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RESEARCH

Pal(l)adian Arithmetic as Revealed in the Palazzo Della Torre, Verona

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Abstract Room ratios in Palladio's design for the Palazzo Della Torre mostly ignore his own canonical recommendations and none of the rooms exemplify his rules for room heights. Proportionately, however, the scheme, in plan and elevation, is a brilliant celebration of the cube root, just three years after Cardano published the solution to the cubic equation using methods passed to him by Tartaglia. Daniele Barbaro, Tartaglia and Cardano were all known to each other, and it seems most likely that Palladio would have taken a personal interest in the matter. The cube root that underpins the proportional scheme is Delian, that is, the cube root of 2 cited in Vitruvius. Palladio derives other roots of 2 in anticipation of the arithmetics which emerged in the early seventeenth century for the equal temperament musical scale. Of course, it must be understood that only rational convergents to the cube root of 2 are used. The relationship of room plan and elevation ratios in Palazzo Della Torre is illustrated by using the technique shown in Barbaro *La Practica della Perspecttiva* in which three-dimensional objects are unfolded to make two-dimensional "nets", but figures are not used.

Keywords Andrea Palladio · Palazzo Della Torre · Mathematical means · Pythagorean arithmetic · Renaissance architecture · Doubling the cube · Leonardo da Vinci · Number theory

Palladio, in Book I, chapter XXI of his *Four Books on Architecture*, sets out

> seven types of room that are the most beautiful and well-proportioned and turn out better: they can be made circular, though these are rare; or square; or their

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> length will equal the diagonal of the square of the breadth; or a square and a third; or a square and a half; or a square and two-thirds; or two squares (Palladio 1997, p. 57).

That is, he defines, apart from the circle, rectangles of ratios 1/1, $\sqrt{2}/1$, 4/3, 3/2, 5/3, and 2/1. In a previous publication, I have pointed out that between the extremes of 1/1 and 2/1; 4/3 is the harmonic mean, $\sqrt{2}/1$ is the geometric mean, 3/2 the arithmetic mean and 5/3 the contra-harmonic mean (March 2003, p. 11).

These ratios have also been identified with musical intervals in just intonation: the unison, 1/1; perfect fourth, 4/3; augmented fourth/diminished fifth, $\sqrt{2}/1$: perfect fifth, 3/2; major sixth, 5/3; perfect octave, 2/1. Wittkower (1998) promoted this analogy. However, one interval that is noticeably missing from Palladio's account is the major third, 5/4. This ratio is included in Serlio's seven-part canon where a square and a quarter replaces Palladio's circle:

> There are many rectangular proportions. I shall set down here, however, the seven principle ones which the architect can make use of for various things and can adapt to many situations—that which will not serve in one place could serve for another—since he will know how to use them (Hart and Hicks 1996, p. 30).

5/4 is not one of the eleven classical means together enumerated by Nicomachus and Pappus between the extremes 1 and 2 (Heath 1981, p. 87).

Also, in his Book I, chapter XXIII, Palladio sets out three methods to determine the heights of rooms: effectively the arithmetic, geometric, and harmonic means of their lengths and breadths. He concludes:

> These heights are related to each other in the following way: the first is greater than the second and this is greater than the third; so we should make use of each of these heights depending on which one will turn out well to ensure that most of the rooms of different sizes have vaults of an equal height and those vaults will still be in proportion to them, so that they turn out to be beautiful to the eye and practical for the floor or pavement which will go above them because they will all end up on the same level. There are other heights for vaults which do not come under any rule, and the architect will make use of these according to his judgement and practical circumstances (Palladio 1997, pp. 58–59).

Let the length and breadth be x and y:

The arithmetic mean is $(x + y)/2$;
The geometric mean is $\sqrt{(xy)}$;
The harmonic mean is $2xy/(x + y)$.
It is an elementary exercise to show $((x + y)/2) > \sqrt{(xy)} > (2xy/(x + y))$.

Palladio gives numerical examples:

In the first case, 12 and 6 to give the arithmetical mean: $(12+6)/2=9.$
In the second, 9 and 4 to give the geometric mean: $\sqrt{(9\cdot4)}=\sqrt{36}=6.$
In the third, 12 and 6 to give the harmonic mean: $2(12\cdot6)/(12+6)=8.$

Above, Palladio changes dimensions for the geometric example: "one should take note that it will not always be possible to calculate the height with whole numbers" (1997, p. 58). For example, taking the dimensions 12 and 6, the geometric mean is $\sqrt{(12\cdot6)}=\sqrt{72}=6\sqrt{2}$, that is, the diagonal of square of sides 6. This issue is briefly discussed by Vitruvius in the Introduction to Book IX, where it is stated that "nobody can discover this [the value] by calculation" (Vitruvius 2009, p. 243). The diagonal of a square of sides 10 is examined and the diagonal is estimated to be between 14 and 15. In Barbaro's commentary on Vitruvius, an illustration shows a 5 × 5 square with a diagonal 7 1/14 (1567: 351). Doubled, to compare with the 10 × 10 Vitruvian example, the diagonal becomes 14 1/7. This implies an estimate of 99/70 for $\sqrt{2}$ (Fig. 1).

Such arithmetical computations, extraction of roots, were known at the time among the numerate. Several methods were used. Here, the relationship between the three means above using the numbers 1 and 2 is taken:

$$(1+2)/2>\sqrt{1\cdot2}>2(1\cdot2)/(1+2)$$

or

$$3/2>\sqrt{2}>4/3.$$

It is known that a number lying between rational numbers p/q and p'/q' is $(p+p')/(q+q')$. Further, if p/q is a convergent value to $\sqrt{N}$, Nq/p will be a companion convergent since $(p/q)\cdot(Nq/p)=N$.

$(3+4)/(2+3)=7/5$; the square of this number, 49/25, is less than 2. The rational number 10/7 squared is greater than 2. The square root 2 must lie between these numbers, that is $(7+10)/(5+7)=17/12$; when squared, this is greater than 2, while 24/17 is less than 2. Again, $\sqrt{2}$ must lie between these values $(17+24)/(12+17)=41/29$. Its companion is 58/29. Between these is the number $(41+58)/(29+41)=99/70$. This is the value Barbaro illustrates, while 7/5,

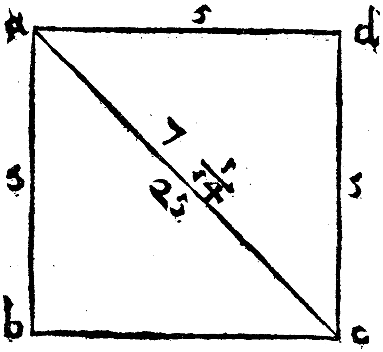

Fig. 1 Detail from (Barbaro 1567, p. 351)

17/12 and 24/17 are rational values for $\sqrt{2}$ that Palladio explicitly uses in plans in *The Four Books*. Palladio uses the convergent 26/15 in the Villa Rotunda for $\sqrt{3}$. This is arrived at in similar way: 2/1, 3/2, 5/3, 9/5, 7/4, 12/7, 19/11, 33/19, 26/15,…. He uses 7/4 and 12/7 elsewhere, and 19/11 in the Palazzo Della Torre. Vitruvius goes on to discuss the cube root of 2 in the context of the Delian problem, the doubling of the cube (2009, p. 247). Leonardo da Vinci, in the *Codex Atlanticus*, notes that a cube of sides 4 has a volume of 64, while one with sides 5 has a volume of 125, just short of 128, twice the volume of the first (Fig. 2). In his own words, the side of the double cube would need to be "5 and a certain inexpressible fraction, which is easy to make but hard to say" (Reti 1974, p. 73). This was around 1500.

The ratio 5/4 occurs in Vitruvius, Book II, chapter 3: "So a brick which is five palms square is called a *pentadoron*, and that four palms square, a *tetradoron*; public buildings are constructed with *pentadora*, and private buildings with *tetradora*" (Vitruvius 2009, p. 43). In his commentary Barbaro illustrates these as cubes, making one the double volume of the other (Barbaro 1567, p. 75) (Fig. 3).

By mid-century the Welsh physician and mathematician Robert Recorde had computed the doubling of a cube with sides 3 feet as requiring sides "3 feet and 77/100 and 1/7 of 1/160" (Recorde 1969). Recorde was in the court of Edward VI during the time Barbaro was Venetian Ambassador from 1548 to 1551. The approach, like the result, was untidy. It seems evident that a similar method to that for the extraction of square roots might apply to the extraction of cube roots. It has been shown that 5/4 is less than the cube root of 2 and it is evident that 4/3 is greater. The cube root of 2 must lie between these two extremes. $(5 + 4)/(4 + 3) = 9/7$ is such a value and it is greater. The solution must lie between this upper value and the lower 5/4. Such a value is 14/11. This too is greater, so $(5 + 14)/(4 + 11) = 19/15$ is a better convergent, but still larger than required.

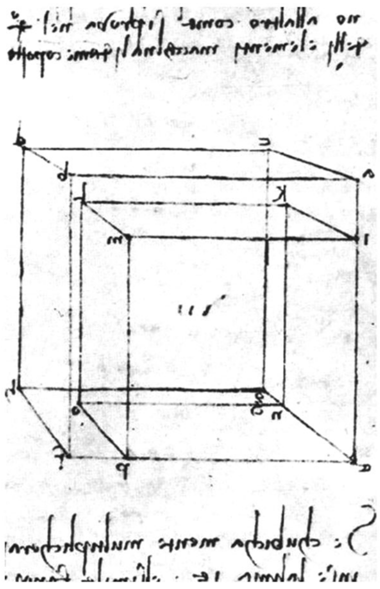

Fig. 2 Leonardo da Vinci, detail of *Codex Atlanticus*, fol. 161r

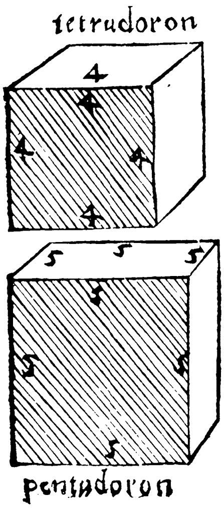

Fig. 3 Detail from (Barbaro 1567, p. 77)

(5 + 19)/(4 + 15) = 24/19 is yet another improvement, and so on, mediating between upper and lower estimates. A modern reader may check on these values by resort to the decimal system, still half a century away from Palladio's day. To the fifth decimal place:

4/3	cubed is	64 /27	2.37037	upper
5/4	cubed is	125 / 64	1.95312	lower
9/7	cubed is	729 / 343	2.12536	upper
14/11	cubed is	2744 / 1331	2.06161	upper
19/15	cubed is	6859 / 3375	2.03230	upper
24/19	cubed is	13824 / 6859	2.01545	upper
29/23	cubed is	24389 / 12167	2.00457	upper
34/27	cubed is	39304 / 19683	1.99686	lower
63/50	cubed is	250047 / 125000	2.00038	upper
⋮				

Palladio uses 5/4, 19/15, and 24/19 as rational convergents for cube root 2 in Palazzo Della Torre, together with various composite ratios. In summary, the rational convergents for:

the square root 2: $3/2, 4/3, 7/5, 10/7, \mathbf{17/12}, \mathbf{24/17}, \ldots$
the square root 3: $2/1, 3/2, 5/3, \mathbf{9/5}, \mathbf{7/4}, \mathbf{12/7}, \mathbf{19/11}, \ldots$
the cube root 2: $2/1, 4/3, \mathbf{5/4}, \mathbf{9/7}, \mathbf{14/11}, \mathbf{19/15}, 24/19, \ldots$

Ratios used for floor plans in the *Four Books* are shown in bold (March 1998, p. 278). Early convergents happen to belong to the Palladian canon.

Room Proportions in Palazzo Della Torre

Figure 4 shows the plan of the Palazzo Della Torre as depicted in Palladio's *Four Books*; Fig. 5 shows the schematic plan on with the following analysis is based.

On the ground floor, on entry, the principle room (labelled [1] in Fig. 5) is P.30 × P.19 and P.24 high, (where P. is a *piede vicentino*). The next room [2] in the *enfilade* is P.19 × P.15, then [3] P.19 × P.11, then [4] P.19 × P. 19, and round the corner an un-dimensioned room, then across the entrance from the street [5] P.19 × P.17. All these rooms are ostensibly P.24 high. Apart from two square corner rooms, none of the remaining rooms conform to Palladio's canon stated so clearly in the *Four Books*. On ascending the grand oval staircase—the type of which is attributed to Marc'Antonio Barbaro, in Book I, chapter XXVIII (Palladio 1997, p. 67)—the first rooms on arrival [6], on either side of the vestibule, are dimensioned P.22 ½ by P.18, a ratio of 5/4. Then up again is the great hall [7], spanning over the courtyard P.34 × P.32 and again P.24 high. In summary, each room is defined dimensionally by length L, width W, and height H. For comparison, the recommended largest and smallest heights given by Palladio's method using the arithmetic and harmonic means of length and width are given, H_A and H_G:

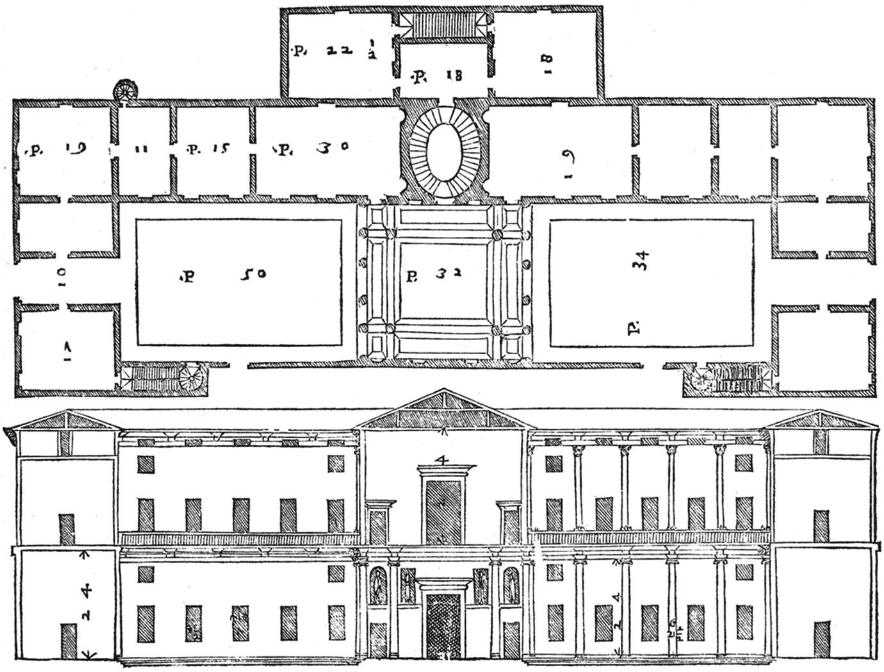

Fig. 4 The plan of the Palazzo Della Torre (Palladio 1997, p. 87)

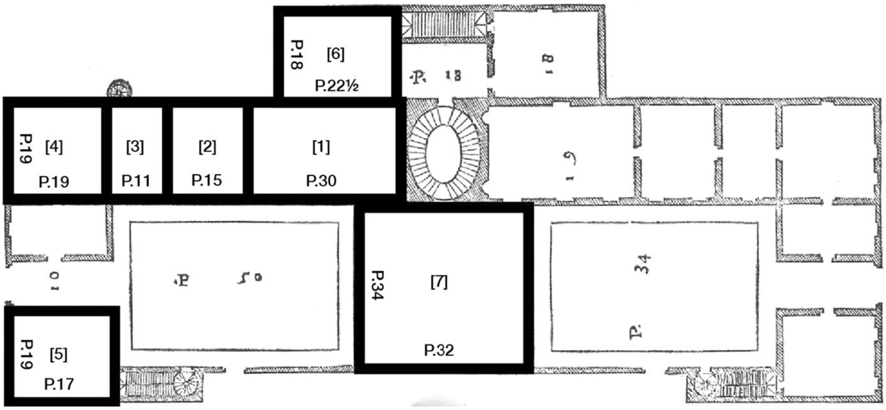

Fig. 5 The schematic key plan used in the analysis

Room	L, W, H	H_A	H_H
[1]	30, 19, 24	24 1/2	23 13/49
[2]	19, 15, 24	17	16 13/17
[3]	19, 11, 24	15	13 14/15
[4]	19, 19, 24	19	19
[5]	19, 17, 24	18	17 17/18
[6]	22 1/2, 18, ?	20 1/4	20
[7]	34, 32, 24	33	32 32/33

It is seen that only room [1] satisfies the recommendation closely. Lower ceilings are suggested for rooms [2] to [5], especially room [3]. Room [6] is not given a ceiling height, but note that the harmonic mean is a whole number. The great hall exceeds the stated ceiling height by almost a third. However, Palladio's practical advice is to level the ceilings for the sake of level floors above, and he appears to take the ceiling of the first room [1] as key. This is acceptable, it appears, in all the rooms except the smallest [3]. Here there is an external spiral staircase, and the fenestration indicates a possible mezzanine. Likewise, in room [1] the fenestration suggests the possibility of an open gallery at mezzanine level. It should also be noted that the Ionic columns are P.24 high.

The rational ratios of floor plans F, and the walls (long L and short S) are set out in parallel with their cube and square root proxies:

[1]	F	L	S	F	L	S
[2]	30/19*	5/4	24/19	$2^{2/3}/1$	$2^{1/3}/1$	$2^{1/3}/1$
[3]	19/15	24/19	8/5*	$2^{1/3}/1$	$2^{1/3}/1$	$2^{2/3}/1$ *
[4]	19/11	24/19	24/11*	$3^{1/2}/1$	$2^{1/3}/1$	$3^{1/\cdot}.\ 2^{1/3}/1$*
[5]	19/19	24/19	24/19	1/1	$2^{1/3}\ 2^{1/3}/1$	$2^{1/3}/1$
[6]	19/17*	24/19	24/17	$2^{1/6}/1$	$2^{1/3}/1$	$2^{1/2}/1$
[7]	5/4	?	?	$2^{1/3}/1$	?	?
[8]	17/16*	17/12	4/3	$3/2^{2/3}/1$ *	$2^{1/2}/1$	4/3

The composite ratios are set out below, using lines to indicate multiplication in the contemporary manner (cfr. the Latin edition of Barbaro 1567, pp. 83–86, a detail of which is given in Fig. 6):

[1*]	30/19	19 15 2 1 39 19	is a rational convergent to	$2^{1/3}$ 1 2 1 2 $2^{1/3}$	or	$2^{2/3}/1$
[2*]	8/5	5 4 2 1 8 5	is a rational convergent to	$2^{1/3}$ 1 2 1 3 $2^{1/3}$	or	$2^{2/3}/1$
[3*]	24/11	19 11 24 19 24 11	is a rational convergent to	$3^{1/2}$ 1 $2^{1/3}$ 1 $3^{1/2}\cdot 2^{1/3}$ 1	or	$3^{1/2}\cdot 2^{1/3}/1$
[5*]	19/17	24 19 24 17 19 17	is a rational convergent to	$2^{1/3}$ 1 $2^{1/2}$ 1 $2^{1/2}$ $2^{1/3}$	or	$2^{1/6}/1$
[7*]	17/16	4 3 17 12 17 16	is a rational convergent to	4 3 $2^{1/2}$ 1 $3\cdot 2^{1/2}$ 4	or	$3/2^{3/2}$


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Secundus compoſitionis modus eſt in quo ratio primi ad ſecundum, conſtat ex ratione tertii ad ſextum, atque ratione quinti ad quartum.

Subtripla.	3—9
ſeſquialtera.	6—4
Subdupla	18—36

Tertius modus eſt in quo ratio primi termini ad tertium conſtat ex ratione ſecundi ad quartum, & quinti ad ſextum.

ſubdupla	2—4
ſubſeſquialtera	6—9
Subtripla. 1. ad 3.	12—36

Vndecimus modus eſt, in quo tertii ad quartum comparatio fit ex rationibus primi ad ſecundum, & ſexti ad quintum.

Subdupla	1—2
Seſquialtera	9—6
Subſeſquitertia ut 3. ad 4.	9—12

Duodecimus modus eſt in quo tertii ratio ad quartum fit ex rationibus primi ad quintum, & ſexti ad ſecundum. 10

Subſextupla	1—6
Quadrupla ſeſquialtera	9—2
Subſeſquitertia	9—12

Fig. 6 Detail from Daniele Barbaro, *M. Vitruvii Pollionis de architectura libri decem: cum commentariis Danielis Barbari*, Venice (1567, p. 84) showing examples of composite ratios

This ratio can also be expressed $\sqrt{9}/\sqrt{8}$, that is to say the geometric mean between the unison, 1/1, and the major second, 9/8.

The room height in [3] is twice that of the width. It is suggested above that there could be a mezzanine. Within the system of proportioning revealed in the Palazzo Della Torre, above, a hypothetical height of P.15 is proposed for room [3], leaving a reasonable height of P.9 for the mezzanine including its floor structure. The long wall is then proportioned to the cube root of 2, 19/15, and the short wall to the ratio 15/11. This latter ratio is also to found in the floor plan of the Villa Rotunda. This ratio has a beautiful symmetry: the square root of three to the cube root of two.

[3*] 15/11

19 15		$2^{1/3}$ 1	
×	is a rational convergent to	×	
19 11		$3^{1/2}$ 1	
15 11		$3^{1/2}$ $2^{1/3}$	or $3^{1/2} / 2^{1/3}$

That room [6] is the only rectangular room with a whole number geometric mean height, P.20, suggests that this might be explored further. Such a height would match the unmarked second storey room shown in section at the street entrances, which is less than P.24. The wall ratios are then 20/18 = 9/8 and 22 ½/20 = 10/9. The latter ratio is used later in the Olympic room of the Villa Barbaro at Maser. These ratios are associated with the major and minor tones of the then contemporary just intonation scale. A value in between these two is (10 + 9)/(9 + 8) = 19/17, the floor plan ratio of room [5]. 9/8 does not seem to be a ratio Palladio uses in his palazzi and villa plans in the *Four Books*. Nevertheless, in musical theory of the period it was a matter of dispute as to whether the tone might be divided into two equal parts, semitones (Palisca 1985, pp. 88–110). An approximation was accepted by some. They argued that doubling the tone 9/8 × 2 = 18/16, while 18/16 = (18/17) · (17/16), and that 18/17 was a minor semitone, 17/16 a major semitone. Twelve minor semitones just fall short of the octave. Twelve major semitones exceed the octave. Indeed, 18/17 was generally accepted by lutenists and luthiers for tuning purposes. It is noteworthy, that the arithmetic shown in this Palazzo preludes musicians' quest for equal temperament later in the century, in which roots of 2—the cube root, in particular—played a key part. Musical intervals implicit in Palazzo Della Torre include:

$2^{1/6}/1$	major second	19/17
$2^{1/3}/1$	major third	5/4, 19/15, 24/19
$2^{1/2}/1$	augmented fourth	17/12, 24/17
$2^{2/3}/1$	minor sixth	8/5, 30/19

The sequence of rooms [1], [2], [3], [4] that form the *enfilade* are themselves proportionally related, not just within themselves individually, but between themselves as the diagram in Fig. 7 shows. The sequence can be seen as a play on ratios involving just the numbers 2 and 3: the *Dyad* and *Triad* in Pythagorean arithmetic. The play involves the first even number and the first odd: understood to be female and male. The *Monad*, 1, was not counted to be a number (Fig. 7).

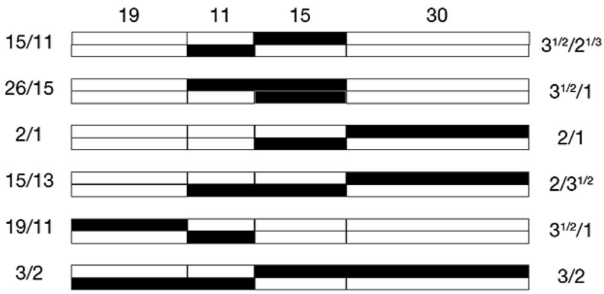


Fig. 7 Proportional relationships of rooms [1], [2], [3] and [4] in the *enfilade*

Palladio also indicates two details: the window dimensions and the diameter of the Ionic columns. The window has dimensions 7 ¾ by 3 ½, a ratio of 31/14. This can be thought of as a square 14/14 and a rectangle 17/14. In turn, the ratio 17/14 may be derived as a composite using convergents already recognized above:

17/14 24 17

X

24 14 (12/7)

17 14

is a rational convergent to

$2^{1/2}$ 1

X

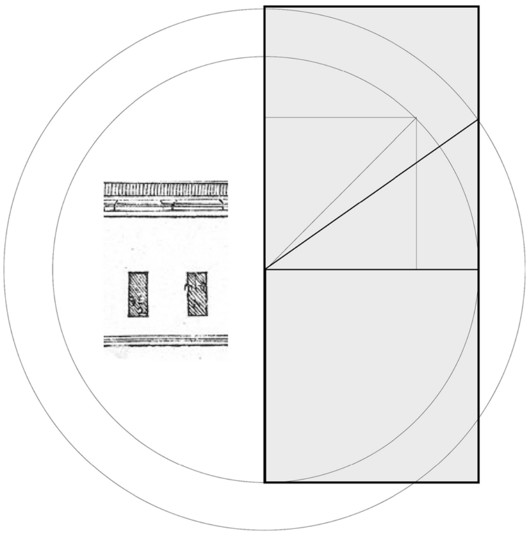
$3^{1/2}$ 1

$3^{1/2}$ $2^{1/2}$ or $3^{1/2} / 2^{1/2}$

The geometric reconstruction show its base in the $\sqrt{1}$, $\sqrt{2}$, $\sqrt{3}$ Pythagorean triangle (Fig. 8). Yet Palladio bypasses geometric construction by arithmetically using rational convergents.

The diameter of the Ionic column is recorded as 2@1 ½ (the @ symbol is closest to that used in the original figure). Now 1 ½ inches is an eighth of a Vicentine foot

Fig. 8 Geometric reconstruction of the window dimensions



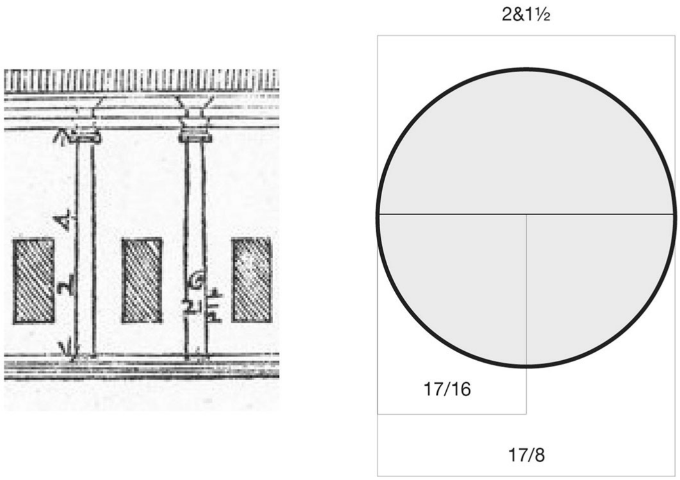


Fig. 9 *Left* Detail of the elevation of Villa Della Torre (Palladio 1997, p. 87) showing the column width and height; *right* manipulation of the diameter

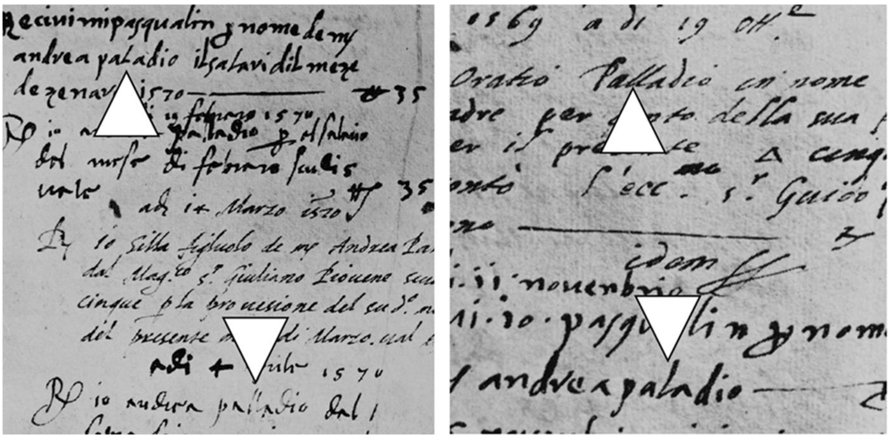


Fig. 10 Pal(l)adio is spelt with one *l* or two *ll*

(*piede vicentino*). A simple manipulation shows the diameter to be 17/8 feet, and the radius to be 17/16 feet, which relates directly to the floor proportion of the great hall, above, $(3/2^{3/2})$ (Fig. 9). Only, there is a problem. The height of the Palladian Ionic order is supposed to be nine times the diameter of the lowest part of the column (Palladio 1997, pp. 32–33). With a diameter P.17/8, the height would fall short at P. 19@1 ½, not P.24 as shown. One-ninth of P.24 is P.2@8, two feet eight inches. Two feet and an eighth, or two feet eight? *Ottavo* or *otto*?

Elsewhere I have noted that Pal(l)adio is both spelt with one ‘l’ or two ‘ll’ (March 1998, pp. 239–243) (Fig. 10). In the frieze of the Tempietto Barbaro at Maser,

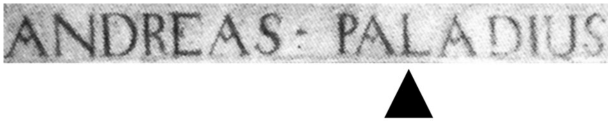


Fig. 11 The name "Andreas Paladius" spelt with one *l* on the frieze of the Tempietto Barbaro at Maser

A=1 **K**=10 **T**=100	**B**=2 **L**=20 **V**=200	**C**=3 **M**=30 **X**=300
D=4 **N**=40 **Y**=400	**E**=5 **O**=50 **Z**=500	**F**=6 **P**=60
G=7 **Q**=70	**H**=8 **R**=80	**I**=9 **S**=90

Fig. 12 The "nine square" method for converting letters of the Roman alphabet into numbers

supervised by Marc'Antonio Barbaro, the name is carved in stone 'PALADIVS' (Fig. 11).

Letters of the Roman alphabet could be converted into numbers using the 'nine square' method current in knowing circles at the time (Fig. 12).

ANDREAS sums to 32, PALADIVS to 34 using digits only. These are the dimensions of the great hall. The floor area of the great hall is $34 \cdot 32 = 1088$. This happens to be the number of VITRVVIVS using digits, tens and hundreds. From the inside of one entrance to the other entrance—that is, the length of the whole courtyard—is P.132 ($50 + 32 + 50$). This is a number for PALLADIVS computed in triangular numbers—one of the not uncommon methods. Further, the first rooms [6] to be entered from the grand stair have a floor area of 405, the number of PALLADIVS using digits, tens and hundreds:

A N D R E A S   P A L A D I V S
1 + 4 + 4 + 8 + 5 + 1 + 9 = 32   6 + 1 + 2 + 1 + 4 + 9 + 2 + 9 = 34

V I T R V V I V S
200 + 9 + 100 + 80 + 20 + 200 + 9 + 200 + 90 = 1088 = 32 x 34

P A L L A D I V S
21 + 1 + 3 + 3 + 1 + 10 + 45 + 3 + 45 = 132 (50+32+50)

60 + 1 + 20 + 20 + 1 + 4 + 9 + 200 + 90 = 405 ($22\frac{1}{2}$ x 18)

Pal(l)adio received his Latin name when he was with Count Gian Giorgio Trissino. In its time, it is not improbable that Trissino performed some alphanumeric computations to arrive at a name relating his protégé to Vitruvius. Wittkower draws attention to Giuseppe Gualdo:

> Palladio's contemporary, [who] wrote in his reliable life of the architect, that 'when Trissino noticed that Palladio was a very spirited young man with much inclination for mathematics, he decided in order to cultivate his genius to explain Vitruvius to him, … (Wittkower 1998, p. 62).

Background

Palazzo Della Torre is no more. It was bombed during WWII in January 1945 (Zorzi 1965; Puppi 1975). Branko Mitrović (2004) provides an axonometric reconstruction of the scheme and argues convincingly for three-dimensional analyses of Palladio's architecture. Pythagorean arithmetics were a standard texts among humanists (March 1998, 2008). The eleven means of Nicomachus/Pappus are enumerated in (Heath 1981). The means are computed between the six ratios between a,b,c (a > b>c) where b is a mean and the ratios between positive differences A = b−c, B = a−c, C = a−b. Between the extremes a = 2/1 and c = 1/1 the means are, in the order presented in Heath (1981, p. 87):

1	C/A=a/a=b/b=c/c = 1		3/2	(arithmetic)
2	C/A=a/b=b/c		√2/1	(geometric)
3	C/A=a/c		4/3	(harmonic)
4	C/A=c/a		5/3	(subcontrary to harmonic)
5	C/A=c/b		(1+ √5)/2	(first contra-geometric)
6	C/A=b/a		(√17-1)/2	(second contra-geometric)
7	B/A=a/c		3/2	
8	B/C=a/c		3/2	
9	B/A=b/c		(1+ √5)/2	
10	B/C=b/c		1/1	
11	B/C-a/b		4/3	
12	B/A=a/b	is illusory since it gives	a=b.	

Of Palladio's canon, 3/2 appears three times, 4/3 twice, 5/3 and $\sqrt{2}/1$ once. Items 5 and 9 have means equal to the golden section. If the golden section had any aesthetic value at the time, surely this Pythagorean arithmetic relationship would have been noted and grasped.

The Palazzo Della Torre is assumed to have been planned in 1551, and was still under construction in 1568, at the death of Count Giovanni Battista Della Torre. In 1545 Cardano published *Artis magnae sive de regulis algebraicis liber unus,* or *Ars magna* (Cardano 1993): a book considered to be one of the great books of the Renaissance and a significant landmark in the history of mathematics. From 1535 onwards there had been a public rumpus over the authorship of the solution to the cubic equation; something Luca Pacioli, at the turn of the century, in his *Summa de arithmetica, geometria, proportioni et proportionalità* of 1494 had declared could not be done and was as impossible as squaring the circle. At root it was rivalry between two mathematicians, Niccolò Tartaglia in Venice and Girolamo Cardano in Milan. Tartaglia taught mathematics at Verona, Brescia and Venice. In the course of what Oystein Ore, in his foreword to the translation of Cardano's *Ars Magna*, describes as 'one of the most violent feuds in the history of science' (Cardano 1993, p. ix), public notices*, cartelli,* were published over several months; in 1548 a contest was held in Santa Maria del Giardino dei Minori Osservanti, Milan; challenges were arbitrated by Don Ferrante di Gonzaga, governor of Milan; the victor by default, Cardano's secretary, Ludovico Ferrari, was thought to have been announced and rewarded (Jayawardene 2008). The Venetian, defeated, slunk home. It was not a matter that interested persons could ignore, especially in the Venetian Republic. At mid-century, it would not be unreasonable to suggest that cubes and cubic roots were in the air among the numerate in the Republic, including Verona. Is it possible that proportionality in Palazzo Della Torre celebrates contemporary mathematical advances? Or at the very least, is it a paean to the Vitruvian story about the doubling of the altar at Delos? (Vitruvius 2009, p. 147).

The fourteenth-century Aristotelian polymath Nicole Oresme established the use of fractional exponents in *De proportionibus proportionum* around the mid-fourteenth century (Oresme 1966). The notion could not have been unfamiliar two centuries later. In the presentation here, modern symbolism is used.

Mathematics in sixteenth-century Italy was two-faced. One face turned towards the future as Cardano does in *Ars Magna* with his acceptance of the square roots of negative numbers before the later understanding of complex numbers (Rose 1975). The other face looked back and played to occult themes—hermetic, cabalistic, neo-Platonic, Pythagorean (Yates 1983; Copenhaver 1992; Allen 1994). Yet even a progressive like Cardano had his conservative side as an astrologer (Grafton 1999). The expulsion of Jews from Spain in 1492 led to a substantial migration to the Venetian Republic. Frances Yates (1933) tells of the Jewish influence on the Venetian friar, Francesco Giorgi. Both the Greek and Hebrew languages use their alphabets for numbers. That is to say they do not have separate symbols for numerals. It is not surprising that alphanumeric transformations are common in both (Heath 1921; Cajori 1993). Johann Reuchlin, in his *De arte cabalistica* of 1516 (Reuchlin 1983) had polished his Latin with Ermolao Barbaro (Geiger 1964), uncle to Daniele Barbaro, with whom Palladio was collaborating on the edition of Vitruvius. In 1531 Henry Cornelius Agrippa presented a nine-square table to enable Latin words and names to be converted into numbers using the 23-letter Latin alphabet (Agrippa 2009). It should also be noted that Arabic languages were alphanumeric. Venetians would have been familiar with this through trade with the Ottomans and North Africa (Ifrah 1985).

This paper has indicated one method of computing convergent rational values for roots. Others exist. The one chosen is derived from Fowler (1999). It was always possible to use square and cube tables with parallel columns, one with a simple value and the other with the values multiplied by N, the number of the root required. Thus, for N = 2 with cubes

x	x^3	$2x^3$
2	8	16
3	27	54
4	64	126
5	125	250
⋮		

from which $(4/3)^3$ is seen to exceed 2, while $(5/4)^3$ falls short. It is also possible that among the secrets held by masons were root tables. In discussing proportionality in Frank Lloyd Wright's early work, I drew attention to a carpenter's manual containing exactly such tables with the same convergents used above (March 1995; Anonymous 1899). Pal(l)adio, born Andrea di Pietro della Gondola, had been trained in the trades.

References

Agrippa, Henry Cornelius. 2009. *Three Books of Occult Philosophy* (1531). Trans. James Freake, ed. Donald Tyson. Woodbury MN: Llewellyn Publications.

Allen, Michael J. B. 1994. *Nuptial Arithmetic*. London: University of California Press.

Anonymous. 1899. *Steel Square and its Uses: a Complete Up-to-Date Encyclopedia on the Practical Uses of the Steel Square, Showing How It Can Be Used by the Carpenter in his Daily Work.* River Forest: Duke Hill

Barbaro, Daniele. 1567. *I Dieci Libri dell'architettura di M. Viruvio tradotti e commentati da Mons. Daniel Barbaro eletto Patriarca d'Aquileia, da lui riveduti & ampliati; & hora in piu commoda forma ridotti.* Venice: Francesco de' Franceschi Senese e Giovanni Chrieger Alemano Compagni. Latin edition: *M. Vitruvvii Pollionis De Architectura Libri Decem cum commentariis,* Venice, 1567.

Cajori, Florian. 1993. *A History of Mathematical Notation* (1928). New York: Dover.

Cardano, Girolamo. 1993. *Ars Magna or the Rules of Algebra* (1545). T. Richard Witmer, trans. New York: Dover.

Copenhaver, Brian P. 1992. *Hermetica*. Cambridge: Cambridge University Press.

Fowler, David. 1999. *The Mathematics of Plato's Academy: A New Reconstruction*. Oxford: Clarendon Press.

Geiger, L. 1964. *Johann Reuchlin sein Leben und seine Werke*. Nieuwkoop: De Graaf.

Grafton, Anthony. 1999. *Cardano's Cosmos: The Worlds and Works of a Renaissance Astrologer*. Cambridge MA: Harvard University Press.

Hart, Vaughan, and Peter Hicks. 1996. *Sebastiano Serlio on Architecture*, vol. I. New Haven: Yale University Press.

Heath, Thomas. 1981. *A History of Greek Mathematics* (1921). New York: Dover.

Ifrah, Georges. 1985. *From One to Zero*. New York: Viking Penguin Inc.

Jayawardene, S. A. 2008. Ferrari, Ludovico. *Complete Dictionary of Scientific Biography*. http://www.encyclopedia.com. Retrieved 6 Oct. 2014.

March, Lionel. 1995. Sources of characteristic spatial relations in Frank Lloyd Wright's decorative designs. In: *Frank Lloyd Wright: The Phoenix Papers*, ed. Paul K. Zygas, 12–49. Tucson: University of Arizona Press.

March, Lionel. 1998. *Architectonics of Humanism: Essays on Number in Architecture*. London: Academy Editions.

March, Lionel. 2003. Foreword. In: Silvio Belli, *On Ratio and Proportion: the common properties of quantity*, trans. and eds. Stephen R. Wassell and Kim Williams, 7–13. Florence. Kim Williams Books.

March, Lionel. 2008. Palladio, Pythagoreanism and Renaissance Mathematics. *Nexus Network Journal* **10**, 2: 227–243.

Mitrović, Branko. 2004. *Learning from Palladio*. New York: W. W. Norton & Company.

Oresme, Nicole. 1966. *De proportionibus proportionum* and *Ad pauca respicientes* (ca.1350). Edward Grant, ed. and trans. Madison: The University of Wisconsin Press.

Palisca, Claude V. 1985. *Humanism in Italian Renaissance Musical Thought*. New Haven: Yale University Press.

Palladio, Andrea. 1997. *The Four Books of Architecture.* Robert Tavernor and Richard Schofield, trans. Cambridge MA: MIT Press.

Puppi, Lionello. 1975. *Andrea Palladio*. London. Phaidon Press Limited.

Recorde, Robert. 1969. *The Whetstone of Witte* (1557). New York: Da Capo Press.

Reuchlin, Johann. 1983. *De arte cabalistica* (1516). Trans. Martin and Sarah Goodman. New York: Arabis Books.

Reti, Ladislao ed. 1974. *The Unknown Leonardo*. New York: Abradale Press.

Rose, Paul Lawrence. 1975. *The Italian Renaissance of Mathematics*. Geneva: Librairie Droz.

Vitruvius. 2009. *On Architecture.* Richard Schofield, trans. London: Penguin Classics.

Wittkower, Rudolf. 1998. *Architectural Principles in the Age of Humanism (1949).* London: Academy Editions.

Yates, Frances A. 1983. *The Occult Philosophy in the Elizabethan Age*. London: ARK.

Zorzi, Giangiorgio. 1965. *Le opere pubbliche e i palazzi privati di Andrea Palladio.* Venice: Neri Pozza.

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