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The Golden Section

To Renaissance mathematician Luca Pacioli, it was the Divine Proportion. To German astronomer Johannes Kepler, it was a precious jewel. The only proportion to increase simultaneously by geometric progression and by simple addition, the Golden Section achieves unity among diverse elements in remarkably efficient ways. We explore the Golden Ratio $1:\phi$, also known as the Golden Mean, and its appearance in the regular pentagon and other geometric constructions.

I Introduction

To Renaissance mathematician Luca Pacioli, it was the Divine Proportion. To German astronomer Johannes Kepler it was a precious jewel.[1] The only proportion to increase simultaneously by geometric progression and by simple addition, the Golden Section achieves unity among diverse elements in remarkably efficient ways. Though not without its detractors, its appearance has been observed in nature, design and architecture; from Egyptian pyramids and the Parthenon of ancient Greece to Le Corbusier's Modular system; from sunflowers and daisies in the plant world to spiral shells beneath the seas.

Definition:

When a line is divided into two unequal parts (*a* and *b*) such that the shorter part relates in length to the longer part in the same way as the longer part relates to the whole ($a:b :: b:a + b$), the result is a **Golden Section**:[2]

SHORTER:LONGER :: LONGER:SHORTER + LONGER (or WHOLE)

$a : b :: b : a + b$

The only ratio capable of generating proportions consisting of just two terms (*a* and *b*), the Golden Section is signified by the Greek letter *phi* or ϕ (= $\sqrt{5}/2 + 1/2$), after the Greek sculptor Phidias, and translates numerically to an incommensurable ratio of 1:1.618034..... or $1:\phi$. The reciprocal, $1/\phi$ ($\sqrt{5}/2 - 1/2$), equals 0.618034….

The ϕ number series $1/\phi^3, 1/\phi^2, 1/\phi, 1, \phi, \phi^2, \phi^3$... increases simultaneously by geometric proportion ($1/\phi : 1 :: 1 : \phi$) and by simple addition ($1/\phi + 1 = \phi$).

$$1/\phi^3,\ 1/\phi^2,\ 1/\phi,\ 1,\ \phi,\ \phi^2,\ \phi^3...$$

$$.236…,\ .382…,\ .618...,\ 1,\ 1.618...,\ 2.618...,\ 4.236….$$

The Golden Section, ϕ, is also called the Golden Ratio, the Golden Mean and the "extreme and mean" ratio.[3]

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II The pentagon

ϕ is an incommensurable number that cannot be stated as a whole number fraction, but appears with absolute precision in the geometry of a pentagon.

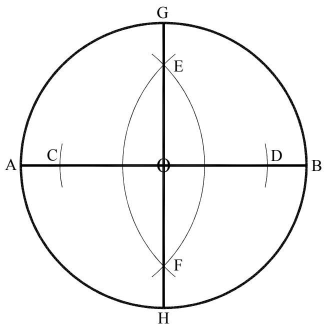

Fig. 1

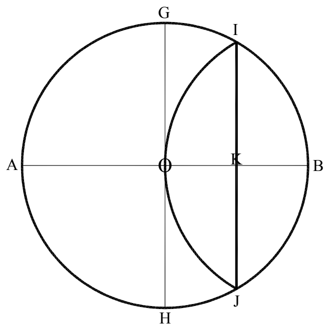

Fig. 2

- With a compass, draw a circle.
- Draw the horizontal diameter AB through the center of the circle.
- Set the compass at an opening that is slightly smaller than half the radius of the circle. Place the compass point at the center of the circle (point O). Draw arcs that cross the horizontal diameter on the left and right, at points C and D.
- Set the compass at an opening that is slightly larger than before. Place the compass point at C. Draw an arc above and below, as shown.
- With the compass at the same opening, place the compass point at D. Draw an arc above and below, as shown.
- Locate points E and F where the two arcs intersect.
- Draw the line EF through the center of the circle.
- Extend the line EF in both directions to the circumference of the circle (points G and H).

Lines AB and GH locate the horizontal and vertical diameters of the circle (fig. 1).

***

- Locate point B at the right end of the horizontal diameter (AB).
- Place the compass point at B. Draw an arc of radius BO that intersects the circle at points I and J.
- Connect points I and J.
- Locate point K, where the line IJ intersects the horizontal diameter (AB).

Point K divides the radius OB in half (fig. 2).

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- Connect points K and G.

- Place the compass point at K. Draw an arc of radius KG that intersects the horizontal diameter (AB) at point L (fig. 3).

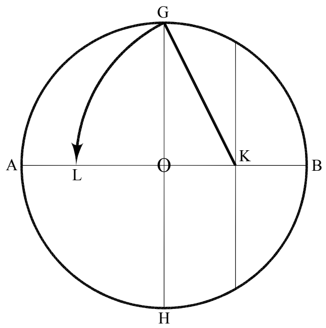

Fig. 3

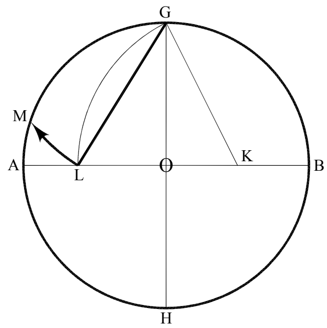

Fig. 4

- Connect points G and L.
- Place the compass point at G. Draw an arc of radius GL that intersects the circle at point M, as shown (fig. 4).

***

- Connect points G and M.

Line GM locates the side of a regular pentagon inscribed within the circle.

- Place the compass point at M. Draw an arc of radius MG that intersects the circle at point N, as shown.
- Place the compass point at N. Draw an arc of the same radius that intersects the circle at point P.
- Place the compass point at P. Draw an arc of the same radius that intersects the circle at point Q.
- Connect the five points G, M, N, P and Q.

The result is a regular pentagon inscribed within the circle (fig. 5).

***

- Draw the diagonals NG and PG within the pentagon.

If the side (PN) of the pentagon is 1, the diagonal (NG) equals ϕ (= $\sqrt{5}/2+1/2$ or 1.618034...).

The side and diagonal of any regular pentagon are in the ratio 1:ϕ (fig. 6).

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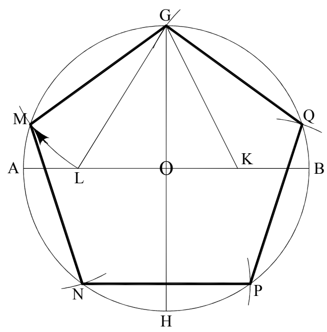
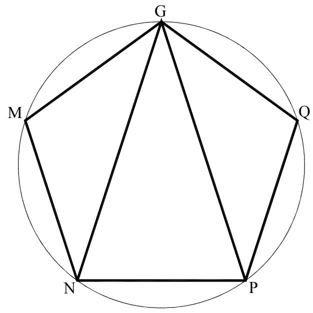




PN : NG :: 1 : Φ

Fig. 5 Fig. 6

Definition:

The Greek for "pentagon" is *pentagônon* (from *penta-* "five" + *gônia* "angle") [Liddell 1940, Simpson 1989].

Analysis by the Pythagorean Theorum. To understand the pentagon's inherent ϕ proportions, we must consider the ½:1 or 1:2 right triangle that initiates its construction.

- Locate the right triangle GOK.

The side GO coincides with the radius of the original circle and equals 1. The side OK coincides with half the radius and equals ½ .

By the Pythagorean Theorem,

$GO^2 + OK^2 = KG^2$
$1^2 + (½)^2 = (5/4) = KG^2$
$KG = \sqrt{5}/2$ (fig. 7.)[4]

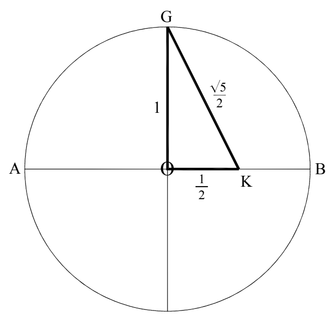


Fig. 7

- Place the compass point at K. Draw an arc of radius KG that intersects the horizontal diameter (AB) at point L, as shown.

$KG = KL = \sqrt{5}/2$
$OL = KL – KO = \sqrt{5}/2 - ½ = 1/\phi$
$BL = KL + BK = \sqrt{5}/2 + ½ = \phi$
$LA = OA – OL = 1 – (\sqrt{5}/2 - ½) = (3 - \sqrt{5})/2 = 1/\phi^2$
LA:OL :: OL:OG :: OG:LB
$1/\phi^2 : 1/\phi :: 1/\phi : 1 :: 1 : \phi$

- Locate the right triangle LOG.

The side LO equals $1/\phi$. The side OG equals 1.

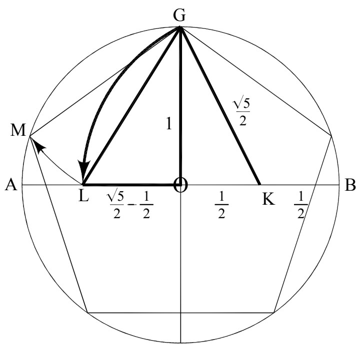
By the Pythagorean Theorem,

$LO^2 + OG^2 = GL^2$

$(1/\phi)^2 + (1)^2 = (1/\phi^2 + 1) = GL^2$

$GL = \sqrt{(1/\phi^2 + 1)}$

The hypotenuse GL equals the side (GM) of the pentagon inscribed within the original circle (fig. 8).



LA : OL :: OL : OG :: OG : LB
$\frac{1}{\Phi^2} : \frac{1}{\Phi} :: \frac{1}{\Phi} : 1 :: 1 : \Phi$

Fig. 8

III Golden ratios in the pentagon

Each segment of a regular pentagonal system relates to the others according to a variable of ϕ.

- Draw the pentagon (GMNPQ) and its five diagonals.

If the side (PN) of the pentagon is 1, the diagonal (NG) equals ϕ.

But if the segment SP equals 1:

$RS = 1/\phi$
$PN = \phi$

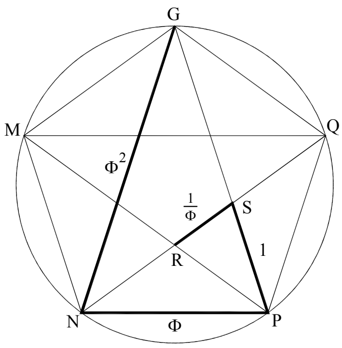
$NG = \phi^2$

The segments increase simultaneously by geometric proportion and by simple addition (fig. 9).

RS:SP :: SP:PN :: PN:NG
$1/\phi : 1 :: 1 : \phi :: \phi : \phi^2$
.618...:1 :: 1:1.618... :: 1.618…: 2.618....

At the same time,

$RS + SP = PN$
$1/\phi + 1 = \phi$
$.618\ldots + 1 = 1.618\ldots$
$SP + PN = NG$
$1 + \phi = \phi^2$
$1 + 1.618\ldots = 2.618\ldots$



RS : SP :: SP : PN :: PN : NG

$\frac{1}{\Phi} : 1 :: 1 : \Phi :: \Phi : \Phi^2$

RS + SP = PN

$\frac{1}{\Phi} + 1 = \Phi$

Fig. 9

IV The Golden Triangle

Definition:
The **Golden** or **Sublime Triangle** is an isosceles triangle formed by two diagonals and one edge of a regular pentagon. It contains one 36° angle and two 72° angles. The Golden Triangle divides into a 108°-36°-36° isosceles triangle and a reciprocal that is proportionally smaller in the ratio $1 : 1/\phi$.[5]

- Draw the pentagon (GMNPQ) and its five diagonals.
- Locate the Golden 36°-72°-72° triangle GPN.
- Locate the line PT, as shown.

Line PT divides the major triangle (GPN) into a 108°-36°-36° triangle (TGP) and a reciprocal (PNT) that is proportionally smaller in the ratio 1 : $1/\phi$.

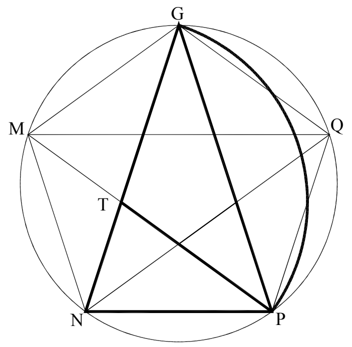


Fig. 10

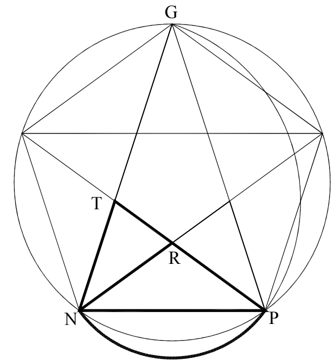


Fig. 11

- Place the compass point at T. Draw an arc of radius TG that intersects the line GP at points G and P (fig. 10).

***

- Locate the Golden 36°-72°-72° triangle PNT.
- Locate the line NR, as shown.

Line NR divides the major triangle (PNT) into a 108°-36°-36° triangle (RPN) and a reciprocal (NTR) that is proportionally smaller in the ratio 1 : $1/\phi$.

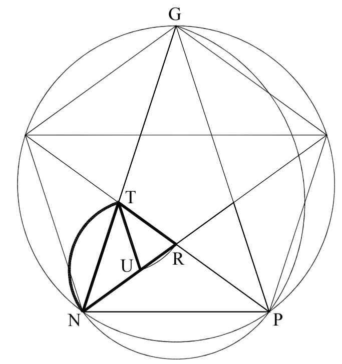


Fig. 12

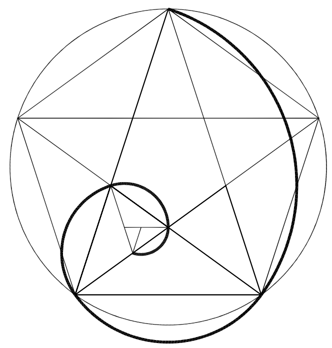


Fig. 13

- Place the compass point at R. Draw an arc of radius RP that intersects the line PN at points P and N (fig. 11).

***

- Locate the Golden 36°-72°-72° triangle NTR.

- Place the compass point at T. Draw an arc of radius TR that intersects the line RN at point U, as shown.
- Locate the line TU.
- Place the compass point at U. Draw an arc of radius UN that intersects the line NT at points N and T (fig. 12).

***

- Repeat the process continually, as shown.

The result is an equiangular spiral in the ratio 1 : ϕ (fig. 13).

V How to draw a Golden Mean rectangle

The Golden Mean proportion may take many forms and expressions. For example, the diagonal of half a square yields a rectangle in the ratio 1:ϕ.

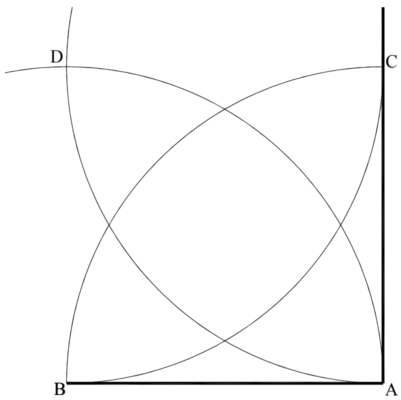


Fig. 14

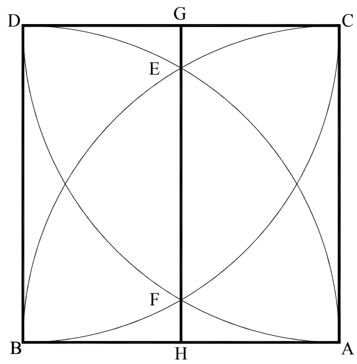


Fig. 15

- Draw a horizontal baseline AB equal in length to one unit.
- From point A, draw an open-ended line perpendicular to line AB, that is slightly longer in length.
- Place the compass point at A. Draw a quarter-arc of radius AB that intersects the line AB at point B and the open-ended line at point C.
- Place the compass point at B. Draw a quarter-arc (or one slightly longer) of the same radius, as shown.
- Place the compass point at C. Draw a quarter-arc (or one slightly longer) of the same radius, as shown.
- Locate point D, where the two quarter-arcs, taken from points B and C, intersect.
- Place the compass point at D. Draw a quarter-arc of the same radius that intersects the line CA at point C and the line BA at point B (fig. 14).

***

- Connect points B, D, C and A.

The result is a square (BDCA) of side 1.

- Locate points E and F where the quarter-arcs intersect.
- Draw the line EF.
- Extend the line EF in both directions to points G and H on the square.

Line GH divides the square (BDCA) in half (fig. 15).

***

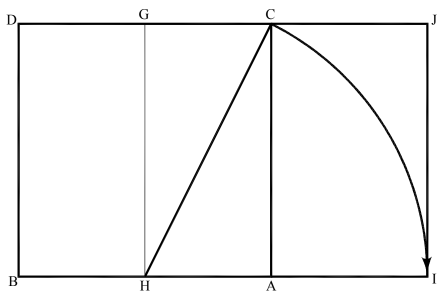
- Connect points H and C.
- Place the compass point at H. Draw an arc of radius HC that intersects the extension of line BA at point I.
- From point I, draw a line perpendicular to line IB that intersects the extension of line DC at point J.

The rectangle (JIBD) that results is in the ratio 1:ϕ.[6]

JI:IB :: 1:ϕ

The major 1:ϕ rectangle (JIBD) divides into a square (DCAB) and a reciprocal (CJIA) that is proportionally smaller in the ratio 1 : 1/ϕ.[7]

The long side (IB) of the major 1 : ϕ rectangle (JIBD) equals the sum of the short (CJ) and long (JI) sides of the reciprocal (CJIA) (fig. 16.).



CJ : JI :: JI : IB

$\frac{1}{\Phi}$: 1 :: 1 : Φ

CJ + JI = IB

$\frac{1}{\Phi} + 1 = \Phi$

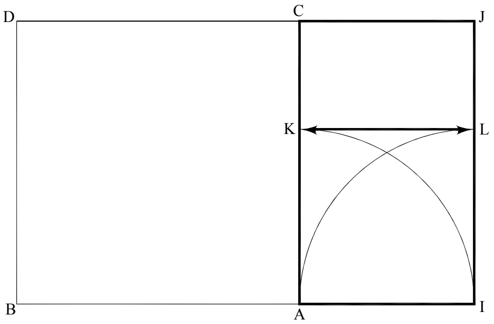
Fig 16

VI The rectangle of whirling squares

- Place the compass point at A. Draw a quarter-arc of radius AI that intersects the line CA at point K.

- Place the compass point at I. Draw a quarter-arc of radius IA that intersects the line JI at point L.
- Connect points K and L.

The line KL divides the major $1{:}\phi$ rectangle (CJIA) into a square (AKLI) and a reciprocal (KCJL) that is proportionally smaller in the ratio $1 : 1/\phi$ (fig. 17).



KC : CJ :: CJ : JI :: JI : IB

$\frac{1}{\Phi^2} : \frac{1}{\Phi} :: \frac{1}{\Phi} : 1 :: 1 : \Phi$

Fig. 17

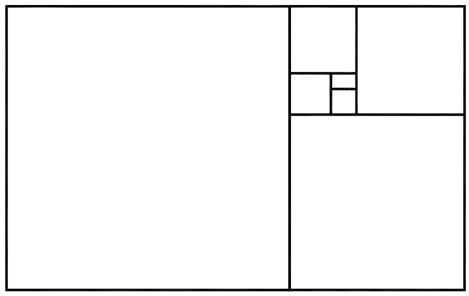


Fig. 18

- Repeat the process continually, as shown.

The result is rectangle of whirling squares (fig. 18).

***

- Place the compass point at C. Draw a quarter-arc of radius CD that intersects the line BI at point A.
- Place the compass point at K. Draw a quarter-arc of radius KA that intersects the line IJ at point L.
- Place the compass point at M. Draw a quarter-arc of radius ML that intersects the line JD at point N.
- Repeat the process continually to reveal a spiral of quarter-arcs whose radii decrease in the ratio $1 : 1/\phi$.

The quarter-arcs decrease toward a fixed point of origin (the pole or eye), but never touch it (fig. 19).

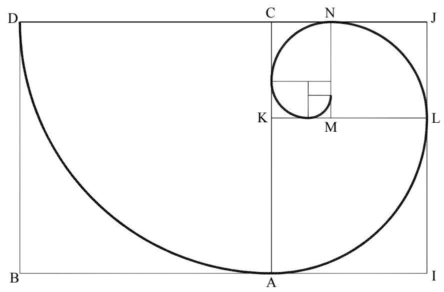
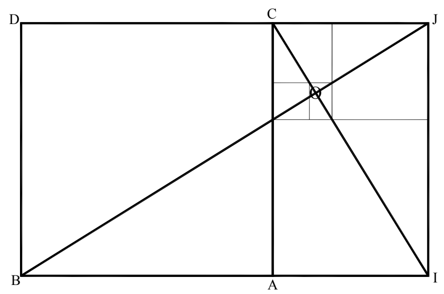


Fig. 19

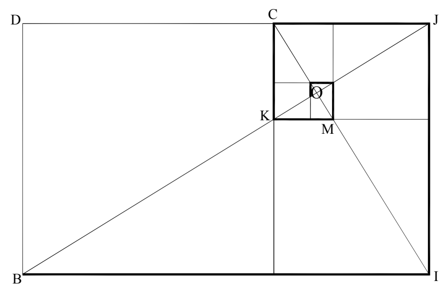
- Draw the diagonal BJ of the major rectangle JIBD
- Draw the diagonal IC of the reciprocal CJIA.

The diagonals (BJ and IC) intersect at 90° at point O (fig. 20).



IC : BJ :: 1 : Φ

Fig. 20



OK : OC :: OC : OJ :: OJ : OI :: OI : OB :: 1 : Φ

MK : KC :: KC : CJ :: CJ : JI :: JI : IB

$\frac{1}{\Phi^3} : \frac{1}{\Phi^2} :: \frac{1}{\Phi^2} : \frac{1}{\Phi} :: \frac{1}{\Phi} : 1 :: 1 : \Phi$

MK + KC = CJ

$\frac{1}{\Phi^3} + \frac{1}{\Phi^2} = \frac{1}{\Phi}$

Fig. 21

- Locate the equiangular spiral of straight-line segments MK, KC, CJ, JI and IB.
- Locate the pole or eye of the spiral at the intersection of the diagonals BJ and IC (point O).
- Locate the spiral's radii vectors OK, OC, OJ, OI and OB.[8]

The radii vectors are separated by equal angles (90°). Their lengths increase in the ratio $1{:}\phi$. The sum of two adjacent radii vectors equals the length of the next larger vector.

The equiangular spiral BIJCKM decreases in the ratio $1 : 1/\phi$ toward a fixed point of origin (the pole at point O), but never touches it (fig. 21).

VII How to divide a line in Golden Section

In previous examples, the ϕ ratio was obtained by adding a new length of $1/\phi$ to a line of one unit. In this construction, the line of one unit is divided, in the ratio $1/\phi^2 : 1/\phi$ or $1 : \phi$.

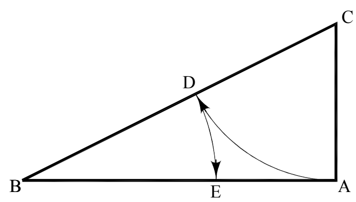
- Draw a baseline (AB) equal to 1.
- From point A, draw a line AC perpendicular to line AB, equal to half the length of AB.
- Connect points C and B.

The result is a right triangle CAB with short and long sides in the ratio ½ :1 or 1:2.

- Place the compass point at C. Draw an arc of radius CA that intersects the hypotenuse BC at point D.
- Place the compass point at B. Draw an arc of radius BD that intersects the long side AB at point E.

Point E divides the side AB in Golden Section

If the line AB is 1, segments AE and EB equal $1/\phi^2$ and $1/\phi$, respectively (fig. 22).[9]



$$AE : EB :: EB : AB$$

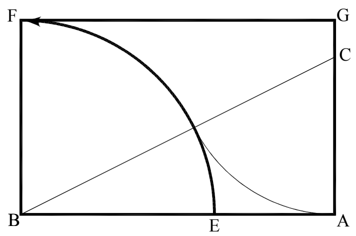
$$\frac{1}{\Phi^2} : \frac{1}{\Phi} :: \frac{1}{\Phi} : 1$$

$$AE + EB = AB$$

$$\frac{1}{\Phi^2} + \frac{1}{\Phi} = 1$$

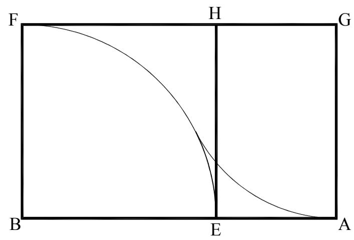
Fig. 22

- Place the compass point at B. Draw an arc of radius BE that intersects a line drawn from point B, perpendicular to line BA, at point F.
- From point F, draw a line perpendicular to line FB that intersects the extension of line AC at point G.
- Connect points G, A, B and F.



GA : AB :: $\frac{1}{\Phi}$: 1

Fig. 23



HG : GA :: GA : AB

$\frac{1}{\Phi^2} : \frac{1}{\Phi} :: \frac{1}{\Phi} : 1$

HG + GA = AB

$\frac{1}{\Phi^2} + \frac{1}{\Phi} = 1$

Fig. 24

The result is a $1/\phi$:1 or 1:ϕ rectangle (GABF) (fig. 23).

***

- From point E, draw a line perpendicular to line BA that intersects the line FG at point H.

The major $1/\phi$: 1 rectangle (GABF) divides into a square (FHEB) and a reciprocal (HGAE) that is proportionally smaller in the ratio 1 : $1/\phi$ (fig. 24).

VIII How to draw a pentagon from a square

In this construction, we draw a regular pentagon from a square of side 1.[10]

- Draw a square (ABCD) of side 1.
- Place the compass point at A. Draw a semicircle of radius AB that intersects the extension of line BA at point E.
- Place the compass point at B. Draw a semicircle of the same radius that intersects the extension of line AB at point F (fig. 25).

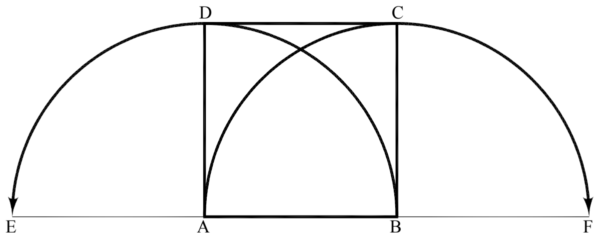


Fig. 25

- Place the compass point at C. Draw a quarter-arc of radius CB that intersects the square at points B and D.
- Place the compass point at D. Draw a quarter-arc of the same radius that intersects the square at points A and C.
- Locate points G and H where the quarter-arcs and semicircles intersect, as shown.
- Draw the line GH.
- Extend the line GH in both directions to points I and J on the square.

Line IJ divides the square (ABCD) in half (fig. 26).

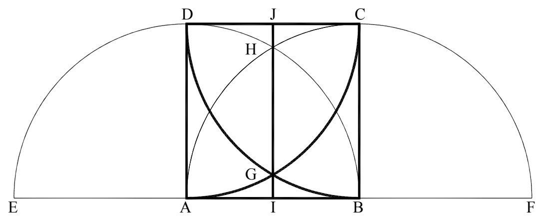


Fig. 26

- Connect points I and C.
- Draw a semicircle of radius IC that intersects the line EF at points K and L (fig. 27).

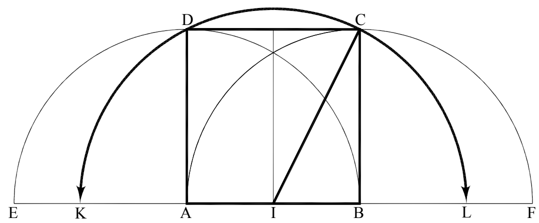


Fig. 27

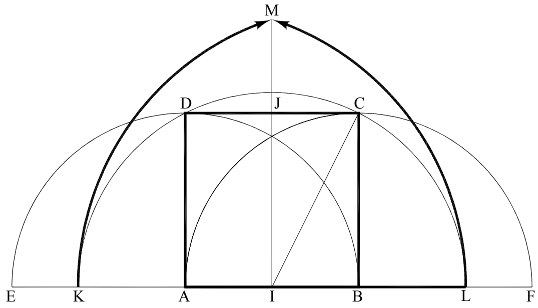


Fig. 28

- Place the compass point at A. Draw an arc of radius AL that intersects the extension of line IJ at point M.
- Place the compass point at B. Draw an arc of the same radius that intersects the extension of line IJ at point M (fig. 28).

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- Locate points N and O, as shown.
- Connect points A, B, O, M and N.

The result is a regular pentagon whose side of 1 equals the side of the square (ABCD) (fig. 29).

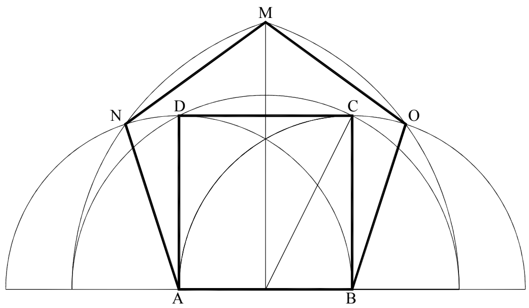
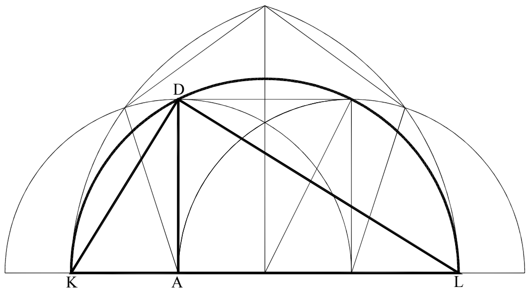


Fig. 29.

Analysis by the Theorum of Thales. The construction for drawing a pentagon from a square is based on the ϕ relationships that result when a square is inscribed within a semicircle.[11] The construction further demonstrates the Theorem of Thales; that any triangle inscribed within a semicircle is right-angled.[12]

- Locate the semicircle drawn on the diameter KL.
- Locate points K, D and L.
- Connect points K, D and L.

The result is a right triangle inscribed within the semicircle.



$$AK : AD :: AD : AL$$
$$\tfrac{1}{\Phi} : 1 :: 1 : \Phi$$

Fig. 30

The Theorem of Thales states that within a semicircle, a perpendicular line (DA) drawn from any point (D) along the perimeter to the diameter is the mean proportional or geometric mean of the two line segments (AK and AL) that result on the diameter (fig. 30).

IX History of the Golden Ratio

The "extreme and mean" ratio appears as early as Euclid, if not before, and is recognized as a mathematical principle by art and architectural theorists of the Renaissance such as Leon Battista Alberti, Sebastiano Serlio, Albrecht Dürer and Luca Pacioli.[13] But the origin and history of its actual use in art and architecture are rigorously debated. Opponents are careful to distinguish ϕ as a mathematical principle from its design application. Marcus Frings [2002] and others argue that the "extreme and mean" ratio does not appear in Vitruvius's canon of proportion, and therefore architectural theorists of the Renaissance who rediscovered Vitruvian principles are unlikely to have adopted it. Pacioli's name for ϕ is *Divina proportione*, the title of his 1509 treatise where, in the first book (*Compendium de divina proportione*), he discusses the philosophical and theological aspects of the ratio. But the second book, *Tractato de l'architectura*, a treatise on architecture, does not advocate its use in design practice. Few dispute that the Golden Section has appeared architecturally for aesthetic purposes since the mid-nineteenth century, when Adolf Zeising and Gustav Theodor Fechner introduced it to architectural theory [Frings 2002, 9-20; Herz-Fischler 1998, 149-151, 171-172; March 2001, 85-86; Padovan 1999, 304-308; Scholfield 1958, 98-99].

And yet, claims for the Golden Section have been made in architectural works from prehistory, including Neolithic stone circles, Egyptian pyramids, the Parthenon of ancient Greece, and at least one Palladian villa [Critchlow 1982, 87; Fletcher 1995, 9, 17, 23; Fletcher 2000, 73-85; Hambidge 1924, xvi-xvii, 1, 7]. Since the Renaissance, humanists and builders have published exact geometric constructions in art, architectural and building treatises, including, in the sixteenth century, drawings for a pentagon in Dürer's *Underweysung der messung* (*The Painter's Manual*) and in Serlio's compendium on geometry in *Trattato di architettura* (*On Architecture*). A related construction for a decagon appears in Alberti's fifteenth-century *De re aedificatoria* (*On the Art of Building in Ten Books*). In the eighteenth century, Peter Nicholson, Batty Langley, Sébastien Le Clerc and others illustrated exact constructions in manuals for architects and builders [Dürer 1977: II, 144-146; Serlio 1996: I, 29 (fol. 20); Alberti 1988; VII, 196; Le Clerc 1742, 112-3, 180-1; Langley 1726, 11 and pl. 1, fig. XXX; Nicholson 1809, 14 and pl. 13].

X Symbolism of the pentad

In a previous column, we introduced the numbers 1, 2, 3 and 4 of the tetractys, noting their connection to the archetypes of Monad, Dyad, Triad and Tetrad and the qualities of Unity, Multiplicity, Harmony and Body or Form [Fletcher 2005b, 177]. The Pythagorean tradition of identifying numbers with qualities and values extends to the number "five," or Pentad, whose names include Wedding, Marriage, Justice and Light.[14]

"Five" is a circular number and a spherical number, returning to itself in the last digit when raised to the second and third powers (5 x 5 = 25 and 5 x 5 x 5 = 125). The pentad invokes the circle in another way, as the center point that unites the four cardinal directions. The Pythagoreans associated the pentad with the immutable fifth element of ether that comprehends the other four elements. Likewise the fifth regular solid, or dodecahedron, stands for the zodiac and totality.

One name for the pentad is Marriage because it unites the first distinct "species of numbers"—the triad ("three"), which is the first odd or masculine number, and the dyad ("two"), which is the first even or feminine number. The pentad may symbolize the hierogamy, or sacred marriage, of heaven and earth. Another name for the pentad is Justice because, as the middle of the numbers in the decad (1, 2, 3, 4, 5, 6, 7, 8, 9) it achieves equality and balance [Taylor 1972, 189-191].

In a previous column, we equated the hexad, or number "six," with the great world or macrocosm of the universe [Fletcher 2005a, 143]. In similar fashion, the pentad symbolizes the microcosm of humanity or man. The pentagon may represent the human figure with head upright and arms and legs outstretched, while "five" is often associated with the digits of the human hand. Karl Menninger notes that the connection between the number "five" and the "fingers" and "hand" is cross-cultural. The Gothic for "five" is *fimf* and is likely related to the Gothic for "finger," which is *figgrs*. The Slavic for "five" is *pētj* and is likely related to the Slavic for "fist," which is *pēsti*. The Egyptian word for "five" is the same as for "hand" [Menninger 1977, 148-149, Simpson 1989].

XI The Golden Ratio in nature and human body

The Golden Ratio is often linked with the growth of living organisms and has been observed in numerous living forms, including the human anatomy. Allowing for unique and individual differences, the overall proportions of the human face generally conform to a Golden Mean rectangle, while the fingers divide at the joints in ϕ progression (1, ϕ, ϕ^2, ϕ^3...).

Some believe that the tradition of rendering the human body according to ϕ dates from ancient Egypt through modern times with Le Corbusier's Modulor system.[15] Generally speaking, at birth, the navel divides an infant's total height in half. As the infant develops, the navel appears to "rise," eventually marking a Golden Section in the adult male figure. Meanwhile, the half division "lowers" to the place of the sexual organs. But the location of the groin can produce a new ϕ division, when the arms are raised directly overhead.

From these two ϕ divisions—of one's height from head to toe, at the navel; and of one's height with arms directly overhead, at the groin—Le Corbusier derives two intertwined "red" and "blue" scales that comprise the Modulor. The system develops from a square, whose side of 1 equals the height from the floor to a person's navel; and from a double square, whose long side of 2 equals the person's total height with arms raised overhead. The "red" series develops by adding a Golden Section ($1/\phi$) to the square of side 1, locating the top of the head. The "blue" series develops by subtracting a Golden Section ($1/\phi$) from the long length of the double square, locating the groin. The Modulor is expressed in whole numbers and may be adapted to a person of any height or applied to any design situation [Le Corbusier 1980, 50-58, 63-67] (fig. 31).

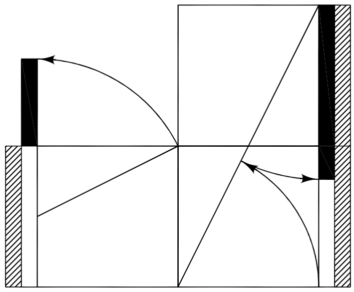
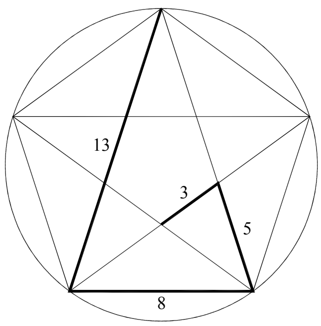


Fig. 31. Le Corbusier's Modulor

XII The Fibonacci number series

The thirteenth-century Italian mathematician Leonardo of Pisa, who is also called Fibonacci, popularized the Hindu–Arabic decimal system first introduced to the West by al-Khwarizmi. But Fibonacci is best known for the whole number series that bears his name. Fibonacci numbers simulate a true ϕ progression, increasing by geometric proportion and simple addition, simultaneously.[16]



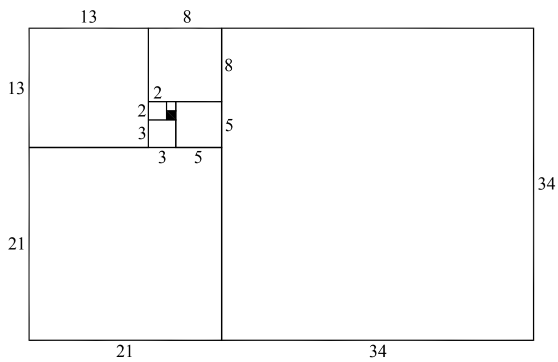


Fig. 32

Fig. 33

The Fibonacci number series reads: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144....

Similar to a true ϕ progression, each number is the sum of the preceding two: 1+1=2; 1+ 2=3; 2+3=5; 3+5=8; 5+8=13… .

At the same time, each successive ratio of adjacent numbers oscillates first above, then below the true value of ϕ (1.618034…).

13/8 (1.62500) is greater than ϕ.
21/13 (1.61538) is less than ϕ.

34/21(1.61904) is greater than ϕ, but closer in value.
55/34 (1.61764) is less than ϕ, and closer still.

The larger the numbers, the more accurately the ratios they form approximate the true value of ϕ. Thus, Fibonacci numbers may represent the segments of a regular pentagonal system (fig. 32) or a rectangle of whirling squares (fig. 33).[17]

Fibonacci numbers can only approximate the incommensurable value of ϕ, but Theodore Cook observes that a double Fibonacci series expresses an exact ϕ progression [Cook 1979, 420]:

$$\phi^2 = 1+\phi$$
$$\phi^3 = 1+2\phi$$
$$\phi^4 = 2+3\phi$$
$$\phi^5 = 3+5\phi$$
$$\phi^6 = 5+8\phi$$

Fibonacci numbers regularly appear in the natural world, as in the display of petals about the center of flowers. The primrose has 5 petals; the ragwort, 13 petals; the daisy, 21 or 34 petals; and the Michelmas daisy, 55 or 89 petals, all Fibonacci numbers.

The seeds of a sunflower arrange about the center in a pattern of opposing sets of curves. In some varieties, 34 long curves spiral clockwise, while 55 short curves spiral in the opposite direction. In others, 55 and 89 curves spiral to the right and left, respectively. When the pattern reaches its spatial limit, a new arrangement of opposing curves may appear, but it, too, follows the Fibonacci sequence. Thus, a sunflower that begins with 34 and 55 curves about the center may change to a new arrangement of 55 and 89.

XIII Application: common grass-of-Parnassus (Parnassia palustris)

A photograph by Karl Blossfeldt presents a common grass-of-Parnassus (*Parnassia palustris*) with the outer pieces removed, leaving the inner parts of the flower to display a complex pattern of five-fold symmetry (fig. 34).

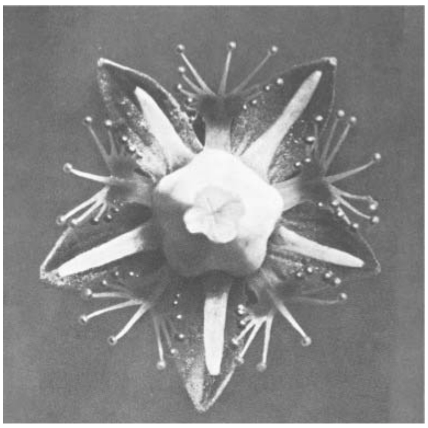


Fig. 34

No single flower conforms exactly to an idealized geometric pattern, but the common grass-of-Parnassus expresses the dynamic symmetry of pentagons with remarkable precision (fig. 35a-d).[18] Whether such geometry is inherent in nature or merely the product of human perception, it reaches beyond the rigid perfection of universal principles to rich and subtle methods of applying abstract rules.

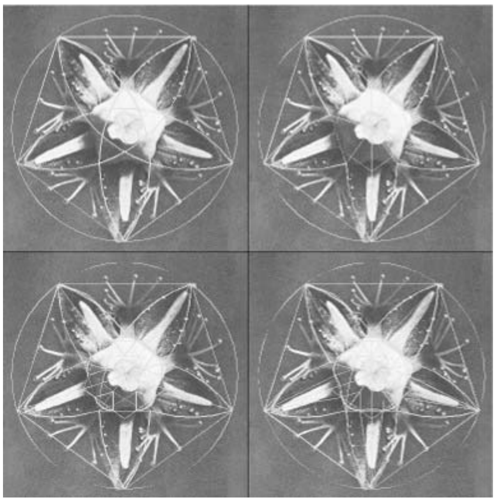


Fig. 35a-d. Photograph: © 2005 Karl Blossfeldt Archiv / Ann u. Jürgen Wilde, Köln / Artists Rights Society (ARS), New York. Geometric overlays by Rachel Fletcher. *Reproduction, including downloading of Karl Blossfeldt's wordks is prohibited by copyright laws and international conventions without the express written permission of Artists Rights Society (ARS), New York.*

Notes

1. Kepler's *Mysterium Cosmographicum* states that "there are two treasure houses of geometry: one, the ratio of the hypotenuse in a right-angled triangle to the sides [the Pythagorean Theorem], and the other, the line divided in the mean and extreme ratio…. The former…can rightly be compared to a mass of gold: the second, on proportional division, can be called a jewel" [Kepler 1981, 133, 143].
2. If the whole is equal to 1, the proportion translates to [$1/\phi^2 : 1/\phi :: 1/\phi : (1/\phi^2 + 1/\phi)$ or 1]. The Golden Section may also express the relationship between a whole line (*a*) and its longer part (*b*), such that the whole relates in length to the longer part in the same way as the longer part relates to the whole minus the longer part ($a : b :: b : a - b$). If the whole is equal to 1, the proportion translates to [$1 : 1/\phi :: 1/\phi : (1 - 1/\phi)$ or ϕ^2].
3. For a full account of names for the Golden Section, see [Herz-Fischler 1998, 164-170].
4. For more on the Pythagorean Theorem, see [Fletcher 2005b, 152-154].
5. The reciprocal of a major triangle is a figure similar in shape, but smaller in size, such that the short side of the major triangle equals the long side of the reciprocal.
6. Note the similarity between this construction and that of the pentagon. If the side of the square (DCAB) is 1, the diagonal HC of the half -square is $\sqrt{5}/2$. HC = HI = $\sqrt{5}/2$. BI = BH + HI = $\frac{1}{2} + \sqrt{5}/2 = \phi$. AI = HI – HA = $\sqrt{5}/2 - \frac{1}{2} = 1/\phi$.
7. The reciprocal of a major rectangle is a figure similar in shape, but smaller in size, such that the short side of the major rectangle equals the long side of the reciprocal. The diagonal of the

reciprocal and the diagonal of the major rectangle intersect at right angles [Hambidge 1967, 30, 131]; see [Fletcher 2004, 103].

8. The radius vector is the variable line segment drawn to a curve or spiral from a fixed point of origin (the pole or eye) [Simpson 1989]; see also [Fletcher 2004, 105].
9. Note the similarity between this construction and those of the pentagon and the $1 : \phi$ rectangle. If the short and long sides of the right triangle CAB are ½ and 1, respectively, the hypotenuse BC equals $\sqrt{5}/2$. $BD=BC–DC=\sqrt{5}/2–½=1/\phi$. BD=BE. AB=1. $AE=AB–EB=1–1/\phi=1/\phi^2$.
10. This construction appears in [Ghyka 1977, 35].
11. In fig. 27, if the side of the square (ABCD) is 1, the diagonal IC of the half -square is $\sqrt{5}/2$. IC = IL = IK. AI = ½. AL = AI + IL = $½ + \sqrt{5}/2 = \phi$. $AK=IK–IA=\sqrt{5}/2–½=1/\phi$. Therefore, AK = $1/\phi$, AD = 1, and AL = ϕ.
12. For more on the Theorem of Thales, see [Fletcher 2004, 107].
13. One Euclidean construction divides a line into "extreme and mean ratio," such that the short segment is in ratio to the longer as the longer is in ratio to the whole (or $1 : \phi$) [1956, II: 267 (bk. VI, prop. 30); II: 188 (bk. VI, def. 3)]. Book XIII describes the five regular Platonic solids inscribed within a sphere. Of the five, the icosahedron and the dodecahedron involve the pentagon and its inherent "extreme and mean" proportions [1956, III: 453 (bk. XIII, prop. 8)]. Thomas Heath traces Euclids's division of a line in "extreme and mean" ratio to the early Pythagoreans from whom, Heath assumes, an exact construction of a regular pentagon evolved [Heath 1981, 168.] Keith Critchlow observes a prehistoric application of the ϕ ratio in carvings on a Neolithic sphere housed in Edinburgh, which display the symmetry of a dodecahedron [Critchlow 1982, 149].
14. Of the connection between the pentad and light, the modern Platonist Thomas Taylor says that light is the consequence of circular motion, following the four processes of length, breadth, depth and the "sameness" of the sphere itself [Taylor 1972, 188-189]. Also, Apollo, the god of light, personifies five qualities—omnipotence, omniscience, omnipresence, eternity and unity [Cooper 1978, 116].
15. R. A. Schwaller de Lubicz advocates the Egyptian "tradition" of dividing the human figure in Golden Section at the navel, with the caveat that a small portion of the crown is subtracted from the total height [Schwaller de Lubicz 1998, 313, 325-326, 341-343].
16. Fibonacci's main work, *Liber abaci*, uses the number series 1, 1, 2, 3, 5, 8, 13… to calculate the month-to-month progeny of a single pair of rabbits, such that each pair produces a new pair every month [Huntley 1970, 158-159].
17. To construct the rectangle of whirling squares, begin with a square of side 1. Place a new square of side 1 adjacent to it. The result is a double square. Place a new square of side 2 adjacent to the long side of the double square. The result is a 2 x 3 rectangle. Place a new square of side 3 adjacent to the long side of the 2 x 3 rectangle. The result is a 3 x 5 rectangle. Repeat the process, indefinitely.
18. The first step of the analysis (fig. 35a) replicates Dürer's construction for a five-pointed star, suggesting his appreciation of ϕ as a dynamic proportional system [Dürer 1977: II, 155-156].

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Rachel Fletcher is a theatre designer and geometer living in Massachusetts, with degrees from Hofstra University, SUNY Albany and Humboldt State University. She is the creator/curator of two museum exhibits on geometry, "Infinite Measure" and "Design By Nature". She is the co-curator of the exhibit "Harmony by Design: The Golden Mean" and author of its exhibition catalog. In conjunction with these exhibits, which have traveled to Chicago, Washington, and New York, she teaches geometry and proportion to design practitioners. She is an adjunct professor at the New York School of Interior Design. Her essays have appeared in numerous books and journals, including "Design Spirit", "Parabola", and "The Power of Place". She is the founding director of Housatonic River Walk in Great Barrington, Massachusetts, and is currently directing the creation of an African American Heritage Trail in the Upper Housatonic Valley of Connecticut and Massachusetts.