Rachel Fletcher

Musings on the Vesica Piscis

Geometer Rachel Fletcher explains the geometry, symbolism, and applications of the vesica piscis

Any two circles can intersect to produce an almond shape, but when two circles of identical size intersect such that the center of one lies on the circumference of the other, the result is a *vesica piscis*. Related by geometry to the triangle, the vesica signifies the mediation of opposites and is associated in Christian symbolism with the Trinity. We explore its inherent three-ness through geometric constructions, producing equiangular spirals and other proportional systems, while we examine the elementary geometric Theorem of Thales and the Law of Similar Triangles. We begin at the beginning, by drawing a circle.

Circle

- With a compass, draw a circle, at any radius (Fig. 1).

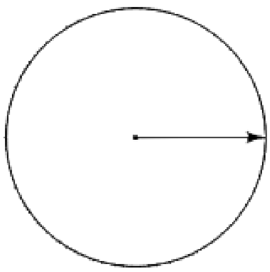


Fig. 1

DEFINITION:
The **circle** is the set of points in a plane that are equally distant from a fixed point in the plane. The fixed point is called the **center**. The given distance is called the **radius**. The totality of points on the circle is called the **circumference**.

"Circle" is from the Latin *circulus*, which means "small ring" and is the diminutive of the Latin circus and the Greek *kuklos*, which mean "a round" or "a ring." The Latin for "circumference" is *circumferentia* (from *circum* "round, about" + *ferre* "to bear"), which is a late literal translation of the Greek *periphereia*, which means "the line around a circular body" or "periphery." The Latin for "center" is *centrum*, from the Greek *kentron*, which means "sharp point" and originally meant the "stationary point of a pair of compasses" [Hoad 1996, Liddell 1940, Simpson 1989].

Compass is the term for the two-legged drafting tool used to draw arcs and circles, as well as the navigational instrument that locates geographic directions. The history of the word is uncertain. Possibly, "compass" is from the Old French *compas*, from *compasser* (based on the Latin *com-* "together" + *passus* "pace"), which means "to go around, measure, divide equally." The Greek word for "compass" is *diabêtês*, which is taken from the verb *diabainô*, "to stand with legs apart." The Latin word for *diabêtês* is *circinus*, which is from *circa*, "round." [Hoad 1996, Liddell 1940, Simpson 1989].

The circle stands for unity, oneness, eternity, wholeness, and completeness. The only form that encloses all other radially symmetrical or regular figures, it may connote pre-form, the genesis of form, origins or beginnings.

The circle's circumference is closed and continuous and as such conveys continuous cycles of endings and beginnings. Circles may signify cycles of time such as: phases of the sun and moon; cycles of light and dark; and perpetual rhythms of sleeping and waking, birth and death, growth and decay, systole and diastole, and inhalation and exhalation. In its totality, the circle suggests the timeless whole. The moving point along the circle conveys the passage of time.

Circles are measured relative to the incommensurable value *pi* (π = 3.1415927…).[1] In contrast to the square, whose perimeter and area can be measured in finite whole numbers, the circle may symbolize heavenly, transfinite or transcendental realms.

The circle expresses justice and democracy, since all of its points are equally distant from the center. A communal form, it imparts no sense of social hierarchy. The circle may symbolize the collective—the ring that "unites man through the infinite chain of hands."[2]

Some Christian churches, tribal ritual spaces, and the instinctive way in which children gather to play—all take the form of the circle, drawing upon its magical and protective qualities and its sense of center and place. As a sacred space, the circle orients to the horizon and to the cosmic edges of the universe. Mircea Eliade observes that one gathers in a circle to distinguish what is known (cosmos) from what is unknown (chaos); to "'found the world' and to live in a real sense" [Eliade 1959, 23].

Vesica Piscis

- With a compass, draw a circle, then a second circle of equal radius, such that the center of the second circle lies on the circumference of the first. The vesica piscis is the womb-like area that is shared between the two circles (Fig. 2).

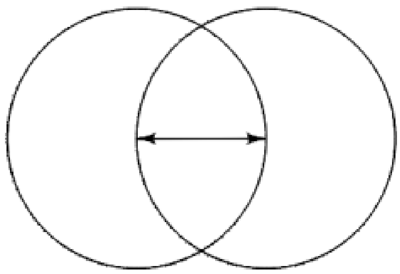


Fig. 2

DEFINITION:
The **vesica piscis** is the intersection of two identical circles, such that the center of one circle lies on the circumference of the other.

The vesica piscis signifies the mediation of two distinct entities; the complementariness of polar opposites, as when two extremes complete and depend upon one another to exist. One circle may signify the breath of spirit, which is eternal; the other may signify the body physical, which is forever changing and adapting, The vesica piscis itself symbolizes that which mediates spirit and body; or the psyche or soul.

In another way, the vesica piscis may represent the phenomenon of color, which Goethe understands to mediate "light and its absence." He notes that we perceive "all the varieties of hues" when "the greatest brightness…acts near the greatest darkness" [Goethe 1970, 5, 206].

The pisces symbol

"Vesica piscis" is the Latin *vesica* "bladder" + *piscis* "fish" [Hoad 1996]. Commonly used in Christian symbolism, it resembles the graphic symbol for Pisces, the twelfth and last sign of the Zodiac, which is signified by two partial circles that are distinct, yet bound by a line. Historically, the Christian Era coincides with the Piscean Age, when the sun's entry into the constellation Pisces marked the first day of spring and the new year.

As a Christian symbol, the vesica piscis may signify Christ Incarnate, who mediates heaven and earth, or humanity and the divine. In Medieval Christian art, Christ is commonly depicted emerging from a vesica piscis to portray the entry of transcendent form into the physical world and made flesh [Williams 2001]. The vesica piscis can also signify the womb—in Christianity, the womb of the Virgin from which Christ emerges. The proportions of the vesica piscis appear in the Gothic arch and underlie rectangular floor plans of numerous churches and chapels, such as the Mary Chapel in Glastonbury Abbey.

Some vernacular cultures combine images of sun and moon in the form of a vesica piscis. The full disk of the sun, which radiates its own permanent golden light, represents the unifying state of "solar" consciousness in which reality is perceived as eternal and One. In contrast to the sun, the sharp and cutting edge of the crescent moon may signify the division of unity into different parts. The moon continually changes as it progresses through various phases. Its silvery light is the surface reflection of other bodies. Together, the sun and moon convey complementary polarities of self and other, sameness and difference, and one whole and many parts (Fig. 3).

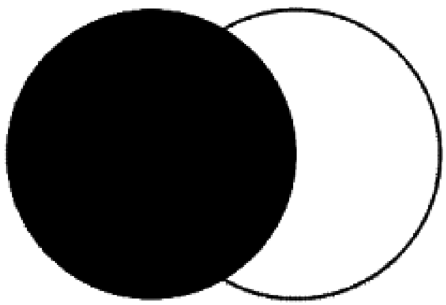


Fig. 3

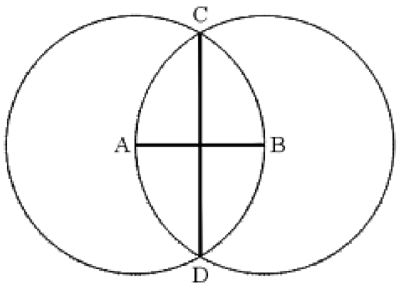
Equilateral Triangle and the Ratio 1 : √3

- Draw a vesica piscis from two circles of radius 1 (AB).
- Locate its short and long axes (AB and CD).

The short, horizontal axis equals the radius of the vesica's generating circles. The short and long axes are in the ratio l : √3, or 1: 1.7320508….. If the short axis (AB) equals 1, the long axis (CD) equals √3 (Fig. 4).

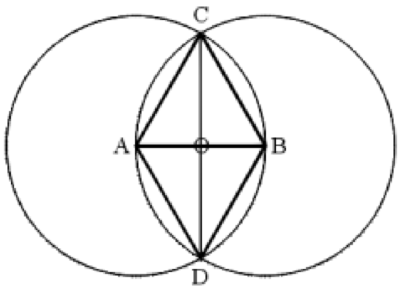
- Connect points ABC, above and ABD, below.

The result is two equilateral triangles.[3] If the half side (OB) of the equilateral triangle (ABC) is 1, the altitude (OC) equals √3.[4] (Fig. 5).



AB:CD :: 1:√3

Fig. 4



OB:OC :: 1:√3

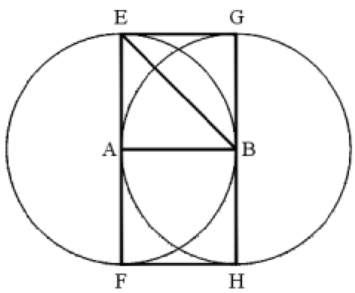
Fig. 5

The vesica piscis shares a common geometry with the equilateral triangle and may signify the Holy Trinity and other triadic relationships.

The Square and the Ratio 1 : √2

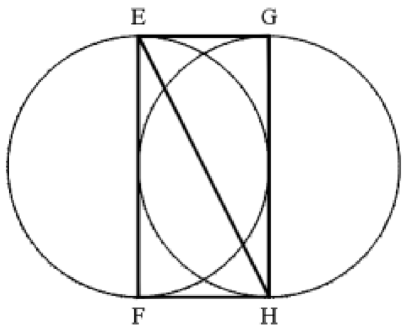
- Draw a vesica piscis from two circles of radius 1 (AB).
- Locate the vertical diameters (EF and GH) of each generating circle.
- Connect points ABGE, above, and ABHF, below. The result is two squares.
- Draw the diagonal BE through the square ABGE.

If the side (AB) of the square is 1, the diagonal (BE) equals √2, or 1.4142135... (Fig. 6).



AB:BE :: 1:√2

Fig. 6



FH:HE :: 1:√5

Fig. 7

DEFINITION:
The **diagonal** is the straight line joining two nonadjacent vertices of a plane figure, or two vertices of a polyhedron that are not in the same face. The Greek for "diagonal" is *diagônios* (from *dia* "across" + *gônia* "angle"), which means "from angle to angle" [Liddell 1940, Simpson 1989].

The Golden Section and the Ratio 1 : ϕ

- Draw a vesica piscis from two circles of radius 1.
- Locate the vertical diameters (EF and GH) of each generating circle.
- Connect points FHGE. The result is a double square.
- Draw the diagonal HE.

If the side (FH) is 1, the diagonal (HE) equals √5, or 2.236067... (Fig. 7).

- Draw the horizontal axis IJ through the centers of the two circles (points A and B).
- From the midpoint of diagonal EH (point O), draw a line OH. If the short axis of the vesica (AB) is 1, segment OH equals (√5/2).
- Place the compass point at O. Draw an arc of radius OH, intersecting the horizontal axis IJ on the right, at point K.

If the short axis of the vesica (AB) is 1, segment AK equals *phi* (ϕ = √5/2 + l/2) or 1.618034…. Segment BK equals the reciprocal of ϕ; in other words 1/ϕ or (√5/2 - l/2) or 0.618034…. The ratio 1 : ϕ is known as the Golden Section, or the “extreme and mean” ratio (Fig. 8).

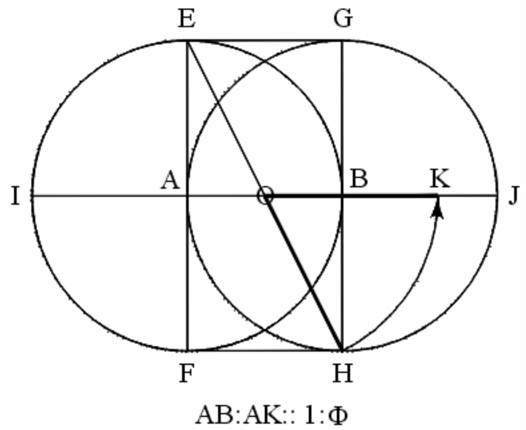


Fig. 8

Incommensurable Ratios and Dynamic Symmetry

DEFINITION:
"Incommensurable" is an adaptation of the medieval Latin *incommensurabilis* (*in-* "not" + *com-* "together" + *mensura* "a measure"). Quantities that lack a common measure or factor are **incommensurable** [Simpson 1989].

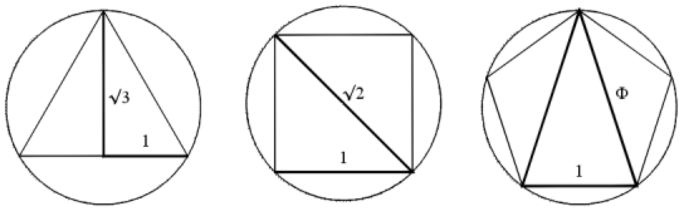


Fig. 9

Like the ratio 1: π (pi), incommensurable ratios such as l : √2, l : √3, and 1 : ϕ, cannot be expressed precisely in finite whole numbers, but their absolute values are present in simple geometric forms. The half-side and altitude of any equilateral triangle are in the ratio l : √3. The side and diagonal of any square are in the ratio l : √2. The side and diagonal of any regular pentagon are in the ratio 1 : ϕ (Fig 9).

Incommensurable ratios may organize space so that the same proportion persists continually through endless divisions. This quality of continuity, which Jay Hambidge calls "dynamic symmetry," is unique to the incommensurables and implies that every level of form, from the micro- to the macrocosmic, may be united through measure and proportion [Hambidge 1960; 1967].

How to Generate a 1 : √3 Proportional System with a Vesica Piscis

- Draw a vesica piscis from two circles of radius 1 (AB).
- Locate the short and long axes (AB and CD).

The short and long axes of the vesica piscis equal 1 and √3 (Fig. 10).

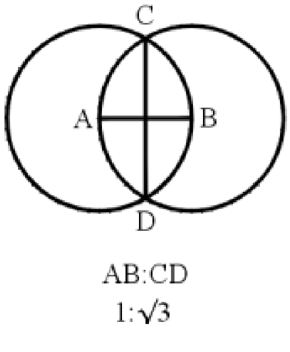


Fig. 10

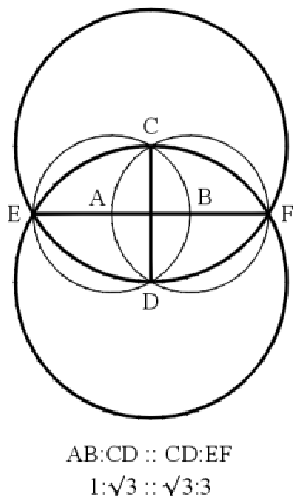


Fig. 11

Draw a new vesica piscis from two circles of radius CD:

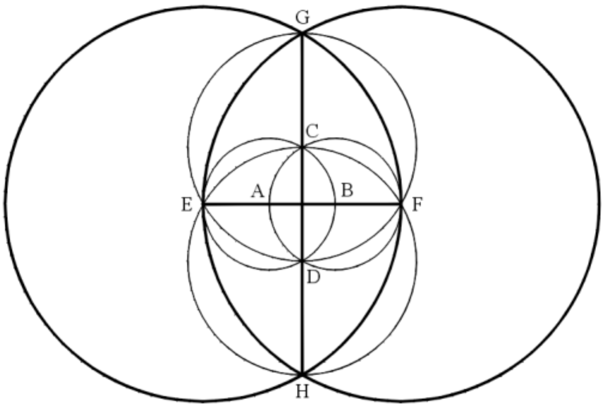
- Place the compass point at C. Draw a circle of radius CD.
- Then, place the compass point on D. Draw a second circle of radius DC.

The short and long axes of the new vesica (CD and EF) equal √3 and 3 (Fig. 11).

Draw a new vesica piscis from two circles of radius EF:

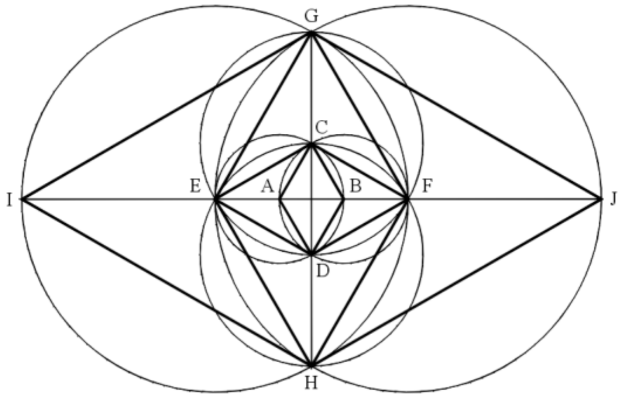
- Place the compass point at E. Draw a circle of radius EF.
- Then, place the compass point on F. Draw a second circle of radius FE.

The short and long axes of the new vesica (EF and GH) equal 3 and 3 √3 (Fig. 12).



AB:CD :: CD:EF :: EF:GH
1:√3 :: √3:3 :: 3:3√3
Fig. 12

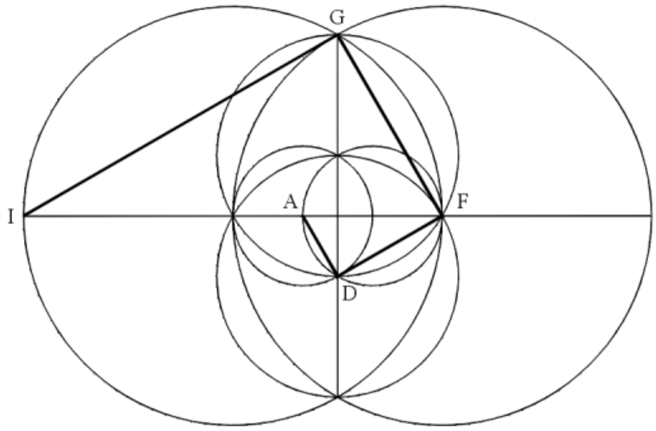
- Within each vesica piscis, draw two equilateral triangles (Fig. 13).



AB:CD :: CD:EF :: EF:GH
1:√3 :: √3:3 :: 3:3√3
Fig. 13

- Connect sides AD, DF, FG, and GI from each successive set of triangles.

The spiral that results follows a 1: √3 geometric progression (Fig. 14).

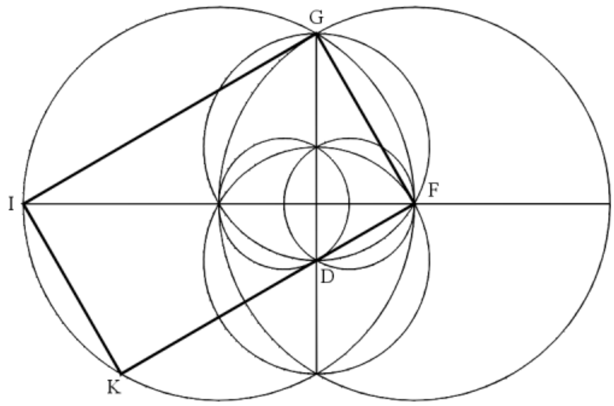


AD:DF :: DF:FG :: FG:GI
1:√3 :: √3:3 :: 3:3√3
Fig. 14

The 1 : √3 Rectangle

- From point F, draw a line through point D, until it intersects the large circle on the left at point K.
- Connect points FGIK.

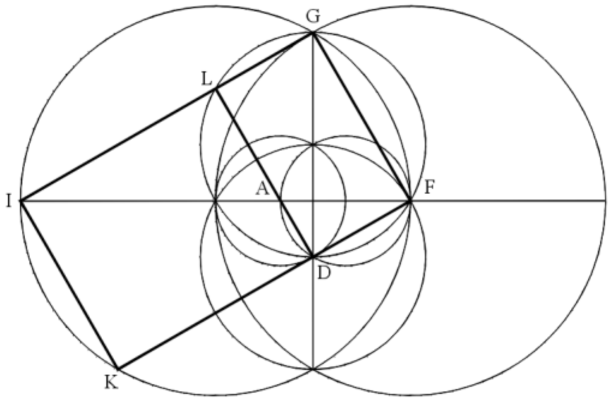
The result is a rectangle (FGIK) with short and long sides in the ratio l : √3 (Fig. 15).



FG:GI :: 1:√3
Fig. 15

- From point D, draw a line through point A, until it intersects line GI at point L.

The result is a smaller rectangle (DFGL) in the ratio 1/√3 : 1 or l : √3. Rectangle DFGL is the reciprocal of the whole rectangle FGIK (Fig. 16).

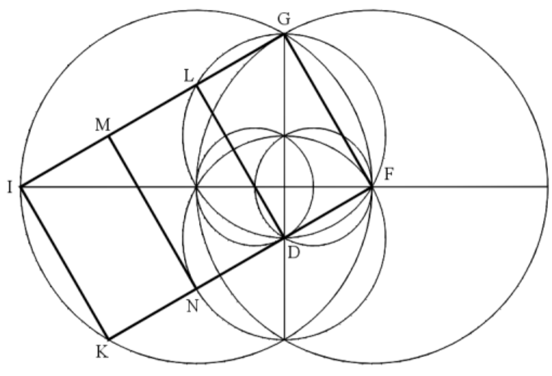


DF:FG :: FG:GI
1/√3 : 1 :: 1:√3
Fig. 16

DEFINITION:
The **reciprocal** of a major rectangle is a figure similar in shape, but smaller in size, such that the short side of the major rectangle equals the long side of the reciprocal. The diagonal of the reciprocal and the diagonal of the major rectangle intersect at right angles [Hambidge 1967, 30, 131].

- From point N, draw a line that is perpendicular to line FK and intersects line GI at point M.

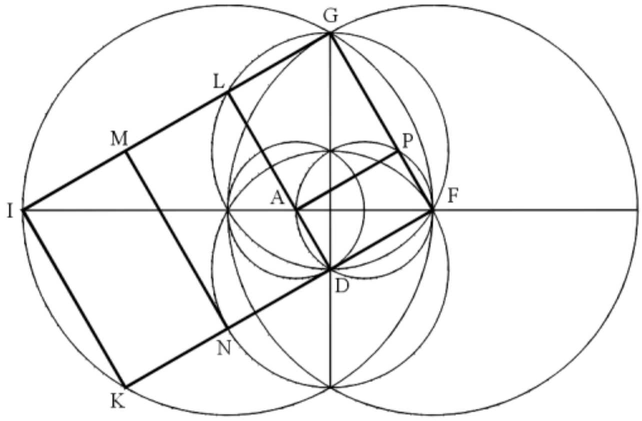
The rectangles NDLM and KNMI that result are each in the ratio 1/√3 : 1 or l : √3. The major l : √3 rectangle FGIK divides into three reciprocals that are proportionally smaller in the ratio l : √3 (Fig. 17).



DF:FG :: FG:GI
1/√3 : 1 :: 1:√3
Fig. 17

- From point A, draw a line that is perpendicular to line LD and intersects line GF at point P.

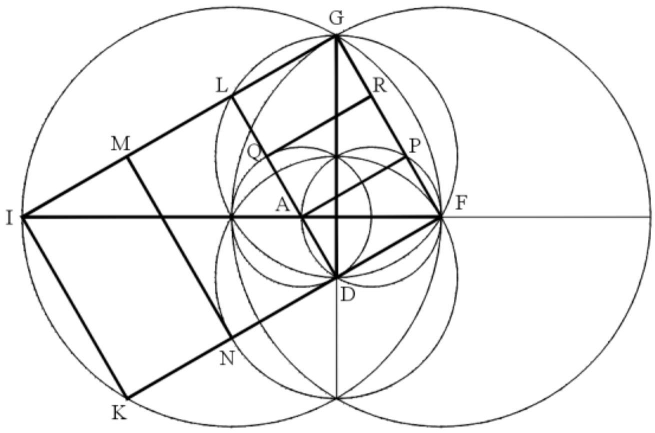
The result is a smaller rectangle (ADFP) in the ratio 1/3 : 1/√3 or l : √3 (Fig. 18).



AD:DF :: DF:FG :: FG:GI
1/3 : 1/√3 :: 1/√3 : 1 :: 1:√3
Fig. 18

- From point Q, draw a line that is perpendicular to line LD and intersects line GF at point R.

The rectangles QAPR and LQRG that result are each in the ratio 1/3 : 1/√3 or l : √3. The major l : √3 (or 1/√3 : 1) rectangle DFGL divides into three reciprocals that are proportionally smaller in the ratio l : √3 (Fig. 19).



AD:DF :: DF:FG :: FG:GI
1/3 : 1/√3 :: 1/√3 : 1 :: 1:√3
Fig. 19

Such constructions illustrate how incommensurable ratios replicate consistently through endless spatial divisions, even as the identical ratio remains present in the relationship of one level to the next. A l x √3 rectangle of any size divides into three proportionally smaller reciprocals in the ratio l : √3. If the process continues indefinitely, the side lengths of successively larger rectangles form a perfect geometric progression (l ,√3 , 3, 3√3…).

"Dynamic symmetry" is the name given by Jay Hambidge to this proportioning principle, which he finds in l x √3 and other root rectangles, and observes in the human figure, in plant life, and in classical Greek and other forms of art. Hambidge associates dynamic symmetry with our perception of beauty, noting "its power of transition and movement from one form to another...It produces the only perfect modulating process in any of the arts" [Hambidge 1967, xv-vxi].

Of symmetry, Hambidge says, "using the word in the Greek sense of analogy; literally it signifies the relationship which the composing elements of form in design, or in any organism of nature, bear to the whole. In design, it is the thing which governs the just balance of variety in unity" [Hambidge 1967, xii].

We will now explore how dynamic symmetry and l : √3 proportions manifest in other ways.

- Locate the diagonal (IF) of the rectangle FGIK.
- Locate the diagonal (GD) of the reciprocal DFGL.

GD: IF :: 1: √3. The diagonals IF and GD intersect at 90° at point O (Fig. 20).

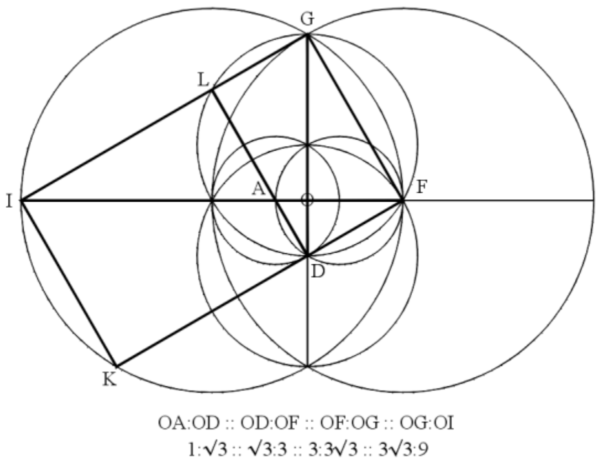


Fig. 20

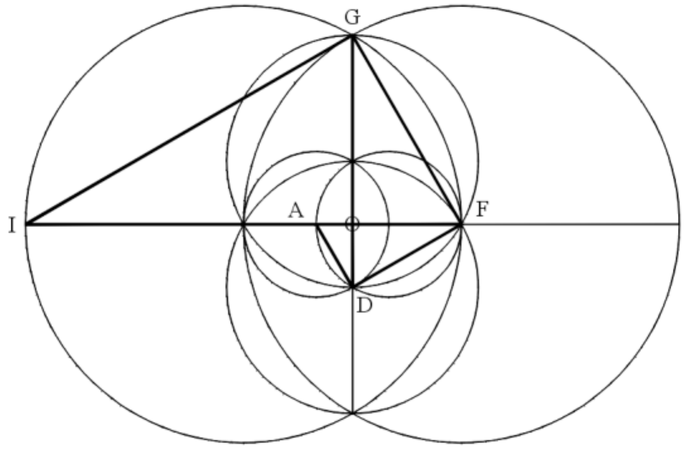
The 1 : √3 Equiangular Spiral

DEFINITION:
The Latin word for "radius" is *radius*, which means "staff," "spoke of a wheel," "measuring rod" or "ray." "Vector" is from the Latin *vehere,* "to carry." The **radius vector** is the variable line segment drawn to a curve or spiral from a fixed point of origin (the **pole** or **eye**) [Harper 2001, Simpson 1989].

An equiangular spiral is a spiral curve in which distinct radii vectors emanating from the pole at equal angles to one another are in continual proportion. Between any three consecutive radii vectors, the middle vector is the mean proportional or geometric mean of the other two. The spiral that results grows in size continuously without changing its shape. It is also called a logarithmic spiral or proportional spiral.[5]

- Locate the equiangular spiral of straight-line segments AD, DF, FG and GI.
- Locate the pole of the spiral at point O.
- Locate the radii vectors OA, OD, OF, OG, and OI.

The radii vectors are separated by equal angles (90°). Their lengths increase in a 1: √3 geometric progression. Equiangular spirals such as IGFDA decrease continuously towards the pole, but never touch it (Fig. 21).



OA:OD :: OD:OF :: OF:OG :: OG:OI

$1:\sqrt{3} :: \sqrt{3}:3 :: 3:3\sqrt{3} :: 3\sqrt{3} : 9$

Fig. 21

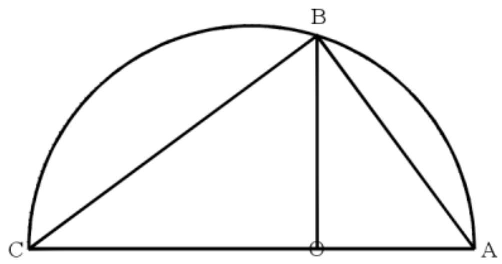
The Theorem of Thales and the Law of Similar Triangles

- Draw a semi-circle on the diameter CA.
- Locate a point (B) anywhere along the perimeter.
- From point B, draw lines to points C and A.

The triangle CBA is a right triangle.

- From point B draw a line (BO) that is perpendicular to the diameter CA.

Line BO is the mean proportional or geometric mean of lines OA and OC (Fig. 22).



OA:OB :: OB:OC

Fig. 22

Thales of Miletus (ca. 624-547 B.C.), considered the first Greek mathematician, is thought to have learned geometry from the Egyptians. He is credited with the theorem that any triangle inscribed within a semi-circle is right-angled. The **Theorem of Thales** states that within a semi-circle, as in Fig. 22, a perpendicular line (BO) drawn from any point (B) along the perimeter to the diameter is the mean proportional or geometric mean of the two line segments (OA and OC) that result on the diameter, that is, OA: OB :: OB:OC.

Triangles are similar that have corresponding angles equal and corresponding segments proportional. The **Law of Similar Triangles** states that two triangles are similar if they have two angles and one side equal.

Proof that triangles BOA, COB and CBA are similar

- Compare triangles BOA and CBA:

angle BOA = angle CBA; angle BAO = angle CAB; side AB = side AB
therefore, triangle BOA ~ triangle CBA

- Compare triangles COB and CBA:

angle COB = angle CBA; angle OCB = angle BCA; side BC = side BC
therefore, triangle COB ~ triangle CBA

- Triangle BOA ~ triangle CBA
- Triangle COB ~ triangle CBA

Therefore, triangle BOA ~ triangle COB ~ triangle CBA

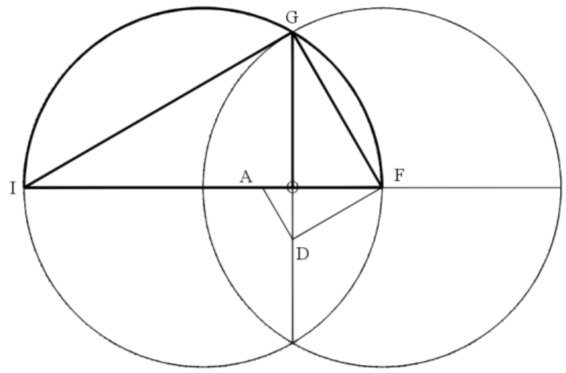
Equiangular spirals demonstrate the Theorem of Thales and the Law of Similar Triangles. In Fig. 21:

- Locate the equiangular spiral of straight-line segments AD, DF, FG and GI.
- On line IF, construct a semi-circle.
- Locate point G on the perimeter of the semi-circle.
- From point G, draw lines to points F and I (lines GF and GI of the spiral).

The triangle IGF is a right triangle.

- From point G on the semi-circle, draw a line GO perpendicular to line IF.

Triangles GOF, IOG and IGF are similar. Line OG is the mean proportional or geometric mean of lines OF and OI (Fig. 23).



OF:OG :: OG:OI

1:√3 :: √3:3

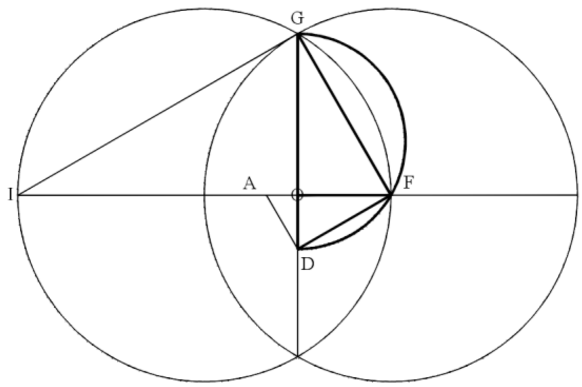
Fig. 23

- Locate the equiangular spiral of straight-line segments AD, DF, FG and GI.
- On line GD, construct a semi-circle.
- Locate point F on the perimeter of the semi-circle.
- From point F, draw lines to points D and G (lines FD and FG of the spiral).

The triangle GFD is a right triangle.

- From point F on the semi-circle, draw a line FO perpendicular to line GD.

Triangles FOD, GOF and GFD are similar. Line OF is the mean proportional or geometric mean of lines OD and OG (Fig. 24).



OD:OF :: OF:OG

1:√3 :: √3:3

Fig. 24

Application: Bramante's Tempietto

Palladio says of the sixteenth-century Italian architect Donato Bramante that he was the "first to make known that good and beautiful architecture which had been hidden from the time of the ancients til now" [Palladio 1997, IV, xvii, 276]. Bramante's Doric style Tempietto ("Little Temple"), designed for the Church of San Pietro in Montorio in Rome, commemorates the crucifixion of St. Peter (Fig. 25). It is the only Renaissance work that appears in Book Four of Palladio's *I quattro libri* (*The Four Books on Architecture*), which is otherwise dedicated to classical temples of the ancients.

Sebastiano Serlio's *Trattato di architettura* (*On Architecture*) presents an elevation of the Tempietto that appears to unfold from a vesica piscis proportioned to the temple's dome [Serlio 1544, III, xlviii, 48] (Fig. 25).

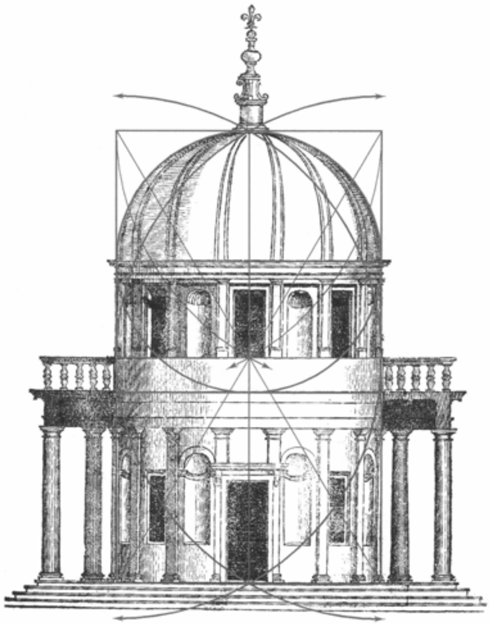


Fig. 25

- Draw the circle that traces the dome's exterior surface.
- Draw a square about the circle.
- From the top edge of the square, draw a downward pointing equilateral triangle.

The bottom apex of the triangle locates the top of the balcony rail and the base of the second story.

- At this location, draw a vesica piscis from two circles whose radii equal the diameter of the circle of the dome.

The base of the vesica piscis locates the floor level of the temple. The top of the vesica piscis locates the top of the dome.

A 1: √3 rectangle encloses the vesica piscis, and contains two equilateral triangles.

Notes

1. The circle's circumference, c, is the product of π and twice the radius, r ($c = \pi 2\ r$). Its area is the product of π and the square of the radius ($a = \pi r^2$).
2. R. Schwarz, *Von Bau der Kirche* (*The Church Incarnate*), 24. Cited in [Norberg-Schultz 1972, 20].
3. For proof, see Euclid, Book I, Prop. 1: "On a given finite straight line to construct an equilateral triangle" [Euclid 1956, I: 241-242].
4. For proof, apply the Pythagorean theorem ($OB^2 + OC^2 = BC^2$)
5. [Thompson 1992, 748-758]; [Hambidge 1967, 5-6]. D'Arcy Thompson describes equiangular spirals as "any plane curve proceeding from a fixed point (or pole), and such that the vectorial area of any sector is always a gnomon to the whole preceding figure" [1992, 763]. "Gnomon," the post that marks the time of day by the shadow it casts on a sundial, is from the Greek *gnômôn,* which means "indicator" or "interpreter" and specifically "the pointer of the sundial" or "carpenter's square" [Liddell 1940, Simpson 2005]. In mathematics, the gnomon is the shape which, when added to a figure, produces the same figure, but larger; as when an "L" shape added to a square produces a larger square. In similar fashion, the equiangular spiral is a "growing structure… built up of successive parts, similar in form, magnified in geometrical progression, and similarly situated with respect to a center of similitude" [Thompson 1992, 763].

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About the Geometer

Rachel Fletcher is a theatre designer and geometer living in Massachusetts, with degrees from Hofstra University, SUNY Albany and Humboldt State University. She is the creator/curator of two museum exhibits on geometry, "Infinite Measure" and "Design By Nature". She is the co-curator of the exhibit "Harmony by Design: The Golden Mean" and author of its exhibition catalog. In conjunction with these exhibits, which have traveled to Chicago, Washington, and New York, she teaches geometry and proportion to design practitioners. She is an adjunct professor at the New York School of Interior Design. Her essays have appeared in numerous books and journals, including "Design Spirit", "Parabola", and "The Power of Place". Her design/consulting credits include an outdoor mainstage for Shakespeare & Co. in Lenox, Massachusetts and the Marston Balch Theatre at Tufts University.