3

Some Properties of the Euler Equations

In this chapter we apply the mathematical tools presented in Chap. 2 to analyse some of the basic properties of the time–dependent Euler equations. As seen in Chap. 1, the Euler equations result from neglecting the effects of viscosity, heat conduction and body forces on a compressible medium. Here we show that these equations are a system of *hyperbolic* conservations laws and study some of their mathematical properties. In particular, we study those properties that are essential for finding the solution of the Riemann problem in Chap. 4. We analyse the eigenstructure of the equations, that is, we find eigenvalues and eigenvectors; we study properties of the characteristic fields and establish basic relations across rarefactions, contacts and shock waves. It is worth remarking that the process of finding eigenvalues and eigenvectors usually involves a fair amount of algebra as well as some familiarity with basic physical quantities and their relations. For very complex systems of equations finding eigenvalues and eigenvectors may require the use of symbolic manipulators. Useful background reading for this chapter is found in Chaps. 1 and 2.

3.1 The One–Dimensional Euler Equations

Here we study the one–dimensional time–dependent Euler equations with an ideal Equation of State, using conservative and non–conservative formulations. The basic structure of the solution of the Riemann problem is outlined along with a detailed study of the elementary waves present in the solution. We provide the foundations for finding the exact solution of the Riemann problem in Chap. 4.

3.1.1 Conservative Formulation

The conservative formulation of the Euler equations, in differential form, is

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$$\mathbf{U}_t + \mathbf{F}(\mathbf{U})_x = \mathbf{0} \;, \tag{3.1}$$

where $\mathbf{U}$ and $\mathbf{F}(\mathbf{U})$ are the vectors of conserved variables and fluxes, given respectively by

$$\mathbf{U} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \equiv \begin{bmatrix} \rho \\ \rho u \\ E \end{bmatrix} \;, \quad \mathbf{F} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} \equiv \begin{bmatrix} \rho u \\ \rho u^2 + p \\ u(E+p) \end{bmatrix} \;. \tag{3.2}$$

Here ρ is density, p is pressure, u is particle velocity and E is total energy per unit volume

$$E = \rho(\frac{1}{2}u^2 + e) \;, \tag{3.3}$$

where e is the *specific internal energy* given by a caloric Equation of State (EOS)

$$e = e(\rho, p) \;. \tag{3.4}$$

For ideal gases one has the simple expression

$$e = e(\rho, p) = \frac{p}{(\gamma - 1)\rho} \;, \tag{3.5}$$

with $\gamma = c_p/c_v$ denoting the *ratio of specific heats*. From the EOS (3.5) and using equation (1.36) of Chap. 1 we write the *sound speed* a as

$$a = \sqrt{(p/\rho^2 - e_\rho)/e_p} = \sqrt{\frac{\gamma p}{\rho}} \;. \tag{3.6}$$

The conservation laws (3.1)–(3.2) may also be written in quasi–linear form

$$\mathbf{U}_t + \mathbf{A}(\mathbf{U})\mathbf{U}_x = \mathbf{0} \;, \tag{3.7}$$

where the coefficient matrix $\mathbf{A}(\mathbf{U})$ is the *Jacobian matrix*

$$\mathbf{A}(\mathbf{U}) = \frac{\partial \mathbf{F}}{\partial \mathbf{U}} = \begin{bmatrix} \partial f_1/\partial u_1 & \partial f_1/\partial u_2 & \partial f_1/\partial u_3 \\ \partial f_2/\partial u_1 & \partial f_2/\partial u_2 & \partial f_2/\partial u_3 \\ \partial f_3/\partial u_1 & \partial f_3/\partial u_2 & \partial f_3/\partial u_3 \end{bmatrix} \;.$$

Proposition 3.1 (Jacobian Matrix). *The Jacobian matrix* $\mathbf{A}$ *is*

$$\mathbf{A}(\mathbf{U}) = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{1}{2}(\gamma-3)(\frac{u_2}{u_1})^2 & (3-\gamma)(\frac{u_2}{u_1}) & \gamma - 1 \\ -\frac{\gamma u_2 u_3}{u_1^2} + (\gamma-1)(\frac{u_2}{u_1})^3 & \frac{\gamma u_3}{u_1} - \frac{3}{2}(\gamma-1)(\frac{u_2}{u_1})^2 & \gamma(\frac{u_2}{u_1}) \end{bmatrix} \;.$$

Proof. First we express all components f_i of the flux vector $\mathbf{F}$ in terms of the components u_i of the vector $\mathbf{U}$ of conserved variables, namely $u_1 \equiv \rho$, $u_2 \equiv \rho u$, $u_3 \equiv E$. Obviously $f_1 = u_2 \equiv \rho u$. To find f_2 and f_3 we first need to express the pressure p in terms of the conserved variables. From (3.3) and (3.5) we find

$$p = (\gamma - 1)[u_3 - \frac{1}{2}(u_2^2/u_1)] \ .$$

Thus the flux vector can be written as

$$\mathbf{F}(\mathbf{U}) = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} \equiv \begin{bmatrix} u_2 \\ \frac{1}{2}(3-\gamma)\frac{u_2^2}{u_1} + (\gamma-1)u_3 \\ \gamma\frac{u_2}{u_1}u_3 - \frac{1}{2}(\gamma-1)\frac{u_2^3}{u_1^2} \end{bmatrix} \ .$$

By direct evaluation of all partial derivatives we arrive at the sought result.

Exercise 3.2. Write the Jacobian matrix $\mathbf{A}(\mathbf{U})$ in terms of the the sound speed a and the particle velocity u.

Solution 3.3.

$$\mathbf{A}(\mathbf{U}) = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2}(\gamma-3)u^2 & (3-\gamma)u & \gamma-1 \\ \frac{1}{2}(\gamma-2)u^3 - \frac{a^2 u}{\gamma-1} & \frac{3-2\gamma}{2}u^2 + \frac{a^2}{\gamma-1} & \gamma u \end{bmatrix} \ . \quad (3.8)$$

Often, the Jacobian matrix is also expressed in terms of the *total specific enthalpy* H, which is related to the *specific enthalpy* h and other variables, namely

$$H = (E+p)/\rho \equiv \frac{1}{2}u^2 + h \ , \quad h = e + p/\rho \ . \quad (3.9)$$

The Jacobian matrix may also be written as

$$\mathbf{A}(\mathbf{U}) = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2}(\gamma-3)u^2 & (3-\gamma)u & \gamma-1 \\ (\gamma-1)u^3 - \gamma u E/\rho & \gamma E/\rho - \frac{3}{2}(\gamma-1)u^2 & \gamma u \end{bmatrix} \ . \quad (3.10)$$

Proposition 3.4 (The Homogeneity Property). *The Euler equations (3.1)–(3.2) with the* ideal–gas *EOS (3.5) satisfy the homogeneity property*

$$\mathbf{F}(\mathbf{U}) = \mathbf{A}(\mathbf{U})\mathbf{U} \ . \quad (3.11)$$

Proof. The proof of this property is immediate. By multiplying the Jacobian matrix (3.8) by the vector $\mathbf{U}$ in (3.2) we identically reproduce the vector $\mathbf{F}(\mathbf{U})$ of fluxes in (3.2).

This remarkable property of the Euler equations forms the basis for numerical schemes of the *Flux Vector Splitting* type studied in Chap. 8. Note that the relationship between the flux $\mathbf{F}$, the coefficient matrix $\mathbf{A}$ and the conserved variables $\mathbf{U}$ for the Euler equations is identical to that for linear systems with constant coefficients, see Sect. 2.4 of Chap. 2. This property is also satisfied by the Euler equations with an Equation of State that is slightly more general than (3.5). See Steger and Warming [463] for details.

Proposition 3.5. *The eigenvalues of the Jacobian matrix* $\mathbf{A}$ *are*

$$\lambda_1 = u - a \; , \; \lambda_2 = u \; , \; \lambda_3 = u + a \tag{3.12}$$

and the corresponding right eigenvectors are

$$\mathbf{K}^{(1)} = \begin{bmatrix} 1 \\ u - a \\ H - ua \end{bmatrix} , \; \mathbf{K}^{(2)} = \begin{bmatrix} 1 \\ u \\ \frac{1}{2}u^2 \end{bmatrix} , \; \mathbf{K}^{(3)} = \begin{bmatrix} 1 \\ u + a \\ H + ua \end{bmatrix} . \tag{3.13}$$

Proof. Use of the expression (3.8) for $\mathbf{A}$ and the *characteristic polynomial*

$$|\mathbf{A} - \lambda \mathbf{I}| = 0 \; ,$$

lead to

$$\begin{array}{l} (\lambda - u)(\gamma u - \lambda)\left[(2u - \gamma u - \lambda\right] + \\ (\lambda - u)\left[-a^2 - (\gamma - 1)u^2 + (\gamma - 1)\gamma u^2\right] + \Delta = 0 \; , \end{array}$$

where

$$\Delta = \frac{1}{2}(\gamma u - \lambda)(1 - \gamma)u^2 - \frac{1}{2}(\gamma - 1)u^2\left[(1 - 2\gamma)\lambda + \gamma u\right] \; .$$

Manipulations show that Δ also contains the common factor $(\lambda - u)$, which implies that $\lambda_2 = u$ is a root of the characteristic polynomial and thus an eigenvalue of $\mathbf{A}$. After cancelling $(\lambda - u)$ the remaining terms give

$$\lambda^2 - 2u\lambda + u^2 - a^2 = 0 \; ,$$

with real roots

$$\lambda_1 = u - a \; , \; \lambda_3 = u + a \; .$$

Therefore the eigenvalues are: $\lambda_1 = u - a$, $\lambda_2 = u$, $\lambda_3 = u + a$ as claimed. To find the right eigenvectors we look, see Sect. 2.1 of Chap. 2, for a vector $\mathbf{K} = [k_1, k_2, k_3]^T$ such that

$$\mathbf{AK} = \lambda \mathbf{K} \; .$$

By substituting $\lambda = \lambda_i$ in turn, solving for the components of the vector $\mathbf{K}$ and selecting appropriate values for the scaling factors we find the desired result.

The eigenvalues are all real and the eigenvectors $\mathbf{K}^{(1)}$, $\mathbf{K}^{(2)}$, $\mathbf{K}^{(3)}$ form a complete set of *linearly independent eigenvectors*. We have thus proved that the time–dependent, one–dimensional Euler equations for ideal gases are *hyperbolic*. In fact these equations are *strictly hyperbolic*, because the eigenvalues are all real and *distinct*, as long as the sound speed a remains positive. Hyperbolicity remains a property of the Euler equations for more general equations of state, as we shall see in Chap. 4 for covolume gases.

3.1.2 Non–Conservative Formulations

The Euler equations (3.1)–(3.2) may be formulated in terms of variables other than the conserved variables. For smooth solutions all formulations are equivalent. For solutions containing shock waves however, non–conservative formulations give incorrect shock solutions. This point is addressed via the shallow water equations and the isothermal equations in Sect. 3.3 of this chapter. In spite of this, non–conservative formulations have some advantages over their conservative counterpart, when analysing the equations, for instance. Also, from the numerical point of view, there has been a recent revival of the idea of using schemes for non–conservative formulations of the equations. See e.g. Karni [278] and Toro [508], [517].

Primitive–Variable Formulations

For smooth solutions the equations may be formulated, and solved, using variables other than the conserved variables. For the one–dimensional case one possibility is to choose a vector $\mathbf{W} = (\rho, u, p)^T$ of *primitive* or *physical* variables, with p given by the equation of state. Expanding derivatives in the first of equations (3.1)–(3.2), the mass equation, we obtain

$$\rho_t + u\rho_x + \rho u_x = 0 \; . \tag{3.14}$$

By expanding derivatives in the second of equations (3.1)–(3.2), the momentum equation, we obtain

$$u \left[\rho_t + u\rho_x + \rho u_x\right] + \rho \left[u_t + u u_x + \frac{1}{\rho} p_x\right] = 0 \; .$$

Use of (3.14) followed by division through by ρ gives

$$u_t + u u_x + \frac{1}{\rho} p_x = 0 \; . \tag{3.15}$$

In a similar manner, the energy equation in (3.1)–(3.2) can be rearranged so as to use (3.14) and (3.15). The result is

$$p_t + \rho a^2 u_x + u p_x = 0 \; . \tag{3.16}$$

Thus, in quasi–linear form we have

$$\mathbf{W}_t + \mathbf{A}(\mathbf{W})\mathbf{W}_x = \mathbf{0} \,, \tag{3.17}$$

where

$$\mathbf{W} = \begin{bmatrix} \rho \\ u \\ p \end{bmatrix} \,, \quad \mathbf{A}(\mathbf{W}) = \begin{bmatrix} u & \rho & 0 \\ 0 & u & 1/\rho \\ 0 & \rho a^2 & u \end{bmatrix} \,. \tag{3.18}$$

Proposition 3.6. *The system (3.17)–(3.18) has real eigenvalues*

$$\lambda_1 = u - a \,,\ \lambda_2 = u \,,\ \lambda_3 = u + a \,, \tag{3.19}$$

with corresponding right eigenvectors

$$\mathbf{K}^{(1)} = \alpha_1 \begin{bmatrix} 1 \\ -a/\rho \\ a^2 \end{bmatrix} \,,\ \mathbf{K}^{(2)} = \alpha_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \,,\ \mathbf{K}^{(3)} = \alpha_3 \begin{bmatrix} 1 \\ a/\rho \\ a^2 \end{bmatrix} \,. \tag{3.20}$$

where $\alpha_1, \alpha_2, \alpha_3$ are scaling factors, or normalisation parameters, see Sect. 2.1 of Chap. 2. The left eigenvectors are

$$\left.\begin{aligned} \mathbf{L}^{(1)} &= \beta_1(0, 1, -\tfrac{1}{\rho a}) \,, \\ \mathbf{L}^{(2)} &= \beta_2(1, 0, -\tfrac{1}{a^2}) \,, \\ \mathbf{L}^{(3)} &= \beta_3(0, 1, \tfrac{1}{\rho a}) \,, \end{aligned}\right\} \tag{3.21}$$

where $\beta_1, \beta_2, \beta_3$ are scaling factors.

Proof. (Left to the reader).

Exercise 3.7. Verify that by choosing appropriate normalisation parameters $\alpha_1, \alpha_2, \alpha_3$ and $\beta_1, \beta_2, \beta_3$ in (3.20) and (3.21) respectively, the left and right eigenvectors $\mathbf{L}^{(j)}$ and $\mathbf{K}^{(j)}$ of $\mathbf{A}(\mathbf{W})$ are *bi–orthonormal*, that is

$$\mathbf{L}^{(j)} \cdot \mathbf{K}^{(i)} = \begin{cases} 1 \text{ if } i = j \,, \\ 0 \text{ otherwise} \,. \end{cases} \tag{3.22}$$

Characteristic Equations

Recall that the eigenvalues $\lambda_1 = u - a$, $\lambda_2 = u$, $\lambda_3 = u + a$ define characteristic directions $\mathrm{d}x/\mathrm{d}t = \lambda_i$ for $i = 1, 2, 3$. For a set of partial differential equations (3.17) a *characteristic equation* says that in a direction $\mathrm{d}x/\mathrm{d}t = \lambda_i$, $\mathbf{L}^{(i)} \cdot \mathrm{d}\mathbf{W} = 0$, or in full

$$\mathbf{L}^{(i)} \cdot \begin{bmatrix} \mathrm{d}\rho \\ \mathrm{d}u \\ \mathrm{d}p \end{bmatrix} = 0 \, . \tag{3.23}$$

By expanding (3.23) for $\mathbf{L}^{(1)}, \mathbf{L}^{(2)}, \mathbf{L}^{(3)}$ we obtain the *characteristic equations*

$$\left.\begin{array}{l} \mathrm{d}p - \rho a \, \mathrm{d}u = 0 \text{ along } \mathrm{d}x/\mathrm{d}t = \lambda_1 = u - a \, , \\ \\ \mathrm{d}p - a^2 \, \mathrm{d}\rho = 0 \text{ along } \mathrm{d}x/\mathrm{d}t = \lambda_2 = u \, , \\ \\ \mathrm{d}p + \rho a \, \mathrm{d}u = 0 \text{ along } \mathrm{d}x/\mathrm{d}t = \lambda_3 = u + a \, . \end{array}\right\} \tag{3.24}$$

These differential relations hold true *along characteristic* directions. For numerical purposes, linearisation of these equations provides ways of solving the Riemann problem for the Euler equations, approximately; see Sect. 9.3 of Chap. 9.

Entropy Formulation

The entropy s can be written as

$$s = c_v \ln(\frac{p}{\rho^\gamma}) + C_0 \, , \tag{3.25}$$

where C_0 is a constant. From this equation we obtain

$$p = C_1 \rho^\gamma e^{s/c_v} \, , \tag{3.26}$$

where C_1 is a constant. Now, if in the primitive–variable formulation (3.17) we use entropy s instead of pressure p we have the new vector of unknowns

$$\mathbf{W} = (\rho, u, s)^T \, , \tag{3.27}$$

and a corresponding new way of expressing the governing equations.

Proposition 3.8. *The entropy s satisfies the following PDE*

$$s_t + u s_x = 0 \, . \tag{3.28}$$

Proof. From (3.25) and the expression (3.6) for the sound speed a we have

$$s_t = \frac{c_v}{p} \left[p_t - a^2 \rho_t \right] \, , \quad s_x = \frac{c_v}{p} \left[p_x - a^2 \rho_x \right] \, .$$

But from (3.16) $p_t = -\rho a^2 u_x - u p_x$, and hence $s_t + u s_x = 0$, as claimed.

The significance of the result is that

$$s_t + u s_x = \frac{\mathrm{d}s}{\mathrm{d}t} = 0 \, , \tag{3.29}$$

and so *in regions of smooth flow*, the entropy s is constant along particle paths $\mathrm{d}x/\mathrm{d}t = u$. Hence, along a particle path one has the *isentropic law* given by

$$p = C\rho^{\gamma} , \tag{3.30}$$

where $C = C(s_0)$ is a function of the *initial entropy* s_0 and is constant along the path so long as the flow remains smooth; see Sect. 1.6.2 of Chap. 1. In general of course, C changes from path to path. When solving the Riemann problem the initial entropy can be computed on the initial data of the Riemann problem, which is piece–wise constant. If C is the same constant throughout the flow domain we speak of *isentropic flow*, or sometimes, *homentropic flow*. This leads to the special set of governing equations (1.109)–(1.110) presented in Chap. 1. The governing equations for the entropy formulation, written in quasi–linear form, are

$$\mathbf{W}_t + \mathbf{A}(\mathbf{W})\mathbf{W}_x = \mathbf{0} , \tag{3.31}$$

with

$$\mathbf{A}(\mathbf{W}) = \begin{bmatrix} u & \rho & 0 \\ a^2/\rho & u & \frac{1}{\rho}\frac{\partial p}{\partial s} \\ 0 & 0 & u \end{bmatrix} . \tag{3.32}$$

Proposition 3.9. *The eigenvalues of system (3.31)–(3.32) are*

$$\lambda_1 = u - a \, , \; \lambda_2 = u \, , \; \lambda_3 = u + a \tag{3.33}$$

and the corresponding right eigenvectors are

$$\mathbf{K}^{(1)} = \begin{bmatrix} 1 \\ -a/\rho \\ 0 \end{bmatrix} , \; \mathbf{K}^{(2)} = \begin{bmatrix} -\frac{\partial p}{\partial s} \\ 0 \\ a^2 \end{bmatrix} , \; \mathbf{K}^{(3)} = \begin{bmatrix} 1 \\ a/\rho \\ 0 \end{bmatrix} . \tag{3.34}$$

Proof. (Left to the reader).

3.1.3 Elementary Wave Solutions of the Riemann Problem

Here we describe the structure of the solution of the Riemann problem as a set of *elementary waves* such as rarefactions, contacts and shock waves, see Sect. 2.4.4 of Chapt. 4. Each of these elementary waves are studied in detail. Basic relations across these waves are established. Such relations will be used in Chap. 4 to *connect* all unknown states to the data states and thus find the complete solution of the Riemann problem.

The Riemann problem for the one–dimensional, time dependent Euler equations (3.1)–(3.2) with data $(\mathbf{U}_{\mathrm{L}}, \mathbf{U}_{\mathrm{R}})$ is the IVP

$$\left.\begin{array}{l} \mathbf{U}_t + \mathbf{F}(\mathbf{U})_x = \mathbf{0} , \\ \mathbf{U}(x,0) = \mathbf{U}^{(0)}(x) = \begin{cases} \mathbf{U}_{\mathrm{L}} & \text{if } x < 0 , \\ \mathbf{U}_{\mathrm{R}} & \text{if } x > 0 . \end{cases} \end{array}\right\} \tag{3.35}$$

The physical analogue of the Riemann problem is the *shock–tube problem* in Gas Dynamics, in which the velocities u_{L} and u_{R} on either side of the diaphragm, here idealised by an initial discontinuity, are zero. Shock tubes and shock–tube problems have played, over a period of more than 100 years, a fundamental role in fluid dynamics research.

The structure of the similarity solution $\mathbf{U}(x/t)$ of (3.35) is as depicted in Fig. 3.1. There are three waves *associated* with the three characteristic fields

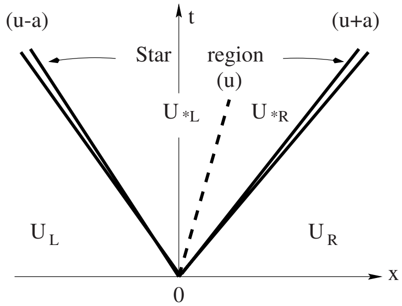

Fig. 3.1. Structure of the solution of the Riemann problem in the x–t plane for the time–dependent, one dimensional Euler equations. There are three wave families associated with the eigenvalues $u - a$, u and $u + a$

corresponding to the eigenvectors $\mathbf{K}^{(i)}$, $i = 1, 2, 3$. We choose the convention of representing the outer waves, when their character is unknown, by a pair of rays emanating from the origin and the middle wave by a dashed line. Each wave family is shown along with the corresponding eigenvalue. The three waves separate four constant states. From left to right these are $\mathbf{U}_{\mathrm{L}}$ (left data state); $\mathbf{U}_{*\mathrm{L}}$ between the 1–wave and the 2–wave; $\mathbf{U}_{*\mathrm{R}}$ between the 2–wave and the 3–wave and $\mathbf{U}_{\mathrm{R}}$ (right data state). As we shall see the waves present in the solution are of three types: rarefaction waves, contact discontinuities and shock waves. In order to identify the types we analyse the characteristic fields for $\mathbf{K}^{(i)}$, $i = 1, 2, 3$; see Sects. 2.4.3 and 2.4.4 of Chap. 2.

Proposition 3.10. *The $\mathbf{K}^{(2)}$–characteristic field is linearly degenerate and the $\mathbf{K}^{(1)}$, $\mathbf{K}^{(3)}$ characteristic fields are genuinely non–linear.*

Proof. For the $\mathbf{K}^{(2)}$–characteristic field we have

$$\nabla\lambda_2(\mathbf{U}) = [\partial\lambda_2/\partial u_1, \partial\lambda_2/\partial u_2, \partial\lambda_2/\partial u_3] = [-u/\rho, 1/\rho, 0] \ .$$

Hence

$$\nabla\lambda_2 \cdot \mathbf{K}^{(2)} = [-u/\rho, 1/\rho, 0] \cdot \begin{bmatrix} 1 \\ u \\ \frac{1}{2}u^2 \end{bmatrix} = 0$$

and therefore the $\mathbf{K}^{(2)}$ characteristic field is linearly degenerate as claimed. The proof that the $\mathbf{K}^{(1)}$ and $\mathbf{K}^{(3)}$ characteristic fields are *genuinely nonlinear* is left to the reader.

The wave associated with the $\mathbf{K}^{(2)}$ characteristic field is a *contact discontinuity* and those associated with the $\mathbf{K}^{(1)}$, $\mathbf{K}^{(3)}$ characteristic fields will either be rarefaction waves (smooth) or shock waves (discontinuities), see Sect. 2.4.4 of Chapt. 4. Of course one does not know in advance what types of waves will be present in the solution of the Riemann problem. The only exception is the middle wave, which is always a contact discontinuity. Fig. 3.2 shows a

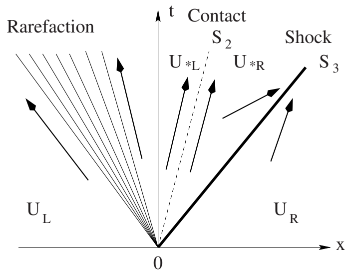

Fig. 3.2. Structure of the solution of the Riemann problem in the x–t plane for the time–dependent, one dimensional Euler equations, in which the left wave is a rarefaction, the middle wave is a contact discontinuity and the right wave is a shock wave

particular case in which the left wave is a rarefaction, the middle wave is a contact and the right wave is a shock wave. For each wave we have drawn a pair of arrows, one on each side, to indicate the characteristic directions of the corresponding eigenvalue. For the rarefaction wave we have

$$\lambda_1(\mathbf{U}_{\mathrm{L}}) \le \lambda_1(\mathbf{U}_{*\mathrm{L}}) \ .$$

The eigenvalue $\lambda_1(\mathbf{U})$ increases monotonically as we cross the rarefaction wave from left to right and the characteristics on either side diverge from the wave; compare with Fig. 2.20 of Chap. 2. For the shock wave, characteristics run into the wave and we have

$$\lambda_3(\mathbf{U}_{*\mathrm{R}}) > S_3 > \lambda_3(\mathbf{U}_{\mathrm{R}}) \ ,$$

which is the *entropy condition*. See Sect. 2.4.4 of Chap. 2. S_3 is the speed of the 3–shock. For the contact wave we have

$$\lambda_2(\mathbf{U}_{*\mathrm{L}}) = \lambda_2(\mathbf{U}_{*\mathrm{R}}) = S_2 \ ,$$

where S_2 is the speed of the contact wave; the characteristics are parallel to the contact wave. Recall that this is what happens for *all characteristic fields* in linear hyperbolic systems with constant coefficients. Next we study each type of waves separately.

Contact Discontinuities

The contact discontinuity in the solution of the Riemann problem for the Euler equations can be analysed by utilising the eigenstructure of the equations. In particular the Generalised Riemann Invariants will reveal which quantities change across the wave. Recall that for a general $m \times m$ hyperbolic system, such as (3.1)–(3.2) or (3.7), with

$$\mathbf{W} = [w_1, w_2, \cdots, w_m]^T \ ,$$

and right eigenvectors

$$\mathbf{K}^{(i)} = \left[k_1^{(i)}, k_2^{(i)}, \cdots, k_m^{(i)}\right] \ ,$$

the $i-$th Generalised Riemann Invariants are the $(m-1)$ ODEs

$$\frac{\mathrm{d}w_1}{k_1^{(i)}} = \frac{\mathrm{d}w_2}{k_2^{(i)}} = \frac{\mathrm{d}w_3}{k_3^{(i)}} = \cdots = \frac{\mathrm{d}w_m}{k_m^{(i)}} \ .$$

Using the eigenstructure (3.12)–(3.13) of the conservative formulation (3.1)–(3.2), for the $\mathbf{K}^{(2)}$–wave we have

$$\frac{\mathrm{d}\rho}{1} = \frac{\mathrm{d}(\rho u)}{u} = \frac{\mathrm{d}E}{\frac{1}{2}u^2} \ . \tag{3.36}$$

Manipulation of these equalities gives

$$p = \text{constant}, \ u = \text{constant}$$

across the contact wave. The same result follows directly by inspection of the eigenvector $\mathbf{K}^{(2)}$ in (3.20) for the primitive–variable formulation (3.17)–(3.18): the wave jumps in ρ, u and p are proportional to the corresponding components of the eigenvector. These are zero for the velocity and pressure. The jump in ρ is in general non–trivial. To conclude: *a contact wave is a discontinuous wave across which both pressure and particle velocity are constant but density jumps discontinuously as do variables that depend on density, such as specific internal energy, temperature, sound speed, entropy, etc.*

Rarefaction Waves

Rarefaction waves in the Euler equations are associated with the $\mathbf{K}^{(1)}$ and $\mathbf{K}^{(3)}$ characteristic fields. Inspection of the eigenvectors (3.20) for the primitive–variable formulation reveals that ρ, u and p change across a rarefaction wave. We now utilise the Generalised Riemann Invariants for the eigenstructure (3.33)–(3.34) of the entropy formulation (3.31)–(3.32).

Proposition 3.11. *For the Euler equations the Generalised Riemann Invariants across* 1 *and* 3 *rarefactions are*

$$\left.\begin{aligned} I_{\mathrm{L}}(u,a) = u + \tfrac{2a}{\gamma-1} &= \mathit{constant} \\ s &= \mathit{constant} \end{aligned}\right\} \mathit{across}\ \lambda_1 = u - a\ , \tag{3.37}$$

$$\left.\begin{aligned} I_{\mathrm{R}}(u,a) = u - \tfrac{2a}{\gamma-1} &= \mathit{constant} \\ s &= \mathit{constant} \end{aligned}\right\} \mathit{across}\ \lambda_3 = u + a\ . \tag{3.38}$$

Proof. Across a wave associated with $\lambda_1 = u - a$ wave we have

$$\frac{\mathrm{d}\rho}{1} = \frac{\mathrm{d}u}{-a/\rho} = \frac{\mathrm{d}s}{0}\ .$$

Two meaningful relations are

$$u + \int \frac{a}{\rho}\,\mathrm{d}\rho = \text{constant and } s = \text{constant.} \tag{3.39}$$

Similarly, across the $\lambda_3 = u + a$ wave we have

$$u - \int \frac{a}{\rho}\,\mathrm{d}\rho = \text{constant and } s = \text{constant.} \tag{3.40}$$

In order to reproduce (3.37) and (3.38) we need to evaluate the integrals in (3.39) and (3.40). First we note that by inspection of the eigenvectors $\mathbf{K}^{(1)}$ and $\mathbf{K}^{(3)}$ the condition of constant entropy across the respective waves is immediate. We may therefore use the isentropic law (3.30) with the constant C evaluated at the appropriate data state (constant). Thus the integral is as found for the isentropic equations in Sect. 2.4.3 of Chap. 2, that is

$$\int \frac{a}{\rho}\,\mathrm{d}\rho = \frac{2a}{\gamma - 1}\ ,$$

and thus equations (3.37)–(3.38) are reproduced.

To summarise: *a rarefaction wave is a smooth wave associated with the* 1 *and* 3 *fields across which* ρ, u *and* p *change. The wave has a fan–type shape and is enclosed by two bounding characteristics corresponding to the Head and the Tail of the wave. Across the wave the Generalised Riemann Invariants apply.* The *solution within* the rarefaction will be given in Chap. 4, where the full solution of the Riemann problem is presented.

Shock Waves

Details on the Physics of shock waves are found in any book on Gas Dynamics. We particularly recommend Becker [35], Anderson [10], Landau and Lifshitz [297]. The specialised book by Zeldovich and Raizer [599] is highly recommended.

In the context of the one–dimensional Euler equations, shock waves are discontinuous waves associated with the genuinely non–linear fields 1 and 3. All three quantities ρ, u and p change across a shock wave. Consider the $\mathbf{K}^{(3)}$ characteristic field and assume the corresponding wave is a right–facing shock wave travelling at the constant speed S_3; see Fig. 3.3. In terms of the primitive variables we denote the state ahead of the shock by $\mathbf{W}_{\rm R} = (\rho_{\rm R}, u_{\rm R}, p_{\rm R})^T$ and the state behind the shock by $\mathbf{W}_* = (\rho_*, u_*, p_*)^T$. We are interested in deriving relations, across the shock wave, between the various quantities involved. Central to the analysis is the application of the Rankine–Hugoniot conditions. It is found convenient to transform the problem to a new frame

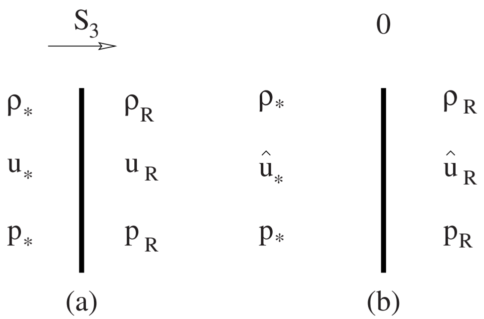

Fig. 3.3. Right–facing shock wave: (a) stationary frame of reference, shock has speed S_3; (b) frame of reference moves with speed S_3, so that the shock has zero speed

of reference moving with the shock so that in the new frame the shock speed is zero. Fig. 3.3 depicts both frames of reference. In the transformed frame (b) the states ahead and behind the shock have changed by virtue of the transformation. Densities and pressures remain unaltered while velocities have changed to the *relative velocities* $\hat{u}_{\rm R}$ and $\hat{u}_*$ given by

$$\hat{u}_* = u_* - S_3 \ , \ \hat{u}_{\rm R} = u_{\rm R} - S_3 \ . \tag{3.41}$$

Application of the Rankine–Hugoniot conditions in the frame in which the shock speed is zero gives

$$\rho_* \hat{u}_* = \rho_{\rm R} \hat{u}_{\rm R} \ , \tag{3.42}$$

$$\rho_* \hat{u}_*^2 + p_* = \rho_{\rm R} \hat{u}_{\rm R}^2 + p_{\rm R} \ , \tag{3.43}$$

$$\hat{u}_*(\hat{E}_* + p_*) = \hat{u}_{\rm R}(\hat{E}_{\rm R} + p_R) \ . \tag{3.44}$$

By using the definition of total energy E and introducing the *specific internal energy* e the left–hand side of (3.44) may we written as

$$\hat{u}_* \rho_* \left[\frac{1}{2} \hat{u}_*^2 + (e_* + p_*/\rho_*) \right]$$

and the right–hand side of (3.44) as

$$\hat{u}_{\rm R}\rho_{\rm R}\left[\frac{1}{2}\hat{u}_{\rm R}^2 + (e_{\rm R} + p_{\rm R}/\rho_{\rm R})\right] .$$

Now we use the *specific enthalpy* h and write

$$h_* = e_* + p_*/\rho_* \ , \quad h_{\rm R} = e_{\rm R} + p_{\rm R}/\rho_{\rm R} . \tag{3.45}$$

Use of equations (3.42) and (3.44) leads to

$$\frac{1}{2}\hat{u}_*^2 + h_* = \frac{1}{2}\hat{u}_{\rm R}^2 + h_{\rm R} . \tag{3.46}$$

By using (3.42) into (3.43) we write

$$\rho_*\hat{u}_*^2 = (\rho_{\rm R}\hat{u}_{\rm R})\hat{u}_{\rm R} + p_{\rm R} - p_* = (\rho_*\hat{u}_*)\frac{\rho_*\hat{u}_*}{\rho_{\rm R}} + p_{\rm R} - p_* .$$

After some manipulations we obtain

$$\hat{u}_*^2 = \left(\frac{\rho_{\rm R}}{\rho_*}\right)\left[\frac{p_{\rm R} - p_*}{\rho_{\rm R} - \rho_*}\right] . \tag{3.47}$$

In a similar way we obtain

$$\hat{u}_{\rm R}^2 = \left(\frac{\rho_*}{\rho_{\rm R}}\right)\left[\frac{p_{\rm R} - p_*}{\rho_{\rm R} - \rho_*}\right] . \tag{3.48}$$

Substitution of (3.47)–(3.48) into (3.46) gives

$$h_* - h_{\rm R} = \frac{1}{2}(p_* - p_{\rm R})\left[\frac{\rho_* + \rho_{\rm R}}{\rho_*\rho_{\rm R}}\right] . \tag{3.49}$$

Assuming the specific internal energy e is given by the the caloric equation of state (3.4), it is then more convenient to rewrite the energy equation (3.49) using (3.45). We obtain

$$e_* - e_{\rm R} = \frac{1}{2}(p_* + p_{\rm R})\left[\frac{\rho_* - \rho_{\rm R}}{\rho_*\rho_{\rm R}}\right] . \tag{3.50}$$

Note that up to this point no assumption on the general caloric EOS (3.4) has been made. In what follows, we derive shock relations that apply to ideal gases in which the ideal caloric EOS (3.5) is assumed. By using (3.5) into (3.50) and performing some algebraic manipulations one obtains

$$\frac{\rho_*}{\rho_{\rm R}} = \frac{(\frac{p_*}{p_{\rm R}}) + (\frac{\gamma-1}{\gamma+1})}{(\frac{\gamma-1}{\gamma+1})(\frac{p_*}{p_{\rm R}}) + 1} . \tag{3.51}$$

This establishes a useful relation between the density ratio ρ_*/ρ_{R} and the pressure ratio p_*/p_{R} across the shock wave.

We now introduce Mach numbers

$$M_{\mathrm{R}} = u_{\mathrm{R}}/a_{\mathrm{R}} \ , \ M_{\mathrm{S}} = S_3/a_{\mathrm{R}} \ , \tag{3.52}$$

where M_{R} is the Mach number of the flow ahead of the shock, in the original frame; M_{S} is the *shock Mach number*. Manipulation of equations (3.48), (3.51) and (3.52) leads to expressions for the density and pressure ratios across the shock as functions of the relative Mach number $M_{\mathrm{R}} - M_{\mathrm{S}}$, namely

$$\frac{\rho_*}{\rho_{\mathrm{R}}} = \frac{(\gamma+1)(M_{\mathrm{R}} - M_{\mathrm{S}})^2}{(\gamma-1)(M_{\mathrm{R}} - M_{\mathrm{S}})^2 + 2} \ , \tag{3.53}$$

$$\frac{p_*}{p_{\mathrm{R}}} = \frac{2\gamma(M_{\mathrm{R}} - M_{\mathrm{S}})^2 - (\gamma-1)}{(\gamma+1)} \ . \tag{3.54}$$

The shock speed S_3 can be related to the density and pressure ratios across the shock wave. In terms of the pressure ratio (3.54) we first note the following relationship

$$M_{\mathrm{R}} - M_{\mathrm{S}} = -\sqrt{\left(\frac{\gamma+1}{2\gamma}\right)\left(\frac{p_*}{p_{\mathrm{R}}}\right) + \left(\frac{\gamma-1}{2\gamma}\right)} \ .$$

This leads to an expression for the shock speed as a function of the pressure ratio across the shock, namely

$$S_3 = u_{\mathrm{R}} + a_R\sqrt{\left(\frac{\gamma+1}{2\gamma}\right)\left(\frac{p_*}{p_{\mathrm{R}}}\right) + \left(\frac{\gamma-1}{2\gamma}\right)} \ . \tag{3.55}$$

Note that as the shock strength tends to zero, the ratio p_*/p_{R} tends to unity and the shock speed S_3 approaches the characteristic speed $\lambda_3 = u_{\mathrm{R}} + a_{\mathrm{R}}$, as expected. We can also obtain an expression for the *particle velocity* u_* behind the shock wave. From (3.42) we relate u_* to the density ratio across the shock, namely

$$u_* = (1 - \rho_{\mathrm{R}}/\rho_*)S_3 + u_{\mathrm{R}}\rho_{\mathrm{R}}/\rho_* \ . \tag{3.56}$$

Example 3.12 (Shock Wave). Consider a shock wave of shock Mach number $M_{\mathrm{S}} = 3$ propagating into the atmosphere with conditions (ahead of the shock) $\rho_{\mathrm{R}} = 1.225$ kg/m^3, $u_{\mathrm{R}} = 0$ m/s, $p_{\mathrm{R}} = 101\,325$ Pa. Assume the process is suitably modelled by the ideal gas EOS (3.5) with $\gamma = 1.4$. From the definition of sound speed (3.6) we obtain $a_{\mathrm{R}} = 340.294$ m/s. As the shock Mach number $M_{\mathrm{S}} = 3$ is assumed (a parameter) then equation (3.52) gives the shock speed as $S = 1020.882$ m/s. From equation (3.53) we obtain $\rho_* = 4.725$ kg/m^3. From equation (3.54) we obtain $p_* = 1\,047\,025$ Pa and from equation (3.56) we obtain $u_* = 756.2089$ m/s.

Remark 3.13. Shock relations (3.53), (3.54) and (3.56) define a state

$$(\rho_*, u_*, p_*)^T$$

behind a shock for given initial conditions $(\rho_R, u_R, p_R)^T$ ahead of the shock and a chosen shock Mach number M_S, or equivalently a shock speed S_3. The shock is associated with the 3–wave family. These relations can be useful in setting up test problems involving a single shock wave to test numerical methods.

The analysis for a 1–shock wave (left facing) travelling with velocity S_1 is entirely analogous. The state ahead of the shock (left side now) is denoted by $\mathbf{W}_\mathrm{L} = (\rho_\mathrm{L}, u_\mathrm{L}, p_\mathrm{L})^T$ and the state behind the shock (right side) by $\mathbf{W}_* = (\rho_*, u_*, p_*)^T$. As done for the 3–shock we transform to a stationary frame of reference. The relative velocities are

$$\hat{u}_\mathrm{L} = u_\mathrm{L} - S_1 \ , \ \hat{u}_* = u_* - S_1 \ . \tag{3.57}$$

Mach numbers are

$$M_\mathrm{L} = u_\mathrm{L}/a_\mathrm{L} \ , \ M_\mathrm{S} = S_1/a_\mathrm{L} \ . \tag{3.58}$$

The density and pressure ratio relationship is

$$\frac{\rho_*}{\rho_\mathrm{L}} = \frac{(\frac{p_*}{p_\mathrm{L}}) + (\frac{\gamma-1}{\gamma+1})}{(\frac{\gamma-1}{\gamma+1})(\frac{p_*}{p_\mathrm{L}}) + 1} \ . \tag{3.59}$$

In terms of the relative Mach number $M_\mathrm{L} - M_\mathrm{S}$ the density and pressure ratios across the left shock can be expressed as follows

$$\frac{\rho_*}{\rho_\mathrm{L}} = \frac{(\gamma+1)(M_\mathrm{L} - M_\mathrm{S})^2}{(\gamma-1)(M_\mathrm{L} - M_\mathrm{S})^2 + 2} \ , \tag{3.60}$$

$$\frac{p_*}{p_\mathrm{L}} = \frac{2\gamma(M_\mathrm{L} - M_\mathrm{S})^2 - (\gamma-1)}{(\gamma+1)} \ . \tag{3.61}$$

The shock speed S_1 can be obtained from either (3.60) or (3.61). In terms of the pressure ratio (3.61) we have

$$M_\mathrm{L} - M_\mathrm{S} = \sqrt{\left(\frac{\gamma+1}{2\gamma}\right)\left(\frac{p_*}{p_\mathrm{L}}\right) + \left(\frac{\gamma-1}{2\gamma}\right)} \ ,$$

which leads to

$$S_1 = u_\mathrm{L} - a_L\sqrt{\left(\frac{\gamma+1}{2\gamma}\right)\left(\frac{p_*}{p_\mathrm{L}}\right) + \left(\frac{\gamma-1}{2\gamma}\right)} \ . \tag{3.62}$$

Note that as the shock strength tends to zero, the ratio p_*/p_L tends to unity and the shock speed S_1 approaches the characteristic speed $\lambda_1 = u_\mathrm{L} - a_\mathrm{L}$, as expected. The particle velocity behind the left shock is

$$u_* = (1 - \rho_{\mathrm{L}}/\rho_*)S_1 + u_{\mathrm{L}}\rho_{\mathrm{L}}/\rho_* \ . \tag{3.63}$$

Shock relations (3.60), (3.61) and (3.63) define a state $(\rho_*, u_*, p_*)^T$ behind a shock for given initial conditions $(\rho_L, u_L, p_L)^T$ ahead of the shock and a chosen shock Mach number M_S, or equivalently a shock speed S_1. The shock is associated with the 1–wave family.

3.2 Multi–Dimensional Euler Equations

In the previous section we analysed the one–dimensional, time–dependent Euler equations. Here we study a few basic properties of the two and three dimensional cases. In differential conservation–law form the three–dimensional equations are

$$\mathbf{U}_t + \mathbf{F}(\mathbf{U})_x + \mathbf{G}(\mathbf{U})_y + \mathbf{H}(\mathbf{U})_z = \mathbf{0} \ , \tag{3.64}$$

with

$$\left.\begin{array}{l} \mathbf{U} = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ E \end{bmatrix} , \quad \mathbf{F} = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ \rho uw \\ u(E+p) \end{bmatrix} , \\ \\ \mathbf{G} = \begin{bmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ \rho vw \\ v(E+p) \end{bmatrix} , \quad \mathbf{H} = \begin{bmatrix} \rho w \\ \rho uw \\ \rho vw \\ \rho w^2 + p \\ w(E+p) \end{bmatrix} . \end{array}\right\} \tag{3.65}$$

Here E is the total energy per unit volume

$$E = \rho\,(\frac{1}{2}\mathbf{V}^2 + e) \ , \tag{3.66}$$

where $\frac{1}{2}\mathbf{V}^2 = \frac{1}{2}\mathbf{V}\cdot\mathbf{V} = \frac{1}{2}(u^2+v^2+w^2)$ is the *specific kinetic energy* and e is *specific internal energy* given by a caloric equation of state (3.4).

The corresponding integral form of the conservation laws (3.64) is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\int\int_V \mathbf{U}\,\mathrm{d}V + \int\int_A \mathcal{H}\cdot\mathbf{n}\,\mathrm{d}A = \mathbf{0} \ , \tag{3.67}$$

where V is a *control volume*, A is the boundary of V, $\mathcal{H} = (\mathbf{F}, \mathbf{G}, \mathbf{H})$ is the tensor of fluxes, $\mathbf{n}$ is the outward unit vector normal to the surface A, $\mathrm{d}A$ is an area element and $\mathcal{H}\cdot\mathbf{n}\,\mathrm{d}A$ is the flux component normal to the boundary A. The conservation laws (3.67) state that the time–rate of change of $\mathbf{U}$ inside volume V depends only on the *total flux* through the surface A, the boundary of the control volume V. Numerical methods of the finite volume type, see Sect. 16.7.3 of Chap. 16, are based on this formulation of the equations. For details of the derivation of integral form of the conservation laws see Sects. 1.5 and 1.6.1 of Chap. 1.

In the next section we study some properties of the two–dimensional Euler equation in conservation form

3.2.1 Two–Dimensional Equations in Conservative Form

The two–dimensional version of the Euler equations in differential conservative form is

$$\mathbf{U}_t + \mathbf{F}(\mathbf{U})_x + \mathbf{G}(\mathbf{U})_y = \mathbf{0} \; , \tag{3.68}$$

with

$$\mathbf{U} = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ E \end{bmatrix} , \quad \mathbf{F} = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ u(E+p) \end{bmatrix} , \quad \mathbf{G} = \begin{bmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ v(E+p) \end{bmatrix} . \tag{3.69}$$

Eigenstructure

Here we find the Jacobian matrix of the x–split equations, its eigenvalues and corresponding right eigenvectors. We also study the types of characteristic fields present.

Proposition 3.14. *The Jacobian matrix* $\mathbf{A}(\mathbf{U})$ *corresponding to the flux* $\mathbf{F}(\mathbf{U})$ *is given by*

$$\mathbf{A}(\mathbf{U}) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -u^2 + \frac{1}{2}(\gamma-1)\mathbf{V}^2 & (3-\gamma)u & -(\gamma-1)v & \gamma-1 \\ -uv & v & u & 0 \\ u\left[\frac{1}{2}(\gamma-1)\mathbf{V}^2 - H\right] & H-(\gamma-1)u^2 & -(\gamma-1)uv & \gamma u \end{bmatrix} . \tag{3.70}$$

The eigenvalues of $\mathbf{A}$ *are*

$$\lambda_1 = u - a \; , \quad \lambda_2 = \lambda_3 = u \; , \quad \lambda_4 = u + a \; , \tag{3.71}$$

with corresponding right eigenvectors

$$\left.\begin{array}{c} \mathbf{K}^{(1)} = \begin{bmatrix} 1 \\ u-a \\ v \\ H-au \end{bmatrix} , \quad \mathbf{K}^{(2)} = \begin{bmatrix} 1 \\ u \\ v \\ \frac{1}{2}\mathbf{V}^2 \end{bmatrix} , \\ \mathbf{K}^{(3)} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ v \end{bmatrix} , \quad \mathbf{K}^{(4)} = \begin{bmatrix} 1 \\ u+a \\ v \\ H+ua \end{bmatrix} . \end{array}\right\} \tag{3.72}$$

Proof. Exercise.

Rotational Invariance

We next prove an important property, called the *rotational invariance* of the Euler equations. The property allows the proof of hyperbolicity in time for the two–dimensional equations (3.68)–(3.69) and can also be used for computational purposes to deal with domains that are not aligned with the Cartesian directions, see Sect. 16.7.3 of Chap. 16. We first note that outward unit vector $\mathbf{n}$ normal to the surface A in the two–dimensional case is given by

$$\mathbf{n} \equiv (n_1, n_2) \equiv (\cos\theta, \sin\theta) \;, \tag{3.73}$$

where θ is the angle formed by x–axis and the normal vector $\mathbf{n}$; θ is measured in an anticlockwise manner and lies in the range $0 \le \theta \le 2\pi$. Fig. 3.4 depicts the situation. The integrand of the surface integral in (3.67) becomes

$$(\mathbf{F}, \mathbf{G}) \cdot \mathbf{n} = \cos\theta \mathbf{F}(\mathbf{U}) + \sin\theta \mathbf{G}(\mathbf{U}) \;. \tag{3.74}$$

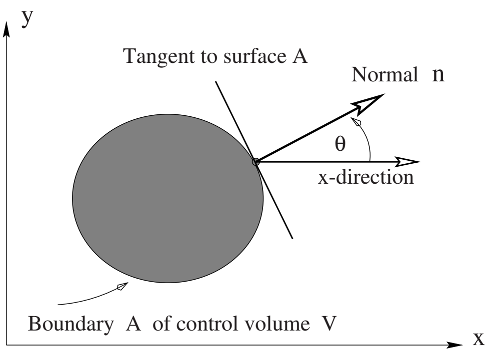

Fig. 3.4. Control volume V on x–y plane; boundary of V is A, outward unit normal vector is $\mathbf{n}$ and θ is angle between the x–direction and $\mathbf{n}$

Proposition 3.15 (Rotational Invariance). *The two–dimensional Euler equations (3.68)–(3.69) satisfy the rotational invariance property*

$$\cos\theta \mathbf{F}(\mathbf{U}) + \sin\theta \mathbf{G}(\mathbf{U}) = \mathbf{T}^{-1}\mathbf{F}\left(\mathbf{T}\mathbf{U}\right) \;, \tag{3.75}$$

for all angles θ and vectors $\mathbf{U}$. Here $\mathbf{T} = \mathbf{T}(\theta)$ is the rotation matrix and $\mathbf{T}^{-1}(\theta)$ is its inverse, namely

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \;, \; \mathbf{T}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \;. \tag{3.76}$$

Proof. First we calculate $\hat{\mathbf{U}} = \mathbf{T}\mathbf{U}$. The result is

$$\hat{\mathbf{U}} = \mathbf{T}\mathbf{U} = [\rho, \rho\hat{u}, \rho\hat{v}, E]^T \ ,$$

with $\hat{u} = u\cos\theta + v\sin\theta$, $\hat{v} = -u\sin\theta + v\cos\theta$. Next we compute $\hat{\mathbf{F}} = \mathbf{F}(\hat{\mathbf{U}})$ and obtain

$$\hat{\mathbf{F}} = \mathbf{F}(\hat{\mathbf{U}}) = \left[\rho\hat{u}, \rho\hat{u}^2 + p, \rho\hat{u}\hat{v}, \hat{u}(E+p)\right]^T \ .$$

Now we apply $\mathbf{T}^{-1}$ to $\mathbf{F}(\hat{\mathbf{U}})$. The result is easily verified to be

$$\mathbf{T}^{-1}\hat{\mathbf{F}} = \begin{bmatrix} \rho\hat{u} \\ \cos\theta\left(\rho\hat{u}^2 + p\right) - \sin\theta\left(\rho\hat{u}\hat{v}\right) \\ \sin\theta\left(\rho\hat{u}^2 + p\right) + \cos\theta\left(\rho\hat{u}\hat{v}\right) \\ \hat{u}(E+p) \end{bmatrix} = \cos\theta\mathbf{F} + \sin\theta\mathbf{G} \ .$$

This is clearly satisfied for the first and fourth components. Further manipulation show that it is also satisfied for the second and third flux components and the proposition is thus proved.

Hyperbolicity in Time

Here we use the rotational invariance property of the two–dimensional time dependent Euler equations to show that the equations are *hyperbolic in time.*

Definition 3.16 (Hyperbolicity in time). *System (3.68)–(3.69) is hyperbolic in time if for all admissible states* $\mathbf{U}$ *and real angles* θ*, the matrix*

$$\mathbf{A}(\mathbf{U}, \theta) = \cos\theta\mathbf{A}(\mathbf{U}) + \sin\theta\mathbf{B}(\mathbf{U}) \qquad (3.77)$$

is diagonalisable. Here $\mathbf{A}(\mathbf{U})$ *and* $\mathbf{B}(\mathbf{U})$ *are respectively the Jacobian matrices of the fluxes* $\mathbf{F}(\mathbf{U})$ *and* $\mathbf{G}(\mathbf{U})$ *in (3.68).*

Proposition 3.17. *The two–dimensional Euler equations (3.68)–(3.69) are hyperbolic in time.*

Proof. We want to prove that the matrix $\mathbf{A}(\mathbf{U}, \theta)$ in (3.77) is *diagonalisable*, see Sect. 2.3.2 of Chap. 2. That is we want to prove that there exist a diagonal matrix $\mathbf{\Lambda}(\mathbf{U}, \theta)$ and a non–singular matrix $\mathbf{K}(\mathbf{U}, \theta)$ such that

$$\mathbf{A}(\mathbf{U}, \theta) = \mathbf{K}(\mathbf{U}, \theta)\mathbf{\Lambda}(\mathbf{U}, \theta)\mathbf{K}^{-1}(\mathbf{U}, \theta) \ . \qquad (3.78)$$

By differentiating (3.75) with respect to $\mathbf{U}$ we have

$$\mathbf{A}(\mathbf{U}, \theta) = \cos\theta\mathbf{A}(\mathbf{U}) + \sin\theta\mathbf{B}(\mathbf{U}) = \mathbf{T}(\theta)^{-1}\mathbf{A}\left(\mathbf{T}(\theta)\mathbf{U}\right)\mathbf{T}(\theta) \ .$$

But the matrix $\mathbf{A}(\mathbf{U})$ is diagonalisable, it has four linearly independent eigenvectors $\mathbf{K}^{(i)}(\mathbf{U})$ given by (3.72). Therefore we can write

$$\mathbf{A}(\mathbf{U}) = \mathbf{K}(\mathbf{U})\mathbf{\Lambda}(\mathbf{U})\mathbf{K}^{-1}(\mathbf{U}) \; ,$$

where $\mathbf{K}(\mathbf{U})$ is the non–singular matrix the columns of which are the right eigenvectors $\mathbf{K}^{(i)}(\mathbf{U})$, $\mathbf{K}^{-1}(\mathbf{U})$ is its inverse and $\mathbf{\Lambda}(\mathbf{U})$ is the diagonal matrix with the eigenvalues $\lambda_i(\mathbf{U})$ given by (3.71) as the diagonal entries. Then we have

$$\begin{aligned} \mathbf{A}(\mathbf{U},\theta) &= \mathbf{T}(\theta)^{-1}\left\{\mathbf{K}\left(\mathbf{T}(\theta)\mathbf{U}\right)\mathbf{\Lambda}\left(\mathbf{T}(\theta)\mathbf{U}\right)\mathbf{K}^{-1}\left(\mathbf{T}(\theta)\mathbf{U}\right)\right\}\mathbf{T}(\theta) \\ &= \left\{\mathbf{T}(\theta)^{-1}\mathbf{K}\left(\mathbf{T}(\theta)\mathbf{U}\right)\right\}\mathbf{\Lambda}\left(\mathbf{T}(\theta)\mathbf{U}\right)\left\{\mathbf{T}(\theta)^{-1}\mathbf{K}\left(\mathbf{T}(\theta)\mathbf{U}\right)\right\}^{-1} \; . \end{aligned}$$

Hence the condition for hyperbolicity holds by taking

$$\mathbf{K}(\mathbf{U},\theta) = \mathbf{T}^{-1}(\theta)\mathbf{K}\left(\mathbf{T}(\theta)\mathbf{U}\right) \; , \quad \mathbf{\Lambda}(\mathbf{U},\theta) = \mathbf{\Lambda}\left(\mathbf{T}(\theta)\mathbf{U}\right) \; .$$

We have thus proved that the time–dependent, two dimensional Euler equations are hyperbolic in time, as claimed.

Characteristic Fields

Next we analyse the characteristic fields associated with the four eigenvectors given by (3.72).

Proposition 3.18 (Types of Characteristic Fields). *For $i = 1$ and $i = 4$ the $\mathbf{K}^{(i)}(\mathbf{U})$ characteristic fields are genuinely non–linear, while for $i = 2$ and $i = 3$ they are linearly degenerate.*

Proof. The proof that the fields $i = 2$ and $i = 3$ are linearly degenerate is trivial. Clearly

$$\nabla\lambda_2 = \nabla\lambda_3 = (-u/\rho, 1/\rho, 0, 0) \; .$$

By inspecting $\mathbf{K}^{(2)}(\mathbf{U})$ and $\mathbf{K}^{(3)}(\mathbf{U})$ it is obvious that

$$\nabla\lambda_2 \cdot \mathbf{K}^{(2)}(\mathbf{U}) = \nabla\lambda_3 \cdot \mathbf{K}^{(3)}(\mathbf{U}) = 0$$

and therefore the 2 and 3 characteristic fields are linearly degenerate as claimed. The proof for $i = 1, 4$ involves some algebra. The result is

$$\nabla\lambda_1 \cdot \mathbf{K}^{(1)}(\mathbf{U}) = -\frac{(\gamma+1)a}{2\rho} \neq 0 \; , \; \nabla\lambda_4 \cdot \mathbf{K}^{(4)}(\mathbf{U}) = \frac{(\gamma+1)a}{2\rho} \neq 0$$

and thus the 1 and 4 characteristic fields are *genuinely non–linear* as claimed.

In the context of the Riemann problem we shall see that across the 2 and 3 waves both pressure p and *normal* velocity component u are constant. The 2 field is associated with a contact discontinuity, across which density jumps discontinuously. The 3 field is associated with a shear wave across which the *tangential* velocity component jumps discontinuously. The 1 and 4 characteristic fields are associated with shock waves and rarefaction waves.

3.2.2 Three–Dimensional Equations in Conservative Form

Here we extend previous results proved for the two–dimensional equations, to the time–dependent three dimensional Euler equations. Proofs are omitted, they involve elementary but tedious algebra.

Eigenstructure

The Jacobian matrix $\mathbf{A}$ corresponding to the flux $\mathbf{F}(\mathbf{U})$ in (3.64) is given by

$$\mathbf{A} = \frac{\partial \mathbf{F}}{\partial \mathbf{U}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ \hat{\gamma}H - u^2 - a^2 & (3-\gamma)u & -\hat{\gamma}v & -\hat{\gamma}w & \hat{\gamma} \\ -uv & v & u & 0 & 0 \\ -uw & w & 0 & u & 0 \\ \frac{1}{2}u[(\gamma-3)H - a^2] & H - \hat{\gamma}u^2 & -\hat{\gamma}uv & -\hat{\gamma}uw & \gamma u \end{bmatrix} , \quad (3.79)$$

where

$$H = (E+p)/\rho = \frac{1}{2}\mathbf{V}^2 + \frac{a^2}{(\gamma-1)} \, , \mathbf{V}^2 = u^2 + v^2 + w^2 \, , \; \hat{\gamma} = \gamma - 1 \, . \quad (3.80)$$

The x–split one–dimensional system is hyperbolic with real eigenvalues

$$\lambda_1 = u - a \, , \; \lambda_2 = \lambda_3 = \lambda_4 = u \, , \; \lambda_5 = u + a \, . \quad (3.81)$$

The matrix of corresponding right eigenvectors is

$$\mathbf{K} = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ u-a & u & 0 & 0 & u+a \\ v & v & 1 & 0 & v \\ w & w & 0 & 1 & w \\ H - ua & \frac{1}{2}\mathbf{V}^2 & v & w & H + ua \end{bmatrix} . \quad (3.82)$$

We also give the expression for the inverse matrix of $\mathbf{K}$, namely

$$\mathbf{K}^{-1} = \frac{(\gamma-1)}{2a^2} \begin{bmatrix} H + \frac{a}{\hat{\gamma}}(u-a) & -(u + \frac{a}{\hat{\gamma}}) & -v & -w & 1 \\ -2H + \frac{4}{\hat{\gamma}}a^2 & 2u & 2v & 2w & -2 \\ -\frac{2va^2}{\hat{\gamma}} & 0 & \frac{2a^2}{\hat{\gamma}} & 0 & 0 \\ -\frac{2wa^2}{\hat{\gamma}} & 0 & 0 & \frac{2a^2}{\hat{\gamma}} & 0 \\ H - \frac{a}{\hat{\gamma}}(u+a) & -u + \frac{a}{\hat{\gamma}} & -v & -w & 1 \end{bmatrix} . \quad (3.83)$$

Rotational Invariance

We now state the rotational invariance property for the three–dimensional case.

Proposition 3.19. *The time–dependent three dimensional Euler equations are rotationally invariant, that is they satisfy*

$$\cos\theta^{(y)}\cos\theta^{(z)}\mathbf{F}(\mathbf{U})+\cos\theta^{(y)}\sin\theta^{(z)}\mathbf{G}(\mathbf{U})+\sin\theta^{(y)}\mathbf{H}(\mathbf{U})=\mathbf{T}^{-1}\mathbf{F}\left(\mathbf{TU}\right)\ , \tag{3.84}$$

for all angles $\theta^{(y)}$, $\theta^{(z)}$ and vectors $\mathbf{U}$. Here $\mathbf{T}=\mathbf{T}(\theta^{(y)},\theta^{(z)})$ is the rotation matrix

$$\mathbf{T}=\begin{bmatrix}1 & 0 & 0 & 0 & 0\\ 0 & \cos\theta^{(y)}\cos\theta^{(z)} & \cos\theta^{(y)}\sin\theta^{(z)} & \sin\theta^{(y)} & 0\\ 0 & -\sin\theta^{(z)} & \cos\theta^{(z)} & 0 & 0\\ 0 & -\sin\theta^{(y)}\cos\theta^{(z)} & -\sin\theta^{(y)}\sin\theta^{(z)} & \cos\theta^{(y)} & 0\\ 0 & 0 & 0 & 0 & 1\end{bmatrix}\ , \tag{3.85}$$

and is the product of two rotation matrices, namely

$$\mathbf{T}=\mathbf{T}(\theta^{(y)},\theta^{(z)})=\mathbf{T}^{(y)}\mathbf{T}^{(z)}\ , \tag{3.86}$$

with

$$\left.\begin{array}{l}\mathbf{T}^{(y)}\equiv\mathbf{T}^{(y)}(\theta^{(y)})=\begin{bmatrix}1 & 0 & 0 & 0 & 0\\ 0 & \cos\theta^{(y)} & 0 & \sin\theta^{(y)} & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & -\sin\theta^{(y)} & 0 & \cos\theta^{(y)} & 0\\ 0 & 0 & 0 & 0 & 1\end{bmatrix}\ ,\\[2ex] \mathbf{T}^{(z)}\equiv\mathbf{T}^{(z)}(\theta^{(z)})=\begin{bmatrix}1 & 0 & 0 & 0 & 0\\ 0 & \cos\theta^{(z)} & \sin\theta^{(z)} & 0 & 0\\ 0 & -\sin\theta^{(z)} & \cos\theta^{(z)} & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1\end{bmatrix}\ .\end{array}\right\} \tag{3.87}$$

More details of the rotational invariance and related properties of the three–dimensional Euler equations are found in Billett and Toro [64].

3.2.3 Three–Dimensional Primitive Variable Formulation

As done for the one–dimensional Euler equations, we can express the two and three dimensional equations in terms of primitive variables.

Proposition 3.20. *The three–dimensional, time–dependent Euler equations can be written in terms of the primitive variables $\mathbf{W}=(\rho,u,v,w,p)^T$ as*

$$\left.\begin{array}{c}\rho_t + u\rho_x + v\rho_y + w\rho_z + \rho(u_x + v_y + w_z) = 0 \;, \\ \\ u_t + uu_x + vu_y + wu_z + \frac{1}{\rho}p_x = 0 \;, \\ \\ v_t + uv_x + vv_y + wv_z + \frac{1}{\rho}p_y = 0 \;, \\ \\ w_t + uw_x + vw_y + ww_z + \frac{1}{\rho}p_z = 0 \;, \\ \\ p_t + up_x + vp_y + wp_z + \rho a^2(u_x + v_y + w_z) = 0 \;. \end{array}\right\} \tag{3.88}$$

Proof. To prove this result one follows the same steps as for the one–dimensional case leading to equations (3.14)–(3.16).

Equations (3.88) can be written in quasi–linear form as

$$\mathbf{W}_t + \mathbf{A}(\mathbf{W})\mathbf{W}_x + \mathbf{B}(\mathbf{W})\mathbf{W}_y + \mathbf{C}(\mathbf{W})\mathbf{W}_z = \mathbf{0} \;, \tag{3.89}$$

where the coefficient matrices $\mathbf{A}(\mathbf{W})$, $\mathbf{B}(\mathbf{W})$ and $\mathbf{C}(\mathbf{W})$ are given by

$$\mathbf{A}(\mathbf{W}) = \begin{bmatrix} u & \rho & 0 & 0 & 0 \\ 0 & u & 0 & 0 & 1/\rho \\ 0 & 0 & u & 0 & 0 \\ 0 & 0 & 0 & u & 0 \\ 0 & \rho a^2 & 0 & 0 & u \end{bmatrix} \;, \tag{3.90}$$

$$\mathbf{B}(\mathbf{W}) = \begin{bmatrix} v & \rho & 0 & 0 & 0 \\ 0 & v & 0 & 0 & 0 \\ 0 & 0 & v & 0 & 1/\rho \\ 0 & 0 & 0 & v & 0 \\ 0 & 0 & \rho a^2 & 0 & v \end{bmatrix} \;, \tag{3.91}$$

$$\mathbf{C}(\mathbf{W}) = \begin{bmatrix} w & \rho & 0 & 0 & 0 \\ 0 & w & 0 & 0 & 0 \\ 0 & 0 & w & 0 & 0 \\ 0 & 0 & 0 & w & 1/\rho \\ 0 & 0 & 0 & \rho a^2 & w \end{bmatrix} \;. \tag{3.92}$$

Proposition 3.21. *The eigenvalues of the coefficient matrix* $\mathbf{A}(\mathbf{W})$ *in (3.90) are given by*

$$\lambda_1 = u - a \;,\; \lambda_2 = \lambda_3 = \lambda_4 = u \;,\; \lambda_5 = u + a \;. \tag{3.93}$$

with corresponding right eigenvectors

$$
\left.
\begin{array}{c}
\mathbf{K}^{(1)} = \begin{bmatrix} \rho \\ -a \\ 0 \\ 0 \\ \rho a^2 \end{bmatrix}, \; \mathbf{K}^{(2)} = \begin{bmatrix} 1 \\ 0 \\ v \\ w \\ 0 \end{bmatrix}, \; \mathbf{K}^{(3)} = \begin{bmatrix} \rho \\ 0 \\ 1 \\ w \\ 0 \end{bmatrix}, \\[2ex]
\mathbf{K}^{(4)} = \begin{bmatrix} \rho \\ 0 \\ v \\ 1 \\ 0 \end{bmatrix}, \; \mathbf{K}^{(5)} = \begin{bmatrix} \rho \\ a \\ 0 \\ 0 \\ \rho a^2 \end{bmatrix}.
\end{array}
\right\} \tag{3.94}
$$

Proof. The proof involves the usual algebraic steps for finding eigenvalues and eigenvectors. See Sect. 2.1 of Chap. 2.

3.2.4 The Split Three–Dimensional Riemann Problem

When solving numerically the two or three dimensional Euler equations by most methods of the upwind type in current use, one requires the solution of *split* Riemann problems. The x–split, three–dimensional Riemann problem is the IVP

$$
\left.
\begin{array}{l}
\mathbf{U}_t + \mathbf{F}(\mathbf{U})_x = \mathbf{0} \, , \\
\mathbf{U}(x,0) = \mathbf{U}^{(0)}(x) = \begin{cases} \mathbf{U}_{\mathrm{L}} & \text{if } x < 0 \, , \\ \mathbf{U}_{\mathrm{R}} & \text{if } x > 0 \, , \end{cases}
\end{array}
\right\} \tag{3.95}
$$

where

$$
\mathbf{U} = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ E \end{bmatrix}, \quad \mathbf{F}(\mathbf{U}) = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ \rho u w \\ u(E+p) \end{bmatrix}. \tag{3.96}
$$

The structure of the similarity solution is shown in Fig. 3.5 and is almost identical to that for the one–dimensional case shown in Fig. 3.1. Both pressure and *normal* particle velocity u are constant in the *Star Region*, across the middle wave. There are two new characteristic fields associated with $\lambda_3 = u$ and $\lambda_4 = u$, arising from the multiplicity 3 of the eigenvalue u; these correspond to two *shear waves* across which the respective tangential velocity components v and w change discontinuously. For the two–dimensional case we proved in Sect. 3.2.1 that the λ_3–field is linearly degenerate. This result is also true for the λ_4–field in three dimensions. The 1 and 5 characteristic fields are genuinely non–linear and are associated with rarefactions or shock waves, just as in the one–dimensional case. By inspecting the eigenvectors $\mathbf{K}^{(1)}$ and $\mathbf{K}^{(5)}$ in (3.94) we see immediately that the Generalised Riemann Invariants across 1 and 5 rarefaction waves give no change in the tangential velocity components v and w across these waves, see Fig. 3.5. In fact this is

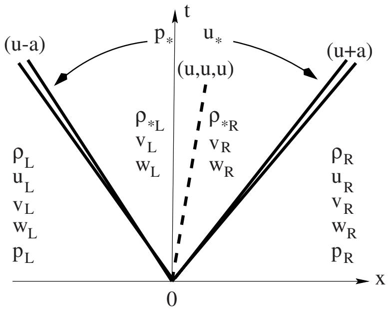

Fig. 3.5. Structure of the solution of the three–dimensional split Riemann problem

also true when these waves are shock waves. Consider a right shock wave of speed S associated with the 5 field. By transforming to a frame of reference in which the shock speed is zero and applying the Rankine–Hugoniot conditions we obtain the same relations (3.42)–(3.44) as in the one–dimensional case plus two extra relations involving v and w. The three relevant relations are

$$\rho_*(u_* - S) = \rho_R(u_R - S) , \tag{3.97}$$

$$\rho_*(u_* - S)(v_* - S) = \rho_R(u_R - S)(v_R - S) , \tag{3.98}$$

$$\rho_*(u_* - S)(w_* - S) = \rho_R(u_R - S)(w_R - S) . \tag{3.99}$$

Application of the shock condition (3.97) into equations (3.98) and (3.99) gives directly $v_* = v_R$ and $w_* = w_R$. A similar analysis for a left shock wave gives an equivalent result. Hence the tangential velocity components v and w remain constant across the non–linear waves 1 and 5, irrespective of their type.

Therefore finding the solution of the Riemann problem for the split three–dimensional equations is fundamentally the same as finding the solution for the corresponding one–dimensional Riemann problem. The solution for the extra variables v and w could not be simpler: it consists of single jump discontinuities across the shear waves from the values v_L, w_L on the left data state to the values v_R, w_R on the right data state. This simple behaviour of the tangential velocity components in the solution of split Riemann problems is sometimes incorrectly modelled by some *approximate* Riemann solvers.

3.3 Conservative Versus Non–Conservative Formulations

The specific purpose of this section is first to make the point that under the assumption of smooth solutions, *conservative* and non–conservative formulations are not unique. It is vitally important to scrutinise the conservative formulations carefully, as these may be conservative purely in a mathematical sense. The key question is to see what the *conserved quantities* are in the formulation and whether the conservation statements they imply make *physical sense*. The second point of interest here is to make the reader aware of the fact that in the presence of shock waves, formulations that are *conservative purely in a mathematical sense* will produce wrong shock speeds and thus wrong solutions. We illustrate these points through the one–dimensional shallow water equations, see Sect. 1.6.3 of Chap. 1,

$$\begin{bmatrix} \phi \\ \phi u \end{bmatrix}_t + \begin{bmatrix} \phi u \\ \phi u^2 + \frac{1}{2}\phi^2 \end{bmatrix}_x = \mathbf{0} \; . \tag{3.100}$$

They express the physical laws of conservation of mass and momentum. Under the assumption of smooth solutions we can expand derivatives so as to write the equations in primitive–variable form

$$\phi_t + u\phi_x + \phi u_x = 0 \; , \tag{3.101}$$

$$u_t + uu_x + \phi_x = 0 \; . \tag{3.102}$$

It is tempting to derive new conservation–law forms of the shallow water equations starting from equations (3.101)–(3.102). One possibility is to keep the mass equation as in (3.101) and re–write the momentum equation (3.102) as

$$u_t + (\frac{1}{2}u^2 + \phi)_x = 0 \; . \tag{3.103}$$

Now we have an alternative *conservative* form of the shallow water equations, namely

$$\begin{bmatrix} \phi \\ u \end{bmatrix}_t + \begin{bmatrix} \phi u \\ \frac{1}{2}u^2 + \phi \end{bmatrix}_x = \mathbf{0} \; . \tag{3.104}$$

Mathematically, see Chap. 2, this is a system of conservation laws. It expresses conservation of mass, as in (3.101), and *conservation of particle speed* u. Physically, this second conservation law does not make sense. A critical question is this : *can we use the conservation–law form* (3.104) for the shallow water equations. The anticipated answer is : *yes we can, if and only if solutions are smooth.* In the presence of shock waves formulations (3.100) and (3.104) lead to different solutions, as we now demonstrate.

Without loss of generality we consider a right facing shock wave in which the state ahead of the shock is given by the variables ϕ_R, u_R.

Proposition 3.22. *A right–facing shock wave solution of (3.100) has shock speed*

$$\left.\begin{aligned} S &= u_R + Q/\phi_R \; , \\ Q &= \left[\tfrac{1}{2}\left(\phi_* + \phi_R\right)\phi_*\phi_R\right]^{\frac{1}{2}} \; , \end{aligned}\right\} \tag{3.105}$$

while a right–facing shock wave solution of (3.104) has speed

$$\left.\begin{aligned} \hat{S} &= u_R + \hat{Q}/\phi_R \; , \\ \hat{Q} &= \left[\tfrac{2}{\phi_* + \phi_R}\right]^{\frac{1}{2}} \phi_*\phi_R \; . \end{aligned}\right\} \tag{3.106}$$

Proof. This is left to the reader as an exercise. Use contents of Chap. 2 and those of Sect. 3.1.3 of Chap. 3.

Remark 3.23. Clearly the shock speeds S and $\hat{S}$ are equivalent only when $\phi_* \equiv \phi_R$, that is when the shock wave is trivial. In general

$$\hat{S} \leq S \qquad (3.107)$$

and thus shock solutions of the *new* (incorrect) conservation laws (3.104) are slower than shock solutions of the *conventional* (correct) conservation laws (3.100). Note also that the conservative form (3.104) is non–unique.

Consider now the isothermal equations of Gas Dynamics, see Sect. 1.6.2 of Chap. 1. In conservation–law form these equations read

$$\begin{bmatrix} \rho \\ \rho u \end{bmatrix}_t + \begin{bmatrix} \rho u \\ \rho u^2 + a^2 \rho \end{bmatrix}_x = \mathbf{0} \; , \qquad (3.108)$$

where the sound speed a is constant. These conservation laws state that mass and momentum are conserved, which is in accord with the laws of conservation of mass and momentum studied in Chap. 1. Let us now assume that solutions are sufficiently smooth so that partial derivatives exist; we expand derivatives and after some algebraic manipulations obtain the primitive–variable formulation

$$\rho_t + u\rho_x + \rho u_x = 0 \; , \qquad (3.109)$$

$$u_t + uu_x + \frac{a^2}{\rho}\rho_x = 0 \; . \qquad (3.110)$$

This is a perfectly acceptable formulation, valid for smooth flows.

New conservation laws can be constructed, starting from the primitive formulation (3.109)–(3.110) above. One such possible system of conservation laws is

$$\begin{bmatrix} \rho \\ u \end{bmatrix}_t + \begin{bmatrix} \rho u \\ \frac{1}{2}u^2 + a^2 ln \rho \end{bmatrix}_x = \mathbf{0} \; . \qquad (3.111)$$

Mathematically, these equations are a set of conservation laws, see Sects. 2.1 and 2.4 of Chap. 2. Physically however, they are useless, they state that mass and *velocity* are conserved !

Exercise 3.24. Using the contents of Sect. 3.1.3 for isolated shock waves, compare the shock solutions of the two conservative formulations (3.108) and (3.111). Which gives the fastest shock ? Find other *conservative* formulations corresponding to (3.109)–(3.110).