines of curvature (LCP) of the final surface. In this way, we propose a process that follows a bottom-up generative system based on Monge surfaces, which offers wide possibilities of formal exploration, while imposing geometrical-constructive properties that are very useful in fabrication and assembly processes.

Keywords Monge surfaces · Carved surfaces · Moulding surfaces · Developable surfaces · Rotation-minimizing frame

✉ Francisco Gonzalez-Quintial
francisco.gonzalez@ehu.eus

Andrés Martin-Pastor
archiamp@us.es

1 Department of Architecture, Escuela Técnica Superior de Arquitectura (ETSA), Universidad del País Vasco (UPV-EHU), Donostia, Spain

2 Department of Graphic Engineering, Escuela Técnica Superior de Ingeniería de la Edificación (ETSIE), Universidad de Sevilla (US), Seville, Spain

Published online: 22 July 2024

Introduction

There has been a boom in the construction of architectural forms based on double curvature surfaces in recent years. This has produced both theoretical advances from a geometrical point of view and interesting built examples. These developments have led to the rationalisation of the geometries used, especially in terms of constructability. In these developments we can distinguish the use of shapes obtained by using two types of systems, known as bottom-up and top-down. This classification can both be applied in the case of obtaining a global shape and when approaching the resolution of an architectural form.

In terms of the general shape and approaches to the global form, on the one hand we can speak of imposed shapes, which are the result of a purely aesthetic and formal intention. In these cases, questions of optimization are left aside. Alternatively, forms are rationally adapted to a function by applying form finding processes. This mainly relates to optimised structural functioning in response to the external forces to which the final architectural form will be subjected. One example is the application of the principle of funicularity, which has been extensively used since it was developed in the 19th century as a system of graphic statics. This was used to generate forms that solve the issues in transmitting a structure's own weight. Another application is the use of minimal surfaces, where the tension is uniform, for generating surfaces that work mainly in traction.

When working at a more detailed scale, the approach is similar. Thus, in top-down processes, questions of optimization for construction are left aside in favour of a faithful reproduction of the global shape. This can lead, in certain cases, to very complex constructive processes which demand the application of highly sophisticated technical means. These constitute their own fields of research and development. On the other hand, in bottom-up processes, geometrical fundamentals are considered from the first phases of the design process with the intention of simplifying construction and optimising material usage. Some examples of this include the application of systems based on geodesic curves as the basis for the use of flexible linear elements for the resolution of gridshell-type shells, and enclosures constructed with panelling based on flat quadrilaterals built with rigid materials or developable strips that allow the use of materials, which mainly bend in one direction.

A bottom-up process clearly imposes greater limits on the generation of built forms than top-down processes. This is highly relevant from an architectural point of view. However, in a top-down process fidelity to form can, in certain cases, give rise to problems with constructive resolution. This can lead to a costly building process which may also involve certain compromises with respect to the original design proposal.

The above trade-off in the resolution of shapes is an important issue. In response, this paper demonstrates a process that offers a high degree of flexibility for formal exploration, while also maintaining some invariable geometric principles that facilitate the constructive resolution of double curvature shapes in a rationalised way. The approach described remains bottom-up throughout its development. This is true in terms of obtaining both the global form and when taking advantage of the geometric properties that underlie the different stages involved in the constructive resolution of the projected form.

Monge Surfaces. Geometric Foundations

The term 'Monge surface' refers to the form described as follows by the French mathematician and geometrician Gaspard Monge in his classic text *Application de l'Analyse a la Geometrie par G. Monge* (1807):

> XXV. - De la surface courbe dont toutes les normals sont tangentes à une même surface développable quelconque.
> [Of a curved surface where all the normals are tangent to a unique single developable surface.]

Monge did not, of course, name the surface after himself, but he did describe it and demonstrate its geometric properties. Darboux (1893) and Raffy (1901) subsequently carried out further exploration. Returning to Monge's original text, two different approaches to the system of *'génération'* of this surface can be identified. We focus specifically on the first form of generation, which is described as follows:

> …the proposed surface can be understood to have been generated by the motion of an arbitrary plane curve, constant in shape and size, whose plane rotates without slipping over a developable surface (Monge 1849) (Fig. 1).

A Monge surface is thus defined by the movement of a plane curve generatrix along a directrix curve, in such a way that the generatrix remains contained in a plane normal to the directrix and is displaced without torsion (Fig. 2). Defining the surface by means of the generatrix and the directrix provides indisputable advantages, as in this way the surface can be characterised by only two curves. From a mathematical approach, the condition of zero torsion between consecutive generatrices is linked to the concept of 'rotation minimising frame', which will be discussed later.

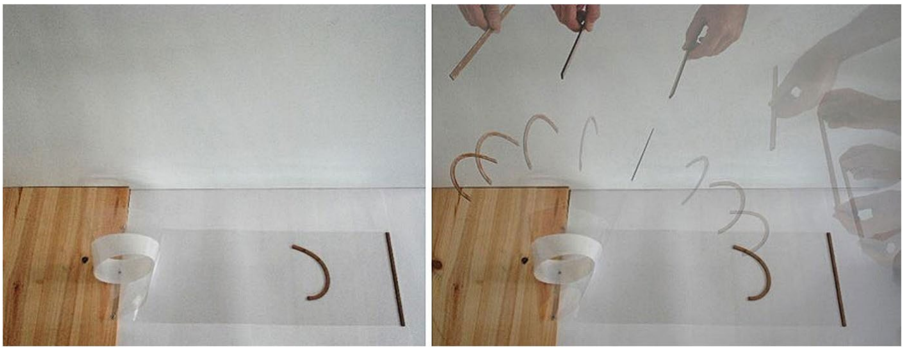

Fig. 1 A Monge surface generated by moving a curve contained in a plane, which rotates without slipping over a developable surface

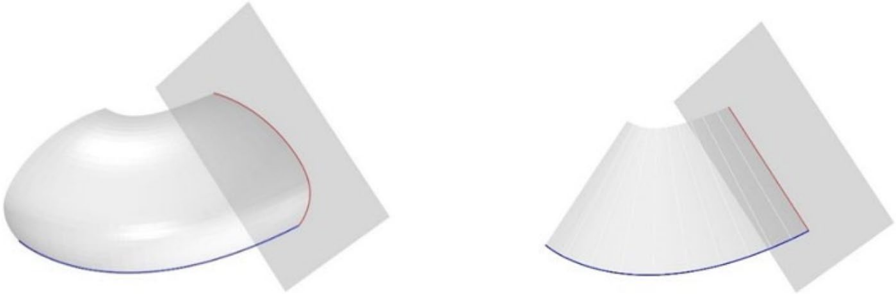

Fig. 2 Monge surface generated by the torsion-free movement of a generatrix along a directrix perpendicular to it (left). The zero-torsion movement is evident when the generatrix is a straight line, as it generates a developable ruled surface (right)

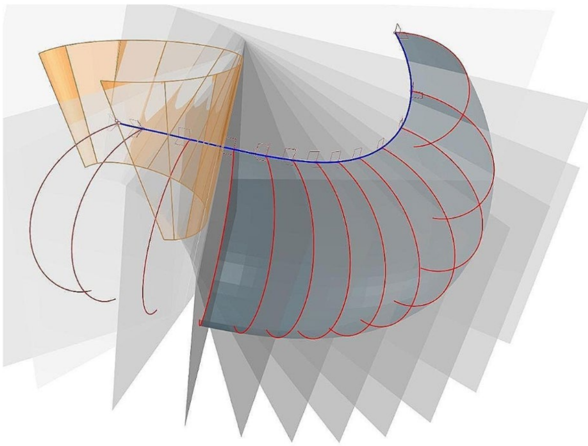

Fig. 3 Polar surface (orange) formed by the planes normal to the directrix curve (blue) and to which the normals of the Monge surface are tangent

It is important to point out that the developable surface referred to in Monge's text, formed by the envelope of the planes whose intersection defines the generatrices, is specifically the polar surface of the directrix (Fig. 3). This relationship demands some attention since the polar surface of a curve is easily definable (Izquierdo Asensi 1996).

A polar surface is any one of the three developable surfaces which, together with the tangential and rectifying surfaces, are generated from a curved directrix -planar or not- in space, by means of the movement of the Frenet frame along it. In the case of the tangential surface, it is the movement of the osculating plane for rectifying the surface of the binormal plane. In the particular case of the polar surface, the surface is generated by the movement of a plane normal to the curve. The consecutive intersections of these infinitely close normal planes form the ruled generatrices of this continuous developable surface, which is called the polar surface.

A polar surface is also generated by the movement of a straight line parallel to the binormal of the curve at a point, through the centre of the osculating circle of the curve, corresponding to that same point. These binormals coincide with the tangential lines of the polar surface (Fig. 4).

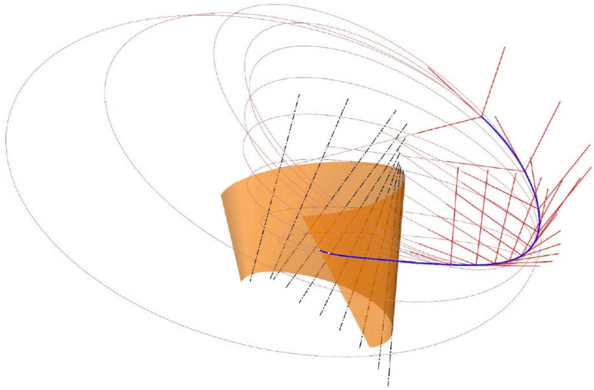

Fig. 4 Given a non-planar curve (blue) and its associated polar surface (orange), we can draw its binormal lines. If we make lines parallel to the binormal lines pass through the centres of the osculating circles corresponding to the discrete point of the curve through which we have obtained the binormal, the developable surface that forms the sweep of these lines coincides with the same polar surface

Among the applications of the polar surface of a curve, we highlight the generation of curves parallel with respect to this initial curve. Based on this property, a Monge surface can be defined as that generated by any set of parallel curves in space (Martín-Pastor González-Quintial 2024).

Objectives

It is important to keep the purely applied character of Monge's geometrical work in mind. There is a clear link with the concept of 'constructive geometry' (Sakarovitch 2019) which remains relevant today when considering the intimate relationship between geometric processes and digital fabrication. Monge surfaces share, amongst others which are discussed below, one basic characteristic identified in the original description:

> VII.…Through each point of any curved surface there always pass two lines of curvature which intersect at right angles on the surface,… From this it follows, first, that the generatrix, in all its positions, is the line of one of the curvatures of the surface generated; secondly, that this line of curvature is always flat (Monge 1849).

These two families of curves on the surface have a direct relationship with the originating curves, generatrix and directrix, and are families of conjugate curves, specifically, they form a PCL network, with all the advantages that this entails (Sotomayor 2007; Schiftner et al. 2013). The generating curve and all its rotations in space constitute one of the families of curves and its conjugate, the trajectory curve, and any of its parallels to the surface derive from the other.

The approach we describe in this paper takes advantage of the geometric-constructive properties underlying the PCL of Monge surfaces to generate them systematically, with a view to their application in architecture.

From a morphological point of view, a Monge surface is strongly conditioned by whether the directrix is a flat or a warped curve (Fig. 5). In the system we present here, however, there are no differences between the two at a generative level.

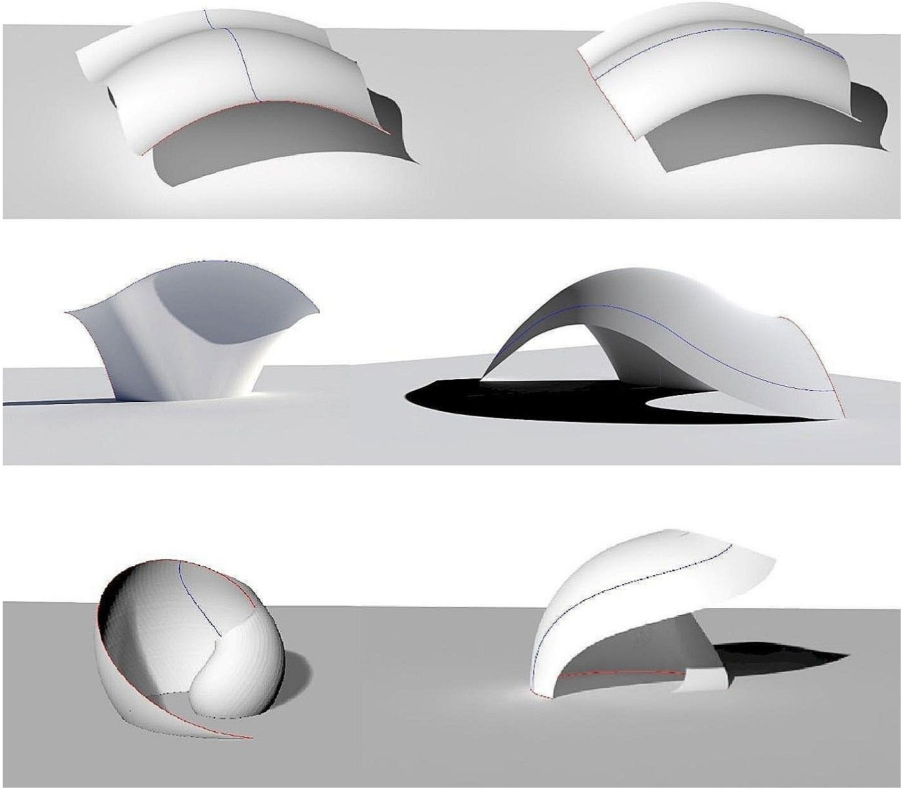

Fig. 5 Different surfaces obtained through the procedure described by Monge

The digital graphic tools intimately related with these processes of generation open up very promising avenues for the formal exploration of these surfaces. 'Augmented graphic thinking', a concept discussed in depth by (Martín-Pastor, Vargas-Peña 2024) articulates the theoretical and methodological foundations on which we have based our own approach to the problem.

Monge and Polar Surfaces. Contextualisation

The current literature includes a number of references to Monge surfaces. Some sources name them explicitly, while others refer to the same geometrical concept under a different name.

In texts that refer to generation defined as 'sweeping a profile along a curved path' (Pottmann et al. 2010) we identify a direct allusion to the geometrical concept of the Monge surface. Pottmann calls Monge surfaces with flat directrix 'moulding surfaces' and those with warped directrix 'generalised moulding surfaces'. In the latter case, Pottman uses the concept of 'rotation minimising frame' to explain the generation of these surfaces.

In the *Encyclopedia of Analytical Surfaces*, Krivoshapko and Ivanov (2015) describe Monge surfaces as 'carved surfaces of general type'. They classify them into a series of types and describe their mathematical formulation. Finally, they specify that "carved surfaces of general type and Monge surfaces are the same surfaces" (Krivoshapko and Ivanov 2015: 199). However, they always use a plane curve as a directrix. This limits the geometric repertoire which is actually possible with Monge development. Krivoshapko and Ivanov describe a 'kinematic model', which we identify as an iteration of Monge's first formulation. Their work includes an extensive list of references to the use of these surfaces for the construction of shells.

Mesnil et al. (2015) also identify Monge surfaces as 'generalised moulding surfaces' and work with both the idea of the kinematic model and the synthetic concept of the 'rotation minimising frame'. These two concepts are also found in Brander and Gravesen (2018).

There are also some theoretical proposals for concrete uses of this type of surface in architecture (Filipova and Rynkovskaya 2017). Recently, Gil-Oulbe and Ndomilep (2020a; 2020b) have engaged in mathematical research on Monge surfaces, and they use a set of terms related to 'carved surfaces', similar to those which appears in the Encyclopedia of Analytical Surfaces.

The term 'polar surface'' appears in Monge's own work, *Mémoire sur les développées*, *les rayons de courbure et les différents genres d'inflexions des courbes a double courbure*, a text written in 1771 and published in 1785. This text also contains the earliest known graphic representation of this surface (Fig. 6).

Very little literature addresses polar surfaces, including the texts *Geometry and its Applications in Arts*, *Nature and Technology* (Glaeser 2012) and the abovemen-

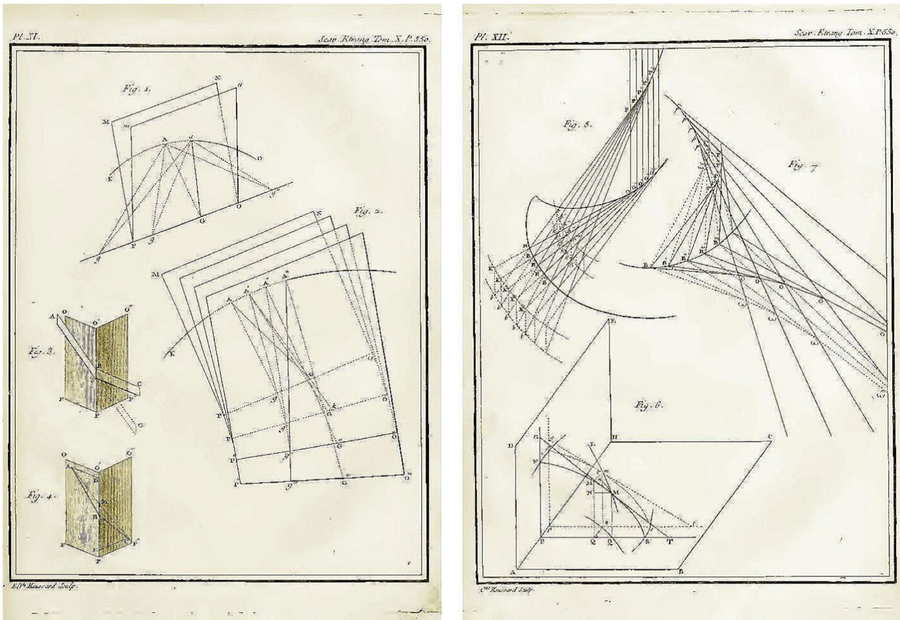

Fig. 6 Monge Surfaces. Image: Monge 1785: Pl. XI, XII

tioned *Encyclopedia of Analytical* Surfaces (Krivoshapko and Ivanov 2015). Nor are they explicitly mentioned in major reference texts on developable surfaces including *Developable Surfaces: Their History and Application* (Lawrence 2011), *Parametric Geometry of Curves and Surfaces* (Lastra 2021) or *Architectural Geometry* (Pottmann et al. 2010). Krivoshapko and Ivanov do not use the term ‘polar surface’ for purely terminological reasons, employing instead the terms ‘evolute surfaces’, ‘directix surfaces’ and ‘fixed axoid’. Gil-Oulbé and Ndomilep use the terms ‘fixed axoid’ and ‘fixed directrix torso’ implicit in Carved Surfaces (2020a; 2020b).

Beyond the significant variation in the terminology used, it is clear that, in the absence of a unifying concept of the polar surface, Monge surfaces have been approached from different perspectives. It is also evident that Monge surfaces with warped directrix have not been studied in depth, nor is there a significant volume of work on the morphological or constructive possibilities for the application of these surfaces in architecture.

Generating Monge Surfaces through Graphical Models

Through the direct transcription of the system described by Monge, we can deconstruct the process of generating this surface and rebuild it following a graphical procedure.

A plane R normal to a non-planar curve c not straight at a point A1, intersects with another consecutive plane N, normal to the curve c at B1, along a line that we can define as the axis of rotation e1 of the plane R with respect to the immediately following N. Between the planes R and N there is only one rotation, there is no torsion, and there is no rotation of the binormal around the tangent between these two consecutive positions. Thus, a point A1 defined in the plane R, rotates around the axis e1, intersection of the planes R and N, up to the position B1 without any torsion (Fig. 7).

As has been stated already, the intersection of successive planes normal to a curve generates a ruled developable surface called a polar surface. Described in another way, the Monge surface is generated by the movement, or more properly the rota-

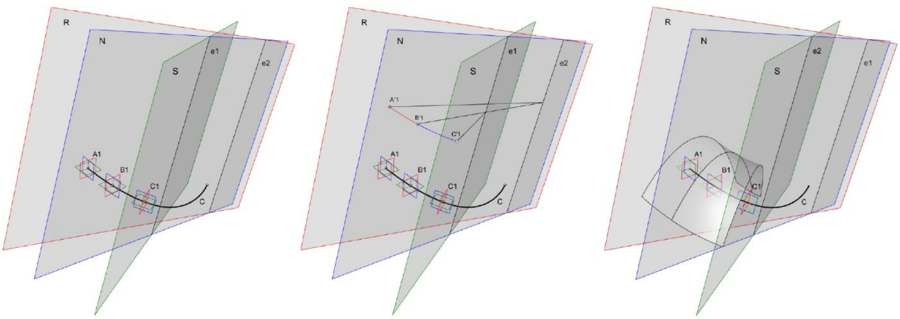

Fig. 7 Successive normal planes to a curve c intersect in lines e1, e2 (left). The axes act as rotation axes of the points contained in the normal planes (centre). A Monge surface is generated by the successive movement of the infinite number of generatrix curves contained in the consecutive planes normal to the directrix (right)

tion, of the plane tangent to the polar developable surface on which we can place any planar curve.

We can now return to the two-by-two consecutive normal planes R, N, S along a curve C that we name the directrix. If we define a curve contained in the first of the plans that we name the generatrix, we obtain a second curve as a product of the rotation of the generatrix in the consecutive plane. This curve is analogous to the movement of the point we rotate around, taking the intersection of the contiguous planes or, in other words, the ruling of the developable, as the axis of rotation. The infinite number of successive generatrix curves contained in the consecutive planes normal to the directrix finally determine the Monge surface.

The surface obtained in its continuous form is a double curvature surface. With a view to application in architecture, it is useful to look for a way to discretise this surface. If we discretise the original generatrix curve at a given number of points and repeat the procedure iteratively, we obtain a discrete surface formed by semi-discrete developable strips (Fig. 8).

We literally transcribed this system of geometric generation into a graphic algorithm through Grasshopper. This algorithm systematised the entire process of generation and the different discretization systems. Starting from two curves, generatrix and directrix, this algorithm can generate three-dimensional models and their flat development, with a focus on the production of physical models (Fig. 9).

Applications for Constructing Architectural Forms

As stated, the process begins with two curves, one necessarily planar, generatrix, and the other directrix, which can be planar or not. First, if both generating and directrix curves are continuous curves, or alternatively, if we use infinite normal planes in the

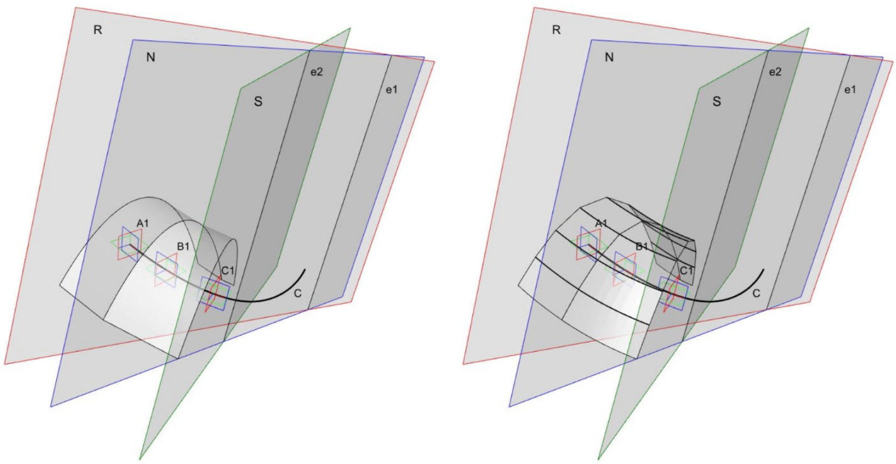

Fig. 8 Discretization of the generating curve. The result is a series of developable semi-discrete surfaces or strips

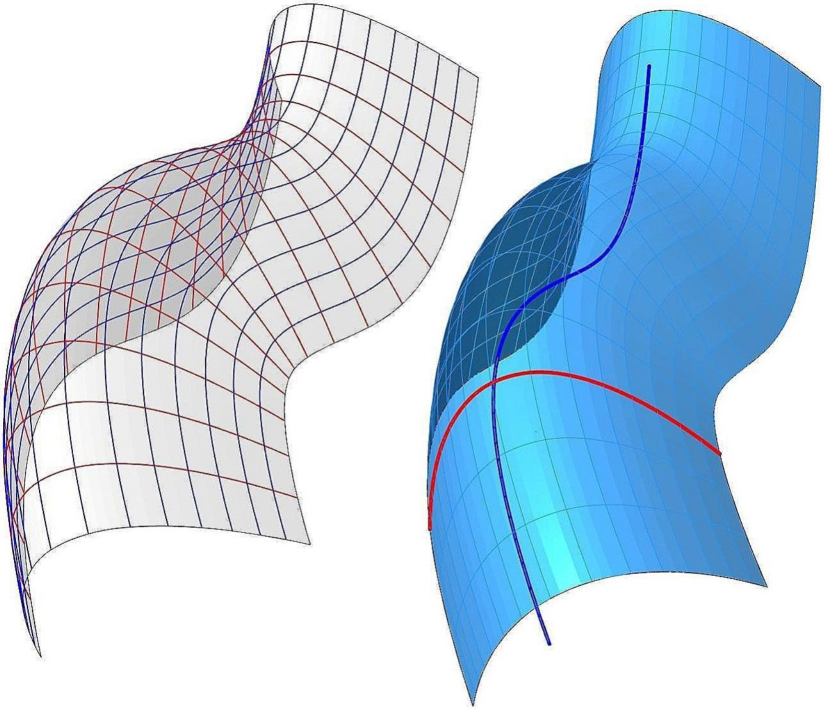

Fig. 9 Generating an algorithm-based Monge surface. Starting from a generatrix (red) following a non-planar directrix (blue)

definition of the polar surface and we make the rotation of the generating curve along the directrix continuous, the surface obtained will be of double curvature (Fig. 10a).

Depending on the degree of discretisation of the curves on the surface generated, we can distinguish different cases of discretisation of the continuous surface.

If the generatrix is polygonal, we obtain a surface that discretises the Monge surface by means of developable strips parallel to the directrix curve, which can be referred to as the meridians of the Monge surface (Fig. 10b).

Note that the parallel curves generated by sweeping the generatrices discretise the directrix curve into arcs of circumference, since the surface generated between normal planes is a surface of revolution. The greater the number of normal planes in the discretisation, the closer this surface will be to the Monge surface.

A second case is produced when the generating curve is continuous and the number of normal planes used in the generation of the surface is discrete. This results in a series of semi-discrete developable sections determined by the parallels of the Monge surface (Fig. 10c).

The third case consists of a discrete directrix curve as well as a discrete number of normal planes. In other words, the parallels on the Monge surface are also discretised as polygonals (Fig. 10d). From this generation we directly obtain a network of flat quadrilaterals, due to the fact that the sections normal to the directrix are spindles of a surface of revolution.

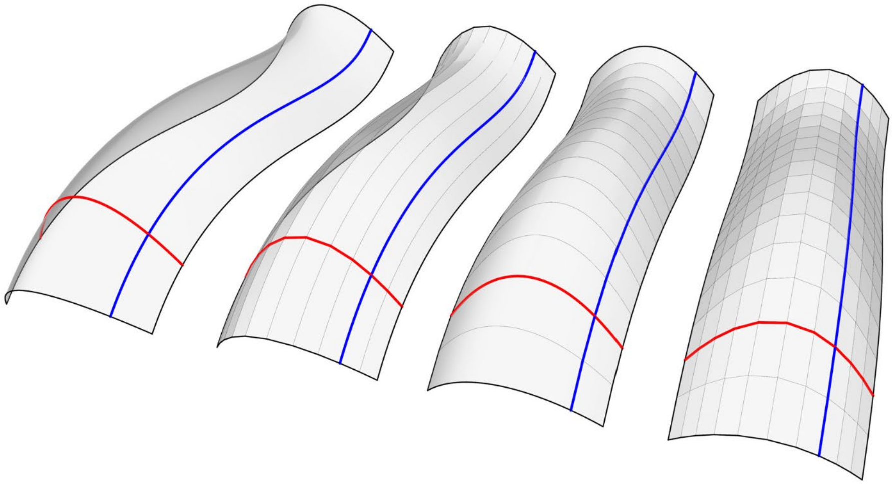

Fig. 10 Different surfaces obtained by the geometric procedure described. From right to left, (**a**) a double curvature Monge surface (**b**) a surface discretized by longitudinal semi-discrete strips, (**c**) a surface discretized by transverse semi-discrete strips, (**d**) a surface discretized by flat quadrilaterals

Building Systems Based on Monge Surfaces

When generating a Monge surface, we control the surface PCL from the earliest stages of generation. While the process for determining the PCL for any free surface is already well understood, for a bottom-up process it is nevertheless important to generate a surface by starting with control over the curves, and not the other way round.

PCL have a number of properties in relation to each other and with respect to the surface. Firstly, the two conjugate families intersect orthogonally, and the normals to the surface along the PCL form the ruled surfaces of a developable surface.

It is possible to discretise a surface using developable strips between two PCL of the same family. This said, there are other procedures for discretisation using developable strips based on networks of conjugate curves (González-Quintial et al. 2015).

We can discretise the surface using planar quadrilateral mesh (PQmeshes). This point needs some clarification. In general, the four points of intersection of four PCL of any surface are not coplanar. This includes Monge surfaces in their continuous form, even if the resulting quadrilateral mesh on the surface has been considered ‘almost flat’. If we want to discretise a surface into PQmeshes from the intersection points of its PCL, a prior rationalisation of the initial surface is necessary. The above procedure, as we have seen, automatically generates a mesh of flat quadrilaterals contained between the discrete generatrices (Fig. 10d).

PCL allows for a discretisation into PQmeshes, which give rise to conicalPQ or circularPQ meshes. These two meshes make it possible to work with special meshes such as ‘mesh with exact offset’, which are very important for the generation of multilayer systems (Pottmann et al. 2007 a).

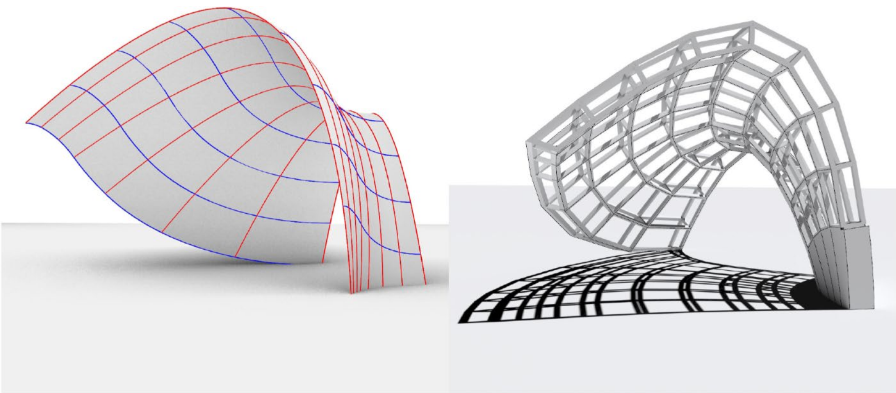

Fig. 11 Discretisation of a Monge surface (left) as a double-layer straight bars model (right)

For Monge surfaces, one of the two PCL families, namely the generatrices, are surface geodesics and plane curves. This gives them remarkably valuable properties for use in construction, including the possibility of placing rectangular strips circumscribed to the surface on the geodesics (Brander and Gravesen 2018).

The geometric properties described above have enormous potential for application in construction. The following is a very succinct description of some of the ways in which Monge surfaces can be applied in structural systems and architectural envelopes.

First, one possible procedure is to generate a straight bar model following the PQMesh discretisation of a Monge surface. This facilitates the use of multilayer systems on the PCL, an issue that has been researched extensively in relation to the possibility of generating meshes with exact offset (Liu et al. 2006; Pottmann et al. 2007a, b, 2008; Mesnil et al. 2018; Jiang et al. 2022; Dellinger et al. 2023). As these are discretised in polygonals, the offset of each PCL generates a PQ strip. These intersect in torsion-free nodes. The disadvantage of these grids is that the bars join nodes forming different angles, which complicates buildability except in some special cases, such as those studied by Mesnil et al. (2015). This bar model does not allow the mechanisation of the node using a 3-axis CNC machine, as the bars are not perpendicular in the node (Fig. 11).

Another interesting application is based on the generation of curved beam-like elements following both PCL families of the surface, perpendicular to the surface. This is a specific solution within the wider field of only bending curved support (Tang et al. 2016). Placing the beams on PCL offers the advantage that all the main planes of the beams intersect at 90° at a single type of joint and this, in turn, ensures that the joint can be machined by a 3-axis CNC milling machine (Fig. 12).

For this type of structural element, the main plane of the beam is perpendicular to the surface, so the edge of the beam is parallel to the surface. All the nodes are orthogonal and identical: the main planes of the beams can be cut at 90° and, in addition, there is only one type of node which can be machined by a 3-axis CNC machine.

A third application can be found in the resolution of architectural envelopes. The generative system described here can produce a wide range of solutions, either

Fig. 12 Digital and physical models of a discretised Monge surface. Developable elements follow the PCL. Image: Piedro Nícolas Paduan Zahernski

by means of strips of developable surfaces, or panelling using flat quadrilaterals (Fig. 13). In this way, a complete construction solution can be produced based on the original generated form, including design, manufacture and assembly.

The 'Rotation Minimising Frame'

Much of the current literature adopts a different approach to the synthetic concept of Monge surfaces to that used in this paper and by Monge himself. In many cases, the concept of 'rotation minimising frame' is used within a mathematical approach. We understand that this issue needs to be clarified, opening a parallel discussion to the primary focus of this article.

The term 'rotation minimising frame' is used in Isogonal Moulding Surfaces (Mesnil et al. 2015), where it is key to holding the movement of the generatrix within this 'rotation minimising frame'. A mathematical approach to defining the 'rotation minimising frame' has received special attention from a computational and mathematical point of view (Farouki 2016; Wang 2008) in the field of Computer Aided Design (CAD).

From a purely geometric point of view, the reference plane on which the generating curve sits, is a plane normal to the directrix, independently of the orientation of the axes that determine it. In general, the plane normal to a curve is the one defined by the line or normal vector and the binormal to the curve, i.e., the normal plane defined in Frenet frame. However, the torsion of the normal plane depends on the relative position of the generatrix curve in its spatial movement along the directrix. If we use the normal plane defined by this Frenet frame:

> For a 3D trajectory y(s), and a 2D contour c(u), the Frenet frame sweep and the rotation minimising sweep yield the same result if and only if y is a plane curve with non-vanishing curvature (Klok 1986).

This definition seems to be a better description of the logic behind the concept of moulding surfaces. Thus, this tracing limits the validity of the use of the normal plane defined by a Frenet frame for obtaining Monge surfaces by means of the use of planar guiding curves. If we wish to obtain a surface, which is not just developable, but also a surface with no torsion, the use of the normal plane according to the Frenet frame is not valid, unless the directrix curve is planar. However, as has already been pointed out, 'there are various ways of defining the plane normal to a curve' (Bishop 1975).

In the development of the synthetic method presented by Monge, this idea of rotation is completely extraneous, since there is no reference to the Frenet trihedron when characterising the normal plane. The condition of producing torsion-free movement is implicit in the movement of the normal plane that moves tangentially without slipping on the polar surface of the directrix, since any segment contained in this plane generates a developable–that is, torsion-free– strip between consecutive ruled lines. In a way, a polar surface can be understood as the graphical equivalent of this reasoning, and this allows us to impose exactly and directly the condition of zero torsion between consecutive generatrices.

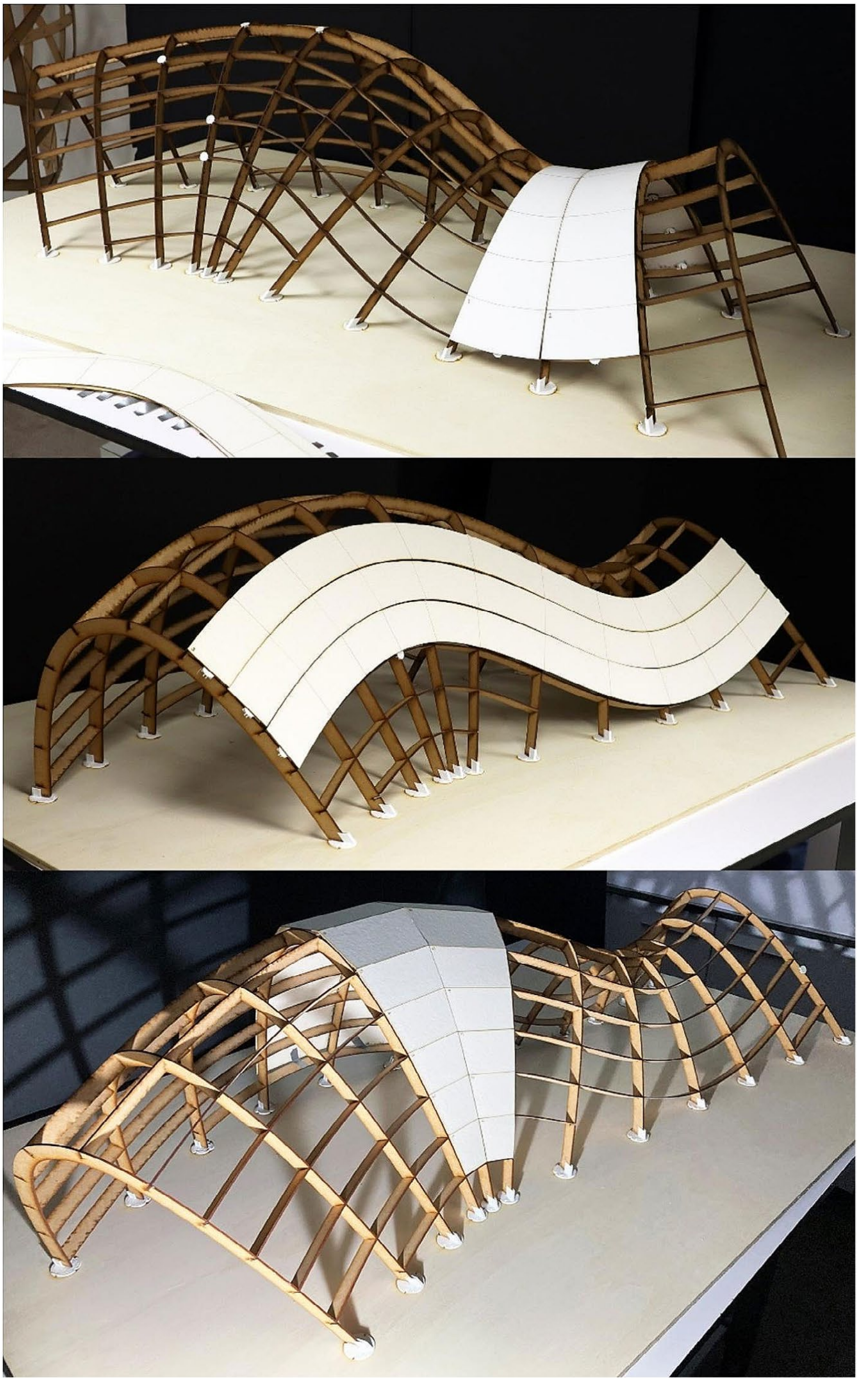

Fig. 13 Different approaches to an envelope system based on semidiscretised strips, parallel (above) meridian (middle) and discretized planar quads (below). Image: Piedro Nícolas Paduan Zahernski

Conclusion

This article highlights the value of Monge's work in defining the surface that now bears his name. It clarifies the geometric processes involved in generating them, and shows the link with polar surfaces. Through the geometric mechanism of generation and its application by means of an algorithm, the paper demonstrates a solution for obtaining the discretization of a double curvature surface, which is directly relevant to questions of buildability in architecture.

We have also shown the important role that polar surfaces can play in the graphical generation of Monge surfaces, and how this represents a graphical alternative to mathematical resolutions which use the so-called 'rotation minimising frame'. This vindicates the importance of these surfaces, which are very rarely included in reference material on geometry in general. Monge surfaces are representative of developable surfaces but, perhaps because no direct practical application has been recognized until now, they have been largely ignored.

This paper, more than anything, offers a starting point. The subject remains open and several related lines of research are currently being investigated. The work of cataloguing the types of Monge surfaces that can be generated in greater depth remains to be done. This includes identifying those which show promise in terms of facilitating construction, and describing their specific properties from the point of view of their application in architecture. The formal exploration of the use of this kind of surface is yet to be carried out.

Funding Open Access funding provided thanks to the CRUE-CSIC agreement with Springer Nature.

Declarations

Conflict of Interest The authors disclose no conflicts of interest.

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