

Descriptive Geometry: From its Past to its Future

Abstract. Descriptive geometry is the science that Gaspard Monge systematized in 1794 and that was widely developed in Europe, up until the first decades of the twentieth century. The main purpose of this science is the certain and accurate representation of three-dimensional shapes on the two-dimensional support of the drawing, while its chief application is the study of geometric shapes and their characteristics, in graphic and visual form. We can therefore understand how descriptive geometry has been, on the one hand, the object of theoretical studies, and, on the other, an essential tool for designers, engineers and architects. Nevertheless, at the end of the last century, the availability of electronic machines capable of representing three-dimensional shapes has produced an epochal change, because designers have adopted the new digital techniques almost exclusively. The purpose of this paper is to show how it is possible to give new life to the ancient science of representation and, at the same time, to endow CAD with the dignity of the history that precedes it.

Introduction

Descriptive geometry is the science that Gaspard Monge systematized in 1794 and that was widely developed in Europe up until the first decades of the twentieth century. The main purpose of this science is the representation, certain and accurate, of shapes of three dimensions on the two-dimensional support of the drawing; while its chief application is the study of the geometric shapes and their characteristics, in a graphic and visual form. We can therefore understand how descriptive geometry has been, on the one hand, the object of theoretical studies, and, on the other, an essential tool for designers, engineers and architects.

Nevertheless, at the end of the last century, the availability of electronic machines, capable of representing three-dimensional shapes, has produced an epochal change, because the designers have adopted the new digital techniques almost exclusively. Furthermore, mathematicians seem to have lost all interest in descriptive geometry, while its teaching in universities has almost disappeared, replaced by a training in the use of CAD software, which mainly has a technical character.

The purpose of this paper is to show how it is possible to give new life to the ancient science of representation and, at the same time, to endow CAD with the dignity of the history that precedes it.

This result may be achieved by verifying and validating some fundamental ideas:

- the idea that descriptive geometry is set within a historical process much wider than the Enlightenment period, a process which goes from Vitruvius to the present day, and that it therefore includes both the compass as well as modern digital technologies;

- the idea that, to the graphic representation methods (perspective, the method of Monge, axonometry, topographic mapping) can today be added the digital methods that are implemented in computer applications (mathematical representation, numerical or polygonal representation);
- the idea that the synergy between the calculation and the visualization of the shape offered by the digital systems, may provide simpler and more general solutions to the classic problems of descriptive geometry;
- the idea that from this view of their historical foundations digital applications should receive a stimulus towards the unification of the terms used for the procedures, the shapes and the operations that the applications offer to the user.

Research groups of five different Italian universities are working on this topic (Rome, Milan, Genoa, Venice and Udine). Here, we present the main lines of this research and the first results.

Apparently, in architectural practices, CAD has replaced descriptive geometry as a tool for the representation of three-dimensional shapes. In universities, the teaching of descriptive geometry is disappearing. The mathematicians have not cared about these studies since the first decade of the last century. Does a future exist for descriptive geometry? Is it possible to give this ancient science a new life?

A look back at history

If we would like to glimpse what the future of a science may be, we must recall its past, since in its past was traced the path that leads, today, towards the future.

In 1794 Gaspard Monge explained, in a course of lectures at the École Normale in Paris, the fundamentals and the first applications of a discipline that then seemed to be totally new; he gave it the name it is known by today: *Géométrie Descriptive*. In the fevered atmosphere of the French Revolution, only very few intellectuals dared to re-evaluate the originality of Monge's work. Joseph Louis Lagrange did it, with plenty of irony, after having attended one of these lessons, exclaiming: *Je ne savais pas que je savais la géométrie descriptive!* ('I didn't know that I knew descriptive geometry').¹ But others understood Lagrange's words as a proof of the clarity of Monge's exposition and pretended not to understand. Michel Chasles also tried to place *Géométrie Descriptive* in its historical perspective,² and he did it with reasoned arguments, but his efforts were not enough to prevent the image of Monge, 'creator' of the science that he baptized, from reaching us in the present time.

Actually, as everyone who has studied history of art knows, descriptive geometry has much older roots. Therefore I think that we should write *Géométrie Descriptive*, in French, when we refer to the science developed by Monge and his school, and write simply 'descriptive geometry' when we allude to the geometric science of representation in its centuries-old journey.

Descriptive geometry teaches to construct and represent shapes of three dimensions and, with these, the objects of all kinds of artistic, planning or production activities. These representations are drawings that are constructed following a geometric code, which permits us to move from the two-dimensional space of the representation to the three-dimensional space of the physical object. Thanks to its ability to create bi-univocal relations between the real space and the imaginary space of the drawing, descriptive geometry also lends itself to many applications, which range from the study of the properties of surfaces, to the creation of spaces and illusory visions.

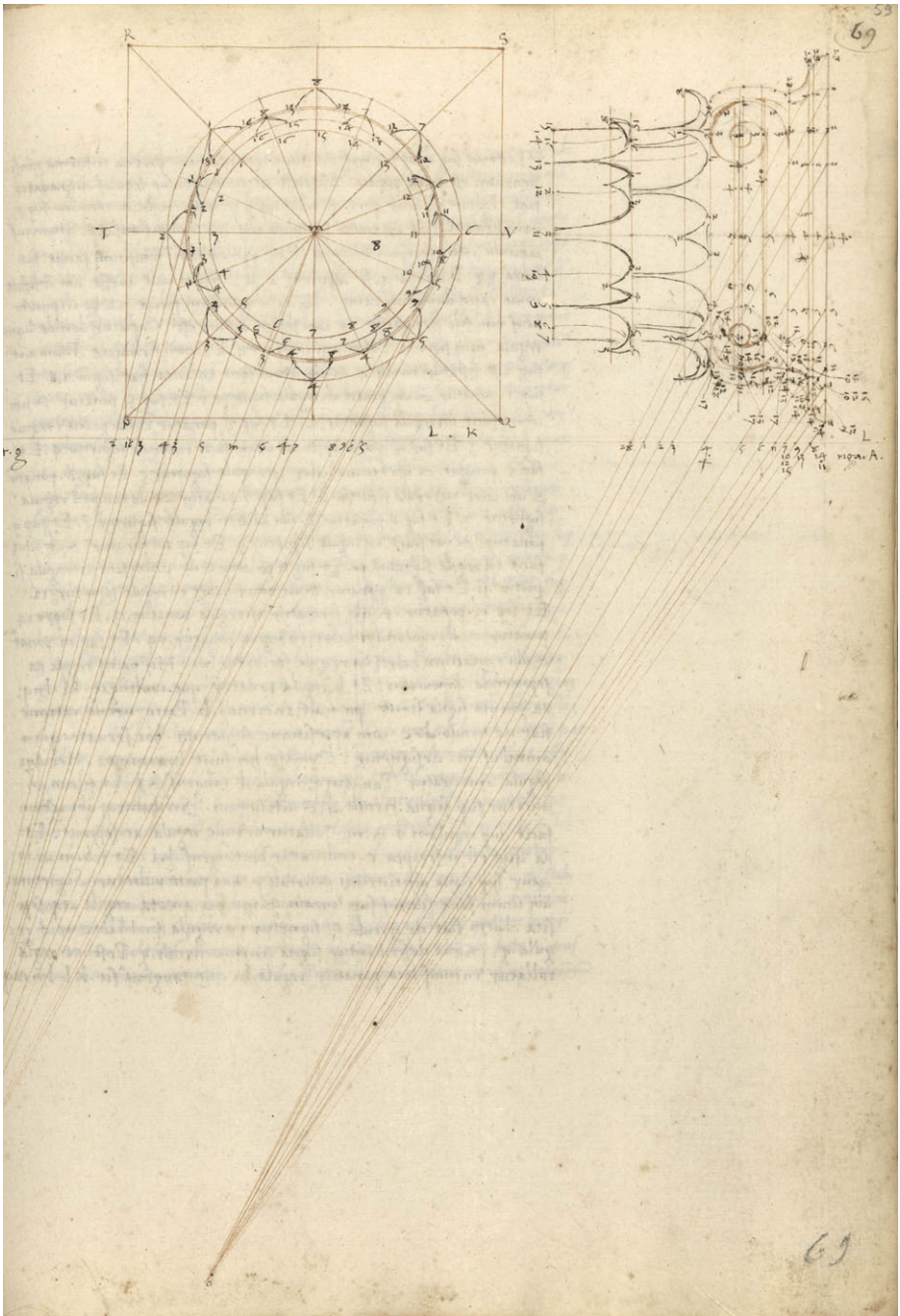


Fig. 1. Fol. 69r of Piero della Francesca's manuscript *De prospectiva pingendi* preserved in Bordeaux. The page represents, in orthogonal projections, the Italic capital, the point of view, the picture plane and a few projection operations of significant points of the capital

Thus, descriptive geometry has strengthened, during the centuries, a fruitful link between art and science.

Now, we could talk for a long time about the wealth of this synthesis of abstraction and manual skill, of reasoning and intuition. And we could cite numerous examples of the outcomes of this synergy throughout the history of art, as well as the history of science. But this discussion, even if fascinating, would turn our attention away from our first objective, which is that of demonstrating that descriptive geometry was not 'invented' by Gaspard Monge in 1794 and that, instead, it has a much older history. For this end, it will be enough to illustrate an example, which will also enable us to better understand the *modus operandi* of the discipline.

Three centuries before Monge (around 1480) Piero Della Francesca composed his famous *De Prospectiva Pingendi*,³ a treatise that teaches how to construct perspectives of objects of three dimensions, using representations of the same objects in plan and elevation. The treatise is divided into three books and contemplates two different methods of constructing perspectives. In the second and third book, in particular, there are minute descriptions of the operations necessary to construct object of remarkable complexity, like buildings, a cross vault, an attic base, a torus (the *mazzocchio* of Paolo Uccello), an Italic capital (fig. 1), an apsidal half-dome divided in coffers. The description is written in an algorithmic form, or better, as a well organized list of graphical operations, all practicable, which, based on certain data, lead to the desired result: the representation of the object as it is perceived by the eye of a man placed in a certain point of observation. If we compare the amount of objective and operating information contained in the text with the number of signs that appear in the supplied small illustrations, we become aware that the graphical description is much less detailed. In other words, the illustration supplied to the text is a mere allusion to a drawing of much bigger dimensions; this is what Piero observes and reconstructs for the reader, proceeding step by step. In fact, Piero enunciates a theory, which is his method of construction of the perspective, and he supports this theory with a series of experiments. The minuteness of the description of each experiment serves to ensure that it is repeatable and that the related theory is thus validated. To be persuaded of what I am saying, it is sufficient to draw one of these drawings again, for instance, the one of the Italic capital (fig. 2).

Without entering into details, I will only examine the flow of Piero's work. In a first phase our scientist-artist explains how to construct the capital in *width and height*, namely in plan and elevation. In other drawings of the treatise, the plan and the elevation are connected to each other by what we today call 'reference lines'.⁴ In the case of the capital, instead, the two drawings are separate, because the complexity of the construction is such that it requires using the format of the paper to the utmost. For that reason, when we have to construct the elevation of a point when the plan is known, or the plan when the elevation is known, we measure the distance of the point from a common reference line. This proves that Piero connects the two 'projections' of the object, because he is fully aware of the meaning of the reference line and not because of a simple intuition. Piero, in other words, conceives the object placed above the plan and in front of the elevation, as Monge will do three centuries later. In this meticulous construction work, the genesis of the representation of the geometric entities and the genesis of the object, proceed hand in hand. This is the *modus operandi* typical of descriptive geometry: the image emerges as the object takes shape in the mental space of the designer and only if the designer is able to give the object a shape. The geometric construction and the simulation of the physical construction are simultaneous.

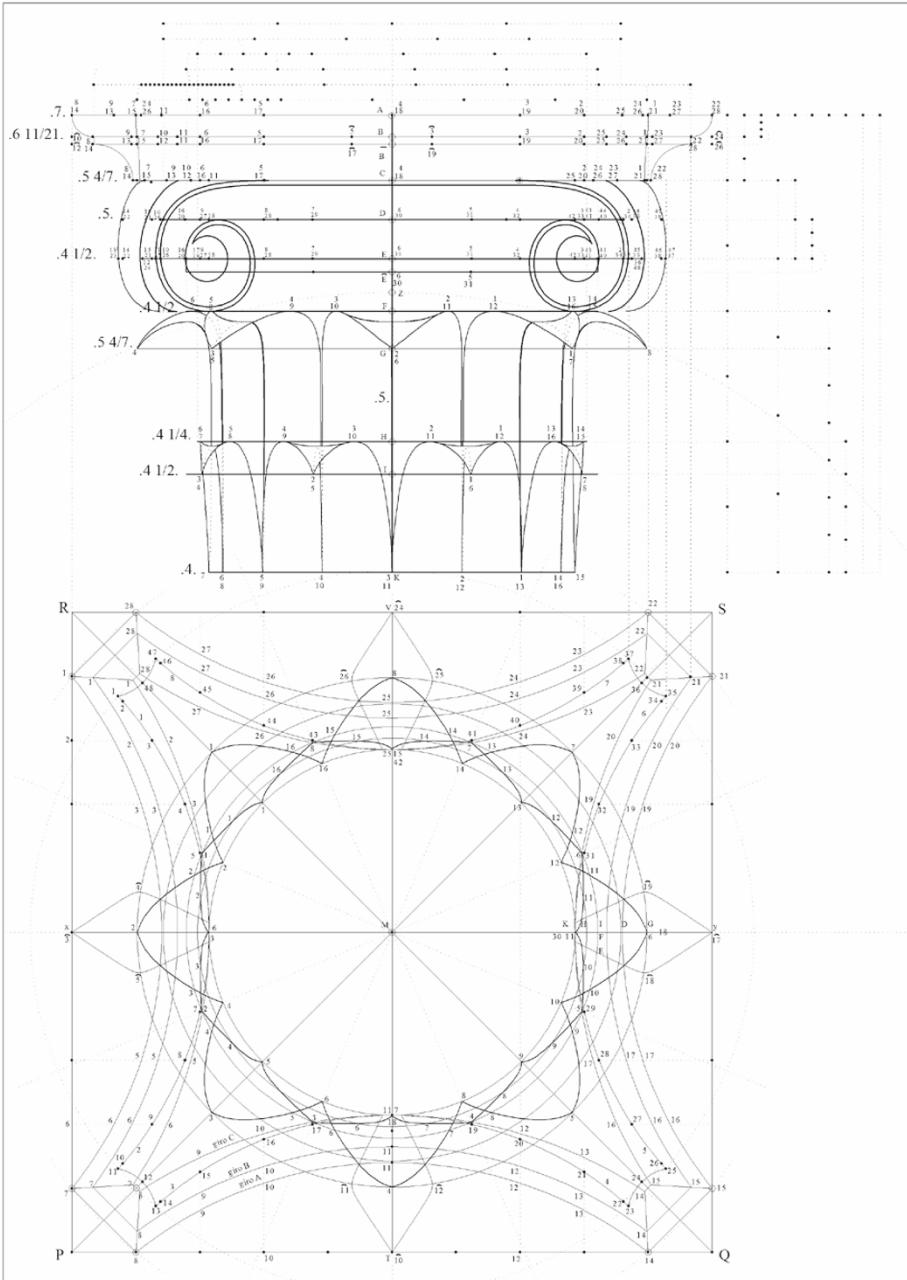


Fig. 2. This drawing, which is readable only in in-folio format, represents the graphic transcription of operations related to the construction of the capital, meticulously described by Piero from fol. 49v to fol. 52r of the manuscript preserved in the Biblioteca Palatina in Parma

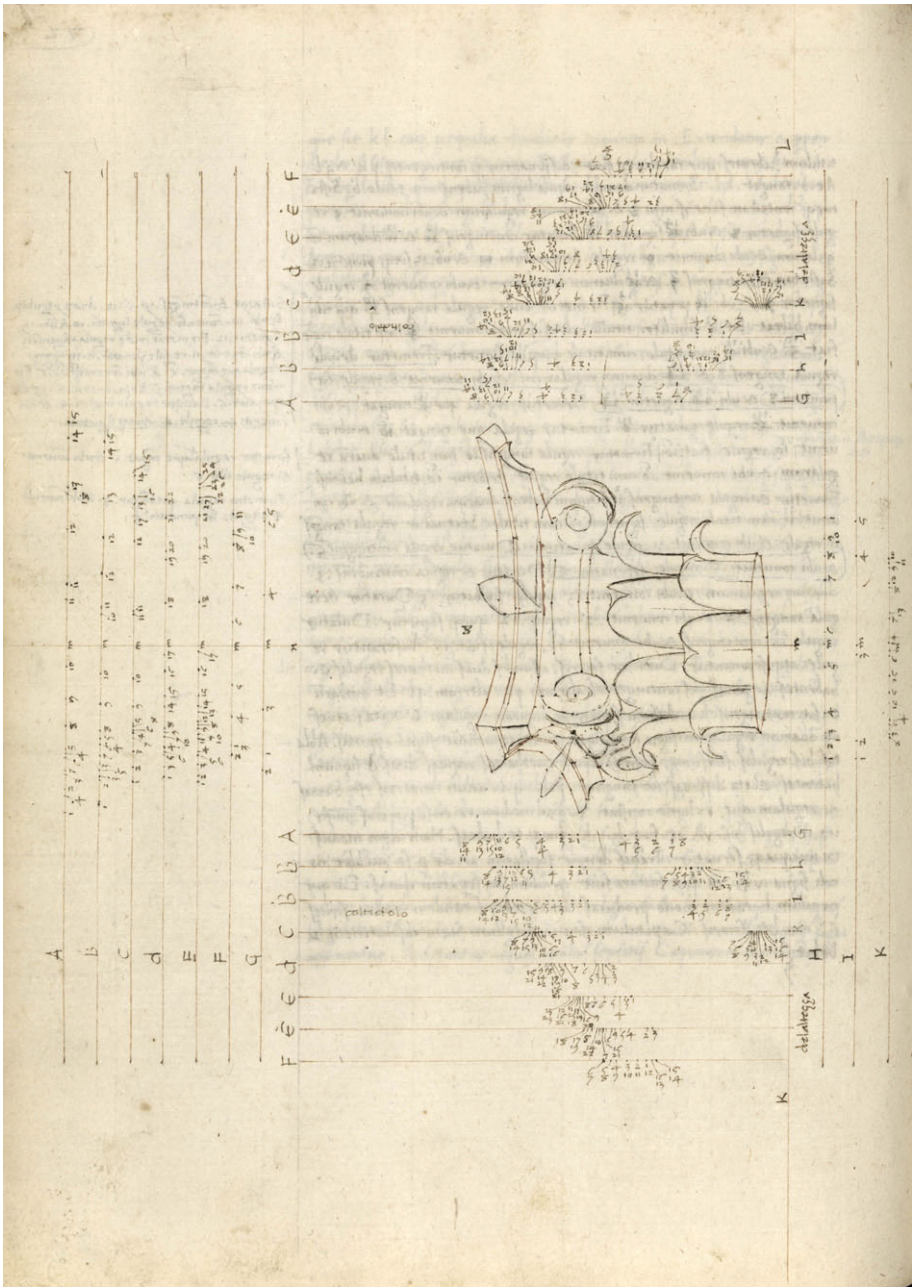


Fig. 3. Fol. 72v of the manuscript preserved in Bordeaux, in which Piero represents the result of the projection operations previously carried out on the capital and experimentally verifies the perspective vision

In the second phase, after having constructed the capital, Piero determines in the space the point of view, the plane surface of the picture plane that will contain the perspective, and he carries out the operations of projection and section that give rise to the perspective illusion. This is another extraordinary moment in the history of descriptive geometry, because, perhaps for the first time, here are described and realized geometric operations carried out in a three-dimensional virtual space, which arises from a representation in plan and elevation. In other words, just as Monge will do three centuries later, Piero uses the two associated orthogonal projections not only in order to generate two significant images, but also to work on the model that these images are able to evoke.

The operations described will generate a crop of experimental data, which are the ‘coordinates’ of the points in which the visual beams meet the picture plane.

In the third and final phase, the data collected in the second phase are methodically written on paper by means of special strips of paper and wood and produce, we would say today, a point cloud that describes the perspective of the capital (fig. 3).

Topicality of descriptive geometry

At this point, I think it is necessary to define a question that is of an ethical nature, so as not to lead to misunderstandings: I don’t want to belittle Gaspard Monge’s role in the history of descriptive geometry, nor, even less, in the history of science. I don’t want to subtract from Monge any of the credit that was given to him. I only wish to give back to descriptive geometry its past as well as its functionalities, and to show how these functionalities are still topical today.

In fact, if we consider *Géométrie Descriptive* as an offspring of the Age of Enlightenment and of the Industrial Revolution, we might also legitimately claim that, in comparison with the era of the computer, this science has outlived its time. But, if we instead consider descriptive geometry for what it is in itself – a science that is rooted in the past, even before Piero’s time, and rooted in the art of thinking and creating space, more than connected to the techniques of production –, then we will realize that this science has not yet exhausted its life cycle and that it still deserves to be considered, studied and developed.

Today, as everyone knows, computers enable us to create three-dimensional models of objects and of geometric shapes. They can also automatically generate the Mongian projections, and not only, of those objects. This technique is called ‘computer-aided design’ (CAD).

Nowadays, as in 1794, crowds of students attend our universities to learn the art of imagining the objects of the future: houses, furnishings, cities, machines. Now, as then, we are faced with the problem of providing them with theoretical and operating tools useful to practise this art of the invention and pre-figuration of space. Can the CAD take the place of descriptive geometry, or is it instead descriptive geometry that has to integrate the CAD among its tools? And, if we would like to carry out this integration, how could we realize it?

The tools of descriptive geometry

I think that the answer to the first question is clear. Computers are tools. They are sophisticated tools, but analogous to the straightedge and compass, which were, for years, the only mechanical tools admitted in the study of geometry. At this point two questions arise: the first concerns, in general, the role of the tools in geometry; the second whether

it is right to go beyond, once and for all, the constraint that was set by classical geometry on the exclusive use of straightedge and compass.

The role of the tools is tied to the experimental character which is present in geometry in general, and particularly in descriptive geometry. These sciences are founded on the vision and the graphical verification of the shapes while created and studied. When, within the context of an abstract reasoning, we introduce the idea of a right angle or of any shape whatsoever, such as a cone or a round hyperboloid, these ideas immediately give rise to the images that are connected to them. It is therefore impossible to reason about geometry without, at the same time, seeing with the mind what we are reasoning about. And what we imagine can also be drawn and seen clearly and shown to others. Naturally, the outcome of a graphical experience cannot, in itself, be a guarantee of scientific truth, which can only be obtained by means of a correct logic, but the graphical experience supports the reasoning and, above all, stimulates the reason with its allusions.

Monge himself, when defining the second aim of *Géométrie Descriptive*, which is that of studying the properties of the shapes, says that the geometric experience offers numerous examples of the passage 'from the known to the unknown' (*du connu à l'inconnu*).⁵ This affirmation, if we stop a moment to consider it, is surprising. Surprising, because we would expect that the graphic representation of a geometric idea is a way to change this from an embryonic condition of an intuition into the certainty of the image, namely, something that we can see and nearly touch. We would expect, therefore, a passage from the unknown to the known. Monge, instead, goes beyond this passage and highlights the heuristic character of the graphic experience, that is, the moment in which the genesis of the image, which forms itself right before our eyes, suggests, without making explicit, relations, properties and characteristics that the intuition did not suspect.

If it is therefore right to use the drawing tools in geometry, not only to show and to verify, but also to experiment, it is unavoidable to ask ourselves which tools should be allowable.

Now, as we already said, for centuries these tools were confined to the straightedge and compass. How can we explain this dogma of ancient and modern science? According to me, there is only one possible explanation: straightedges and compasses, for centuries, were the only tools able to guarantee an acceptable graphic accuracy, therefore, an acceptable experimental verification. If not, what other reasons could François Viète have had for rejecting, almost with contempt, the solution given by Adriaan Van Roomen to the Apollonian problem?⁶ And yet the solution given by Van Roomen (fig. 4) was simple and general, able to tackle with the same logic the series of complex cases of the problem, and lent itself to being extended to space.

But it had a fault: it made use of geometric loci, the conics, which could not at that time be drawn accurately. These same reasons suggested to Lorenzo Mascheroni a geometry completely solved using only the compasses, because, as he himself says in the preface of his work, *è quasi impossibile ch'essa [la riga] sia così dritta che ne garantisca per tutto il suo tratto della posizione a luogo de' punti, che in essa sono* (It is almost impossible that the rule is so similar to a straight line that it can guarantee that the points which lie on its edge all are aligned).⁷ We could go on at length, with these examples of 'experimental' geometry, and up until the present day: you only have to recall the works and the investigating techniques of H. S. M. Coxeter.⁸

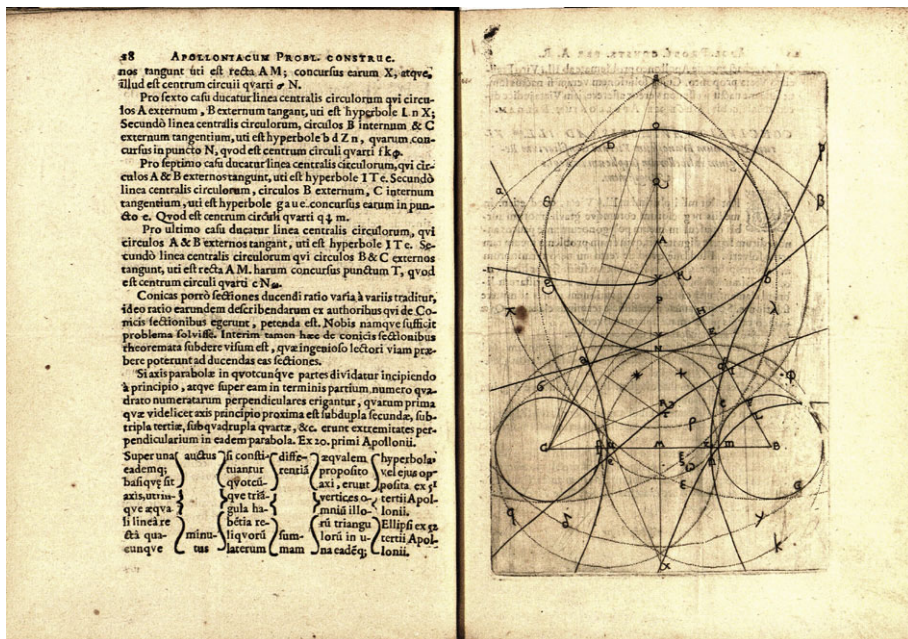


Fig. 4. Adriaan Van Roomen, *Problema Apolloniacum* (1596). In the figure on the right can be seen the hyperbolas that describe the geometric loci of the points equidistant from the given circles and that enable a simple solution of the problem

Well then, as we said, computers are tools. They are tools that, thanks to the synergy with the computation, are able to draw a straight line, a circle, the conics and even much more complex lines, all with the same accuracy. In applications commonly used in industry, this measurement accuracy is on the order of a micron. I would like to recall that traditional technical drawing can achieve, theoretically, the accuracy of two-tenths of a millimetre; thus computers have improved the accuracy of the experiments, which can be performed in geometry, by two size orders.

But, there is something more, because the analogical drawing can only draw lines on plane supports, whereas the digital drawing can draw lines and surfaces in space. Therefore, if (formulating a hypothesis out of its historical context) Piero had had a computer, he could have simplified the second part of his procedure a lot, drawing the projecting lines of the visual pyramid, each one with a single stroke in space. The first and last parts of Piero's experiment, instead, would have kept their laborious character, the first, because it deals with the construction of the capital, which is a problem of curve shapes and skew surfaces, connected by a delicate system of relations, the last, because it translates a discrete system – the point cloud – into a continuous system, with an evident contribution of the interpretation. In all these phases, the role of descriptive geometry is dominant, in spite of the aid of CAD systems, which are purely instrumental.

An outline of a new structure for descriptive geometry: the methods

If, as I believe, descriptive geometry is still the science of representation of space, and computers only a tool at its disposal, we should begin to wonder how the structure of the discipline can and should integrate the new techniques of experimental verification.

Classical descriptive geometry comprises three main parts: the methods of representation, the study of the surfaces, the applications. In each of these parts, the advent of digital drawing has an important role.

To the graphical methods already known, which are the method of Monge, axonometry, perspective and topographic projection, we have to add the digital methods, which are the 'mathematical representation' and the 'numerical representation'.⁹ In fact, if we consider the software dedicated to modelling, or better, to drawing in space, we can recognize two main procedures in representing three-dimensional shapes:¹⁰ the first uses equations and thus describes the curves and the surfaces with continuity; the second uses lists of coordinates of points and rules to connect them, and thus describes the surfaces in a discontinuous or discrete way, approximating them with a polyhedron. The mathematical representation is very accurate and is, for this reason, preferred when metric control of the shape is required. The numerical representation, on the contrary, is imprecise, but easy and quick, and this is why it is preferred when a direct, perceptive, control of the shape is required.

We should not confuse these two methods with the applications that employ them. As a matter of fact, all the applications use both methods, in different measures. For instance, the applications dedicated to industrial production mainly use the mathematical representation, but they generate a polygonal model (numerical) superimposed on the mathematical model, to enable its visualization. In fact, the GPUs (Graphic Processor Unit), which in the hardware are handling the graphics, are not able to process equations, but only numerical representations.

In their turn, the applications dedicated to rendering perspectives, the shadows and chiaroscuro, and to generating animations, mostly use numerical representation, but they also have some mathematical functionalities that make it possible to construct the shapes more rapidly, the basis for subsequent modelling operations.

It is easy to define an analogy between graphical representation methods and those of digital representation if we look not so much at the images that they produce as at the use that architects and artists generally make of them. In the case it is necessary to perform a verification of measurements on the shape, as for instance on the dimensions of an environment system, the architect works using plans and elevations; when instead he wishes to study the outcome in the synthesis of an overall perception, the architect uses perspective for the view from the inside and axonometry for the view from the outside of the planned volumes. We can therefore say that the mathematical representation is analogous to the associated orthogonal projections, because it enables the accurate control of the dimensions; while the numerical representation is analogous to perspective, because it enables accurate control of the view of the object.

Just as we teach and prove geometrically the rules necessary to represent on a plane a three-dimensional object, in a way that it can be re-constructed in space, so, in the descriptive geometry of the future, we can teach the rules necessary to represent in space an object using the descriptions, mathematical or simply numerical, that a machine is able to translate into images in real-time.

An outline of a new structure for descriptive geometry: the study of surfaces

In the past, classical descriptive geometry, when working on a plane, made use exclusively of straightedge and compass. For instance, the solution in space of the Apollonian problem was discussed, in 1812, by Louis Gaultier de Tours, in a *Mémoire* of more than hundred pages,¹¹ in which the theory of the geometric radicals was enunciated

for the first time. All this was intended to ensure that the restrictive use of the circle as a geometric locus is respected. But today, if we accept the use of the conics as well and, in space, the surfaces of revolution generated from them – namely, the hyperboloid, the paraboloid and the ellipsoid – then the Apollonian problem finds a general solution that can be outlined in a few pages.¹² This solution, moreover, has the merit that it can be really carried out, as far as to construct in space the spheres that touch four geometric entities *ad libitum* chosen among points, planes and spheres, as the statement of the problem calls for (fig. 5).

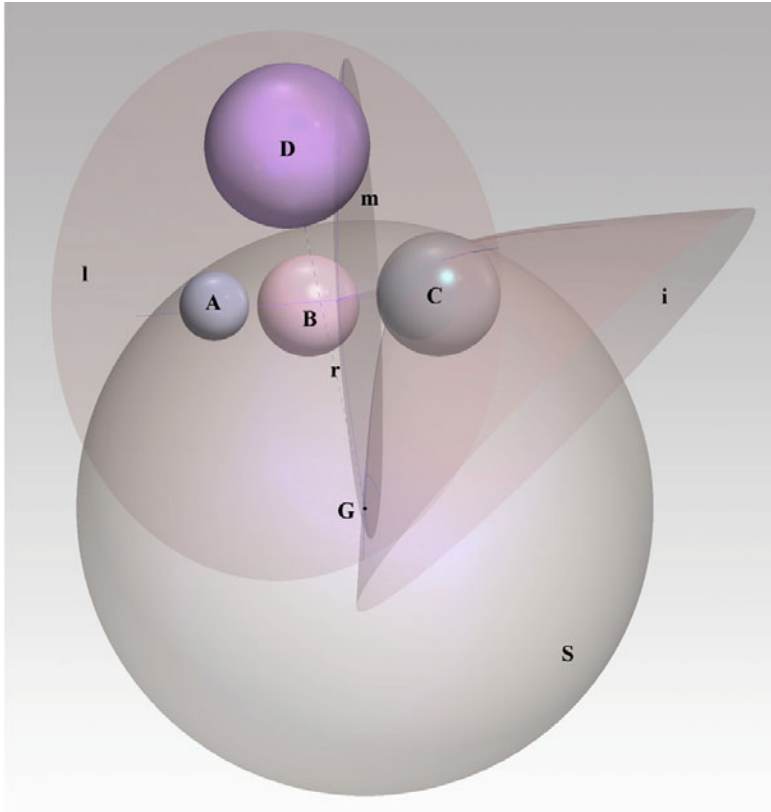


Fig. 5 – The solution of the Problem of Apollonius that uses the mathematical representation applying the method conceived, in the plane, by Adriaan Van Roomen. The sphere S , of radius r , touches the spheres A , B and C with the inner side of the surface, the sphere E with the outer side. The centre G is the common point to the hyperbolas intersection of the hyperboloids of revolution i , l , m . Sixteen solutions to the Problem are possible, but not all are always feasible

Naturally, the Apollonian problem is only one of the numerous examples that we might give of a new way of studying descriptive geometry, a way that uses digital compasses able to draw second-degree curves and surfaces in space.

But the use of the computer also offers other possibilities, which derive from the synergy between the graphic synthesis and the calculation. For instance, the possibility of calculating the centre of mass of a solid can be applied successfully to the construction of the axes of the quadric cone and to the shapes that have a cone-director, like the elliptic hyperboloid of two sheets.¹³ In fact, if we cut the cone with a sphere that has its centre in

the vertex, and we remove the part of the cone that is on the outside of the sphere, the straight line that passes through the vertex and through the barycentre of the shape is the first of the axes, whereas the other two are parallel to the axes of whichever of the ellipses that are obtained cutting the cone with a plane that is perpendicular to the first axis (fig. 6).

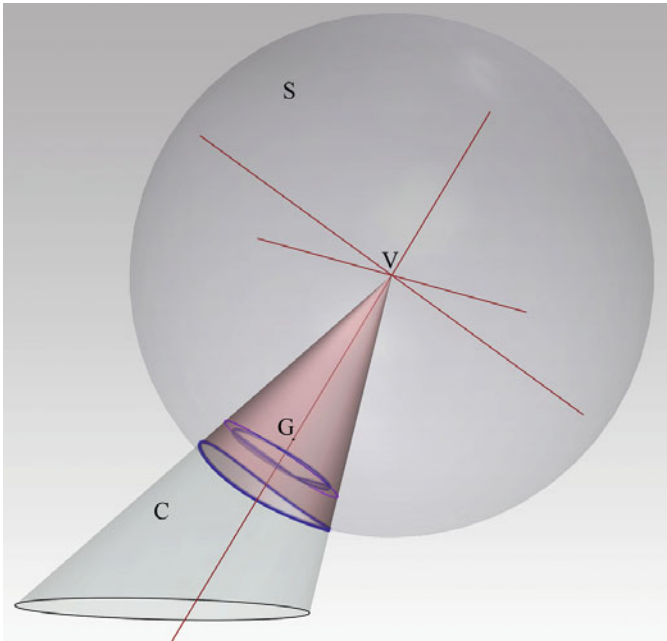


Fig. 6. The construction of the axes of the quadric cone that utilizes the barycentre G of the solid that is obtained by cutting the cone C with a sphere S centred in the vertex V

These functions make available, even at the lower levels of university education, constructions, verifications and concepts that, in the previous literature, are only developed in analytical and not in graphical form. The study of surfaces, which very profitably used the physical models in the past,¹⁴ can therefore count on virtual models today. Unlike physical models – static objects of visual and tactile perception –, virtual models enable all the operations of descriptive geometry, like section operations and geometric and projective transformations.

An outline of a new structure for descriptive geometry: the applications

Classical descriptive geometry has a wide range of applications, many of which are tied to the production of objects, others to the production of images, still others to the study of the history of art. In all these cases, the use of digital representation techniques has given interesting outcomes, for both industry and research. Here I only wish to give a few examples, from among the many that can be recalled.

Today, the study of the Gaussian curvature of the surfaces has accurate descriptions in false colours, which are applied to the control of the continuity of surfaces, within every field of industrial production. The construction of developable surfaces enables the realization of the new plastic forms of the buildings designed by Frank Gehry. Perspective is experiencing second youth in its dynamic and interactive formulation,¹⁵ in

which it is no longer the artist who chooses the point of view: it is the observer who changes it, continually exploring the illusory space.

Finally, we cannot forget the contributions that the new digital descriptive geometry has given, and continue to give, to the study of history, as, for instance, in the case of the Roman paintings of the first century, which are evidence of the knowledge of perspective of the ancients.¹⁶

Descriptive geometry and the education of designers

I do not think I have yet answered thoroughly the question posed at the beginning, namely: Is there still a future for descriptive geometry? Is it possible to give new life to this ancient science?

As a matter of fact, after a superficial review of the question, CAD would seem to be self-sufficient and therefore able to meet the needs of the science, of design and of production, even in the absence of a historical memory. After all, any student who starts to study a modelling application is able, after only a few days, to create three-dimensional shapes.

Indeed, he is able to create them, but not to control them. And it is not by chance that in this empirical approach numerical representation is preferred, with all its approximations. Like clay in a sculptor's hands, the shape represented numerically can be moulded without difficulty, but also without proportions, without measures, without generative laws, in a word, without geometry.

The genesis of a three-dimensional shape, above all when we deal with architecture, is very different. It requires a process that is orderly and guided by reason: the construction.

Let's imagine, for instance, a simple polyhedron like the dodecahedron: it can be constructed using the knowledge of the *mirabili effetti* described by Luca Pacioli,¹⁷ or also moving from the plane to space the development of six of its twelve faces, and then generating the others by symmetry. In both cases, the construction requires a knowledge of inner relations, of rotation operations, of projective relations, all of which belong to descriptive geometry and to its history, and which in no way can be substituted by CAD, because CAD is not a science, but a technique.

We can also mention Piero della Francesca's complex experiment. The construction of the capital is divided into steps that can only be accomplished by following the execution order and the inner relationships, as the stonemason roughly shapes the block of stone from which the capital is obtained, following a pre-determined order of gestures.¹⁸ The mathematical representation demands similar procedures, which cannot be learned from software manuals; they can be learned through the study of descriptive geometry, of its history and its applications.¹⁹

Last, but not least, is the problem of the paradigmatic and syntactic chaos that reigns today in the applications of digital representation. In fact, each of these applications, even if implementing, substantially, the same well-known algorithms, have different names and the commands are placed in different logical positions and in different hierarchies. First of all, this confusion involves a lot of wasted effort and time in order to switch from one software application to another. It also becomes impossible to rapidly compare the performances of the applications on the market. Worst of all, this confusion leads to the impression that the various CAD applications are quite dissimilar because they apply different theories, whereas, on the contrary, they all use the same methods, the ones we mentioned above.

This attitude of the producers, which evidently responds to market logics and market strategies, will endure until the users start to grant a privilege, as is only right, to those products in which it is easy to recognize that logical order and terminology that the history of descriptive geometry has consolidated, as the one which best responds to the needs of science and of art.

Conclusions

Descriptive geometry is considered by many to be an outdated science, perhaps also because it is confused with Monge's *Géométrie Descriptive*. Having removed this mistaken notion, there is still another, namely, that descriptive geometry is the science that teaches how to represent objects of three dimensions on a two-dimensional support. Descriptive geometry has this capability too, but it is only one among several. Indeed, descriptive geometry is, first of all, the science that teaches to construct shapes of three dimensions, by means of a graphic solution that simultaneously controls the metric, formal and perceptive aspects. If we agree on this definition, we can accept the idea that this science is still useful and that it is open to further development.

In the preceding pages, I tried to show that this renewal of the ancient science is possible. It is possible to augment the graphic methods based on central and parallel projections, adding the methods currently used in the digital representation, namely mathematical representation and numerical representation. It is possible and useful to develop the number and the quality of the geometric tools used in construction processes, from the straight line to the circle (straightedge and compasses), to the conics and the quadric surfaces. It is possible and useful to take advantage of the synergy between the synthesis of the images and the analysis of the calculation (as Monge already hoped for), introducing into the construction processes geometric loci whose use, in the past, was only hypothesized, such as, for instance, the barycentre of a solid. It is possible to reassess the wide field of the applications of descriptive geometry, obtaining innovative results, like interactive dynamic perspective. It is possible, and necessary, to normalize the paradigm and the syntagm of the terms that are used in the digital applications, so that it would no longer be as demanding as it is today to change from one system to another and thus make the most of the capabilities of each of them.

Most of all: it is possible to give descriptive geometry a future and to give the digital applications the dignity of the noble history that belongs to them.

Notes

1. See Théodore Olivier, *Additions au Cours de Géométrie Descriptive*, Paris, 1847, Préface, XV.
2. See [Chasles 1837: Note XXIII, Sur l'origine et le développement de la Géométrie descriptive].
3. In all, there exist seven manuscripts of Piero's treatise, which are kept, respectively, in Parma, Reggio Emilia, Milan (two), Bordeaux, London and Paris. The most famous printed edition is the one edited by Giusta Nicco Fasola, which appeared in two editions, in 1942 and in 1984. The Foundation Piero della Francesca is, at the present time, working on the National Critical Edition. See [Besomi, Dalai Emiliani and Maccagni 2009].
4. The term is in quotes because it does not come from Piero's language, even if it belongs to his geometrical conception. Here are meant the lines, perpendicular to the ground line, which pass through the projections, first and second, of a point in space. In Italian, *linee di richiamo* literally translates the expression *ligne de rappel* of the French School and refers to the use that is made of these lines in the constructions.
5. See [Monge 1798: Programme, p. 2].

6. After the publication of the *Mathematical Collections* written by Pappus of Alexandria, translated from Greek by Federico Commandino in 1588, François Viète challenged the mathematicians of the time to give a solution of the Apollonian problem, enunciated in the *Collections*: given three geometric entities chosen among points, straight lines or circles, construct the circumference (or the circumferences) that touches all of them. In 1596 Adriaan Van Roomen published a solution that used conics to construct the centres of the circumferences that met with the conditions posed by the statement. Viète replied in 1600, criticizing, in a very sarcastic tone, Van Roomen's solution. The reason for Viète's irony is in the fact that Van Roomen had failed to observe the rule which imposes the use, in the constructions, of solely the circle and the straight line. See [Viète 1600].
7. See [Mascheroni 1797: Prefazione, VII].
8. Ample evidence of Coxeter's way of working with models of any kind, including physical models, can be found in [Roberts 2006].
9. These denominations are commonly used within the ambit of information technology, but were only recently introduced into the field of descriptive geometry. See [Migliari 2009].
10. In order to avoid useless complications, I am intentionally not considering other methods that may seem to be a hybrid between these two, like the Subdivision Surfaces.
11. See [Gaultier de Tours 1812]. Gaultier's *Mémoire* has been recently analysed and discussed in [Fallavollita 2008].
12. See [Migliari 2008a, 2008b].
13. The procedure, simple and effective, has been found by Marta Salvatore during her studies on the prodromes of descriptive geometry in Amédée François Frézier. See [Salvatore 2008a, 2011].
14. See the catalogue of *Raccolte Museali Italiane di Modelli* [Palladino n.d.].
15. See [Migliari 2008c].
16. See [Migliari 2005a, 2005b].
17. This is the path followed, for instance, by Gino Fano in his *Lezioni di Geometri Descrittiva* [1925]. The procedure is simple and elegant in its graphic realization, but nearly unfeasible in the mathematical representation. In fact, the presence of functions that cannot be calculated exactly, like the square root of five, induces errors which, even if very small, are bigger than the tolerances of the most advanced systems. On the contrary, procedures that simulate the physical construction of the solid, like the one suggested by Gino Loria in his *Metodi Matematici* [1935], are very effective even in the digital field.
18. N. Asgari [1988] has conducted important studies on the algorithms for the working process, in a marble quarry, of the Corinthian capital. On this same topic see also Marco Greco's doctoral dissertation [1996].
19. The application of mathematical algorithms to the study of descriptive geometry and its history has produced many results of remarkable interest. Among these, I would like to recall Camillo Trevisan's studies on stereotomy, on the perspective of the ancients and on axonometry in the nineteenth century [2000a, 2000b, 2005].

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About the author

Riccardo Migliari was born in 1947 and graduated with a degree in Architecture in 1972. He is University Professor in Fundamentals and Applications of Descriptive Geometry since 1990 at the Faculty of Architecture at the 'La Sapienza' University in Rome. His teaching and researching career started right after university graduation and has continued uninterruptedly. He has taught at the Faculties of Architecture 'G. D'Annunzio' in Chieti and at the 'La Sapienza' and the 'Terza Università' in Rome. He also taught Rélevé Instrumental et Photogrammetrie Architecturale at the Post-graduate School of the École Polytechnique d'Architecture et Urbanisme in Algeria, within the International Co-operation Agreement. From 1995 to 2002 he coordinated the Post-graduate School in Architectural and Environmental Survey and Representation and directed the Laboratory of Close Range Photogrammetry at the Department of Representation and Instrumental Survey and a wide range of research activities carried out within this department. He is assiduously engaged in research, particularly in the areas of descriptive geometry and of representation and instrumental survey of architecture. He directed, as Scientific Manager, the architectural survey of the Coliseum in Rome during the preliminary studies for the restoration of the monument undertaken by the Archaeological Superintendence of Rome. From 2003 he has dealt in particular with the renewal of the studies on the scientific representation of space, within in the evolutionary picture of the descriptive geometry, from the projective theory to the digital theory and from the graphical applications to the digital modelling. Since 2008 he has coordinated the "National Laboratory for the Renewal of Descriptive Geometry". He is the author of approximately ninety publications, many of which are monographic. Some of his works can be found at <http://riccardo.migliari.it> and at <http://w3.uniroma1.it/riccardomigliari/Ref/Default.aspx>.