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Daniele Barbaro on Geometric Ratio

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Abstract
Daniele Barbaro examined 'geometric ratio' at length in his 1567 commentary on Vitruvius, explaining the properties of and operations with ratios involving continuous quantities such as lengths. While some of this is lost on us today, his explanations were certainly clear to those of his contemporaries who were studying Euclid thanks to new translations. The purpose of this present paper is to recoup for modern readers some of the notions covered by Barbaro, and to set his explanation into context by briefly reviewing similar treatments by Leon Battista Alberti and Andrea Palladio.

Keywords Daniele Barbaro · Geometric ratio · Denomination · Proportion theory · Renaissance · Palladio · Alberti

Introduction

Of all the excursuses that Daniele Barbaro inserts in his commentary on Vitruvius to explain concepts that he feels are inadequately treated by the Roman author, two of the longest are the explanation of ratios found in Books III and V. He was obviously aware of the importance of the subject not only for mathematicians studying Euclid with the aid of new translations such as those by Bartolomeo Zamberti (published in 1505) and Niccolò Tartaglia (published in 1543), but also for architects, thanks not least to the discussion of ratio in Leon Battista Alberti's *De re aedificatoria* (1999). In what follows I will examine Barbaro's explanation of 'geometric ratios'—that is, those involving continuous quantities—and operations performed with them, found in pp. 98–103 of the 1567 Italian edition of *I dieci libri dell'Architettura tradotti e commentate da Mons. Daniele Barbaro*.[1] Barbaro's aim was to make ratios and

[1] The English translations of quotes from (Barbaro 1567) that appear here are excerpted from (Williams 2019).

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their operations accessible and useful to practicing architects. My aim here is to emphasise the intimate connection between ratio and shape in this treatment. In conclusion, I will briefly place Barbaro's notions of ratio as addressed to architects in the context of Alberti's treatment, which came before him, and Palladio's treatment, which came after.

Barbaro's treatment of 'geometric ratios' in Book III is typical of his approach to all of the subjects that he examines in depth. His explanations are not original; he gleans his material from treatises of authors both ancient and modern, sometimes but not always citing his sources. In this particular case, the treatment of the definitions and classification of ratios draws greatly on Boethius's treatise *De institutione arithmetica*, while the treatment of operations can be related to the earlier traditions of Adelard of Bath (1070–1150), Campanus of Novara (1220?–1296), Jordanus of Nemore (fl. 13th c.), and Nicholas Oresme (1320–1382). I am not going to go into a detailed account of sources here; I only wish here to point out that Barbaro's treatment is the result of his studies and reflects both the legacy of the past and the thought of his day. Barbaro's discussion of ratio, while conceived upon notions of shape useful for architects, remains in a middle ground between theory and practice. It is neither so theoretical that he can do away with numerical values, nor so practical that he can allow himself to rely on figures rather than words. The visual element—really so very important, perhaps the most important of all for architects—is entirely lacking. And interestingly enough, his discussion of ratio and proportion is inserted in the preface to Book III, which concerns the orders, rather than, for example, in the third chapter of Book VI, where Vitruvius specifically discusses room shapes with regard to atria (Vitruvius 2009: 172; Barbaro 1567: 288–289), which might seem the perfect place for an excursus.

The Definition of Geometric Ratios

Barbaro begins his discussion with the definition of ratios:

> [Ratio[2]] is nothing other than the determined relation, respect, or comparison of two quantities comprised within the same genre, as in the case of two numbers, or two solids, or two places, two times, two lines, two planes. … The definition of ratio thus explicated, it is clear that, being found in quantity, some ratios pertain to measures, some to numbers, and some are a mixture of numbers and measures. That which pertains to measures, called 'geometric', will be found in continuous quantities, all of which fall within measure (Barbaro 1567: 98; Williams 2019: 176).

The first thing to note is that Barbaro's definition of 'geometric ratio' differs from our use of the term today. We now use 'geometric ratio' to identify the ratio between

[2] Consistently throughout Book III Barbaro refers to *proportione*, which in modern terms would refer to ratio (that is, a comparison of quantities), and to *proportionalità*, which would refer to proportion (that is, a comparison of ratios). This is consistent with the vocabulary of the day; see (March 1998: 58; Belli 2003: 15). In what follows, for the sake of clarity the modern terms have been adopted.

terms in a geometric series; another term for this is ‘common ratio’. Instead, Barbaro defines geometric ratio as a comparison of measures; in architecture, those measures would be lengths, widths and heights.

Malet (2006) examines the evolution of the Euclidean notions of numbers (which could be multiplied and divided) and magnitudes (which could not) into a synthesis in which the distinction essentially disappeared. According to Malet, the evolution began with the handling of magnitudes by means of numerical measures by Arabic mathematicians, was furthered by Regiomontanus (1436–1476), Niccolò Tartaglia (1499–1557), Christopher Clavius (1538–1612), and Sir Henry Billingsley (?–1606), and concretised by Simon Stevin (1548–1620). Regiomontanus died 38 years before Barbaro was born; Stevin died 50 years after Barbaro died. We can see therefore that Barbaro’s treatment of geometric ratio reflects the thinking about numbers and measures as a synthesis that was current in his day.

I posit that Barbaro thought of ‘geometric ratios’ as embodying shapes; in the case of architecture, such ratios could be used as room shapes. In other words, he is describing ratio as an architect needs to think of it. This kind of ratio is not what we think of as a ‘fraction’. Those who, like me, take particular pleasure in knowing the origin of words will enjoy knowing that, according to the Oxford English Dictionary Online, our English ‘fraction’ comes from ‘Late Middle English: via Old French from ecclesiastical Latin *fractio*(*n*-) ‘breaking (bread)’, from Latin *frangere* ‘to break’. Barbaro’s term for ‘common fractions’ is *rotti volgari*, literally ‘common breakings’. I believe that for Barbaro a geometric ratio such as 3:4 is never a numerical fraction, in the sense of being 3/4 or 0.75 of something, or in the sense of something that is less than a whole. In the case of geometric ratio, even if we are given a whole square, 4:4, and some portion of it is removed, say a quarter, 1:4, the remaining 3:4 is not seen as ‘less than a whole square’ but rather as a rectangle based on an (albeit smaller) unit square and some excess. This distinguishes it even from the harmonic ratios that he treats in Book V, because in that case a whole string is divided into smaller segments to produce individual notes (for instance, a string that plays an octave, 2:1, is ‘broken’ into the ratios of the fifth, 3:2, and the fourth, 4:3).

An interesting clue to Barbaro’s thinking about ratio as shape appears in Book V, in a passage in which he is discussing geometric means:

> It has been proved in the *Arithmetica* that between two square numbers a geometric mean falls proportionally and elsewhere it has been said that those ratios that cannot be **drawn** with a certain and determined number are unknown and irrational (Barbaro 1567: 241–242; Williams 2019: 398, my emphasis).

We might deduce from this that ratios were conceived as being drawn; they are therefore shapes.

Classifying Ratios

Barbaro’s discussion begins by defining the ‘manners’, ‘species’ and ‘subspecies’ of ratio. The two manners of ratio are ratios of equality (a comparison of two equal quantities or magnitudes) and ratios of inequality (a comparison of two unequal

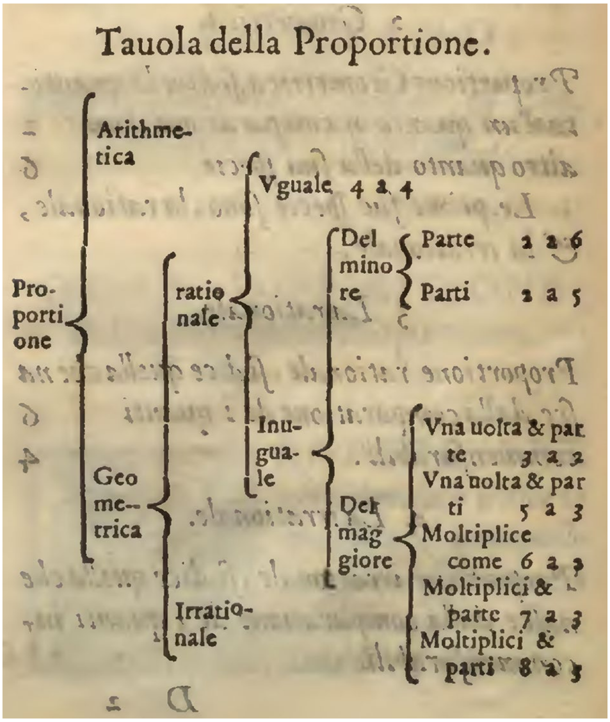


Fig. 1 Silvio Belli's table of the species of ratios (Belli 1573, p. 14; Wassell and Williams 2003: 48)

quantities or magnitudes). Ratios of inequality are further categorised into two species: ratios of greater inequality, in which when the first term of the ratio (the antecedent) is larger than the second term (the consequent), as in 4:2; and ratios of lesser inequality, in which the antecedent of the ratio is smaller than the consequent, as in 2:4. There are then five subspecies of the ratios of greater inequalities: superparticular (one time plus one part); superpartient (one time plus more than one part); multiple (more than one time); multiple superparticular (more than one time plus one part); and multiple superpartient (more than one time plus more than one part). Likewise there are five subspecies of ratios of lesser equality (the reciprocals of the preceding, prefixed with 'sub-': subsuperparticular, subsuperpartient, submultiple, submultiple superparticular, submultiple superpartient).

Barbaro's contemporary Silvio Belli presented these manners, species and subspecies in the form of a table, reproduced in Fig. 1 and translated in Fig. 2.

The species and subspecies presented in the table can be visualised in terms of shape. To begin, the ratios of greater and lesser inequalities are reciprocals of each other. This can be visualised as an inversion of orientation (March 2002: 10) (Fig. 3).

With regard to the superparticular, the statement that 'three goes into four exactly one time plus one-third of it', can be visualised as a square plus one-third of a square (Fig. 4).

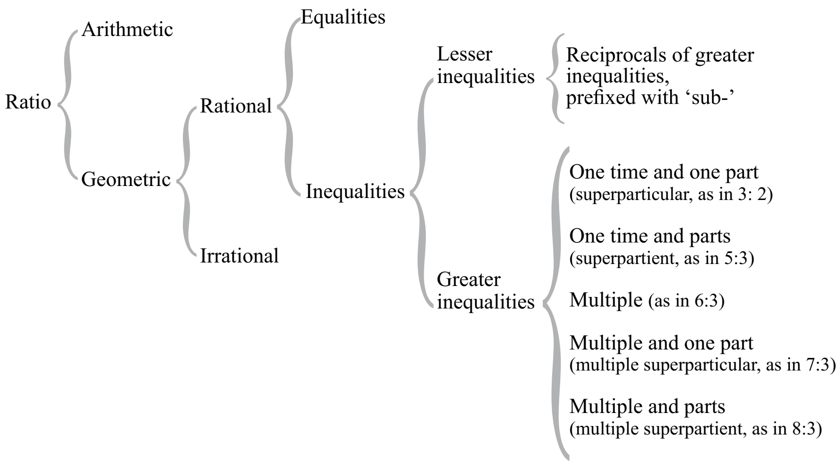


Fig. 2 Silvio Belli's table of the species of ratios translated

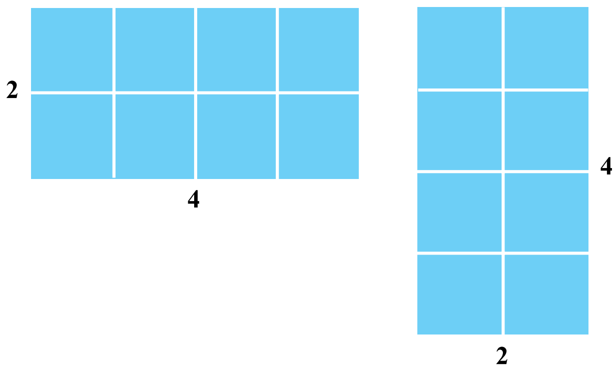


Fig. 3 A graphic interpretation of ratios of greater inequality and ratios of lesser inequality as rectangles with inverted orientations

The superpartient can be visualised as a unit square with the addition of more than one given shapes (Fig. 5).

The multiple is a succession of two or more unit squares. The multiple superparticular would add additional squares to the superparticular visualized in Fig. 4; the multiple superpartient would add additional squares to the superpartient visualized in Fig. 5 (Fig. 6).

The Concept of 'Denomination'

Having established the kinds of ratio, Barbaro goes on to discuss specific ratios. He discusses them by name, and not by number, and this is important. The names assigned to the ratios come from the Latin and indicate the specific numerical

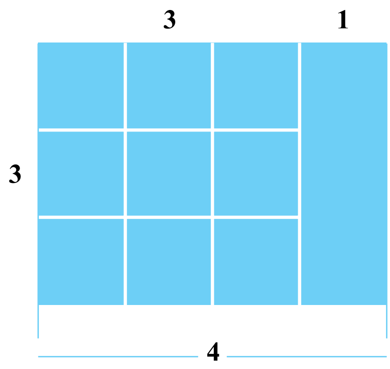


Fig. 4 The superparticular ratio of 4:3 visualized as shapes

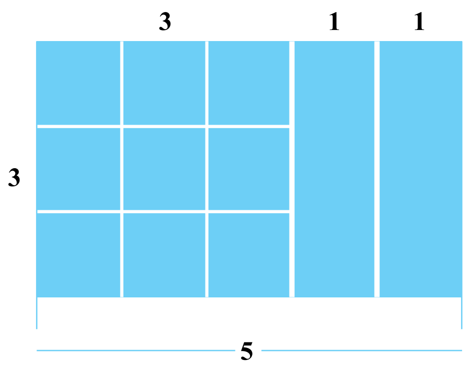


Fig. 5 The superpartient ratio of 5:3 visualized as shapes

relation. For the superparticular, a unity is assumed and a part of it is added. Assuming a unity and adding a half of it results in the 'sesquialteran', from the Latin term meaning 'half of a second', and therefore sesquialteran is one unity plus half of a second unity; in whole number terms, the smallest of these is 3:2. To the given unity a sesquitertian adds a third; a sesquiquartan adds a fourth, and so on. In the case of superpartients, the names are equally indicative: superbipartient adds two parts to the unity (e.g., 5:3); supertripartient adds three parts (e.g., 7:4). To us today these names of ratios might appear to be abstract terminology, but the important point is that each of these terms describes a shape, independent of the lengths that define it. The shape of the sesquialtera is unchanged, regardless of whether its sides are 3:2, 6:4 or 99:66 (Fig. 7).

The name of the shape is its 'denomination'. Denominations are not to be confused with 'denominators' as we use that term today, meaning the consequent of a fraction that indicates the number of equal parts the given quantity is divided into. The denomination is precisely the value that gives the ratio its name. It can be thought of as 'face value', as we use 'denomination' in money today.

Denominations were discussed in the thirteenth- and fourteenth-century works of Jordanus de Nemore, Thomas Bradwardine (ca. 1290–1348), and Oresme among others (Sylla 1984, 2008; Grant 1974: 130ff.). Barbaro tells us why denominations are important:

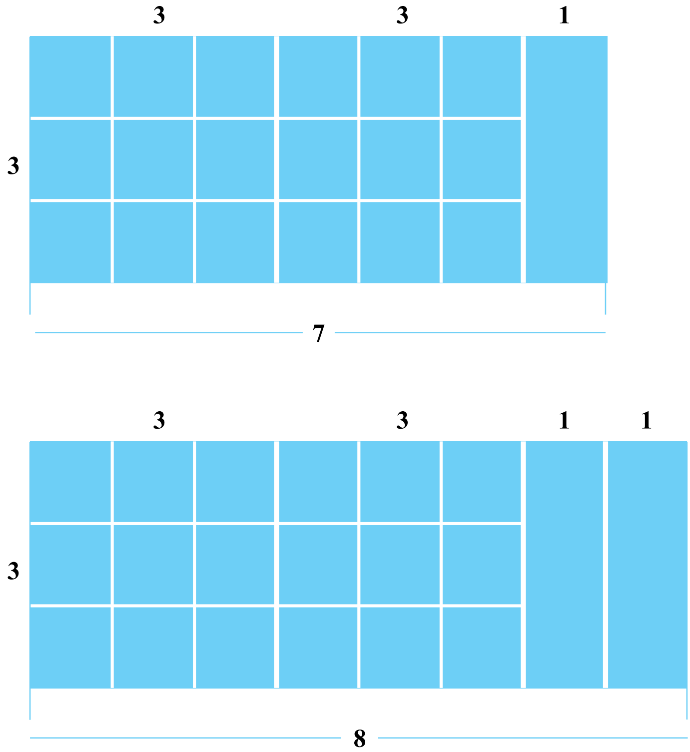


Fig. 6 Multiple superparticular and multiple superpartient ratios visualized as shapes

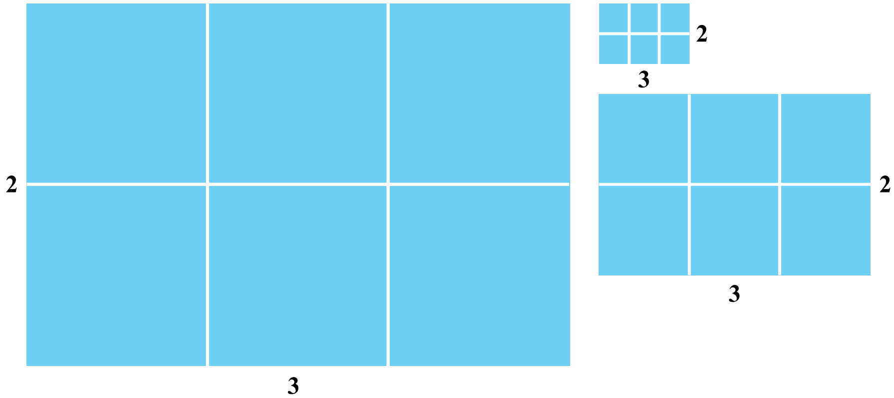


Fig. 7 The shape of sesquialterans, independent of size

> It is necessary to understand how to find the denomination of ratios because this serves to know which ratio is greater and which lesser. Buildings with larger ratios are larger than buildings with smaller ratios; a room of two squares is larger in size than one of a square and a half, since the duple is a larger ratio than the sesquialteran. … From this knowledge [of denominations] … derives this usefulness: being able to know which ratios are among the ratios of greater inequality and which are among those of lesser inequality; which are among the ratios of equalities; and which are similar ratios. Similar ratios are those that have identical denominations; larger ratios are those having greater denominations; smaller ratios, those having lesser denominations, because the denomination is said to be as large as the number that denotes it. So the quadruple is greater than the triple because the quadruple is denominated by four and the triple is denominated by three. So too, the sesquialteran is greater than the sesquitertian because the sesquialteran is denoted by the half and the sesquitertian by the third. So, a triple sesquialteran is greater than a triple sesquitertian, but a triple sesquitertian is greater than a duple sesquialteran; … (Barbaro 1567: 100; Williams 2019: 180–181).

For us, unfamiliar as we are with the concept of denominations, adding numerical examples to this is helpful. A sesquialteran (e.g., 3:2, with a denomination of 1 1/2) is greater than a sesquitertian (e.g., 4:3, with a denomination of 1 1/3). A triple sesquialteran (e.g., 7:2, with a denomination of 3 1/2) is greater than a triple sesquitertian (e.g., 10:3, with a denomination of 3 1/3), but a triple sesquitertian (e.g., 10:3, whose denomination is 3 1/3) is greater than a double sesquialteran (e.g., 5:2, whose denomination is 2 1/2). (It should be noted that in all these comparisons the square taken as unity is the same.) But the concept is clearest of when the ratios are considered as shapes (Fig. 8).

Here we have in a nutshell the importance of denominations: a denomination is a name or designation that allows us to understand at once the relative qualities of a shape. We today, upon hearing 'square' and 'rectangle' immediately conceive a mental image of those shapes: specific in the case of the square; generic in the case of a rectangle. I believe that Barbaro, upon hearing 'sesquialteran' and 'sesquitertian' immediately conceived of them as specifically as we conceive of squares. He may have in fact been able to quickly conjure up mental images of many, more specific ratios: a superbipartient of fourths, or a subsupertripartient of fifths, a skill we are entirely untrained in today.[3]

[3] This brings up the question of how adept artists and architects were at recognizing shape. Vasari, for instance quotes Michelangelo as saying that *bisognava avere le seste negli occhi e non in mano, perché le mani operano e l'occhio giudica: che tale modo tenne ancora nell'architettura* (it is necessary to have a compass in the eye and not in the hand, because the hands work and the eye judges: this holds also for architecture [as in art]) (Vasari 1991: 1256, my translation; cfr. Vasari 2008: 472). It also recalls the experiment done by Gustav Fechner (1997) following on a suggestion by Adolf Zeising (1854) in the attempt to show that the golden rectangle was the recognized as the most pleasing shape. Unfortunately, a discussion of this is beyond the scope of this paper.

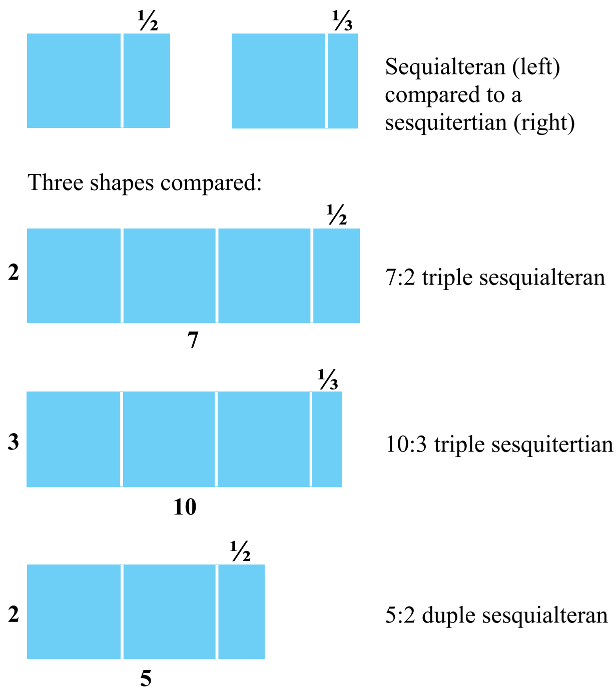


Fig. 8 Comparisons of ratios by shape

Operations with Ratios

> So that what we wish to reveal is well understood, it is useful to say how to reckon the denomination of ratios, how ratios are subtracted and added, and how they are multiplied and divided (Barbaro 1567: 100; Williams 2019: 180).

It is in operations that shapes meet numbers.

Above I mentioned the first reason that denominations are important: because they allow us to conceive the shape. The second reason that denominations are important is seen when we arrive to operations with ratios, because knowing the denominations allows one to know the results of the operations beforehand. In fact, knowing the names—that is, determining the 'denomination'—is the first of the operations with ratios that Barbaro performs.

Determining the Denominations

Denomination both precedes numbers and requires them. You don't need numbers to understand relations between lengths if you have a grasp of the denominations. But to determine the dominations you need numbers. Thus the first operation Barbaro teaches is how to find the denomination of a ratio:

> A brief and expedient rule for finding the numbers by which ratios are denominated is to divide one term of the ratio by the other. The result of such division is always the denomination of the ratio. Dividing is none other than seeing how many times one number goes into another, and with what remainder. From the division and its remainder we can reasonably know the name of each ratio. … (Barbaro 1567: 100; Williams 2019: 180).

Finding the denomination is strictly a mathematical operation: the larger value is divided by the smaller.

In performing operations with ratios, finding the denomination of the single ratios involved in the operations of combining them is followed by finding the denominations of them when combined. It will seem curious to modern readers that there is any need to determine the denomination before calculating value of the two ratios added together, because today we conflate these as a single operation. To Barbaro, however, knowing the combined denomination allowed him to know beforehand what the result of the calculation will be. There is no doubt in my mind that he knew by rote a whole set of combinations: a triple and a duple produce a sextuple; a sesquialteran and a sesquitertian produce a duple; and so forth. The actual measures then followed this. Naturally numbers were necessary. To say that combining a triple to a duple results in a sextuple is to state a truism, while to combine specific lengths results in a specific quantity in relation to a scale.

Finding the combined denomination is likewise a mathematical operation:

> Now to our purpose. In order to combine two ratios together, it is necessary to find the denomination of the ratio produced; then to combine the numbers under that produced ratio. The first operation is done this way: multiply the denomination of one ratio by the denomination of the other so as to find the denomination of them combined. The second operation is done by multiplying the antecedents of each ratio and also multiplying the consequents, observing that this rule serves in the case of similar ratios—that is, when both are ratios of either greater or lesser inequality. Now for an example: take a ratio of nine to three, a triple, and a ratio of four to two, a duple; I want to combine a triple and a duple and see what kind of ratio is formed. First I multiply the denominations, three and two, and find that the product is six. This then will be the denomination of the new ratio; thus from a triple and a duple is born a sextuple. This appears from the multiplication of the numbers of both ratios, since multiplying nine by four gives thirty-six, and three by two gives six, and thirty-six with respect to six comprises the ratio denominated sextuple (Barbaro 1567: 101; Williams 2019: 182–183).

The mathematical operation is that of multiplication. In the example given by Barbaro:

$$\frac{9}{3} \times \frac{4}{2} = \frac{36}{6} = \frac{6}{1}.$$

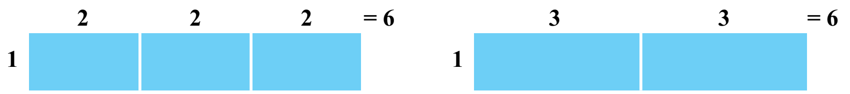


Fig. 9 Two arrays for compounding a duple ratio and a triple

'Adding' and 'Subtracting' Ratios

Barbaro says he will explain the operations of subtraction, addition, multiplication, and division. Although Barbaro uses the term *raccogliere* (to gather, combine, collect) it is apparent that he is talking about compounding ratios. The modern mathematician will object that the only operations involved are the multiplication and division of the numerical components of ratio, but in Barbaro's thinking the composition of ratios is a two-step process involving both number (multiplication and division) and shape (addition and subtraction).

The Italian mathematician Giovanni Battista Benedetti (1530–1590), in his *Diversarum Spectulationum* (1585: 1–7), devised an interesting way of visualizing the composition of ratios.[4] Essentially he gathers the shapes formed by the ratios into arrays or assemblies of rectangles. In what follows I will use Benedetti's method to illustrate Barbaro's explanations and show how shapes are manipulated through composition.

Barbaro gives seven examples of compounding ratios. Three of these involve adding ratios that belong to the same subspecies (adding two multiples, two superparticular, and two superpartients). A fourth involves adding ratios of different subspecies (a subduple and a sesquialteran). The final three are examples of subtraction. We will look at these one by one.

Example 1: Adding Two Multiples

Barbaro first shows us how to add multiples, in this case the duple (2:1) and the triple (3:1). We saw above, based on the determination of the combined denomination, that the result will be a sextuple (6:1). Now using Benedetti's visual method, we will add the shapes. This consists in drawing an array, which can be done in two ways to arrive at identical conclusions: either a 3×1 array of 2×1 rectangles, or a 2×1 array of 3×1 rectangles (Fig. 9).

[4] Benedetti's graphic method is briefly mentioned in (March 1998: 79). Benedetti's 'reformation' of the theory of proportions presented in Euclid's Book V of the *Elements* has been the subject of a study by Enrico Giusti (1991). I don't want to suggest that Barbaro knew of Benedetti's method (Benedetti's *Diversarum speculationum* appeared only after Barbaro's death) or in fact any other graphic method. My aim is to provide a way for us moderns to regain a feeling for ratios as shape.

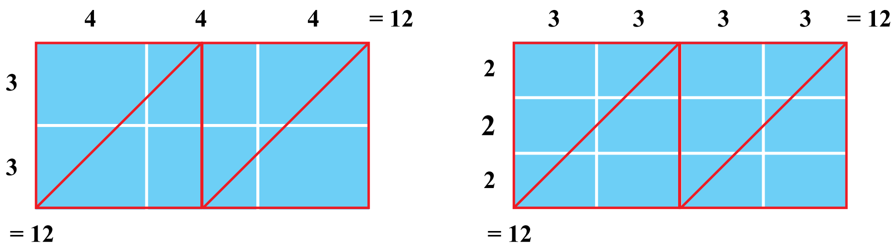


Fig. 10 Two arrays for compounding the sesquialteran and the sesquitertian. The original shapes appear in blue; the resulting two unities are indicated in red (colour figure online)

Example 2: Adding Two Superparticular Ratios

Here we want to add the sesquialteran 3:2 to the sesquitertian 4:3. We know from determining the combined denomination that our result will be a duple:

$$\frac{3}{2} \times \frac{4}{3} = \frac{12}{6} = \frac{2}{1}.$$

To add our shapes, we can draw the following two arrays: a 3×2 array of 4×3 rectangles, or a 4×3 array of 3×2 rectangles (Fig. 10).

Example 3: Adding Two Superpartient Ratios

Here we want to add the superbipartient of thirds 5:3 to the supertripartient of fourths 7:4. We first determine the combined denomination:

$$\frac{5}{3} \times \frac{7}{4} = \frac{35}{12}.$$

Barbaro names this a ‘duple hendecapartient of twelfths’, that is, two unities and eleven parts of twelve (not a particularly familiar ratio). To add our shapes, we can draw the following two arrays: a 5×3 array of 7×4 rectangles, or a 7×4 array of 5×3 rectangles (Fig. 11).

Example 4: Adding a Subduple and a Sesquialteran

Here we want to add the subduple 1:2 and the sesquialteran 3:2. We first determine the combined denomination:

$$\frac{1}{2} \times \frac{3}{2} = \frac{3}{4}.$$

The result is a subsesquitertian. To add these shapes, we can drawing the following two arrays: a 1×2 array of 3×2 rectangles, or a 3×2 array of 1×2 rectangles (Fig. 12).

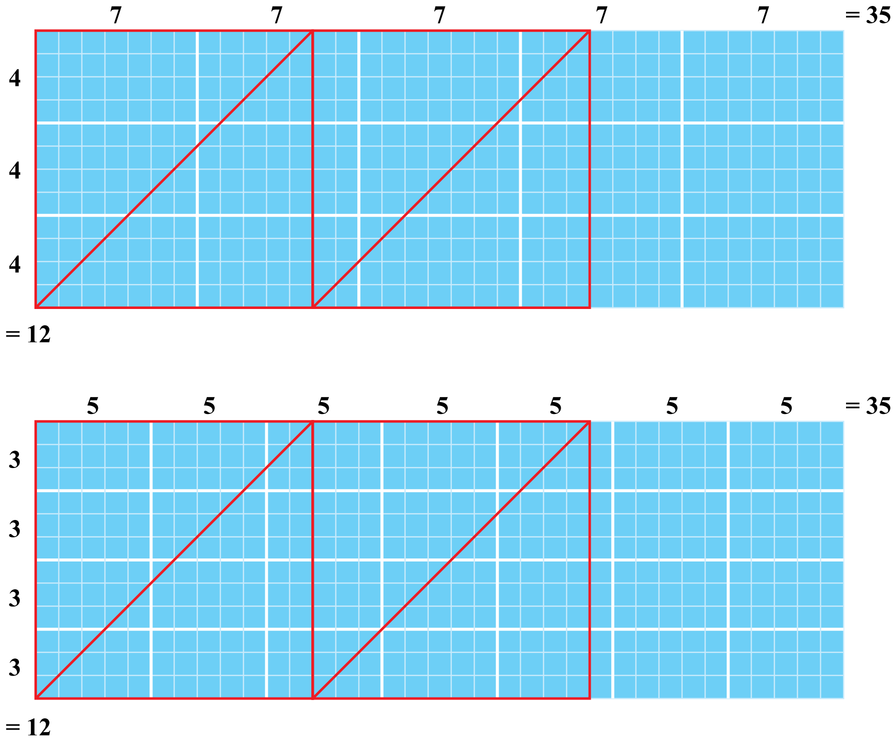
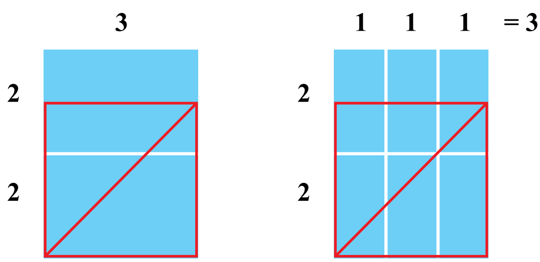


Fig. 11 Two arrays for compounding a superbipartient of thirds and a supertripartient of fourths. The original shapes appear in blue; the resulting unities are indicated in red (colour figure online)

Fig. 12 Two arrays for compounding a subduple and a sesquialteran. The original shapes appear in blue; the resulting unities are indicated in red (colour figure online)



What we notice here is that the orientation of the compound rectangle (in blue) with respect to the unity (in red) has changed: where before the excesses were to the right of the unity (or unities), here the excess appears above it. Thus we see that using the graphic procedure, orientation indicates whether the ratio is one of greater inequality (in our conventions here, a horizontal shape) or one of lesser inequality (a vertical shape).

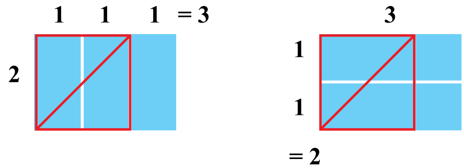


Fig. 13 Two arrays for subtracting a duple from a triple. The original shapes appear in blue; the resulting unities are indicated in red (colour figure online)

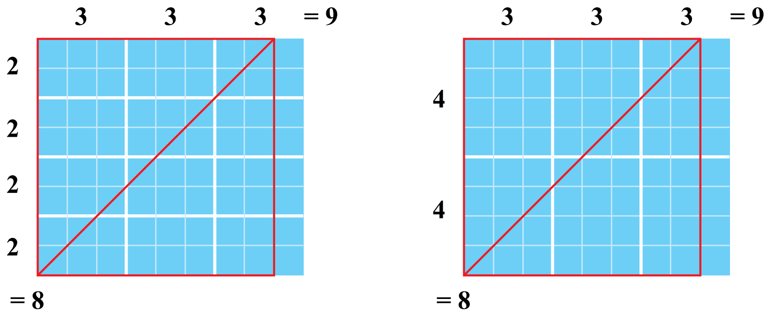


Fig. 14 The arrays for subtracting a superparticular from a sesquialteran. The original shapes appear in blue; the resulting unities are indicated in red (colour figure online)

Example 5: Subtracting Multiples

Now we want to subtract the duple 2:1 from the triple 3:1. We first determine the combined denomination. The mathematical operation is division, which means multiplying one ratio by the inverse of the other:

$$\frac{3}{1} \div \frac{2}{1} = \frac{3}{1} \times \frac{1}{2} = \frac{3}{2}.$$

The result is a sesquialteran. To graphically compute these shapes, we can draw the following two arrays: a 3×1 array of 1×2 rectangles, or a 1×2 array of 3×1 rectangles (Fig. 13).

Example 6: Subtracting One Superparticular from Another

Now we want to subtract the sesquitertian 4:3 from the sesquialteran 3:2. We first determine the combined denomination:

$$\frac{3}{2} \div \frac{4}{3} = \frac{3}{2} \times \frac{3}{4} = \frac{9}{8}.$$

The result is a sesquioctave. The two possible arrays are: a 3×4 array of 3×2 rectangles, and a 3×2 array of 3×4 rectangles (Fig. 14).

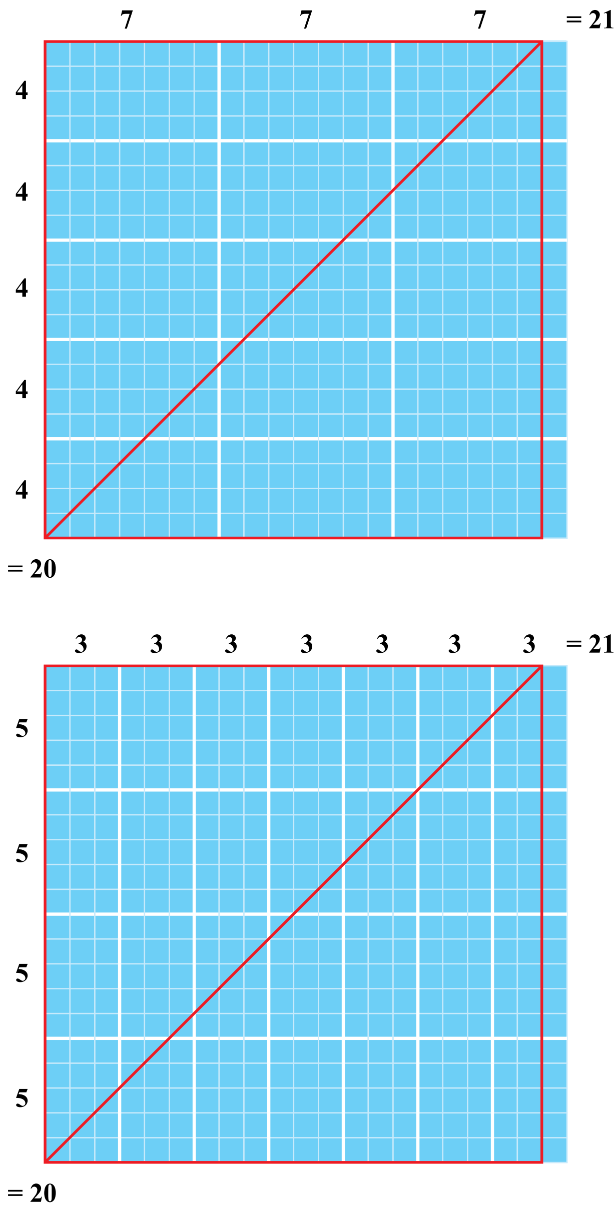


Fig. 15 The arrays for subtracting a superbipartient of thirds from a supertripartient of fourths. The original shapes appear in blue; the resulting unities are indicated in red (colour figure online)

Example 7: Subtracting One Superpartient from Another

Now we want to use the same ratios as in example 3, but this time we will subtract the superbipartient of thirds 5:3 from the supertripartient of fourths 7:4. We first

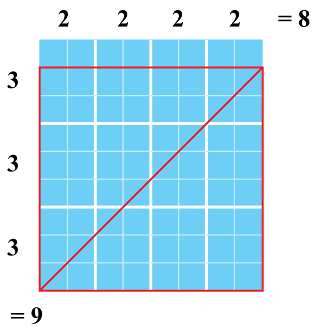


Fig. 16 The array for subtracting a sesquialteran from a superparticular. The original shapes appear in blue; the resulting unities are indicated in red (colour figure online)

determine the combined denomination:

$$\frac{7}{4} \div \frac{5}{3} = \frac{7}{4} \times \frac{3}{5} = \frac{21}{20}.$$

The result is the ratio that Barbaro calls *sesquivegesima* (unity plus another twentieth) (Barbaro 1567: 103). There are again two possible arrays for this: a 3×5 array of 7×4 rectangles, or a 7×4 array of 3×5 rectangles (Fig. 15).

Implications of the Graphic Method

Thanks to these seven examples, we have a good grasp of what it means to add or subtract ratios as shapes. There are two interesting results from experimenting with the graphic method.

One is that the graphic method allows us to contradict a premise that Barbaro stated regarding subtraction:

> When we want to subtract one ratio from another in order to know what ratio remains, it is necessary to begin with this observation: just as in numbers, where we take the lesser from the greater, so too with ratios we take the lesser from the greater (Barbaro 1567: 102; Williams 2019: 184).

But using our graphic method, we can in fact ‘subtract’ the greater from the lesser. For example, we could do the inverse of Barbaro’s example in this particular case of subtracting 3:2 from 4:3:

$$\frac{4}{3} \div \frac{3}{2} = \frac{4}{3} \times \frac{2}{3} = \frac{8}{9}.$$

To show this graphically we construct a 4×3 array of 2×3 rectangles (Fig. 16). What is seen is that the graphic subtraction of a larger ratio from a smaller one results in a rectangle whose orientation is the inverse of that resulting from the subtraction of the smaller from the larger (compare to Fig. 12).

The second interesting property of the graphic method is that it allows us to see how the same ratio (shape) can be constructed by means of various compositions; in other words, it shows what compositions of ratios are equivalent. For instance, the rectangle measuring 24×12 can be achieved in a number of ways: a 1×2 array of

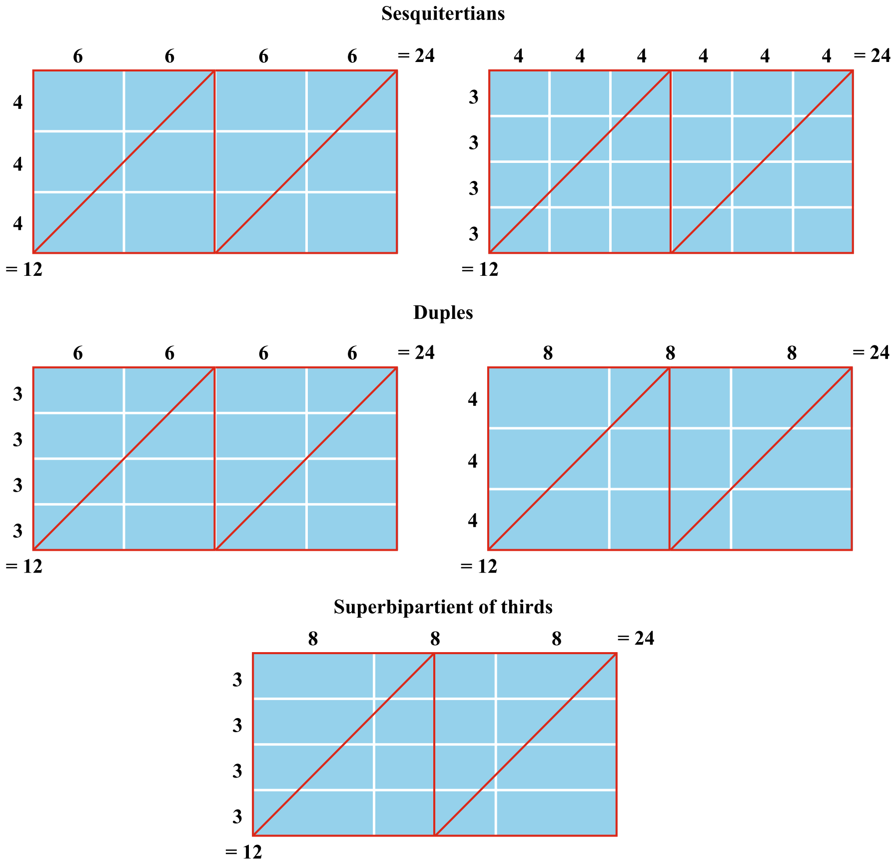


Fig. 17 Examples of equivalent compositions of a duple

12×12 rectangles; a 2×2 array of 12×6 rectangles; a 3×4 array of 8×3 rectangles, a 4×4 array of 6×3 rectangles, and many others, and by the inverses of all of these. This may seem like simple factoring in numerical terms, but for Barbaro, thinking in terms of shape, it meant that the duple can be divided into squares, duples, triples, quadruples, sextuples, sesquialterans, sesquitertians, and duple bipartients of thirds: a whole palette of smaller shapes, each with a particular quality (Fig. 17).

Barbaro in Context

To place Barbaro's treatment in the context of other treatises, here I will briefly compare the presentations of ratio in Barbaro's Vitruvius first with that of Alberti's *De re aedificatoria* (Alberti 1999), and then with that of Palladio's *Quattro Libri* (Palladio 1997). I by no means intend to enter into a discussion of theories of proportion. My aim is to look for linguistic clues to the ways in which these authors conceived of ratio.

Number and Ratio as the Foundation of Beauty

All three authors make it clear that number and reasoned relationships are of fundamental importance in architecture. For Alberti, writing in Book IX, chapter 5, beauty in architecture consisted in what he called *concinnitas*:

> Beauty is a form of sympathy and consonance of the parts within a body, according to definite number, outline, and position, as dictated by *concinnitas*, the absolute and fundamental rule in Nature. This is the main object of the art of building, and the source of her dignity, charm, authority, and worth (Alberti 1999: 301–303).[5]

For Barbaro, beauty in architecture originated in number:

> Thus finding that if some columns were six times higher than the diameter and others ten times, by the innate sentiment that enables us to judge that too much thickness or slenderness was not good, man began to fulfil his role and discourse on what would be pleasing between these extremes. Immediately he devoted himself to the invention of the proportions … It was therefore from number that beauty began to be bestowed (Barbaro 1567: 164–165; Williams 2019: 277).

For Palladio, writing in Book I, chapter 2:

> Beauty will derive from a graceful shape and the relationship of the whole to the parts, and of the parts among themselves and to the whole, because buildings must appear to be like complete and well-defined bodies, of which one member matches another and all the members are necessary for what is required (Palladio 1997: 7).

Ratio in Alberti

The key terms in Alberti's *De re aedificatoria* are *area* (plural *aerae*) and *finitio*, both of which have connotations of 'shape'. The Latin *area* carries a similar meaning to the English, denoting an enclosed space.[6] Alberti uses it when speaking of building sites and rooms; its connection with shape is clear. He first mentions *areae* in Book I, chapter 7, where he describes building sites but also the geometric elements (straight lines, curved lines, angles, circles and parts of circles) that enclose them (Alberti 1999: 19). Brief mention is made of ratios of length and widths in Book VII, chapter 4, where he discusses *areae* used in temples:

[5] For a full discussion of *concinnitas*, see (Tavernor 1985).

[6] See (Alberti 1999: 421, s.v. Area) for a discussion Alberti's use of the term *area*.

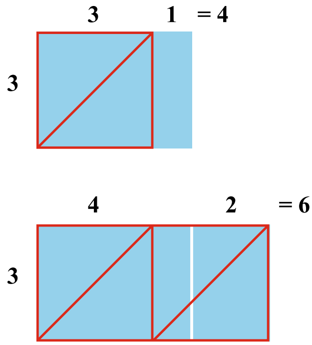


Fig. 18 Alberti's two-step addition of *areae*

> In almost all their quadrangular temples our ancestors would make the length [of the plan] one and a half times the width. Some had a length one and a third times their width, and others a length twice their width (1999: 196).

The true treatment of ratios is found in Book IX, beginning in chapter 5, where he describes *finitio*, translated into English as 'outline', as 'a certain correspondence between the lines that define the dimensions: one dimension being length, another breadth, and the third height' (1999: 305). This is followed by a discussion of the musical ratios constructed from the first four integers (one, two, three and four) and the tonus based on the ratio of nine parts to eight: diapente, sesquialteran, diatessaron, sesquitertian, diapason, etc., terms which, as we have just seen, reappear in Barbaro's text. He then goes on to connect this discussion of *areae* to architecture, beginning chapter 6 with the key treatment that compares *areae* as ratios with relative sizes, and then going on to discuss additions of *areae*. Here his expressions are very similar to those used by Barbaro: two or three smaller *areae* are added to form larger *areae*. The *areae* are cited by denomination before numerical values are given to illustrate the operation and result.

Comparing Alberti's treatment to that of Barbaro, we can note several things. First, Alberti's much more succinct discussion of the ratios and of areas appear to presuppose that the reader is already familiar with ratios and their handling. Barbaro's discussion has the merit of being more clearly didactic, almost like a lesson in mathematics. In Alberti the process of combining areas is straightforward. Take, for instance, the generation of a duple by adding a sesquialteran and a sesquitertian (Alberti 1999: 306). Alberti asks us to take a square of side 3, add to it a rectangle whose short side is equal to one-third of the original square, and then add to that a second rectangle whose short side is equal to one-third of the sum of the previous two sides combined. In Barbaro's terms he is taking an initial unity (3:3) to which he adds a subtriple (1:3), arriving to a sum that is a sesquitertian (4:3). To this he adds a subsesquitalteran (2:3) to arrive at a duple (2:1). In terms of numbers this is:

$$\frac{3}{3}+\frac{1}{3}+\frac{2}{3}=\frac{6}{3}.$$

In terms of shape, this is a two-step process (Fig. 18).

Finally, Barbaro goes beyond the consonant ratios discussed by Alberti and generalizes the discussion of ratio to include non-musical ratios, such as the sextuple (6:1) or 21:20. However, while it seems clear that Alberti and Barbaro are alike in thinking of ratios as shapes (or *areae*, in Alberti's terminology), Alberti makes the specific assignment of shape to architectural form (room shapes), while Barbaro concentrates on theory alone. It can also be remarked that Barbaro seems to have shared Alberti's diffidence towards the use of images, including numerals,[7] as all of his numbers are written out. For the modern reader, this compounds the difficulty of following the text, a difficulty the figures in this present paper are intended to overcome.

Ratio in Palladio

Palladio's treatment of ratios, found in Book II, chapters 21–23 (1997: 56–59) is much more pragmatic than either Alberti's or Barbaro's. We might expect as much, given the practical nature of Palladio's entire treatise. We know that Palladio collaborated with Barbaro on his editions of Vitruvius; presumably they discussed ratios and proportions at some length. In Palladio's treatise there is no discussion of how to compose ratios in general. He limits himself to shape:

> There are seven types of room that are the most beautiful and well proportioned and turn out better: they can be made circular, though these are rare; or square; or their length will equal the diagonal of the square of the breadth; or a square and a third; or a square and a half; or a square and two-thirds; or two squares (Palladio 1997: 57).

In other words, circular, square, the root-2 rectangle ($\sqrt{2}$:1), sesquitertian (4:3), sesquialteran (3:2), superbipartient (3:5), and duple (2:1). But he eschews the Latin terms, making only one mention of a sesquialteran in Book II, Chap. 22, where he describes a vault height: 'nine is to six as six is to four, that is, the *sesquialtera'* (1999: 58). It would thus appear that Palladio took the excursus of Barbaro, distilled from it the essence of what the architect would need in practice, and included only that essence in his own treatise.

It is possible, however, to find vestiges of ratios used to generate lengths. For instance, he says of the Villa Pisani in Montagnana, 'the small rooms and passageway are of equal breadth, their vaults being **two squares high**' (1997: 130, my emphasis). This appears to be a relic of the process of deriving lengths from the sides of rectangles based upon a unit square. Barbaro would have said that the length and width of the plan represented a ratio of equality, and the ceiling height to width was a duple.

[7] Alberti mistrusted all images, including numerals as lending themselves to errors by copyists (he was writing, of course, before print): 'Here I ask those who copy out this work of ours not to use numerals to record numbers but to write their names in full; for example, twelve, twenty, forty and so on, rather than IXX, XX, XL' (Alberti 1999: 200–201); see also (Williams et al. 2010: 3).

Conclusion

The concept of denomination—key to Barbaro's discussion of geometric ratio and Alberti's treatment of ratios—does not in itself constitute proportion theory, but it is an important building block of proportion theory, particularly as proportion theory was thought of in Palladio's day. The impression I have received after comparing the three treatments of Alberti, Barbaro and Palladio is one of successive steps towards making the concepts of *areae* and shapes accessible to practicing architects: from Alberti's Latin treatment without images or digits to represent numbers, to Barbaro's treatment in a modern language (Italian) with the addition of quickly-grasped digits instead of numbers written out, to Palladio's shape descriptions in the text flanked by numbers representing dimensions in plans, sections, and elevations. To sum up, I'd like to paraphrase the comparison of Scott, Amundsen and Shackleton sometimes attributed to explorer and geologist Sir Raymond Priestly: For a succinct description of ratios and their application in *areae*, give me Alberti; for a quick understanding of room shapes, give me Palladio; but when disaster strikes and all hope of understanding and working with ratios is gone, get down on your knees and pray for Barbaro.

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