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BRIESKORN MANIFOLDS AS CONTACT BRANCHED COVERS OF SPHERES

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Abstract

We show that Brieskorn manifolds with their standard contact structures are contact branched coverings of spheres. This covering maps a contact open book decomposition of the Brieskorn manifold onto a Milnor open book of the sphere.

1. Introduction

Brieskorn manifolds have been an interesting source of examples. In the field of topology many exotic spheres can be realized as such manifolds, but also in contact geometry they have provided a rich family of examples. The most prominent ones are the exotic contact structures on $(4n+1)$-spheres ([Ust]).

It has been known for a long time that a Brieskorn manifold $\Sigma(a_0,\dots,a_n) \subset \mathbb{C}^{n+1}$ is an a_0-fold cyclic covering of the unit sphere $\mathbb{S}^{2n-1} \subset \mathbb{C}^n$ branched along the $(2n-3)$-dimensional Brieskorn manifold $\Sigma(a_1,\dots,a_n)$. In this article, we show that this is not only true as smooth manifolds but also in the contact category.

Furthermore, given a $(2n-3)$-dimensional Brieskorn manifold $B := \Sigma(a_1,\dots,a_n)$ there exists a natural (so-called) Milnor open book on $\mathbb{S}^{2n-1}$ that has B as its binding. This open book decomposition can be pulled back by the cyclic branched covering to the Brieskorn manifold $\Sigma(a_0,\dots,a_n)$. In this way it is

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possible to show that the open book of a Brieskorn manifold can be described in an abstract way by using a Milnor open book and taking a power of its monodromy map. One of the goals of this article is to show that the canonical contact structure on a Brieskorn manifold is supported by that open book.

In Section 2, we will state what a contact branched covering is. In Section 3, we recall first the basic definitions regarding open books, and then we show that given an open book decomposition of a contact manifold M, one can easily construct a contact branched covering, with branching locus given by the binding of the open book, such that the covering manifold inherits a natural open book decomposition from the base space M. These notions are applied in Section 4 to Brieskorn manifolds: Theorem 7 proves that there is a contact structure isotopic to the standard one on $\mathbb{S}^{2n-1}$ which is supported by the Milnor open book of $\mathbb{S}^{2n-1}$ with binding a Brieskorn manifold $\Sigma(a_1, \ldots, a_n)$. Finally, Theorem 6 shows that the contact structure of the a_0-fold contact branched covering of $\mathbb{S}^{2n-1}$ as in Lemma 2 is isotopic to the standard contact structure on the Brieskorn manifold $\Sigma(a_0, a_1, \ldots, a_n)$.

We would like to note the following general result ([CNP, Theorem 3.9]). Suppose that X is a complex analytic variety with an isolated singularity at x and f is a complex valued holomorphic function with an isolated singularity at x. Then *the Milnor open book determined by* f on the boundary of a sufficiently small neighborhood of x carries the canonical contact structure on the boundary. Although our result is a special case, our proof contains detailed calculations in coordinates which also describe explicitly the monodromy map in terms of the monodromy map of the Milnor open book on a standard sphere. Since our approach involves the use of contact branched coverings, a similar approach can be employed to determine the monodromy of arbitrary branched coverings explicitly.

2. Contact branched coverings

Branched coverings for contact 3-manifolds were first considered by Gonzalo in [Gon]. He used them to reprove the existence of a contact structure on any oriented 3-manifold. His methods used local charts and were adapted to his special situation. Geiges showed later that a branched covering of a contact manifold of any dimension admits under very natural conditions a contact structure [Gei]. Below we will give a definition of contact branched covers, which coincides essentially with Geiges' construction, and show that up to isotopy it is independent of any choices.

Let (N, α) be a contact manifold, and let $f\colon M \to N$ be a branched covering. The pull-back form $f^*\alpha$ fails to be contact on M, because by definition $\dim(\ker df) = 2$ along the branching locus. This problem can be fixed though by perturbing $f^*\alpha$ slightly.

Lemma 1. *Let $f\colon M \to N$ be a covering branched along $B \subset N$ such that (N, α) and $(B, \alpha|_{TB})$ are contact manifolds. There exists a 1-form γ on M with $d\gamma|_{\ker df} > 0$ along B ($\ker df$ is naturally oriented, because $f^*\alpha$ gives an orientation both for M and $f^{-1}(B)$) such that*

$$f^*\alpha + \varepsilon\,\gamma$$

is a contact form on M for any sufficiently small $\varepsilon > 0$.

Any contact form β_1 on M is isotopic to $f^*\alpha + \varepsilon\,\gamma$ if it lies in a smooth family of 1-forms β_t with $t \in [0,1]$ such that $\beta_0 = f^*\alpha$, and for which β_t is contact for all $0 < t \leq 1$, and for which $d\widetilde{\gamma}|_{\ker df} > 0$, where we have set $\widetilde{\gamma} = \dot{\beta}_t|_{t=0}$.

Definition. $f\colon M \to N$ together with the contact structure given above is called the *contact branched covering* of (N, α) along $(B, \alpha|_{TB})$.

Proof. The existence of such a form γ was proved in [Gei], and the uniqueness of the contact structure can be shown in a similar way. For completeness though, here is the argument: Consider the Taylor expansion of β_t at $t = 0$:

$$\beta_t = f^*\alpha + t\,\widetilde{\gamma} + \mathcal{O}(t^2)\ .$$

We will use this 1-form at time $t_0 = \varepsilon > 0$, where ε will be chosen below. We can form the linear interpolation between β_ε and $f^*\alpha + \varepsilon\,\gamma$ to define the family of 1-forms

$$\alpha_s := f^*\alpha + \varepsilon\,(s\,\gamma + (1-s)\,\widetilde{\gamma}) + (1-s)\,\mathcal{O}(\varepsilon^2) \qquad \text{for } s \in [0,1].$$

The contact condition for this family becomes

$$\begin{aligned}\alpha_s \wedge (d\alpha_s)^n := f^*\left(\alpha \wedge (d\alpha)^n\right) &+ \varepsilon(s\gamma + (1-s)\,\widetilde{\gamma}) \wedge f^*(d\alpha)^n + \\ &+ \varepsilon n f^*\left(\alpha \wedge (d\alpha)^{n-1}\right) \wedge (s d\gamma + (1-s) d\widetilde{\gamma}) + \mathcal{O}(\varepsilon^2).\end{aligned}$$

On the branching locus, the first two terms vanish; the third one is positive for all $s \in [0,1]$ by our assumptions, and by choosing $\varepsilon > 0$ small enough it dominates the $\mathcal{O}(\varepsilon^2)$ part. By continuity there is an open neighborhood U of $f^{-1}(B)$ where the sum of all terms containing an ε factor is positive for any sufficiently small $\varepsilon > 0$. The pull-back $f^*\left(\alpha \wedge (d\alpha)^n\right)$ is positive on the compact set $M - U$, and is thus always larger than $C\ \mathrm{Vol}_M$ for some $C > 0$. We can achieve that the ε terms (by choosing ε still smaller if necessary) are never smaller than $-C\ \mathrm{Vol}_M$. For any sufficiently small $\varepsilon > 0$, it follows that $\alpha_s \wedge d\alpha_s^n > 0$, and thus the corresponding contact structures are isotopic by Gray stability.

□

Note that the definition of a contact branched covering is in analogy with the definition of a symplectic branched covering [Aur]. Furthermore, there is the concept of canonicity of the structure in the symplectic setting too (see [Aur, Proposition 3]).

3. Open books and contact structures

The following definitions are taken from [Gir].

Definition. An *open book* on a closed manifold M is given by a codimension-2 submanifold $B \hookrightarrow M$ with trivial normal bundle, and a bundle $\vartheta\colon (M - B) \to \mathbb{S}^1$. The neighborhood of B should have a trivialization $\mathbb{D}^2 \times B$, where the angle coordinate on the disk agrees with the map ϑ.

The manifold B is called the *binding* of the open book and a fiber $P = \vartheta^{-1}(\varphi_0)$ is called a *page*.

Remark 1. The open set $M - B$ is a bundle over $\mathbb{S}^1$, hence it is diffeomorphic to the mapping torus $P_\Phi := \mathbb{R} \times P/\sim$, where $\sim$ identifies $(t,p) \sim (t+1, \Phi(p))$ for some diffeomorphism Φ of P. Since the neighborhood of the binding has the standard form described above, we can assume that Φ is equal to the identity in some small neighborhood of the binding. By glueing $\mathbb{D}^2 \times B \cong \mathbb{D}^2 \times \partial P$ onto P_Φ in the obvious way, we obtain a manifold diffeomorphic to M.

Definition. A contact structure ξ on M is said to be *supported by an open book* (B, ϑ) of M, if there is a contact form α with $\xi = \ker \alpha$ such that

1. $(B, \alpha|_{TB})$ is a contact manifold.
2. The Reeb field X_{Reeb} of α is transverse to all pages and $d\vartheta(X_{\mathrm{Reeb}}) > 0$. For every $s \in \mathbb{S}^1$, the page $P := \vartheta^{-1}(s)$ is then a symplectic manifold with symplectic form $d\alpha$.
3. Denote the closure of a page P in M by $\overline{P}$. The orientation of B induced by its contact form $\alpha|_{TB}$ should coincide with its orientation as the boundary of $(\overline{P}, d\alpha)$.

Such a contact form is said to be *adapted* to (B, ϑ).

Remark 2. Note that if the binding is connected, point (3) of the definition above holds automatically, because

$$0 < \int_P (d\alpha)^n = \int_B \alpha \wedge (d\alpha)^{n-1} ,$$

by Stokes' theorem. Hence the orientation of B as boundary of P agrees with the one given by the contact form.

Lemma 2. *Let* (N, α) *be a contact manifold that has an open book decomposition* (B, ϑ) *supporting* α*. The* k*-fold cyclic covering* $f\colon M \to N$ *branched over* B *exists, and is a contact manifold adapted to the open book decomposition* $(f^{-1}(B), \sqrt[k]{\vartheta \circ f})$.

Proof. Note that $N - B$ can be written by the remark above as $P_\Phi = \mathbb{R} \times P/\sim$, where $\sim$ identifies (t,p) with $(t+1, \Phi(p))$ for some diffeomorphism Φ of the page P that is the identity in a small neighborhood of ∂P.

Construct M as the mapping torus $P_{\Phi^k} = \mathbb{R} \times P/\sim_k$, where $\sim_k$ identifies (t,p) with $(t+1, \Phi^k(p))$ for the diffeomorphism Φ on P. At the boundary the mapping torus is still diffeomorphic to $\mathbb{S}^1 \times (-\varepsilon, 0] \times \partial P$ such that we can glue in $\mathbb{D}^2 \times B$ to obtain a closed manifold M.

Define the projection $f\colon M \to N$ of the branched covering piecewise:

$$\begin{array}{ccccc} M & \cong & P_{\Phi^k} & \cup_{\mathbb{S}^1 \times \partial P} & \mathbb{D}^2 \times B \\ f \downarrow & & \downarrow f_1 & & \downarrow f_2 \\ N & \cong & P_\Phi & \cup_{\mathbb{S}^1 \times \partial P} & \mathbb{D}^2 \times B \end{array}$$

The map $f_1\colon P_{\Phi^k} \to P_\Phi$ is given by $f_1([t,p]) = [kt, p]$, and the map $f_2\colon \mathbb{D}^2 \times B \to \mathbb{D}^2 \times B$ is given by $f_2(re^{i\varphi}, p) = (g(r)e^{ik\varphi}, p)$, where $g(r)$ is a smooth strictly increasing function on $\mathbb{R}_{\geq 0}$ that is equal to r^k close to zero and equal to r for $r > \delta$ with $\delta > 0$ very small. Then it is clear that f defines a branched covering.

It is clear by Lemma 1 that M supports a contact structure compatible with f. The contact form on M is obtained by taking the pull-back $f^*\alpha$ and adding a small 1-form γ such that $d\gamma|_{\ker df} > 0$. This γ can be chosen to be of the form $\gamma = \varepsilon r^2 \rho(r)\, d\varphi$ on $\mathbb{D}^2 \times B$.

It is also clear that $(f^{-1}(B), \sqrt[k]{\vartheta \circ f})$ is an open book decomposition of M. Since $d\gamma$ vanishes, when restricted to any page, it follows that $\alpha + \gamma$ is supported by this open book.

□

4. Brieskorn manifolds and their canonical contact structures

Before talking about Brieskorn manifolds, we will briefly collect some facts about the sphere: Assume $\mathbb{S}^{2n-1}$ to be embedded in the standard way in $\mathbb{C}^n$. We will denote the points of $\mathbb{C}^n$ by $\mathbf{z} = (z_1, \ldots, z_n)$. The standard contact form on the sphere is

$$\alpha_{\mathrm{std}} = \frac{i}{2} \sum_{j=1}^{n} (z_j\, d\bar{z}_j - \bar{z}_j\, dz_j)\ .$$

Lemma 3. *The 1-form*

$$\beta = \frac{i}{2} \sum_{j=1}^{n} a_j (z_j \, d\bar{z}_j - \bar{z}_j \, dz_j) \ ,$$

with $a_j \in \mathbb{N}$, is isotopic to the standard contact form on $\mathbb{S}^{2n-1} \subset \mathbb{C}^n$.

Proof. The proof works by taking the linear interpolation between β and α_{std}, and checking that all forms in the family are contact. This allows us to use Gray stability.

□

Now, we will explain what a Brieskorn manifold is. Let $f\colon \mathbb{C}^{n+1} \to \mathbb{C}$ be a polynomial of the form

$$f(z_0, z_1, \dots, z_n) = z_0^{a_0} + \cdots + z_n^{a_n} \ ,$$

with fixed numbers $a_0, \dots, a_n \in \mathbb{N}$. It is easy to see that the variety $V_f := f^{-1}(0)$ has a single isolated singularity at $(0, \dots, 0)$. Outside the origin, the equation describes a smooth submanifold of codimension 2, because the matrix

$$\begin{pmatrix} \partial f & \bar\partial f \\ \partial \bar f & \bar\partial \bar f \end{pmatrix} = \begin{pmatrix} a_0 z_0^{a_0-1} & \cdots & a_n z_n^{a_n-1} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & a_0 \bar z_0^{a_0-1} & \cdots & a_n \bar z_n^{a_n-1} \end{pmatrix}$$

has full rank.

Definition. The *Brieskorn manifold* $\Sigma(a_0, \dots, a_n)$ is defined as the intersection

$$\Sigma(a_0, \dots, a_n) := V_f \cap \mathbb{S}^{2n+1} \ .$$

This set is, as its name suggests, a manifold. This can be easily seen by noting that V_f is transverse to $\mathbb{S}^{2n+1}$. Since the sphere has codimension 1, it is enough to find a vector field Z on V_f, which is everywhere transverse to the sphere. The $\mathbb{R}$-action

$$\begin{aligned} \mathbb{R} \times \mathbb{C}^{n+1} &\to \mathbb{C}^{n+1} \\ (z_0, \dots, z_n) &\mapsto (e^{t/a_0} z_0, \dots, e^{t/a_n} z_n) \end{aligned}$$

restricts to the variety V_f, and its infinitesimal generator

$$Z(z_0, \dots, z_n) = \left(\frac{z_0}{a_0}, \dots, \frac{z_n}{a_n} \right)$$

is always transverse to the sphere, because

$$\mathcal{L}_Z(|z_0|^2 + \cdots + |z_n|^2 - 1) = \frac{1}{a_0} |z_0|^2 + \cdots + \frac{1}{a_n} |z_n|^2 \neq 0 \ .$$

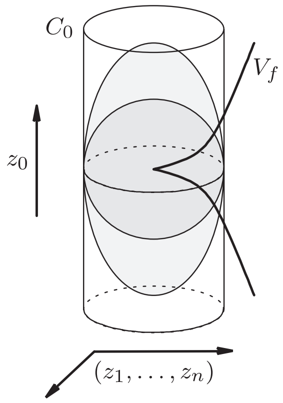

FIGURE 1. *The manifolds* $\Sigma(a_0, \ldots, a_n)$ *and* $\widetilde{\Sigma}(a_0, \ldots, a_n)$ *are obtained by intersecting* V_f *with different hypersurfaces.*

In the rest of the article, we make extensive use of a related manifold: Instead of taking the intersection between V_f and a sphere, define

$$\widetilde{\Sigma}(a_0, \ldots, a_n) := V_f \cap C_0 \ ,$$

where C_0 is the spherical cylinder given by

$$C_0 := \mathbb{C} \times \mathbb{S}^{2n-1} = \{(z_0, z_1, \ldots, z_n) \mid (z_1, \ldots, z_n) \in \mathbb{S}^{2n-1}\} \ .$$

As above it is easy to check that this set is a manifold, because for the defining equation of C_0, we obtain

$$\mathcal{L}_Z(|z_1|^2 + \cdots + |z_n|^2 - 1) = \frac{1}{a_1}|z_1|^2 + \cdots + \frac{1}{a_n}|z_n|^2 \neq 0 \ .$$

The Brieskorn manifold is of course diffeomorphic to $\widetilde{\Sigma}(a_0, \ldots, a_n)$ (see Figure 1). In fact, let

$$R_s := s\,|z_0|^2 + |z_1|^2 + \cdots + |z_n|^2 \ ,$$

then we can define a family of submanifolds Σ_s with $s \in [0, 1]$ by

$$\Sigma_s := V_f \cap R_s^{-1}(1) \ ,$$

where Σ_1 is equal to $\Sigma(a_0, \ldots, a_n)$ and Σ_0 is equal to $\widetilde{\Sigma}(a_0, \ldots, a_n)$.

LEMMA 4. *There is an isotopy* Φ_s *in* V_f *between* $\Sigma(a_0, \ldots, a_n)$ *and* Σ_s.

PROOF. Consider the $\mathbb{R}$-flow above, but let the time-parameter depend on the point that is being mapped, i.e. consider the map

$$\Phi_s\colon (z_0, \ldots, z_n) \mapsto (e^{T/a_0} z_0, \ldots, e^{T/a_n} z_n) ,$$

where $T = T(z_0, \ldots, z_n; s)$ is a function with the following properties: For a point $(z_0, \ldots, z_n) \in \Sigma(a_0, \ldots, a_n)$, we want its image to lie in Σ_s, hence the equation

$$\begin{aligned} 1 &= s\left|e^{T/a_0} z_0\right|^2 + \left|e^{T/a_1} z_1\right|^2 + \cdots + \left|e^{T/a_n} z_n\right|^2 \\ &= s\, e^{2T/a_0} |z_0|^2 + e^{2T/a_1} |z_1|^2 + \cdots + e^{2T/a_n} |z_n|^2 \end{aligned}$$

needs to hold. For any point $(z_0, \ldots, z_n)$ there is a unique solution $T(z_0, \ldots, z_n; s) \geq 0$, because the right-hand side of the equation is a strictly increasing continuous function in T that takes a value less than 1 for $T = 0$.

To prove that the map Φ_s is a bijection, construct a map $\widetilde{\Phi}_s$ analogously to the one above, which maps Σ_s into $\Sigma(a_0, \ldots, a_n)$. It is easy to see that these maps are mutually inverse.

That Φ_s is smooth follows from the fact that T is, and this is proved by checking the inequality:

$$\frac{d}{dT}\left(s e^{2T/a_0} |z_0|^2 + e^{2T/a_1} |z_1|^2 + \cdots + e^{2T/a_n} |z_n|^2 - 1\right) > 0 ,$$

which allows us to apply the implicit function theorem. The map Φ_s is a bijective local diffeomorphism between closed manifolds, hence it is a diffeomorphism.

□

LEMMA 5. *For every Σ_s with $s \in (0, 1]$, the corresponding 1-form*

$$\alpha_s := \frac{i}{2}\Big(s a_0 \left(z_0\, d\bar{z}_0 - \bar{z}_0\, dz_0\right) + a_1 \left(z_1\, d\bar{z}_1 - \bar{z}_1\, dz_1\right) + \cdots + a_n \left(z_n\, d\bar{z}_n - \bar{z}_n\, dz_n\right)\Big)$$

is a contact form, and by Gray stability it follows that every Σ_s (with the exception of $\Sigma_0 = \widetilde{\Sigma}(a_0, \ldots, a_n)$) is contactomorphic to $\Sigma(a_0, \ldots, a_n)$.

PROOF. A long but trivial calculation yields

$$\begin{aligned} &\alpha_s \wedge d\alpha_s^{n-1} \wedge dR_s \wedge df \wedge d\bar{f} \\ &\quad = \Big(s\bar{f} \sum_{j=0}^{n} a_j z_j^{a_j} + s f \sum_{j=0}^{n} a_j \bar{z}_j^{a_j} - 2 a_0 R_s |z_0|^{2(a_0-1)} - 2 s R_s \sum_{j=1}^{n} a_j |z_j|^{2(a_j-1)}\Big)\, \Omega , \end{aligned}$$

with $\Omega := i^n/2(n-1)!\, a_0 \cdots a_n\, dz_0 \wedge d\bar{z}_0 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n$. On Σ_s we have $f = \bar{f} = 0$ and $R_s = 1$, and hence the term is equal to

$$-2\Big(a_0 |z_0|^{2(a_0-1)} + s \sum_{j=1}^{n} a_j |z_j|^{2(a_j-1)}\Big)\, \Omega ,$$

which only vanishes, if both $s = 0$ and $z_0 = 0$, i.e. at points $(0, z_1, \ldots, z_n) \in \Sigma_0 = \widetilde{\Sigma}(a_0, \ldots, a_n)$.

□

Remark 3. Note that by Lemmas 3 and 5 it follows that $(\Sigma(a_1,\dots,a_n),\alpha_1)$ is a contact submanifold of $(\mathbb{S}^{2n-1},\beta)$, and the 1-form α_0 on $\widetilde{\Sigma}(a_0,\dots,a_n)$ is equal to the pull-back $\pi_0^*\alpha_{\mathrm{std}}$ of the standard structure on the sphere under the projection $\pi_0\colon \mathbb{C}^{n+1}\to\mathbb{C}^n, (z_0,\dots,z_n)\mapsto(z_1,\dots,z_n)$.

Theorem 6. *The Brieskorn manifold* $(\Sigma(a_0,\dots,a_n),\alpha_1)$ *is a contact branched cover of the standard sphere* $(\mathbb{S}^{2n-1},\alpha_{\mathrm{std}})$. *More precisely, the map* $\pi_0\colon \mathbb{C}^{n+1}\to\mathbb{C}^n, (z_0,\dots,z_n)\mapsto(z_1,\dots,z_n)$ *induces an* a_0*-fold cyclic branched contact covering*

$$\pi_0\colon \widetilde{\Sigma}(a_0,\dots,a_n)\to(\mathbb{S}^{2n-1},\beta)$$

with branching locus $\Sigma(a_1,\dots,a_n)\subset\mathbb{S}^{2n-1}$.

Note that the latter statement justifies the former, because $(\mathbb{S}^{2n-1},\alpha_{\mathrm{std}})\cong(\mathbb{S}^{2n-1},\beta)$, and $(\Sigma(a_0,\dots,a_n),\alpha_1)$ is contactomorphic to $\widetilde{\Sigma}(a_0,\dots,a_n)$ with the contact structure induced by the branched covering.

Proof. It has been known for a long time that π_0 restricted to $\widetilde{\Sigma}(a_0,\dots,a_n)$ is a branched covering over the sphere. This can be easily seen by noting that $\pi_0(\widetilde{\Sigma}(a_0,\dots,a_n))\subset\mathbb{S}^{2n-1}$, and that this map is surjective follows because a point $(z_1,\dots,z_n)\in\mathbb{S}^{2n-1}$ is covered by $(z_0,z_1,\dots,z_n)\in\widetilde{\Sigma}(a_0,\dots,a_1)$, where z_0 is one of the roots $\sqrt[a_0]{-(z_1^{a_1}+\cdots+z_n^{a_n})}$. Every point of the sphere is covered by a_0 points with the exception of the points on the branching locus $\Sigma(a_1,\dots,a_n)\subset\mathbb{S}^{2n-1}$.

As remarked above, the 1-form α_0 on $\widetilde{\Sigma}(a_0,\dots,a_n)$ is equal to $\pi_0^*\beta$. By adding a small 1-form $\varepsilon\,\gamma$ to α_0 such that $d\gamma|_{\ker d\pi_0}>0$, we obtain a contact form. A possible choice for such a form is

$$\gamma=\frac{i}{2}\,(z_0\,d\bar z_0-\bar z_0\,dz_0)$$

for sufficiently small $\varepsilon>0$, because the kernel of $d\pi_0$ is only non-trivial at $(0,z_1,\dots,z_n)\in\widetilde{\Sigma}(a_0,\dots,a_n)$, and the kernel lies in the z_0-plane.

The only thing left to show is that $(\widetilde{\Sigma}(a_0,\dots,a_n),\alpha_0+\varepsilon\,\gamma)$ is contactomorphic to $(\Sigma(a_0,\dots,a_n),\alpha_1)$. This is most easily seen by using the contact forms $\widetilde{\alpha}_s=\left(\Phi_s^{-1}\right)^*\alpha_s$ on $\widetilde{\Sigma}(a_0,\dots,a_n)$ for $s\in(0,1]$. This is a smooth family of forms that connects to α_0, and the derivative

$$\left.\frac{d}{ds}\right|_{s=0}\widetilde{\alpha}_s=\frac{i}{2}\,(z_0\,d\bar z_0-\bar z_0\,dz_0)$$

has the properties needed to apply Lemma 1. $\square$

Remark 4. It is interesting to consider, whether

$$\alpha_- := \frac{i}{2}\Big(-Ca_0\,(z_0\,d\bar z_0 - \bar z_0\,dz_0) + \sum_{j=1}^n a_j\,(z_j\,d\bar z_j - \bar z_j\,dz_j)\Big)$$

for very large $C > 0$ also gives a contact form. The rationale is that the open book decomposition of such a manifold would have the same pages, but the monodromy map would be inverted.

To check that α_- is a contact form, the following term should not vanish:

$$\begin{aligned}\alpha_- &\wedge (d\alpha_-)^{n-1} \wedge dR_1 \wedge df \wedge d\bar f \\ &= \frac{i^n(n-1)!}{2}\, a_0\cdots a_n\Big(- 2a_0|z_0|^{2(a_0-1)} + 2C\sum_{j=1}^n a_j|z_j|^{2(a_j-1)} - \\ &\qquad - (C-1)\,(a_0-1)\Big(\bar z_0^{a_0}\sum_{j=1}^n a_j z_j^{a_j} + z_0^{a_0}\sum_{j=1}^n a_j\bar z_j^{a_j}\Big)\Big)\, dz_0\wedge\cdots\wedge d\bar z_n.\end{aligned}$$

It is easy to see that this is the case for $a_0 = -1$, i.e. one gets a large set of potentially different contact structures on the sphere. For all Brieskorn manifolds $\Sigma(a_0, a_1, \dots, a_1)$, it is also easy to check that α_- is a contact form. In particular on $\Sigma(k, 2, \dots, 2)$, it can be shown by an explicit computation like the one in [KN] that the open book decomposition uses a k-fold left-handed Dehn twist for the monodromy map, which is indeed the inverse of the standard monodromy.

Unfortunately, for general combinations of integers $a_j \in \mathbb{N}$, it is quite easy to find examples where the contact condition breaks down.

Finally, the following theorem describes a Milnor open book on $\mathbb{S}^{2n-1}$ which supports the contact structure β.

Theorem 7. *Define the polynomial* $f(z_1, \dots, z_n) = z_1^{a_1} + \cdots + z_n^{a_n}$ *on* $\mathbb{C}^n$ *with* $a_j \in \mathbb{N}$. *The sphere* $\mathbb{S}^{2n-1}$ *can be given an open book with binding* $B := \Sigma(a_1, \dots, a_n) := \mathbb{S}^{2n-1} \cap f^{-1}(0)$, *and page fibration*

$$\vartheta\colon \mathbb{S}^{2n-1} - B \to \mathbb{S}^1,\ \mathbf{z} \mapsto \frac{f(\mathbf{z})}{|f(\mathbf{z})|}\ ,$$

with $\mathbf{z} = (z_1, \dots, z_n)$. *The contact form* β *on* $\mathbb{S}^{2n-1}$ *is supported by this open book.*

Proof. Milnor showed in [Mil] that the structure defined in the lemma is an open book. Hence it only remains to show that (B, ϑ) supports the contact form β.

The binding B is a Brieskorn manifold and β is a contact form for such a manifold as proved in Lemma 5.

To show that $d\beta$ is a symplectic form on a page $P_{\vartheta_0} = \vartheta^{-1}(\vartheta_0)$, note that the map

$$e^{it} \cdot (z_1, \dots, z_n) = (e^{it/a_1} z_1, \dots, e^{it/a_n} z_n)$$

is a diffeomorphism from a page P_{ϑ_0} to P_{ϑ_0+t}, and at the same time it is the flow of the Reeb field X_{Reeb} of β:

$$X_{\mathrm{Reeb}} = \frac{d}{dt}(e^{it/a_1} z_1, \dots, e^{it/a_n} z_n) = \sum_{j=1}^{n} \frac{1}{a_j}\left(x_j \frac{\partial}{\partial y_j} - y_j \frac{\partial}{\partial x_j}\right) .$$

One computes that $\iota_{X_{\mathrm{Reeb}}} d\beta = -2d\left(\sum_{j=1}^n |z_j|^2\right)$, and $\beta(X_{\mathrm{Reeb}}) = \sum_{j=1}^n |z_j|^2 = 1$. The Reeb field points in positive direction transversely through any page P_{ϑ_0}, and hence $d\beta|_{P_{\vartheta_0}}$ is non-degenerate.

Finally if the binding B is connected, the orientation of B as boundary of the page $\overline{P}_{\vartheta_0}$ and as contact manifold (B, β) is compatible by Remark 2. If B is non-connected (which is only the case for $\dim B = 1$, because $(2n+1)$-dimensional Brieskorn manifolds are $(n-1)$-connected) each component of $B = \Sigma(a_0, a_1)$ can be written in the form

$$\left\{(e^{i\varphi/a_0}, Ae^{i\varphi/a_1}) \;\middle|\; \varphi \in [0, 2\pi \operatorname{lcm}(a_0, a_1)]\right\} ,$$

where A_1 is an a_1-th root of -1. The φ-parametrization gives the correct orientation, and it follows that the integral of α over any of the N components of B has the same value C. In particular, it follows

$$0 < \int_P d\alpha = \int_B \alpha = NC ,$$

and hence $C > 0$.

□

5. Topological description of the monodromy of the open book of $\Sigma(a_1, \dots, a_n)$

In [Mil] Milnor worked out the topology of the page of the above open book (B, ϑ) of $\Sigma(a_1, \dots, a_n)$ and described the monodromy ψ by its action on $H_1(B)$. Let Ω_a denote the finite cyclic group consisting of all a-th roots of unity and let J denote all linear combinations $(t_1\omega_1, \dots, t_n\omega_n)$ where $\omega_i \in \Omega_{a_i}$, $t_i \geq 0$ $(1 \leq i \leq n)$ and $t_1 + \cdots + t_n = 1$. Then J is a deformation retract of the fiber $P_1 = \vartheta^{-1}(1)$ (op.cit., Lemma 9.2). Here the dimension of P_1 is $2n$. Furthermore, the free Abelian group $H_n(P_1; \mathbb{Z})$ has rank $\mu = (a_1 - 1) \cdots (a_n - 1)$ (op.cit., Theorem 9.1).

It is straightforward to prove the following fact which appears in a more general setting in [A'C, Theorem 3] for $n = 2$.

LEMMA 8. *There is a basis for $H_{n-1}(J)$ in which the $\mu \times \mu$ matrix Ψ for the monodromy ψ of the open book of $\Sigma(a_1, a_2, \ldots, a_n)$ $(n > 1,\ \gcd(a_1, a_2, \cdots, a_n) = 1)$ is*

$$\Psi = A_{a_1-1} \otimes A_{a_2-1} \otimes \cdots \otimes A_{a_n-1}$$

where A_p is the $p \times p$ matrix given by

$$A_p = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ -1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 & 0 \end{bmatrix}.$$

As the last goal, we want to express this monodromy as a product of Dehn twists along Lagrangian spheres. In dimension 3 (i.e. $n = 2$), each circle is Lagrangian on the 2-dimensional pages. Furthermore, in a rational homology sphere the binding determines the open book decomposition up to isotopy (we learned this from [CP]). Hence given the binding in a Brieskorn sphere, any corresponding description of the monodromy in terms of Dehn twists is the solution. This has been described in a purely topological manner, for example, in [AO, Theorem 1]. The question remaining is the relation between the cycles of Dehn twists in these descriptions and the generators of $H_1(J)$ that appear in Lemma 8.

For higher dimensions the problem is more complicated. The skeleton given by Milnor can be made piecewise smooth, and the smooth segments are Lagrangian submanifolds. Unfortunately, we do not yet know how to find proper Lagrangian embeddings of the spheres that constitute the skeleton of a page as a bouquet.

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