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RESEARCH

The Regular Polyhedra: Drawing and Computing in Euclid's day

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Abstract
Can we compute what we cannot draw? How must we draw to produce measurable representations, or visual ones? This research inquires into the relationship between mathematics and figurative representation, and more precisely between drawing and computation. The scientific imagery studied here is the representation of the five platonic solids, discussing various representation techniques from classical antiquity to modern times, and their efficacy to help calculate sizes and proportional ratios. Scholars in history of architectural drawing have too often limited their observations to the very few preserved plans and front views dating back to classical antiquity, without enlarging their investigation to other scientific fields that also rely on drawing as a research tool and communication device. Among these other fields stands the mathematical research, especially solid geometry which deals with objects and entities that have shapes that needs to be somehow drawn in 3D to be studied.

Keywords Platonic solids · Regular polyhedra · Euclidean geometry · Scientific imagery · Architectural drawing · Representation techniques

Preamble

The representations of the five regular polyhedra contained in the early manuscripts of Euclid's *Elements* testify of the many difficulties raised by the question of drawing three-dimensional objects in a time when no graphical conventions existed yet, for doing so.

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The modern drawing methodologies that are standard today and that are currently taught to architecture students to represent 3D spaces and volumes, are the techniques of axonometric projections and perspective views. Both techniques have strengths and weaknesses. Perspectives views are simulations of the human natural view: they produce immediately recognizable images but those are not measurable since they are not scale drawings. Axonometric views, on the other hand, are measurable (when the scale reduction is provided) but they tend to gracelessly distort the object that is shown. None of those two techniques was systematically applied in the drawings illustrating the study of the regular polyhedra contained in the early manuscripts of the Euclid's *Elements.*

This research focuses on the examination of the relationship between theoretical research and geometrical drawing in Euclid's day. Of course, such a study can only be conducted by scrutinising the manuscripts that have been preserved and that are still currently available in libraries. It is now accepted by the scientific community that the most ancient available copy of the *Elements* is the Greek manuscript kept in the Vatican library dating back to the 9th century A.D., some eleven centuries after Euclid's death (after Euclid, 9th century. Vatican).

We may wonder about who the manuscript copyists were: those people who transmitted this ancient knowledge and made its dissemination possible. We have evidence that, most of the time, the manuscripts were produced in two steps. First the text was written. The text copyist used to leave some blank areas on the pages layout in order to provide space for the images. Later, when the text was ready, the illustrations were drawn – presumably – by another copyist. Often the blank spaces were too small, and the drawings would show small, somewhat squeezed, when they did not extend in the margins of the page. The Greek manuscript BL Arundel MS 548, kept in the British Library in London, dating back to the first quarter of the 16th century, is an example of unfinished copy of Euclid's *Elements* in which the images were never drawn as the blank spaces pre-set for illustrations are still empty (after Euclid, 16th century. BL, Arundel) (Fig. 1).

Were the copyists trained in mathematics or were they only trained in calligraphy and drawing; and thus, responsible for the transmission of scientific knowledge and scientific imagery?

We are not going to discuss the quality of the texts of the various manuscripts, and the historical chain of translations that have produced the many still extant copies. This is a subject that has been studied at length by many scholars, editors, commentators, and historians of mathematics. The images on the other hand, have so far received much less attention than the written words, as if they were less important than the literary discourse and a minor aspect of the books. François Peyrard, translator and editor of the French translation of manuscript Vat.Gr.190.pt, published by Patris in Paris in 1819, admits (in a footnote in Book XII) to have picked images in other editions when he was not convinced by those of Vat.Gr.190.pt. (Peyrard 1819).

However, text and images are inseparable from each other in a mathematical demonstration focusing on geometric problems. Furthermore – and this is our concern here – the scientific approach with which the problems are solved shapes the images that support the theory and ease the learning. When solid geometry is addressed, the studied objects and volumes can be shown either in a visual way, or diagrammati-

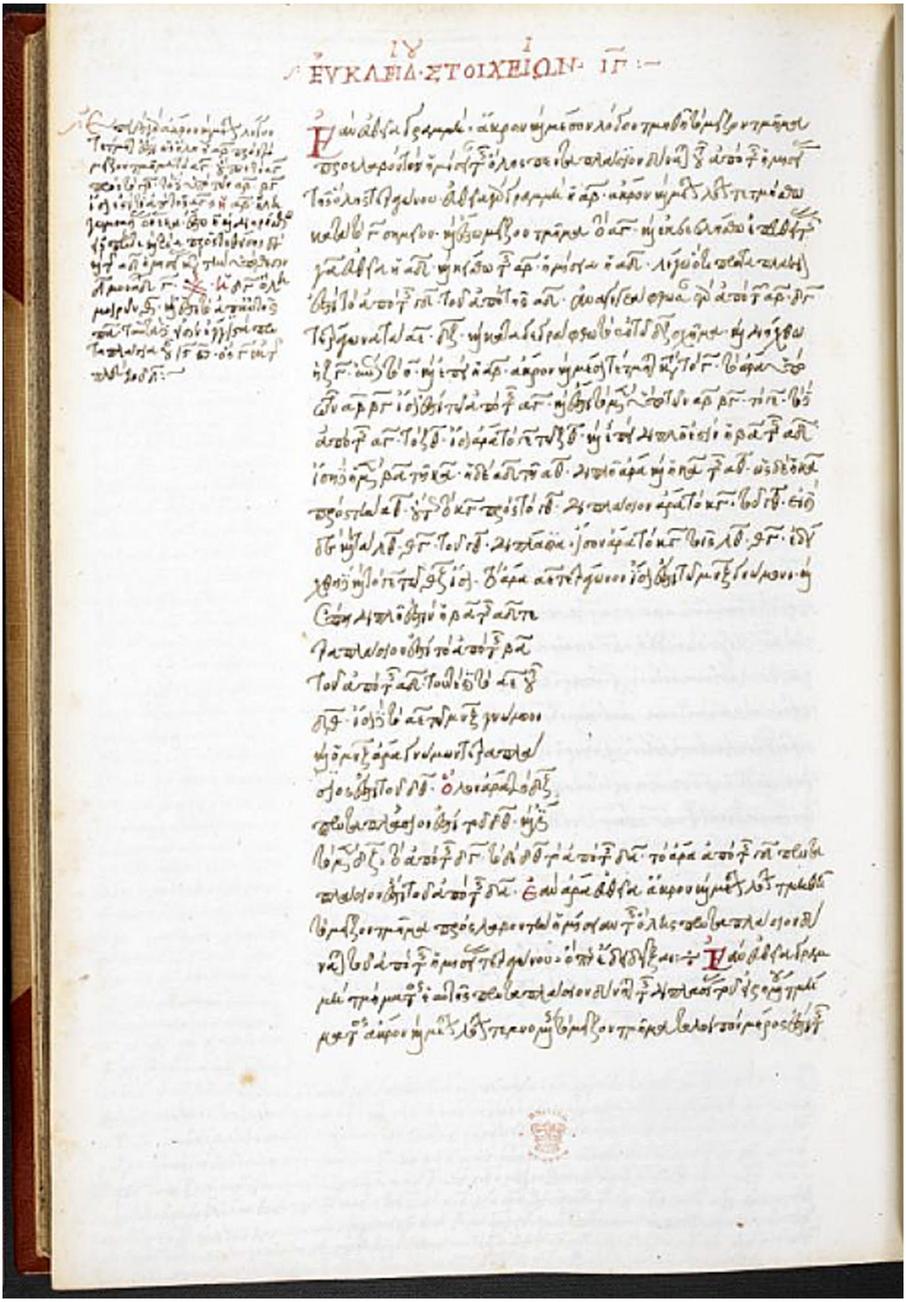

Fig. 1 A page of manuscript BL Arundel MS 548 kept in the British Library showing missing illustrations. Image: BL Arundel MS 548

cally, or they may be explained thanks to a polyptych of multi-views. There are many possible options for representation that depend on what is the true question at stake, how the solution has been reached, and what are the available contemporary techniques for geometric drawing.

Euclidean Geometry and the Structure of the *Elements*

Euclid of Alexandria's treatise, written at the turn of the 4th century B.C., gathered most of the mathematical knowledge that was available at the time in thirteen books. All books are divided in a certain number of 'propositions', each one raising a specific problem which is duly solved. The shortest book – Book II – contains only 14 propositions, while the longest book – Book X – holds as many as 117. Each proposition is illustrated by a graphic which depicts both the problem and the solution. Each book starts with a series of 'definitions' that assert some truths that are not demonstrated or demonstrable.

The *Elements* address issues regarding both arithmetic and geometry. The first nine books are concerned with the theory of numbers, the concept of proportions and planar geometry. Book X addresses the core of the mathematical science of its time: the problem of irrationality and incommensurability. The last three books (XI-XII) are sometimes referred to as 'the solid books' since they address solid geometry with special attention to the study of the five regular polyhedra. Most of the Renaissance editions include two more books (XIV and XV) which are some sort of continuation of book XIII, although they were not by Euclid. Book XIV is attributed to Hypsicles (2nd century B.C.) and book XV was presumably written in the sixth century A.D.

The development of the mathematical demonstrations that we read in the *Elements* makes use of intellectual procedures and computing methods that are no longer familiar to us. Euclid deals with 'magnitudes' which may be either arithmetical or geometrical entities: numbers or shapes (planar or solid). However, numbers may have shapes and shapes may be expressed through numbers. The 'magnitudes' are represented on paper through figures. Scholars of Greek mathematics speak of 'shapeful numbers' or 'figurate quantities'. Hence arithmetical operations can be done by drawing. When numbers are lines, addition means lengthening an initial line; multiplication between two numbers means drawing a rectangle, multiplying three numbers leads to a solid. Therefore, multiplying a number by itself creates a square, and multiplying it once again creates a cube. Indeed, we still speak today of 'square' numbers and 'cubic' numbers, 'square roots' and 'cubic roots'.

Among the main concerns of ancient Greek mathematics stand the concepts of irrationality and incommensurability. The concept of irrationality refers to the quality of a magnitude. The notion of in-commensurability refers to the quality of the ratio between two magnitudes. Two quantities are commensurable if they are linked by plain relationships, such as having a ratio equal to a ratio existing between two natural integers. In Book X Euclid sets out thirteen different kinds of irrationality and commensurability. For instance, quantities may be commensurable-in-length or commensurable according to the squares constructed on their sides. Magnitudes commensurable in length are those that have a common divisor (which may be either rational or irrational). Magnitudes commensurable-in-square are those whose squares are both multiples of a common quantity, such as the side and the diagonal of a square whose squares are in ratio 1 to 2. One type of relationship does not categorically exclude the other: the same quantities may be both commensurable in length and in square. The several kinds of irrational magnitudes bear names – medial, binomial, apotome… – according to how they are obtained: by addition or subtraction of different magnitudes.

In book XIII Euclid shows that the edges of the tetrahedron, octahedron and cube are commensurable-in-square with the diameter of the sphere that contains them, while the edge of the icosahedron is the irrational that is named ‘minor’ and that of the dodecahedron is the irrational that is named ‘apotome’.

Studying the Regular Polyhedra

The first historical description, readable in the dialogue by Plato entitled *Timaeus* (ca. 360 B.C.), focuses on the geometrical shape of the faces of each solid and on how they are put together. Plato defines the regular polyhedra as being an assembly of planar geometrical objects (the faces), according to specific ‘solid angles’. Starting from the planar angle of the single face, the mathematical problem consists in defining the ‘solid angle’ of the final polyhedron. The repeated element for the construction of the volume can therefore be considered either the face itself or the solid angle. While the tetrahedron, the octahedron and the icosahedron all have similar triangular faces, their solid angles differ. These three polyhedra are produced from joining, at each vertex, three, four or five equilateral triangular faces respectively (Fig. 2).

The cube, instead, is defined by its vertices where three square faces meet, and the dodecahedron's solid angle is made by the connection of three pentagonal faces.

Ever since Plato's day, together with other mathematical objects and concepts, the five regular polyhedra have been considered beautiful. They are equiangular and equilateral. They show several kinds of mirroring symmetries: according to cutting planes containing their center; according to their axes or their center. In order to define their size, a single numerical parameter is necessary: the length of their edge. The search for the construction of the regular polhyedra is a search for formal beauty and perfection.

> For to no one will we concede that fairer bodies than these, each distinct of its kind, are anywhere to be seen. (Plato & Bury, 1966) *Timaeus*, 53e)

Following from these descriptions, the next question which is addressed later in time and echoes in Euclid's *Elements* (ca. 300 B.C.) is the quantification of the lengths of the edges of the five regular polyhedra inscribed in a same sphere of given diameter. Book XIII is fully dedicated to this problem. What sort of images are accompanying the text of the five computing problems? The traditional images that are present in the manuscripts up to the Renaissance show that no standards existed in Euclid's day for drawing 3D solids. The five polyhedra are not drawn according to the same kind of

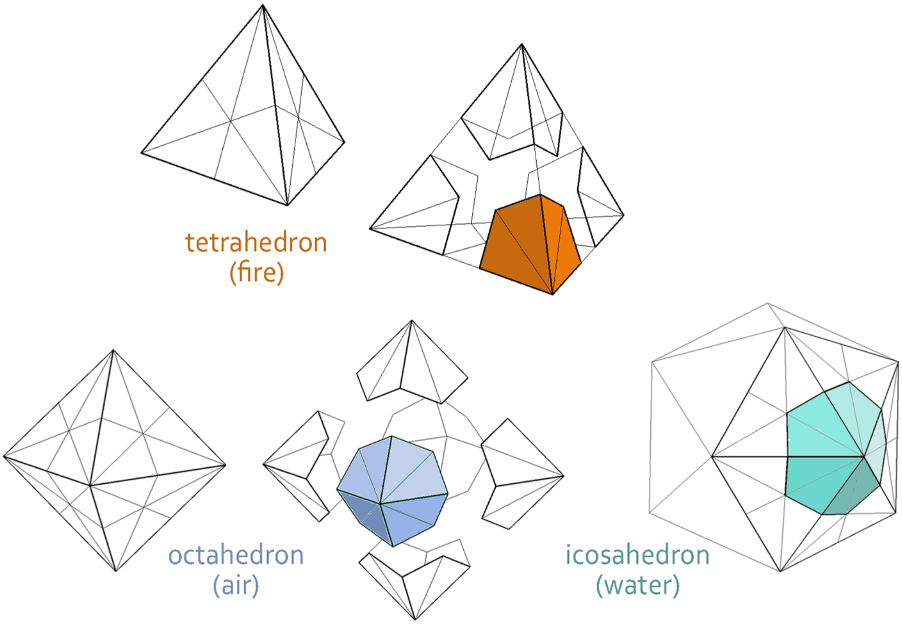

Fig. 2 The solid angles of the tetrahedron, octahedron and icosahedron as envisioned in Plato's day. Drawing: author

geometric projection. The increasing complexity of their shape due to the increasing number of faces is depicted thanks to different kinds of drawings.

Computing the Regular Polyhedra

Can we compute what we cannot draw? Is the act of drawing the successful research methodology? Or is the drawing merely the necessary graphic image that acts as a visual help for the learner? All the images drawn in ancient manuscripts and printed editions of the *Elements* must be read keeping in mind the underlying computation problem. The illustrations are all functional to this very goal: find the proportional ratios of the five solids' edges inscribed in a given sphere.

Ever since classical antiquity the study of the polyhedra started with the building and measuring of 3D models. Plato himself explains that mathematicians used to build models for themselves:

> … The very things which they mould and draw, which have shadows and images of themselves in water… but what they really seek is to get sight of those realities which can be seen only by the mind. (Plato & Shorey, 1969), *Republic*, book VI, 510e)

This fundamental information given us by Plato is the key to the interpretation of Euclid's figures. Indeed, a close examination of the images of the tetrahedron, octahedron and cube, as they show on every edition of the *Elements*, reveals that they are drawings of wooden models of the solids. And the models show more than the mere perfect 'fair bodies' described by Plato: they show the computing problem that Euclidean geometry was struggling to solve. In fact, a plain, perfect tetrahedron would be made of six equal sticks arranged in a triangular pyramid. But since the edge of the solid has to be compared to the diameter of the circumscribing sphere, the study model includes an additional longer stick: the diameter of the sphere itself. The tetrahedron is the only polyhedron which does not entirely embrace the diameter of the sphere: the long stick hangs from one vertex and extends beyond the volume of the pyramid, passing through the opposite base. The builder of the model had to add three additional short sticks in the surface of the 'pierced' triangular base, which have no other function than to hold the diameter stick fixed in its correct position; otherwise the stick would quickly fall apart. And this is this model's image that has been conveyed to posterity in every edition of the *Elements* (Fig. 3).

Similarly for the cube and octahedron. The figures copied in the manuscripts are 2D drawings of a 3D model: they are representations of a previous representation, showing the mathematical problem (computation of incommensurable lengths) in addition to the geometrical object itself. The model of the octahedron too includes the diameter of the sphere, the position of which is secured by the diagonals of the square middle section, whose intersection is the centre of the solid. The model of the cube, includes a diagonal stick on one of the six square faces, recalling the first basic problem of incommensurability between magnitudes, precisely the incommensurability between the side and the diagonal of the square: known as 'Plato's theorem'. The diagonal of the cube, visible in the model, besides being the diameter of the circumscribing sphere, also refers to one of the three classical problems of ancient mathematics: the duplication of the cube. The incommensurability between diagonal of the square and diagonal of the cube is shown here.

The images always show the polyhedra as transparent solids. All their edges are visible and no pictorial effect underlines the depth of the volume. The lines all have the same lineweight; no colour is used, not even shades of grey, to differentiate the front from the back, the far from the close.

The descriptive view of the icosahedron as seen on manuscripts coming from Greek tradition is a special kind of perspective, where the hypothetical visual axis corresponds to a diameter of the sphere. The result recalls a contemporary polar map projection in which two equal parallel circles belonging to a sphere do not plot of the same size. The produced image is a good example of the relationship between the visible and the intelligible in Euclidean imagery. Although the icosahedron does not really show, all of the steps of the procedure for its construction are adequately represented. (Murdoch 1984). The written accompanying text leads the scholar to understand the fact that the radial lines have to be 'seen' as lines that are indeed vertical in respect to the horizontal planar pentagons and parallel circles. All the vertices of the solid are visible, none being hidden, except for the two poles that are aligned. The reading of the image is a mental procedure rather than a visual examination. The drawing is descriptive but not visual and far from being instantly self-explanatory (Fig. 4).

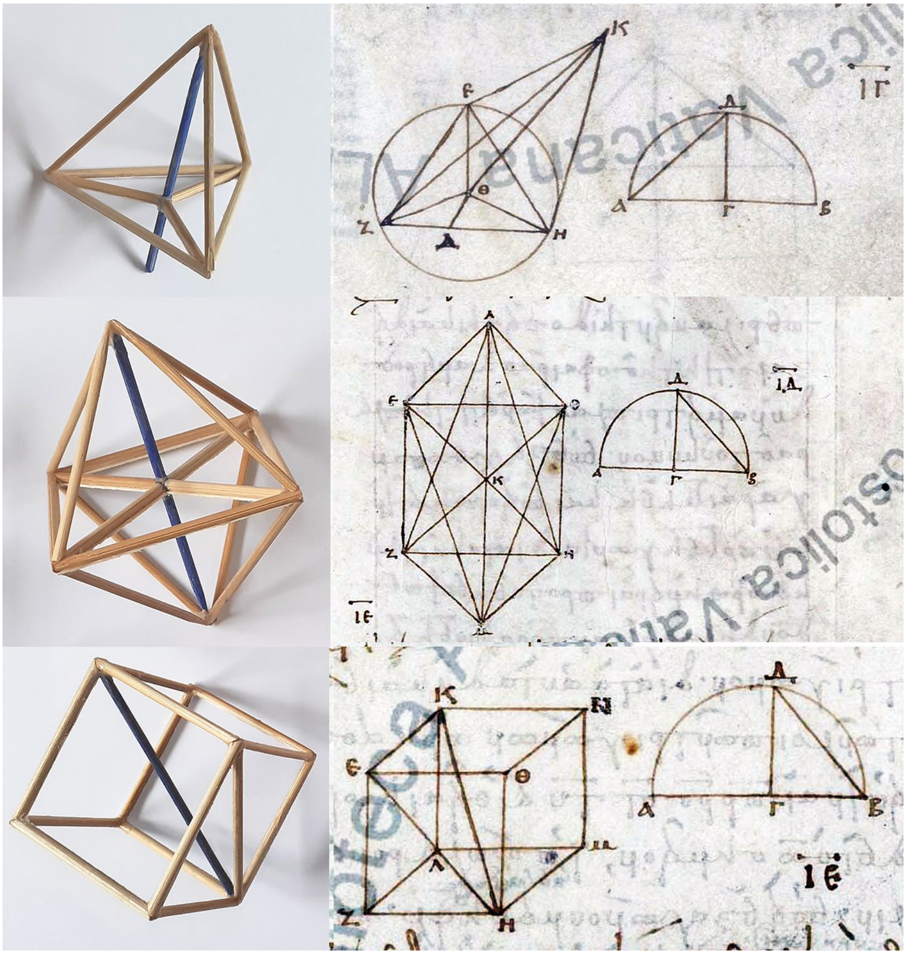

Fig. 3 Wooden models of the tetrahedron, octahedron and cube compared to the traditional representations of the same solids showing in the oldest preserved Greek manuscript Vat.Gr.190.pt, dated 9th century, kept in the Vatican Library. The coloured sticks of the models are the diameters of the circumscribing spheres. Image on the right: Vat.Gr.190.pt c.9th c

The image accompanying the study of the dodecahedron is even less visual than that of the icosahedron. The solid is not compared to its circumscribing sphere but to the cube which it encloses. Nor the dodecahedron or the cube are fully drawn. Most of the times we only see two faces of the cube and one face of the dodecahedron. The rest must be mentally built according to the intellect of the scholar. Nevertheless, the image shows the problem and its solution: the incommensurability between the cube's edge and the dodecahedron's edge: the latter being the 'apotome' of the former (Fig. 5).

Except for the dodecahedron (which is not compared to its circumscribing sphere), the drawings of the four first solids (tetrahedron, octahedron, cube and icosahedron) are always coupled with a planar diagram where the result of the computing problem

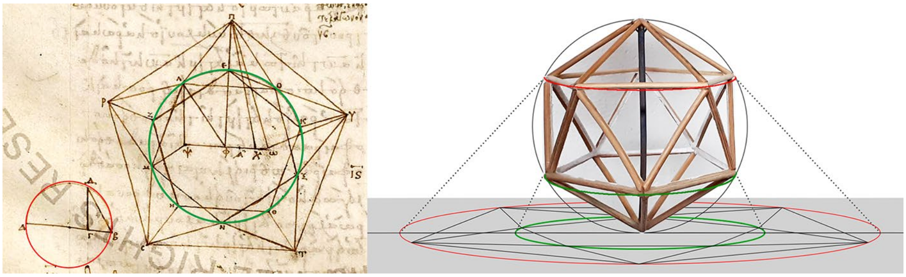

Fig. 4 Left: the icosahedron as shown in the Greek manuscript Vat.Gr.190.pt, dated 9th century. Image: Vat.Gr.190.pt, c.9th century. Right: the principle of the polar map projection, the closest modern technique matching the ancient sketch. Drawing: author

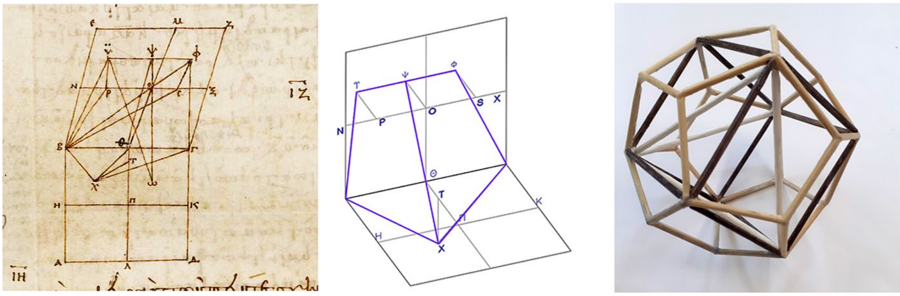

Fig. 5 Left: the dodecahedron as shown in the Greek manuscript Vat.Gr.190.pt, dated 9th century. Image: Vat.Gr.190.pt, c.9th century. Right: the modern digital interpretation and the model showing the nested cube and tetrahedron inside the dodecahedron. Drawing: author

is given. The diagram shows a mere closed semi-circle, which is the image of the sphere and its diameter, containing a line whose length is the edge of the polyhedron which is studied. This diagram is a scaled drawing meant to be measurable since the two lengths (sphere diameter and solid edge) appear in the same scale. This combination of two figures for each problem creates diptychs of images which repeat from manuscript to manuscript (Duvernoy 2023) (Fig. 6).

Drawing the Regular Polyhedra

A quick examination of several manuscripts reveals that errors and inaccuracies in the figures and drawings are recurrent (Murdoch 1984). This takes us back to the question of who the image copyists were. It is obvious that not all of them were trained in geometry or were even fully aware of what they were drawing. The octahedron, for instance is shown in many different ways. Sometimes the view is elongated, sometimes it is shortened. These various distortions, however, recall the many aspects that the shadows cast by the polyhedron on a flat surface may take. Plato's words telling us about the ancient scientific research methodology '… The very things which they mould and draw, which have shadows and images of themselves in water…' (Plato,

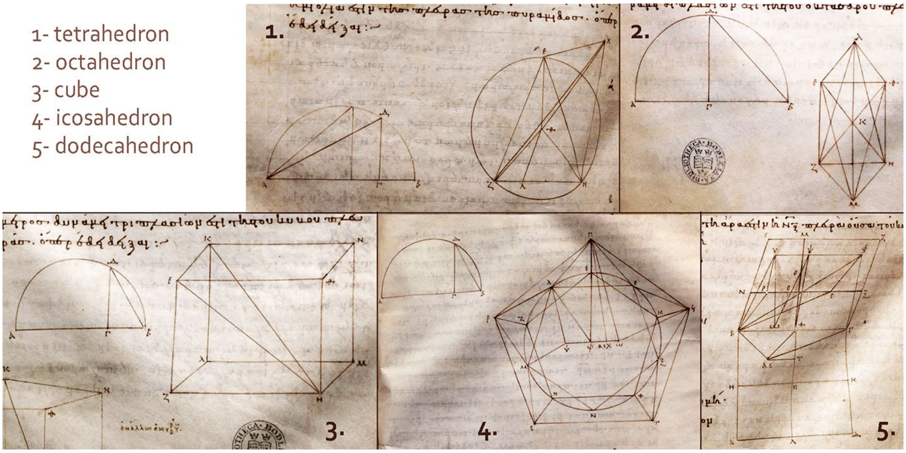

Fig. 6 The five regular polyhedra as they show in manuscript MS D'Orville 301 by Stephen the Clark, dated 9th century. Image: MS D'Orville 301, Bodleian Library, Oxford

Republic VI, 510e once again echo in our ears. Copying the shadows cast by a study model would be an easy way for drawing a 2D view of a complicated solid, and furthermore it would produce exact views from the standpoint of geometric projection since the drawing of shadows belongs to the category of axonometric projections that are largely used in architectural representation today. When looked at from this standpoint, a view of the octahedron where the square of the middle section is distorted into a rectangle, and the vertical axis looks short, is more true and more geometrically correct than a view where the middle section shows as a square – inscribed in a circle – and the vertical axis is exaggeratedly lengthened (Fig. 7).

The first correct visual descriptions of the five polyhedra, close to natural views, are due to Piero della Francesca and are published in his *Libellus de Quinque Corporibus Regularibus* (written sometime between 1480 and 1492) (Williams 2021). Indeed, it is not surprising that the first mathematician to draw correct perspective views is the author of the famous treatise *De Prospectiva Pingendi* (ca. 1482), and also a talented painter. Piero's drawings however illustrate his own book about problems of Euclidean geometry which must not be mistaken for a personal edition of the *Elements*. Unlike Piero's drawings Leonardo da Vinci's sixty drawings of regular and stellate solids published in Luca Pacioli's *De Divina Proportione* (1509) have enjoyed worldwide fame. These drawings mark a turning point in the history of polyhedra illustration. Leonardo most probably prepared them at the request of Luca Pacioli. At that time, they were both living in Milan at the court of Ludovico Sforza and Luca was teaching mathematics to Leonardo (Duvernoy 2008). Leonardo here uses his artistic talent to support the mathematical research by producing pictorial visual help. The depicted polyhedra are not exact perspective constructions: there are 'portraits' of wooden models hanging in front of him (Duvernoy 2023). Indeed, the ribbons hanging the models even show on the pictures. Once again these are representations of former representations: drawings of models. The edges of the solids do not show as lines but show as thick wooden sticks. The depth effect is realistically rendered: the back edges (sticks) are partially hidden behind the front ones, and the

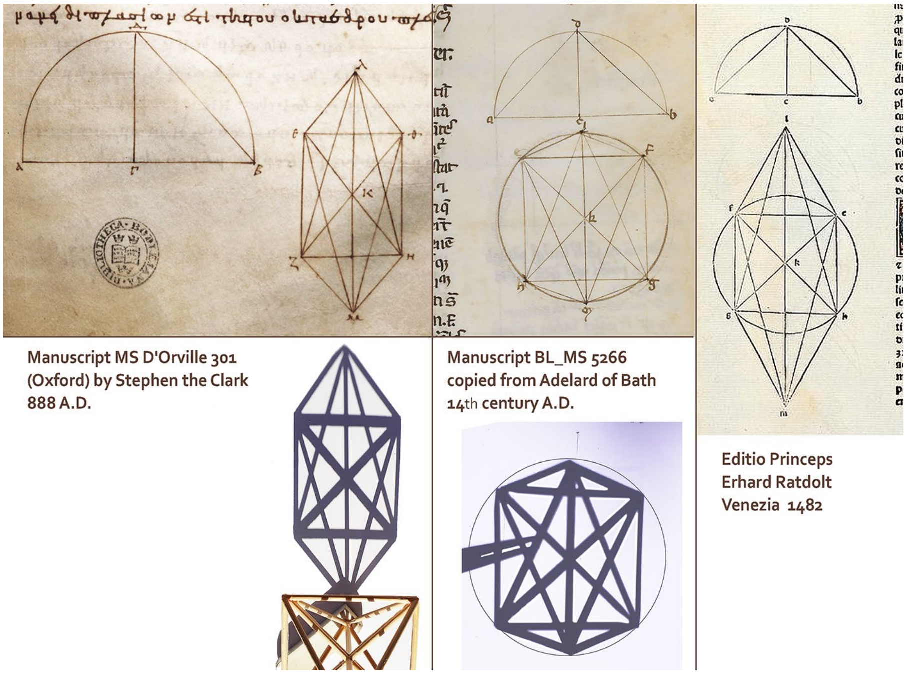

Fig. 7 Drawing shadows. Illustrations of the octahedron. Image: left MS D'Orville 301 by Stephen the Clark, dated 9th century, Bodleian Library, Oxford (after Euclid, 9th century, Bodleian); middle BL MS 5266 copied from Adelard of Bath translation, dating back to the 14th century, British Library (after Euclid, 14th century, BL); right the editio princeps published by Ratdolt in Venice in 1482. The middle image where the six vertices fall along a circle (the sphere) is the most interesting one

tonal values of the colouring enhance the volume. Leonardo's goal is not to demonstrate any mathematical properties. The solids have been 'cleaned' of any Euclidean reference: the diameters of the circumscribing sphere are missing, together with the diagonal of the cube's face and the triangulation of the tetrahedron's base. The images are not measurable. The only purpose of those drawings is to show the polyhedra as they should look, and thus make them recognizable to the neophyte. They are purely visual images. Although they are very well known among mathematicians and other scholars, and are largely available on the web today, they were never published in any other mathematical treatise beside Luca Pacioli's *De Divina Proportione*.

Perspective representation nonetheless creeped into later pre-modern editions of the *Elements*. The first editor to introduce perspective drawings among the illustrations was the Italian mathematician Federico Commandino (1509–1575) (Sorci 2006). Commandino edited two versions of the *Elements*: a first one in Latin (1572) and a second one in Italian (1575), whose influence and importance for the subsequent modern history of the *Elements* is immeasurable (Murdoch 1971). Commandino's work acted as an example for later Italian Renaissance publications. We can cite for instance the 1594 reedition of the Arabic version by Al-Tusi (1201–1274) which was printed by the Typografia Medicea. This press was set up by Ferdinando de Medici (1549–1609), Grand Duke of Tuscany, who started a very ambitious publication plan which was never fully completed. The Al-Tusi reedition was the only Arabic print edition to be published, prior to the nineteenth century and the only printed Arabic edition of the *Elements* to be produced in Europe. It was based on two manuscripts that are still kept today in the Biblioteca Medicea Laurenziana in Florence, Italy (De Young 2012). It is undeniable that the drawings in the original manuscripts are of rather poor quality. The printed reedition instead includes images that come directly from Commandino's work, giving a brand-new modern look to the treatise. We can even notice how some traditional descriptive images of the solids' models are cancelled in favour of the visual perspectives (Duvernoy 2023).

Conclusion

The study of the polyhedra geometry is one of the best examples of interaction between art and science in the Renaissance when the progress in perspective representation and the study of regular and non-regular polyhedra went in parallel. On the one hand perspective representation of the solids became a full topic in itself and gave birth to many art works, including marble pavements and marquetery woodcuts. The solids became a classical exercise when teaching perspective in Italy and abroad. In Germany, the 1560s saw a flurry of publications depicting the solids. (Williams 2021). On the other hand, the investigation on semi-regular polyhedra was triggered by the new study tool.

Scholars in history of architectural drawing have too often limited their observations to the very few preserved plans and front views dating back to classical antiquity, without enlarging their investigation to other scientific fields that also rely on drawing as a research tool and communication device. The illustrations of ancient mathematical treatises may give us some information on the contemporary knowledge in 3D representation. Their interpretation however must take into consideration the purpose of their form: some distortions may have a meaning which is tightly connected to the procedures of computation. Also, partial conclusions about the ancient state-of-the-art of geometric drawing must be restricted to what is present in the books and not to what is missing. The representation technique (axonometry / perspective/ other kinds of projection) is chosen, or even devised, according to its efficacy to prove mathematical properties.

Also, studies on the relationship between art and science, often focus on the impact and influence of mathematics on art, but few have inquired the influence of art on science. What can art do for maths? This research has unveiled how the progress in geometric drawing can support the production of scientific imagery, for the greater ease of the readers.

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Declarations

Conflict of Interest The author states that there is no conflict of interest.

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