Nexus Network Journal (2019) 21:527–545
https://doi.org/10.1007/s00004-018-0413-9

NEXUS NETWORK JOURNAL
Architecture and Mathematics

RESEARCH

# Two Architectures: Two Compared Geometries

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Published online: 29 October 2018


**Abstract**

Two very different cities (Berlin and Copenhagen), with two different stories and two different environments, have two different architectures, namely, the Reichstag building in Berlin and the Treetop Experience in Copenhagen, that are differentially designed. Two geometric shapes, an ellipsoid and a hyperboloid of one sheet, are used to interpret the different architectures. A common architectural element in both architectures is a spiral walkway that allows for the buildings' usability without barriers. In this work, the two observed objects are presented in their respective environment, history and purpose. The shapes are interpreted, which reveals their geometric aspects that are associated with the functional and emotional effects of their architectural forms. Parameterization is the main mathematical tool through which the characteristic shapes are studied and mathematically expressed, using Linear Algebra and the Parametric Geometry of surfaces that are ideally close to the studied architectural objects. Moreover, the free and open-source programming language ASYMPTOTE is used for their virtualization, the model's reliability is assessed, and the analysis of the model's behavior is presented.

**Keywords** Dynamic surfaces · Quadric surfaces · Spiral surfaces · Parametric modeling · Reichstag building · Treetop experience

## Introduction

The geometry of a shape characterizes its beauty, regularity, and functionality, but sometimes it also has an unexpected role. The geometry of a shape determines a feeling, establishes a state of mind, helps to reflect and interpret what is seen, and establishes an emotional relationship with the space and world in which it is located and services.

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Two architectural objects are analyzed herein: the dome roof of the Reichstag building in Berlin, which was designed by Foster (see Fig. 1), and the treetop walk observation tower of Camp Adventure Park, which was designed by the Danish company, EFFEKT (see Fig. 2).

The first shape, the dome, is geometrically a portion of an ellipsoid. Its shape attracts attention to the environment below, namely, the seat of the Bundestag, and protects its history without prohibiting outside light and energy from entering through the chosen materials.

The second shape, the tower, is a hyperboloid of one sheet that, due to its shape, minimizes the environmental impact by simulating the structure of a tree. It forms upwards like a tree and spreads downwards to simulate the entirety of the roots.

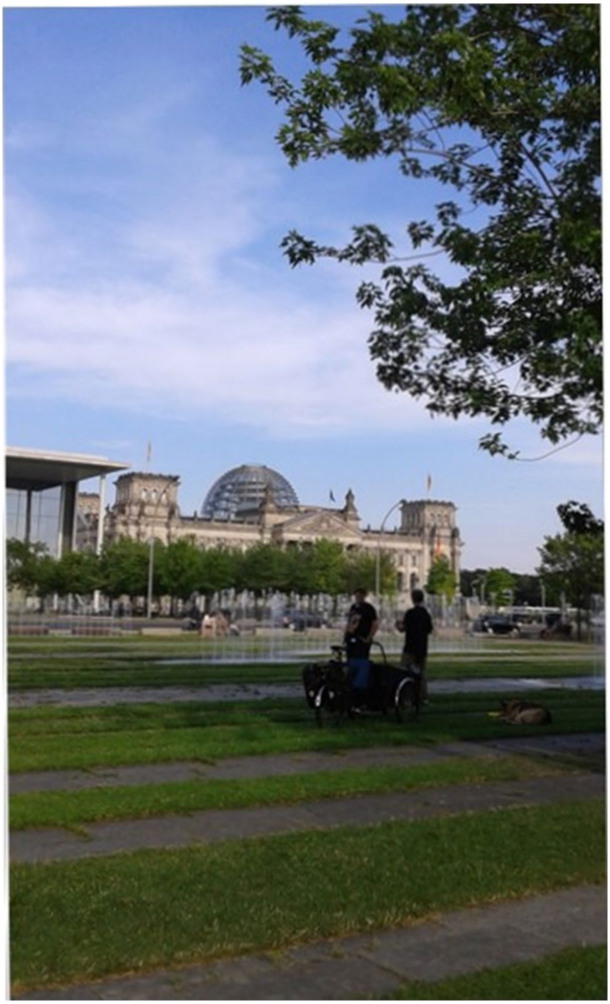


**Fig. 1** Reichstag. (Photo taken by the authors)

**Fig. 2** Treetop experience tower. (Courtesy of the Danish company EFFEKT)

Additionally, in this case, the chosen materials play a decisive role with respect to both its ecological relationship and its disposition that strongly underlines the geometric nature of the shape. The hyperboloid is a ruled surface, and this characteristic is highlighted by the wooden interweaving that builds the architecture.

A common element in the two structures under study is a ribbon-shaped walkway that functions as a seamless and continuous ramp. Visitors walk on the ramps and experience the architectural form by undergoing its effect. In the first case, visitors observe the structure subtended by the dome and virtually experience the frenzy of the work that takes place under it. In the second case, visitors stroll above the trees in close contact with the forest canopy.

In this paper, we carefully and emotionally compare the observations and then apply the structure and precision of mathematics. For the latter, we assess the form using parametric geometry and then use the Asymptote application, which is a computer graphics tool that allows us to illustrate a visual model. We will discuss this process later in this paper.

## The Reichstag Building

Berlin is a city that has suffered a lot, but this suffering is not reflected in its appearance. Berlin has been demolished and renewed, and it is preserved, transformed, and refurbished as it looks to the future. However, the city's history cannot be denied. The Reichstag building has become a symbol of Berlin through which one can strongly feel the influence of its history. Therefore, it is the story of the Reichstag building that has led to it being both a symbol of the city's history and a symbol of its renewal, although it is not without its problems. Let us briefly summarize its most significant historical moments.

Since its inauguration in 1894, the building has witnessed all of the most important historical occurrences of the city, Germany and the whole of Europe. The birth of the Republic was proclaimed from one of its windows in 1918 after the announcement of the abdication of Emperor William II. From the same window in January 1933, Adolf Hitler's seizure of power was announced, which ended the parliamentary democracy in Germany. In May 1945, the red Soviet flag waved from the roof of the Reichstag building as a symbol of the defeat of National Socialist Germany, which unfortunately signaled the beginning of a long, difficult period for Berlin. This period included the city blockade in 1948 and the construction of the Berlin Wall in 1961 in the immediate vicinity of the Reichstag building, where there are still remnants present today.

The Reichstag building has suffered with the city. The building caught on fire in 1933, suffered from devastating bombardments during the war and suffered from an unsuccessful restructuring, with the final demolition of the dome occurring in the 1950s.

However, the building is close to the city and was indeed the soul of the city's rebirth after the fall of the Berlin Wall in 1989. In October 1990, the building hosted the first session of the Federal Parliament of a united Germany and, rightfully, became its official headquarters in June 1991 when the Bundestag decided to move its headquarters to this historic capital.

Even the Reichstag has been renewed. The renovations were designed by the British architect, Norman Foster. During the demolition work, traces of war ruins (including Soviet soldiers' graffiti) and remnants of the restructuring of the 1960s appeared. The decision was eventually made to keep the existing building and to overlap it with a modern architecture that is agile and reflects the building's activities, environmental protection and the preservation of history. Above all, in 1995, the construction of a modern and accessible dome that reflected the courage and radical renewal of the city was begun. In April 1999, Foster symbolically handed over the keys to the renovated building to the Bundestag President.

The Reichstag building is an impressive building with majestic facades. Foster managed to create internal environments that were simultaneously suitable for a modern parliament and open to the world, while maintaining the historical exterior appearance. The various floors are characterized by a structure and colors that take on important roles both in making the environment lively and characterizing the type of service of each environment. The transparency of the internal walls also contributes to lightening the weight of the historical structure, as well as joining the environment with the various activities, thereby making them ideally converge in the plenary hall. Given that the plenary hall is 1200 square meters, it crosses practically the whole building and is visible from all the other floors. The dome, which is also called The Glass Dome, is the most significant part of the building and has become the emblem of the Bundestag (Imhof and Krempel 2001).

The dome is located on a large roof terrace. From this terrace, you can enjoy a view of the whole city and all its modern buildings. Perhaps due the desire to forget the sad period of the city's division, in a few years, these modern buildings will have supplanted all the previous buildings. The dome has a height of 23.5 m and a diameter of 40 m. It weighs 1200 tons, with the steel structures alone weighing 700 tons, and it is covered by two layers of glass interposed by an intermediate layer of tin vinyl.

The shape of the dome can be geometrically interpreted as a portion of an ellipsoid of revolution, which is a classical quadric surface, with semiaxes that measure 23.5 and 20 m, respectively.

Two helical ramps that go to (up) and from (down) the dome's closed platform have a free width of 1.6 m. They allow visitors easy access to the dome and act as a stiffening ring for the dome itself.

All the elements of the dome are supported by the external structure. The external structure consists of a steel skeleton (which is constituted by 14 circumferences on horizontal parallel planes with variable radii and 14 elliptical arches on radial vertical planes) with a glass cover. The components of the skeleton are, from a mathematical point of view, the curves characterizing the mathematical description of the ellipsoid (see the following sections).

At the center of the dome there is a truncated cone that is 2.5 m wide at the base and reaches 16 m in the upper part. The cone (light sculptor) is covered by 360 highly reflective mirrors. The light-maker element is equipped with an automated mobile screen (sun following), which is powered by photovoltaic cells that prevent excess heat or direct sunlight. At night, the screen acts in the opposite direction by transmitting the artificial light of the plenary hall to the outside. Similarly, the cone acts as a hot and fresh air exchanger.

All of this underlines the supporting idea of creating an exemplary public building that respects the ecological balance by supplying energy instead of exploiting it, with an inviting and not Spartan attitude.

## Treetop Experience

Copenhagen is a humane, well-organized, and lively but disciplined city. It does not forget its past, but it has a strong propensity to focus on the future by safeguarding the environment and the well-being of those who live it. The city wants to live its life outdoors despite the inclement climate and move and produce energy through ecological means. Above all, it has a strong desire to live in sync with nature, observe it, bring it closer to the city, and respect it.

The Treetop Experience (Effekt 2018) proposed project designed in this sense. The proponents want to live in a different way with the natural environment and want everyone to enjoy it. The Danish company EFFEKT, which is based in the fields of modern architecture, design and visionary urban development and planning, intends to create an eco-sustainable project with a low environmental impact that will allow for a unique experience inside a natural park. In practice, the Treetop Experience will be located just 1 h from Copenhagen. According to the interviewed designers, the project inauguration is scheduled for the autumn of 2018.

The architectural firm's project will be located in the Haslev forest, which is characterized by a hilly landscape with lakes, streams and swamps. The project is intended to complete Camp Adventure, which is one of the most famous adventure parks in the world.

The project consists (Effekt 2018) of a 600-m-long spiral wooden walkway connected to a 45-m-high wooden observation tower. The tower has a light structure that is constructed by interweaving two sets of 18 wooden ribs each that are wrapped in a steel lattice, which acts as a support without impeding the surrounding view and will end with a viewing platform. As stated by the designers, the interwoven ribs are obtained by twisting the ribs (one set towards the east and one set towards the west) belonging to a cylinder 120 degrees (corresponding to $\frac{2\pi}{3}$ radians). According to a more in-depth analysis by the authors of this paper, the length $l$ of the positioned ribs depends on the maximum width and the height of the tower. Precisely, it is calculated as follows:

$$l = r\sqrt{3 + h^2/r^2}$$

where $2r$ is the maximum width of the tower and $h$ is its height.

With $h = 3r$ (chosen very close to the dimensional ratios of the tower), the ribs are inclined at an angle of $\frac{\pi}{3}$ on the horizontal axis and their respective lengths are equal to $2r\sqrt{3}$.

The shape of the support tower, which is geometrically interpreted as the classic quadric ruled surface of a hyperboloid of one sheet, symbolically follows the shape of a tree. It widens at the base to represent the roots of the tree that are in close contact with the ground, it tightens at the center to simulate the trunk and it attaches at the top to the accompanying crown. Beyond the symbol, the shape helps the structure to slowly narrow and approach the lower trees and young trees that make up the undergrowth. Then, when it widens, it overcomes and embraces all the tallest trees and enjoys a view across the forest. The shape minimizes the environmental impact and makes the structure more stable, higher and more comfortable at the top.

The continuous ramp (see Fig. 3) will be completely accessible from the different design features, including an aviary, ring paths, and bridges, that will augment the visual experience. Through this design, the project has solved the questions of accessibility and disclosure. In this way, anyone, regardless of their physical condition, can fully enjoy the beauty of nature. Needless to say, all of the materials that are used are designed in terms of sustainability.

**Fig. 3** Treetop walkway. (Courtesy of the Danish company EFFEKT)

## Dynamic Surfaces

The geometric shapes that are found in the described architectures can be interpreted as ellipsoid, hyperboloid of one sheet, or striped helicoid dynamic surfaces.

What do we mean by a dynamic surface (Caliò and Scarazzini 1997; Pottmann et al. 2007)? A dynamic surface is a surface that can be obtained through a generative technique that uses the motion and/or continuous deformation of a base curve (the generatrix). The link between the geometric and algebraic interpretations of a generated dynamic surface is given by means of a vector and matrix calculus. It should be remembered that the curve is expressed through a parametric vector (position vector) or a parametric expression and the continuous transformations through a parametric matrix.

Let us highlight what has been stated regarding the shapes that are described in the work.

The *ellipsoid of revolution* can be generated by rotating a half-ellipse of the $yz$-plane around the z-axis.

(i) The parametric equations for the generatrix (ellipse) are as follows:

$$\begin{cases} x = 0 \\ y = a\cos u \\ z = b\sin u \end{cases}, \quad 0 \le u \le 2\pi,\ a, b \in R^{+}$$

(ii) The matrix of revolution (around $z$) is as follows:

$$\begin{bmatrix} \cos v & -\sin v & 0 \\ \sin v & \cos v & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad 0 \le v < 2\pi$$

(iii) The resulting position vector for the ellipsoid is a modification of the position vector of the generatrix, which is given by the product of the rotation matrix and the position vector of the generatrix (expressed by a single column matrix) as follows (see Fig. 4):

$$\begin{bmatrix} \cos v & -\sin v & 0 \\ \sin v & \cos v & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ a\cos u \\ b\sin u \end{bmatrix} = \begin{bmatrix} -a\cos u \sin v \\ a\cos u\cos v \\ b\sin u \end{bmatrix}, \quad 0 \le v < \pi,\ 0 \le u < 2\pi. \tag{1}$$

The *hyperboloid of one sheet* can be generated through a rotation of an arc of hyperbola in the $yz$-plane around the $z$-axis.

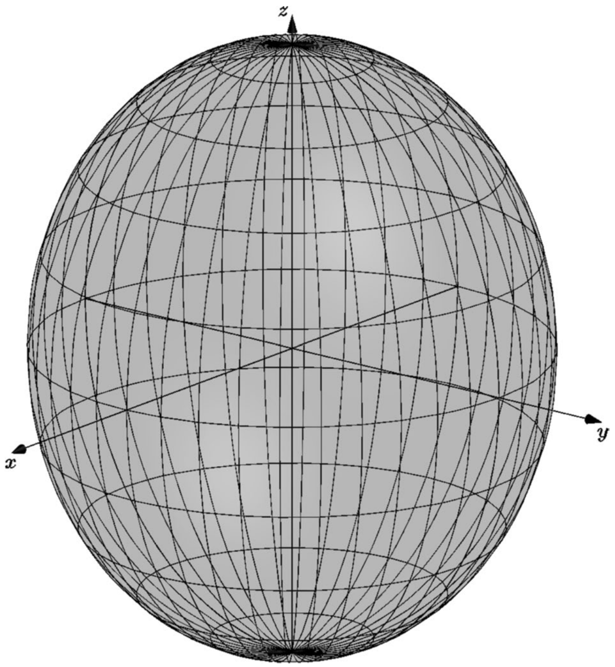



**Fig. 4** Ellipsoid of revolution (with $b > a$)

The parametric equations for the generatrix (hyperbola) are as follows: $\begin{cases} x = 0 \\ y = c\,Chu \\ z = d\,Shu \end{cases}$, $u \in R$, $c, d \in R^+$ where the asymptotes for the hyperbola are given (in the $yz$-plane) by $z = \pm\frac{d}{c}y$, and the functions $Chu$ and $Shu$ are the hyperbolic cosine and sine, respectively.

(i) The resulting position vector for the hyperboloid is given by the modification of the position vector of the generatrix, which is it is determined by the product of the rotation matrix and the position vector of the generatrix as follows (see Fig. 5):

$$\begin{bmatrix} \cos v & -\sin v & 0 \\ \sin v & \cos v & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ c\,Chu \\ d\,Shu \end{bmatrix} = \begin{bmatrix} -cChu \sin v \\ c\,Chu \cos v \\ d\,Shu \end{bmatrix}, \quad 0 \le v < \pi,\ u \in R. \tag{2}$$

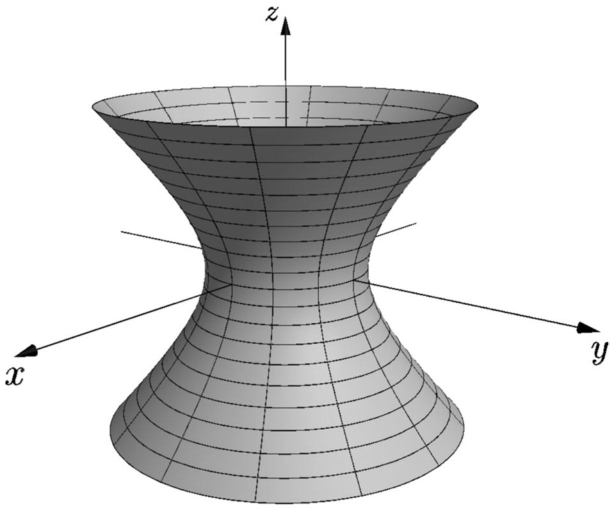



**Fig. 5** Hyperboloid of revolution

A *ruled helicoid*, which is enclosed by the semi-ellipsoid, is obtained by the combination of a rotation with a translation of a segment as follows:

(i) The parametric equations of the *generatrix* segment that lies on the $x$-axis are as follows:

$$\begin{cases} x = a - u \\ y = 0 \\ z = 0 \end{cases}, \quad \frac{3}{4}a \le u \le a,\ a \in R^{+}$$

(ii) The rotation is around the $z$-axis.

$$\begin{bmatrix} \cos v & -\sin v & 0 \\ \sin v & \cos v & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a-u \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} (a-u)\cos v \\ (a-u)\sin v \\ 0 \end{bmatrix}, \quad 0 \le v < 2\pi,\ \tfrac{3}{4}a \le u \le a,\ a \in R^{+}$$

(iii) The translation is along the *directrix*, which is given below. It is defined in such a way that the helicoid has its profile along the ellipsoid and its projection on the $xy$-plane is a specific spiral surface.

$$\begin{cases} x = -\frac{a}{n\pi} v \cos v \\ y = -\frac{a}{n\pi} v \sin v \\ z = b\sqrt{1 - (\frac{v}{n\pi})^2} \end{cases}, \quad 0 \le v < n\pi,\ n \in \mathrm{N}$$

(iv) Finally, the equation is the sum of the rotation matrix multiplied by the position vector of the generatrix curve and the position vector of the directrix curve as follows:

$$\begin{bmatrix} \cos v & -\sin v & 0 \\ \sin v & \cos v & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a-u \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{a}{n\pi} v \cos v \\ -\frac{a}{n\pi} v \sin v \\ b\sqrt{1-(\frac{v}{n\pi})^2} \end{bmatrix}$$

$$= \begin{bmatrix} \left(a-u-a\frac{v}{n\pi}\right)\cos v \\ \left(a-u-a\frac{v}{n\pi}\right)\sin v \\ b\sqrt{1-\left(\frac{v}{n\pi}\right)^2} \end{bmatrix}, \quad 0 \le v < n\pi,\ \frac{3}{4}a \le u \le a,\ n \in N$$

where $a$ and $b$ are the semiaxes of the ellipse generatrix of the ellipsoid, and $n$ is related to the number of spiral arcs of the helicoid (see Figs. 6 and 7).

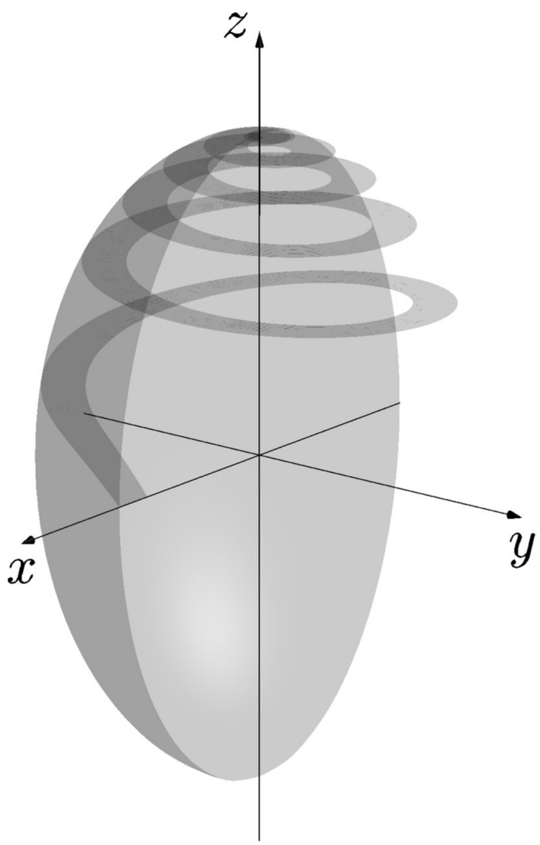
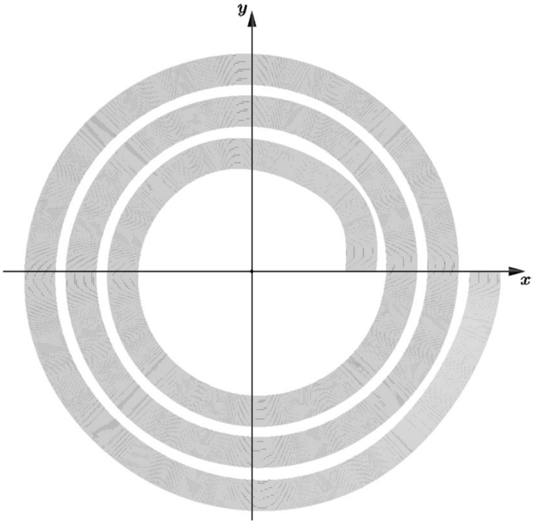



**Fig. 6** Ruled helicoid

**Fig. 7** Horizontal projection




A *ruled helicoid*, which is developed within the hyperboloid, is obtained by the combination of a rotation with a translation of a segment as follows:

(i) The parametric equations for the *generatrix* segment that belongs to the $x$-axis is as follows:

$$\begin{cases} x = c - u \\ y = 0 \\ z = 0 \end{cases}, \quad 0 \le u \le \frac{c}{4},$$

where $c$ is the radius of the circular section at the $d$ level of the hyperboloid.

(ii) The rotation is around the $z$-axis.

$$\begin{bmatrix} \cos v & -\sin v & 0 \\ \sin v & \cos v & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c - u \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} (c - u)\cos v \\ (c - u)\sin v \\ 0 \end{bmatrix}, \quad 0 \le v < 2\pi,\ 0 \le u \le \frac{c}{4}$$

(iii) The translation is along the *directrix*, which is given below. It is defined in such a way that the helicoid has its profile along the hyperboloid and its projection on the $xy$-plane is a particular spiral surface.

$$\begin{cases} x = v\frac{c}{n\pi}\cos v \\ y = v\frac{c}{n\pi}\sin v \\ z = d\sqrt{\left(\frac{v}{n\pi} + 1\right)^2 - 1} \end{cases} \quad 0 \le v < n\pi,\ n \in N.$$

Finally, the equation is the sum of the rotation matrix, multiplied by the position vector of the generatrix curve and the position vector of the directrix curve (see Figs. 8 and 9).

$$\begin{bmatrix} \cos v & -\sin v & 0 \\ \sin v & \cos v & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c-u \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} v\frac{c}{n\pi}\cos v \\ v\frac{c}{n\pi}\sin v \\ d\sqrt{\left(\frac{v}{n\pi}+1\right)^2-1} \end{bmatrix}$$

$$= \begin{bmatrix} \left(c-u+v\frac{c}{n\pi}\right)\cos v \\ \left(c-u+v\frac{c}{n\pi}\right)\sin v \\ d\sqrt{\left(\frac{v}{n\pi}+1\right)^2-1} \end{bmatrix}, \quad 0 \le v < n\pi,\ 0 \le u \le \frac{c}{4},\ n \in N$$

where $n$ is related to the number of spiral arcs of the helicoid.

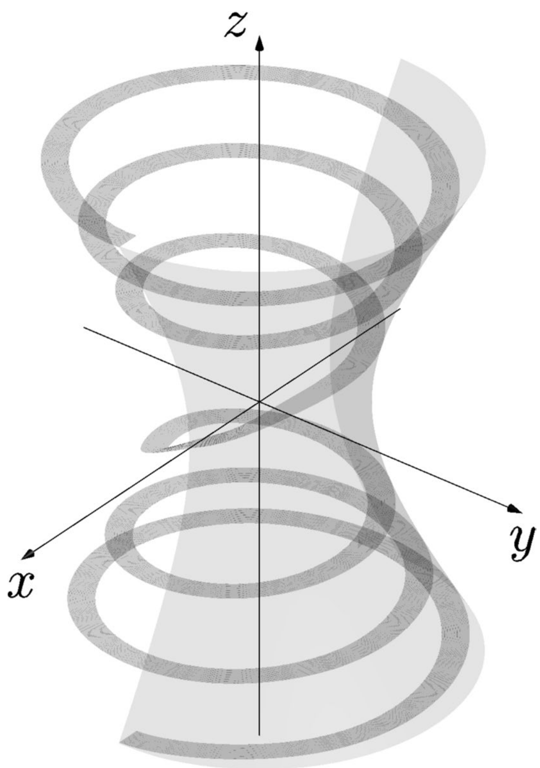
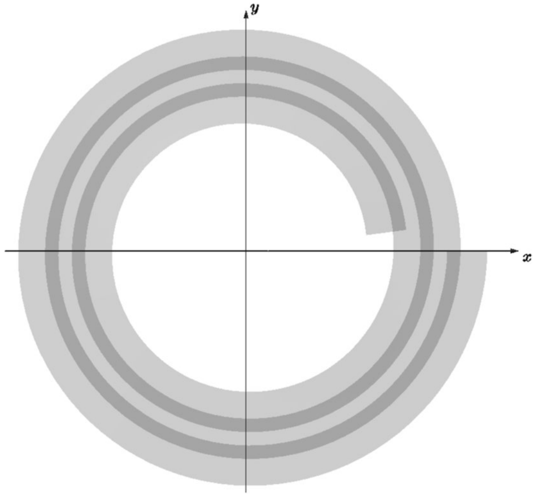



**Fig. 8** Ruled helicoid

Fig. 9 Horizontal projection

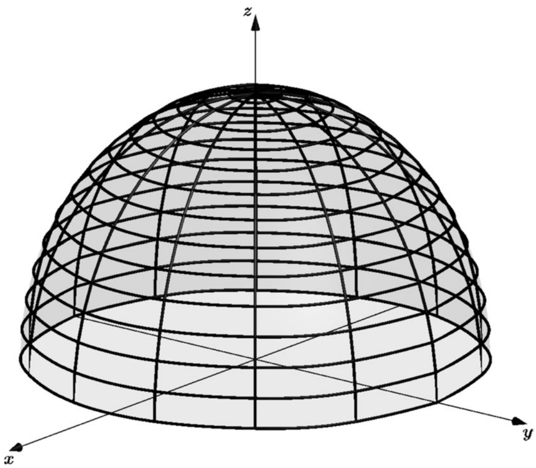



## Virtual Reconstruction of the Two Architectures

By appropriately introducing and modifying some parameters in the classic equations that are presented in the previous paragraphs, it is possible to virtually reconstruct the chosen architectural shapes. Without giving the details of the modifications that were used, we report the results of the simulation that were obtained by the Asymptote computer tool next to the original image. The proportions of the actual measures have been adapted and highlighted.

The results are shown in Figs. 10, 11, 12, 13, 14, 15, 16 and 17 of this section.

Fig. 10 Virtual shape of the dome



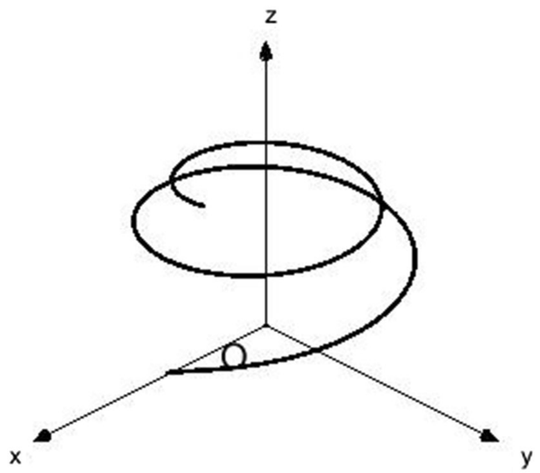



**Fig. 11** Virtual profile of the ramp

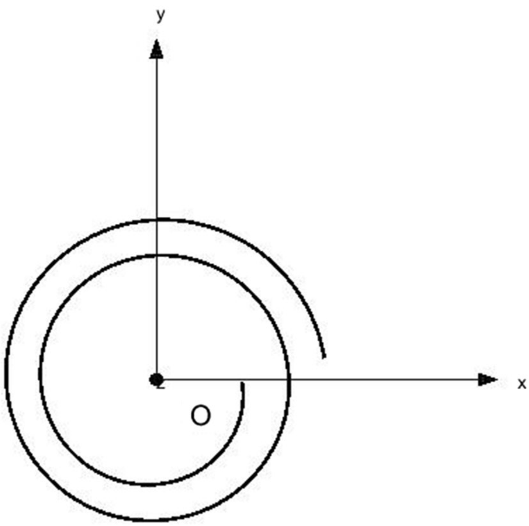



**Fig. 12** Horizontal projection

The original images are snapshots taken by the authors on a recent visit to Berlin (Figs. 18 and 19) and some images that were kindly provided to us by the EFFEKT architectural studio (Figs. 20 and 21).

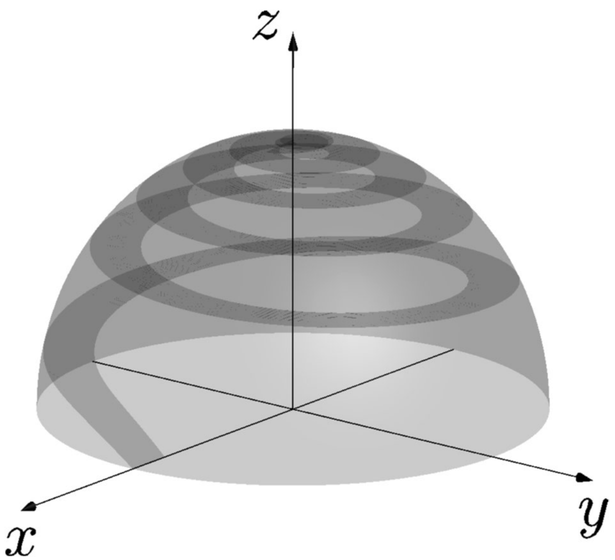
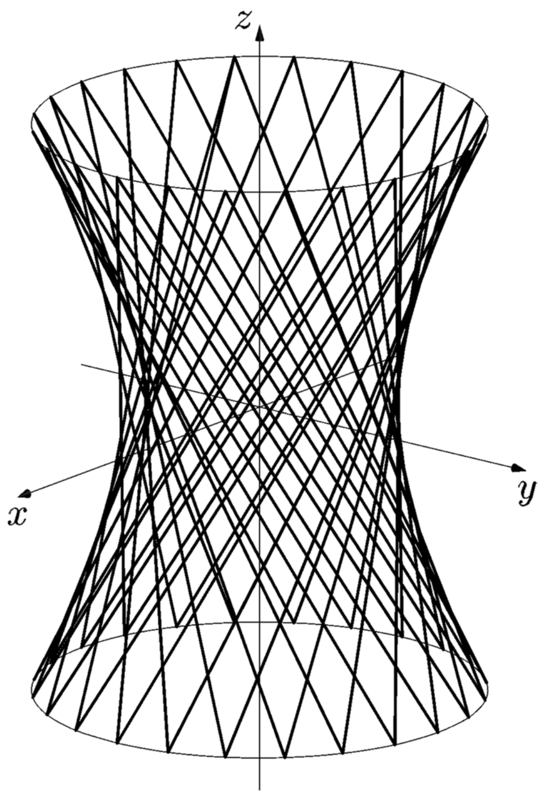



**Fig. 13** Virtual shape of the ramp

**Fig. 14** Virtual reconstruction of the treetop tower



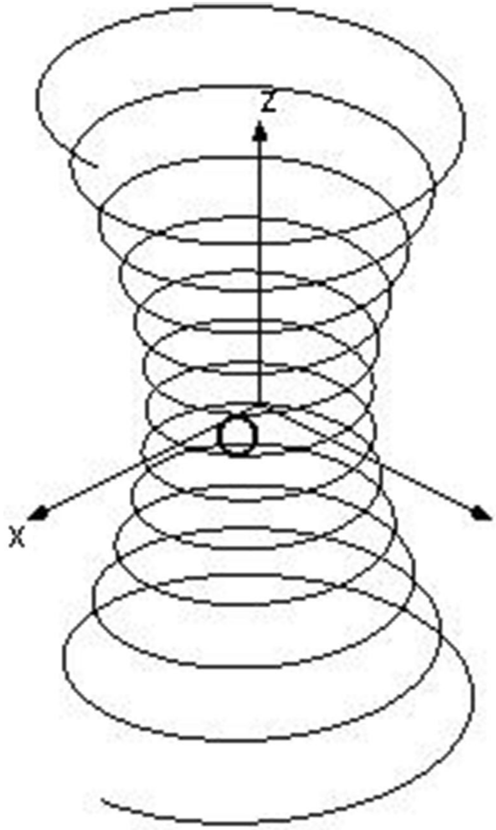



**Fig. 15** Virtual profile of the ramp

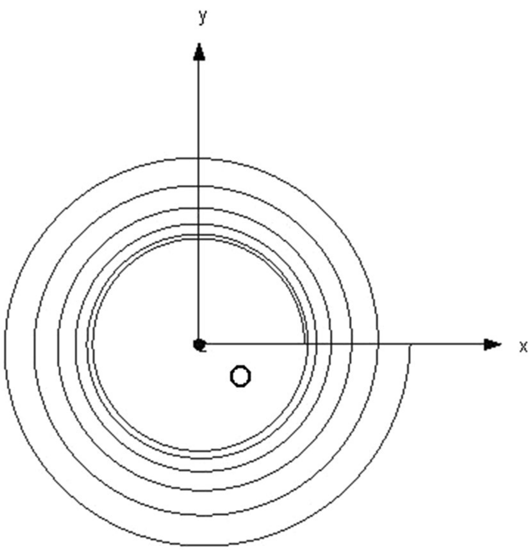



**Fig. 16** Horizontal projection

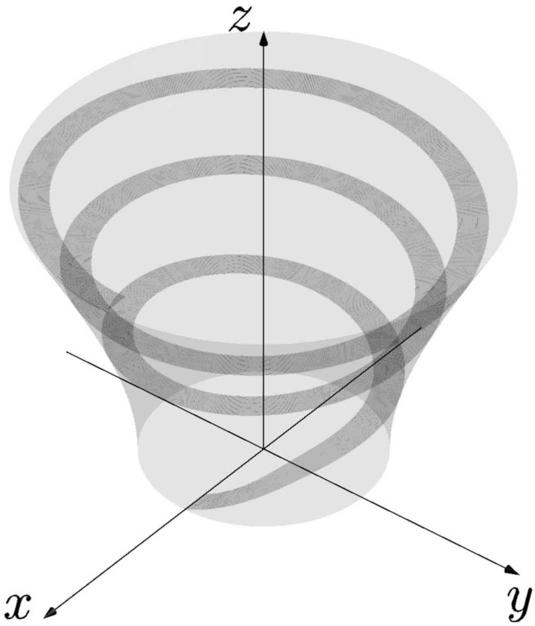



**Fig. 17** Virtual shape of the ramp

**Fig. 18** Reichstag dome. (Photo taken by the authors)

## Conclusions

In this paper, we observed, as we have done for other projects (see Caliò and Marchetti 2015), the geometric shapes of structures. The dome is a convex surface that directs the gaze towards the inside below. The tower is a concave surface that widens the gaze towards the surrounding environment.

We also considered the building materials. One structure has steel ribs with a glass cover, and the other has wooden ribs with a steel mesh cover. Additionally,

**Fig. 19** Interior of the dome. (Photo taken by the authors)

**Fig. 20** Treetop tower project. (Courtesy of the Danish company EFFEKT)

they are seen through different lens. One is life in a closed environment, and the other is life in an open environment.

Finally, we considered the objectives. The dome is an element that was born in the heart of a large city. The dome preserves and protects its history, its memories, and the organizational and political lives of every day. The tower grows in a natural environment and wants to capture and experience every element by committing itself to its valorization and respect.

**Fig. 21** Treetop ramp. (Courtesy of the Danish company EFFEKT)

The spiral walkways bring together the two proposals under the same purpose. The object must be fully usable with a continuous, deliberate path, it must incorporate suggestions that attract attention, and it must be suitable for all people, including children and adults, local and foreigners, scholars and ordinary people and anyone with any physical disability.

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