
Notions on Hyperbolic Partial Differential Equations

In this chapter we study some elementary properties of a class of hyperbolic Partial Differential Equations (PDEs). The selected aspects of the equations are those thought to be essential for the analysis of the equations of fluid flow and the implementation of numerical methods. For general background on PDEs we recommend the book by John [272] and particularly the one by Zachmanoglou and Thoe [596]. The discretisation techniques studied in this book are strongly based on the underlying Physics and mathematical properties of PDEs. It is therefore justified to devote some effort to some fundamentals on PDEs. Here we deal almost exclusively with *hyperbolic* PDEs and *hyperbolic conservation laws* in particular. There are three main reasons for this: (i) The equations of compressible fluid flow reduce to hyperbolic systems, the Euler equations, when the effects of viscosity and heat conduction are neglected. (ii) Numerically, it is generally accepted that the hyperbolic terms of the PDEs of fluid flow are the terms that pose the most stringent requirements on the discretisation techniques. (iii) The theory of hyperbolic systems is much more advanced than that for more complete mathematical models, such as the Navier–Stokes equations. In addition, there has in recent years been a noticeable increase in research and development activities centred on the theme of hyperbolic problems, as these cover a wide range of areas of scientific and technological interest. A good source of up-to-date work in this field is found in the proceedings of the series of meetings on Hyperbolic Problems, see for instance [87], [184], [213]. See also [326]. Other relevant publications are those of Godlewski and Raviart [215], Hörmander [258] and Tveito and Winther [551].

We restrict ourselves to some of the basics on hyperbolic PDEs and choose an informal way of presentation. The selected topics and approach are almost exclusively motivated by the theme of the book, namely the Riemann problem and high-resolution upwind and centred numerical methods.

2.1 Quasi–Linear Equations: Basic Concepts

In this section we study systems of first–order partial differential equations of the form

$$\frac{\partial u_i}{\partial t} + \sum_{j=1}^m a_{ij}(x, t, u_1, \dots, u_m) \frac{\partial u_j}{\partial x} + b_i(x, t, u_1, \dots, u_m) = 0, \quad (2.1)$$

for $i = 1, \dots, m$. This is a system of m equations in m unknowns u_i that depend on space x and a time–like variable t . Here u_i are the *dependent variables* and x, t are the *independent variables*; this is expressed via the notation $u_i = u_i(x, t)$; $\partial u_i / \partial t$ denotes the partial derivative of $u_i(x, t)$ with respect to t ; similarly $\partial u_i / \partial x$ denotes the partial derivative of $u_i(x, t)$ with respect to x . We also make use of subscripts to denote partial derivatives. System (2.1) can also be written in matrix form as

$$\mathbf{U}_t + \mathbf{A}\mathbf{U}_x + \mathbf{B} = \mathbf{0}, \quad (2.2)$$

with

$$\mathbf{U} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} a_{11} & \dots & a_{1m} \\ a_{21} & \dots & a_{2m} \\ \vdots & \vdots & \vdots \\ a_{m1} & \dots & a_{mm} \end{bmatrix}. \quad (2.3)$$

If the entries a_{ij} of the matrix \mathbf{A} are all constant and the components b_j of the vector \mathbf{B} are also constant then system (2.2) is *linear with constant coefficients*. If $a_{ij} = a_{ij}(x, t)$ and $b_i = b_i(x, t)$ the system is *linear with variable coefficients*. The system is still linear if \mathbf{B} depends linearly on \mathbf{U} and is called *quasi–linear* if the coefficient matrix \mathbf{A} is a function of the vector \mathbf{U} , that is $\mathbf{A} = \mathbf{A}(\mathbf{U})$. Note that quasi–linear systems are in general systems of non–linear equations. System (2.2) is called *homogeneous* if $\mathbf{B} = \mathbf{0}$. For a set of PDEs of the form (2.2) one needs to specify the range of variation of the independent variables x and t . Usually x lies in a subinterval of the real line, namely $x_l < x < x_r$; this subinterval is called the *spatial domain* of the PDEs, or just *domain*. At the values x_l, x_r one also needs to specify *Boundary Conditions (BCs)*. In this Chapter we assume the domain is the full real line, $-\infty < x < \infty$, and thus no boundary conditions need to be specified. As to variations of time t we assume $t_0 < t < \infty$. An *Initial Condition (IC)* needs to be specified at the initial time, which is usually chosen to be $t_0 = 0$.

Two scalar ($m = 1$) examples of PDEs of the form (2.1) are the linear advection equation

$$\frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} = 0 \quad (2.4)$$

and the inviscid Burgers equation

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0, \quad (2.5)$$

both introduced in Sect. 1.6.2 of Chap. 1. In the linear advection equation (2.4) the coefficient a (a constant) is the wave propagation speed. In the Burgers equation $a = a(u) = u$.

Definition 2.1 (Conservation Laws). *Conservation laws are systems of partial differential equations that can be written in the form*

$$\mathbf{U}_t + \mathbf{F}(\mathbf{U})_x = \mathbf{0}, \quad (2.6)$$

where

$$\mathbf{U} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}, \quad \mathbf{F}(\mathbf{U}) = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_m \end{bmatrix}. \quad (2.7)$$

\mathbf{U} is called the vector of conserved variables, $\mathbf{F} = \mathbf{F}(\mathbf{U})$ is the vector of fluxes and each of its components f_i is a function of the components u_j of \mathbf{U} .

Definition 2.2 (Jacobian Matrix). *The Jacobian of the flux function $\mathbf{F}(\mathbf{U})$ in (2.6) is the matrix*

$$\mathbf{A}(\mathbf{U}) = \partial \mathbf{F} / \partial \mathbf{U} = \begin{bmatrix} \partial f_1 / \partial u_1 & \dots & \partial f_1 / \partial u_m \\ \partial f_2 / \partial u_1 & \dots & \partial f_2 / \partial u_m \\ \vdots & \vdots & \vdots \\ \partial f_m / \partial u_1 & \dots & \partial f_m / \partial u_m \end{bmatrix}. \quad (2.8)$$

The entries a_{ij} of $\mathbf{A}(\mathbf{U})$ are partial derivatives of the components f_i of the vector \mathbf{F} with respect to the components u_j of the vector of conserved variables \mathbf{U} , that is $a_{ij} = \partial f_i / \partial u_j$.

Note that conservation laws of the form (2.6)–(2.7) can also be written in quasi-linear form (2.2), with $\mathbf{B} \equiv \mathbf{0}$, by applying the chain rule to the second term in (2.6), namely

$$\frac{\partial \mathbf{F}(\mathbf{U})}{\partial x} = \frac{\partial \mathbf{F}}{\partial \mathbf{U}} \frac{\partial \mathbf{U}}{\partial x}.$$

Hence (2.6) becomes

$$\mathbf{U}_t + \mathbf{A}(\mathbf{U})\mathbf{U}_x = \mathbf{0},$$

which is a special case of (2.2). The scalar PDEs (2.4) and (2.5) can be expressed as conservation laws, namely

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, \quad f(u) = au, \quad (2.9)$$

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, \quad f(u) = \frac{1}{2}u^2. \quad (2.10)$$

Definition 2.3 (Eigenvalues). *The eigenvalues λ_i of a matrix \mathbf{A} are the solutions of the characteristic polynomial*

$$|\mathbf{A} - \lambda\mathbf{I}| = \det(\mathbf{A} - \lambda\mathbf{I}) = 0, \quad (2.11)$$

where \mathbf{I} is the identity matrix. The eigenvalues of the coefficient matrix \mathbf{A} of a system of the form (2.2) are called the eigenvalues of the system.

Physically, eigenvalues represent speeds of propagation of information. Speeds will be measured positive in the direction of increasing x and negative otherwise.

Definition 2.4 (Eigenvectors). *A right eigenvector of a matrix \mathbf{A} corresponding to an eigenvalue λ_i of \mathbf{A} is a vector $\mathbf{K}^{(i)} = [k_1^{(i)}, k_2^{(i)}, \dots, k_m^{(i)}]^T$ satisfying $\mathbf{A}\mathbf{K}^{(i)} = \lambda_i\mathbf{K}^{(i)}$. Similarly, a left eigenvector of a matrix \mathbf{A} corresponding to an eigenvalue λ_i of \mathbf{A} is a vector $\mathbf{L}^{(i)} = [l_1^{(i)}, l_2^{(i)}, \dots, l_m^{(i)}]$ such that $\mathbf{L}^{(i)}\mathbf{A} = \lambda_i\mathbf{L}^{(i)}$.*

For the scalar examples (2.9)–(2.10) the eigenvalues are trivially found to be $\lambda = a$ and $\lambda = u$ respectively. Next we find eigenvalues and eigenvectors for a system of PDEs.

Example 2.5 (Linearised Gas Dynamics). The linearised equations of Gas Dynamics, derived in Sect. 1.6.2 of Chap. 1, are the 2×2 linear system

$$\left. \begin{aligned} \frac{\partial \rho}{\partial t} + \rho_0 \frac{\partial u}{\partial x} &= 0, \\ \frac{\partial u}{\partial t} + \frac{a^2}{\rho_0} \frac{\partial \rho}{\partial x} &= 0, \end{aligned} \right\} \quad (2.12)$$

where the unknowns are the density $u_1 = \rho(x, t)$ and the speed $u_2 = u(x, t)$; ρ_0 is a constant reference density and a is the sound speed, a positive constant. When written in the matrix form (2.2) this system reads

$$\mathbf{U}_t + \mathbf{A}\mathbf{U}_x = \mathbf{0}, \quad (2.13)$$

with

$$\mathbf{U} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \equiv \begin{bmatrix} \rho \\ u \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 0 & \rho_0 \\ a^2/\rho_0 & 0 \end{bmatrix}. \quad (2.14)$$

The eigenvalues of the system are the zeros of the characteristic polynomial

$$|\mathbf{A} - \lambda\mathbf{I}| = \det \begin{bmatrix} 0 - \lambda & \rho_0 \\ a^2/\rho_0 & 0 - \lambda \end{bmatrix} = 0.$$

That is, $\lambda^2 = a^2$, which has two real and distinct solutions, namely

$$\lambda_1 = -a, \quad \lambda_2 = +a. \quad (2.15)$$

We now find the right eigenvectors $\mathbf{K}^{(1)}$, $\mathbf{K}^{(2)}$ corresponding to the eigenvalues λ_1 and λ_2 .

The eigenvector $\mathbf{K}^{(1)}$ for eigenvalue $\lambda = \lambda_1 = -a$ is found as follows: we look for a vector $\mathbf{K}^{(1)} = [k_1, k_2]^T$ such that $\mathbf{K}^{(1)}$ is a right eigenvector of \mathbf{A} , that is $\mathbf{A}\mathbf{K}^{(1)} = \lambda_1\mathbf{K}^{(1)}$. Writing this in full gives

$$\begin{bmatrix} 0 & \rho_0 \\ a^2/\rho_0 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} -ak_1 \\ -ak_2 \end{bmatrix},$$

which produces two linear algebraic equations for the unknowns k_1 and k_2

$$\rho_0 k_2 = -ak_1, \quad \frac{a^2}{\rho_0} k_1 = -ak_2. \quad (2.16)$$

The reader will realise that in fact these two equations are equivalent and so effectively we have a single linear algebraic equation in two unknowns. This gives a one-parameter family of solutions. Thus we select an arbitrary non-zero parameter α_1 , a scaling factor, and set $k_1 = \alpha_1$ in any of the equations to obtain $k_2 = -\alpha_1 a/\rho_0$ for the second component and hence the first right eigenvector becomes

$$\mathbf{K}^{(1)} = \alpha_1 \begin{bmatrix} 1 \\ -a/\rho_0 \end{bmatrix}. \quad (2.17)$$

The eigenvector $\mathbf{K}^{(2)}$ for eigenvalue $\lambda = \lambda_2 = +a$ is found in a similar manner. The resulting algebraic equations for $\mathbf{K}^{(2)}$ corresponding to the eigenvalue $\lambda_2 = +a$ are

$$\rho_0 k_2 = ak_1, \quad \frac{a^2}{\rho_0} k_1 = ak_2. \quad (2.18)$$

By denoting the second scaling factor by α_2 and setting $k_1 = \alpha_2$ we obtain

$$\mathbf{K}^{(2)} = \alpha_2 \begin{bmatrix} 1 \\ a/\rho_0 \end{bmatrix}. \quad (2.19)$$

Taking the scaling factors to be $\alpha_1 = \rho_0$ and $\alpha_2 = \rho_0$ gives the right eigenvectors

$$\mathbf{K}^{(1)} = \begin{bmatrix} \rho_0 \\ -a \end{bmatrix}, \quad \mathbf{K}^{(2)} = \begin{bmatrix} \rho_0 \\ a \end{bmatrix}. \quad (2.20)$$

Definition 2.6 (Hyperbolic System). A system (2.2) is said to be *hyperbolic* at a point (x, t) if \mathbf{A} has m real eigenvalues $\lambda_1, \dots, \lambda_m$ and a corresponding set of m linearly independent right eigenvectors $\mathbf{K}^{(1)}, \dots, \mathbf{K}^{(m)}$. The system is said to be *strictly hyperbolic* if the eigenvalues λ_i are all distinct.

Note that strict hyperbolicity implies hyperbolicity, because real and distinct eigenvalues ensure the existence of a set of linearly independent eigenvectors. The system (2.2) is said to be *elliptic* at a point (x, t) if none of the eigenvalues λ_i of \mathbf{A} are real. Both scalar examples (2.9)–(2.10) are trivially hyperbolic. The linearised gas dynamic equations (2.12) are also hyperbolic, since λ_1 and λ_2 are both real at any point (x, t) . Moreover, as the eigenvalues are also distinct this system is strictly hyperbolic.

Example 2.7 (The Cauchy–Riemann Equations). An example of a first-order system of the form (2.2) with t replaced by x and x replaced by y is the Cauchy–Riemann equations

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0, \quad \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 0, \quad (2.21)$$

where $u_1 = u(x, y)$ and $u_2 = v(x, y)$. These equations arise in the study of *analytic functions* in Complex Analysis [379]. When written in matrix notation (2.2) equations (2.21) become

$$\mathbf{U}_x + \mathbf{A}\mathbf{U}_y = \mathbf{0}, \quad (2.22)$$

with

$$\mathbf{U} = \begin{bmatrix} u \\ v \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \quad (2.23)$$

The characteristic polynomial $|\mathbf{A} - \lambda\mathbf{I}| = 0$ gives $\lambda^2 + 1 = 0$, which has no real solutions for λ and thus the system is *elliptic*.

Example 2.8 (The Small Perturbation Equations). In Sect. 1.6.2 of Chap. 1, the small perturbation steady equations were introduced

$$u_x - a^2 v_y = 0, \quad v_x - u_y = 0, \quad (2.24)$$

with

$$a^2 = \frac{1}{M_\infty^2 - 1}. \quad (2.25)$$

$M_\infty = \text{constant}$ denotes the free-stream Mach number and $u(x, y)$, $v(x, y)$ are small perturbations of the x and y velocity components respectively. In matrix notation these equations read

$$\mathbf{U}_x + \mathbf{A}\mathbf{U}_y = \mathbf{0}, \quad (2.26)$$

with

$$\mathbf{U} = \begin{bmatrix} u \\ v \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 0 & -a^2 \\ -1 & 0 \end{bmatrix}. \quad (2.27)$$

The character of these equations depends entirely on the value of the Mach number M_∞ . For subsonic flow $M_\infty < 1$ the characteristic polynomial has complex solutions and thus the equations are of elliptic type. For supersonic flow $M_\infty > 1$ and the system is strictly hyperbolic, with eigenvalues

$$\lambda_1 = -a, \quad \lambda_2 = +a. \quad (2.28)$$

It is left to the reader to check that the corresponding right eigenvectors are

$$\mathbf{K}^{(1)} = \alpha_1 \begin{bmatrix} 1 \\ 1/a \end{bmatrix}, \quad \mathbf{K}^{(2)} = \alpha_2 \begin{bmatrix} 1 \\ -1/a \end{bmatrix}, \quad (2.29)$$

where α_1 and α_2 are two non-zero scaling factors. By taking the values $\alpha_1 = \alpha_2 = a$ we obtain the following expressions for the right eigenvectors

$$\mathbf{K}^{(1)} = \begin{bmatrix} a \\ 1 \end{bmatrix}, \quad \mathbf{K}^{(2)} = \begin{bmatrix} a \\ -1 \end{bmatrix},$$

2.2 The Linear Advection Equation

A general, time-dependent linear advection equation in three space dimensions reads

$$u_t + a(x, y, z, t)u_x + b(x, y, z, t)u_y + c(x, y, z, t)u_z = 0, \quad (2.30)$$

where the unknown is $u = u(x, y, z, t)$ and a, b, c are variable coefficients. If the coefficients are sufficiently smooth one can express (2.30) as a conservation law with source terms, namely

$$u_t + (au)_x + (bu)_y + (cu)_z = u(a_x + b_y + c_z). \quad (2.31)$$

In this section we study in detail the initial-value problem (IVP) for the special case of the linear advection equation, namely

$$\left. \begin{array}{l} \text{PDE: } u_t + au_x = 0, \quad -\infty < x < \infty, \quad t > 0. \\ \text{IC: } \quad \quad \quad u(x, 0) = u_0(x), \end{array} \right\} \quad (2.32)$$

where a is a constant wave propagation speed. The initial data at time $t = 0$ is a function of x alone and is denoted by $u_0(x)$. We warn the reader that for systems we shall use a different notation for the initial data. Generally, we shall not be explicit about the conditions $-\infty < x < \infty; t > 0$ on the independent variables when stating an initial-value problem. The PDE in (2.32) is the simplest hyperbolic PDE and in view of (2.9) is also the simplest hyperbolic conservation law. It is a very useful model equation for the purpose of studying numerical methods for hyperbolic conservation laws, in the same way as the linear, first-order ordinary differential equation

$$\frac{dx}{dt} = \beta, \quad x = x(t), \quad \beta = \text{constant}, \quad (2.33)$$

is a popular model equation for analysing numerical methods for Ordinary Differential Equation (ODEs). Two useful references on ordinary differential equations are Brown [81] and Lambert [296]. In Sect. 15.4 of Chap. 15 we study numerical methods for ODEs in connection with source terms in inhomogeneous PDEs.

2.2.1 Characteristics and the General Solution

We recall the definition of *characteristics* or *characteristic curves* in the context of a scalar equation such as that in (2.32). Characteristics may be defined as curves $x = x(t)$ in the t - x plane along which the PDE becomes an ODE. Consider $x = x(t)$ and regard u as a function of t , that is $u = u(x(t), t)$. The rate of change of u along $x = x(t)$ is

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{dx}{dt} \frac{\partial u}{\partial x} . \quad (2.34)$$

If the characteristic curve $x = x(t)$ satisfies the ODE

$$\frac{dx}{dt} = a , \quad (2.35)$$

then the PDE in (2.32), together with (2.34) and (2.35), gives

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} = 0 . \quad (2.36)$$

Therefore the rate of change of u along the characteristic curve $x = x(t)$ satisfying (2.35) is zero, that is, u is *constant along the curve* $x = x(t)$. The speed a in (2.35) is called the *characteristic speed* and according to (2.35) it is the slope of the curve $x = x(t)$ in the t - x plane. In practice it is more common to use the x - t plane to sketch the characteristics, in which case the slope of the curves in question is $1/a$. The family of characteristic curves $x = x(t)$ given by

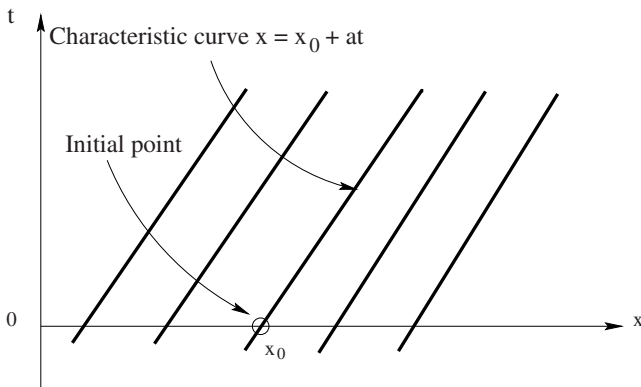


Fig. 2.1. Picture of characteristics for the linear advection equation for positive characteristic speed a . Initial condition at time $t = 0$ fixes the initial position x_0

the ODE (2.35) are illustrated in Fig. 2.1 for $a > 0$ and are a one-parameter family of curves. A particular member of this family is determined when an *initial condition* (IC) at time $t = 0$ for the ODE (2.35) is added. Suppose we set

$$x(0) = x_0 , \quad (2.37)$$

then the single characteristic curve that passes through the point $(x_0, 0)$, according to (2.35) is

$$x = x_0 + at . \quad (2.38)$$

This is also illustrated in Fig. 2.1. Now we may regard the initial position x_0 as a parameter and in this way we reproduce the full one-parameter family

of characteristics. The fact that the curves are *parallel* is typical of linear hyperbolic PDEs with constant coefficients.

Recall the conclusion from (2.36) that u remains constant along characteristics. Thus, if u is given the initial value $u(x, 0) = u_0(x)$ at time $t = 0$, then along the whole characteristic curve $x(t) = x_0 + at$ that passes through the initial point x_0 on the x -axis, the solution is

$$u(x, t) = u_0(x_0) = u_0(x - at) . \quad (2.39)$$

The second equality follows from (2.38). The interpretation of the solution (2.39) of the PDE in (2.32) is this: *given an initial profile $u_0(x)$, the PDE will simply translate this profile with velocity a to the right if $a > 0$ and to the left if $a < 0$. The shape of the initial profile remains unchanged.* The model equation in (2.32) under study contains some of the basic features of wave propagation phenomena, where a wave is understood as some recognisable feature of a disturbance that travels at a finite speed.

2.2.2 The Riemann Problem

By using geometric arguments we have constructed the analytical solution of the general IVP (2.32) for the linear advection equation. This is given by (2.39) in terms of the initial data $u_0(x)$. Now we study a special IVP called the Riemann problem

$$\left. \begin{array}{l} \text{PDE:} \quad u_t + au_x = 0 . \\ \text{IC:} \quad u(x, 0) = u_0(x) = \begin{cases} u_L & \text{if } x < 0 , \\ u_R & \text{if } x > 0 , \end{cases} \end{array} \right\} \quad (2.40)$$

where u_L (left) and u_R (right) are two constant values, as shown in Fig. 2.2. Note that the initial data has a discontinuity at $x = 0$. IVP (2.40) is the simplest initial-value problem one can pose. The trivial case would result when $u_L = u_R$. From the previous discussion on the solution of the general IVP (2.32) we expect any point on the initial profile to propagate a distance $d = at$ in time t . In particular, we expect the initial discontinuity at $x = 0$ to propagate a distance $d = at$ in time t . This particular characteristic curve $x = at$ will then separate those characteristic curves to the left, on which the solution takes on the value u_L , from those curves to the right, on which the solution takes on the value u_R ; see Fig. 2.3. So the solution of the Riemann problem (2.40) is simply

$$u(x, t) = u_0(x - at) = \begin{cases} u_L & \text{if } x - at < 0 , \\ u_R & \text{if } x - at > 0 . \end{cases} \quad (2.41)$$

Solution (2.41) also follows directly from the general solution (2.39), namely $u(x, t) = u_0(x - at)$. From (2.40), $u_0(x - at) = u_L$ if $x - at < 0$ and $u_0(x - at) =$

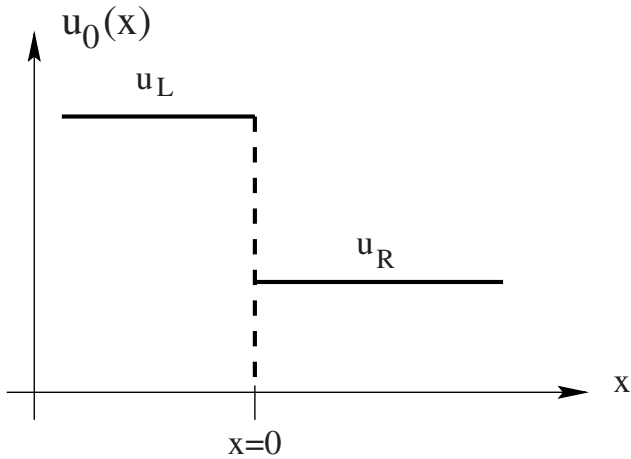


Fig. 2.2. Illustration of the initial data for the Riemann problem. At the initial time the data consists of two constant states separated by a discontinuity at $x = 0$

u_R if $x - at > 0$. The solution of the Riemann problem can be represented in the $x-t$ plane, as shown in Fig. 2.3. Through any point x_0 on the x -axis one can draw a characteristic. As a is constant these are all parallel to each other. For the solution of the Riemann problem the characteristic that passes through $x = 0$ is significant. This is the only one across which the solution changes.

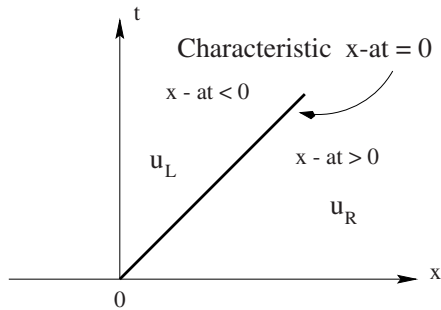


Fig. 2.3. Illustration of the solution of the Riemann problem in the $x-t$ plane for the linear advection equation with positive characteristic speed a

2.3 Linear Hyperbolic Systems

In the previous section we studied in detail the behaviour and the general solution of the simplest PDE of hyperbolic type, namely the linear advection

equation with constant wave propagation speed. Here we extend the analysis to sets of m hyperbolic PDEs of the form

$$\mathbf{U}_t + \mathbf{A}\mathbf{U}_x = \mathbf{0}, \quad (2.42)$$

where the coefficient matrix \mathbf{A} is constant. From the assumption of hyperbolicity \mathbf{A} has m real eigenvalues λ_i and m linearly independent eigenvectors $\mathbf{K}^{(i)}$, $i = 1, \dots, m$.

2.3.1 Diagonalisation and Characteristic Variables

In order to analyse and solve the general IVP for (2.42) it is found useful to transform the dependent variables $\mathbf{U}(x, t)$ to a new set of dependent variables $\mathbf{W}(x, t)$. To this end we recall the following definition

Definition 2.9 (Diagonalisable System). *A matrix \mathbf{A} is said to be diagonalisable if \mathbf{A} can be expressed as*

$$\mathbf{A} = \mathbf{K}\mathbf{\Lambda}\mathbf{K}^{-1} \text{ or } \mathbf{\Lambda} = \mathbf{K}^{-1}\mathbf{A}\mathbf{K}, \quad (2.43)$$

in terms of a diagonal matrix $\mathbf{\Lambda}$ and a matrix \mathbf{K} . The diagonal elements of $\mathbf{\Lambda}$ are the eigenvalues λ_i of \mathbf{A} and the columns $\mathbf{K}^{(i)}$ of \mathbf{K} are the right eigenvectors of \mathbf{A} corresponding to the eigenvalues λ_i , that is

$$\mathbf{A} = \begin{bmatrix} \lambda_1 & \dots & 0 \\ 0 & \dots & 0 \\ \vdots & \vdots & \vdots \\ 0 & \dots & \lambda_m \end{bmatrix}, \quad \mathbf{K} = [\mathbf{K}^{(1)}, \dots, \mathbf{K}^{(m)}], \quad \mathbf{A}\mathbf{K}^{(i)} = \lambda_i\mathbf{K}^{(i)}. \quad (2.44)$$

A system (2.42) is said to be *diagonalisable* if the coefficient matrix \mathbf{A} is diagonalisable. Based on the concept of diagonalisation one often defines a hyperbolic system (2.42) as a system with real eigenvalues and diagonalisable coefficient matrix.

Characteristic variables

The existence of the inverse matrix \mathbf{K}^{-1} makes it possible to define a new set of dependent variables $\mathbf{W} = (w_1, w_2, \dots, w_m)^T$ via the transformation

$$\mathbf{W} = \mathbf{K}^{-1}\mathbf{U} \text{ or } \mathbf{U} = \mathbf{K}\mathbf{W}, \quad (2.45)$$

so that the linear system (2.42), when expressed in terms of \mathbf{W} , becomes *completely decoupled*, in a sense to be defined. The new variables \mathbf{W} are called *characteristic variables*. Next we derive the governing PDEs in terms of the characteristic variables, for which we need the partial derivatives \mathbf{U}_t and \mathbf{U}_x in equations (2.42). Since \mathbf{A} is constant, \mathbf{K} is also constant and therefore these derivatives are

$$\mathbf{U}_t = \mathbf{K}\mathbf{W}_t, \quad \mathbf{U}_x = \mathbf{K}\mathbf{W}_x.$$

Direct substitution of these expressions into equation (2.42) gives

$$\mathbf{K}\mathbf{W}_t + \mathbf{A}\mathbf{K}\mathbf{W}_x = \mathbf{0}.$$

Multiplication of this equation from the left by \mathbf{K}^{-1} and use of (2.43) gives

$$\mathbf{W}_t + \mathbf{\Lambda}\mathbf{W}_x = \mathbf{0}. \quad (2.46)$$

This is called the *canonical form* or *characteristic form* of system (2.42). When written in full this system becomes

$$\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{bmatrix}_t + \begin{bmatrix} \lambda_1 & \dots & 0 \\ 0 & \dots & 0 \\ \vdots & \vdots & \vdots \\ 0 & \dots & \lambda_m \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{bmatrix}_x = \mathbf{0}. \quad (2.47)$$

Clearly the i -th PDE of this system is

$$\frac{\partial w_i}{\partial t} + \lambda_i \frac{\partial w_i}{\partial x} = 0, \quad i = 1, \dots, m \quad (2.48)$$

and involves the *single unknown* $w_i(x, t)$; the system is therefore *decoupled* and is identical to the linear advection equation in (2.32); now the characteristic speed is λ_i and there are m characteristic curves satisfying m ODEs

$$\frac{dx}{dt} = \lambda_i, \quad \text{for } i = 1, \dots, m. \quad (2.49)$$

2.3.2 The General Initial–Value Problem

We now study the IVP for the PDEs (2.42). The initial condition is now denoted by superscript (0), namely

$$\mathbf{U}^{(0)} = (u_1^{(0)}, \dots, u_m^{(0)})^T,$$

rather than by subscript 0, as done for the scalar case. We find the general solution of the IVP by first solving the corresponding IVP for the canonical system (2.46) or (2.47) in terms of the characteristic variables \mathbf{W} and initial condition $\mathbf{W}^{(0)} = (w_1^{(0)}, \dots, w_m^{(0)})^T$ such that

$$\mathbf{W}^{(0)} = \mathbf{K}^{-1}\mathbf{U}^{(0)} \quad \text{or} \quad \mathbf{U}^{(0)} = \mathbf{K}\mathbf{W}^{(0)}.$$

The solution of the IVP for (2.46) is direct. By considering each unknown $w_i(x, t)$ satisfying (2.48) and its corresponding initial data $w_i^{(0)}$ we write its solution immediately as

$$w_i(x, t) = w_i^{(0)}(x - \lambda_i t), \quad \text{for } i = 1, \dots, m. \quad (2.50)$$

Compare with solution (2.39) for the scalar case. The solution of the general IVP in terms of the original variables \mathbf{U} is now obtained by transforming back according to (2.45), namely $\mathbf{U} = \mathbf{K}\mathbf{W}$.

Example 2.10 (Linearised Gas Dynamics Revisited). As a simple example we now study the general IVP for the linearised equations of Gas Dynamics (2.12), namely

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}_t + \begin{bmatrix} 0 & \rho_0 \\ a^2/\rho_0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}_x = \mathbf{0}, \quad u_1 \equiv \rho, \quad u_2 \equiv u,$$

with initial condition

$$\begin{bmatrix} u_1(x, 0) \\ u_2(x, 0) \end{bmatrix} = \begin{bmatrix} u_1^{(0)}(x) \\ u_2^{(0)}(x) \end{bmatrix}.$$

We define characteristic variables

$$\mathbf{W} = (w_1, w_2)^T = \mathbf{K}^{-1}\mathbf{U},$$

where \mathbf{K} is the matrix of right eigenvectors and \mathbf{K}^{-1} is its inverse, both given by

$$\mathbf{K} = \begin{bmatrix} \rho_0 & \rho_0 \\ -a & a \end{bmatrix}, \quad \mathbf{K}^{-1} = \frac{1}{2a\rho_0} \begin{bmatrix} a & -\rho_0 \\ a & \rho_0 \end{bmatrix}.$$

Since $\lambda_1 = -a$ and $\lambda_2 = a$, in terms of the characteristic variables we may write

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}_t + \begin{bmatrix} -a & 0 \\ 0 & a \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}_x = \mathbf{0},$$

or in full

$$\frac{\partial w_1}{\partial t} - a \frac{\partial w_1}{\partial x} = 0, \quad \frac{\partial w_2}{\partial t} + a \frac{\partial w_2}{\partial x} = 0.$$

The initial condition satisfies

$$\begin{bmatrix} w_1^{(0)} \\ w_2^{(0)} \end{bmatrix} = \mathbf{K}^{-1} \begin{bmatrix} u_1^{(0)} \\ u_2^{(0)} \end{bmatrix},$$

or in full

$$w_1^{(0)}(x) = \frac{1}{2a\rho_0} \left[au_1^{(0)}(x) - \rho_0 u_2^{(0)}(x) \right],$$

$$w_2^{(0)}(x) = \frac{1}{2a\rho_0} \left[au_1^{(0)}(x) + \rho_0 u_2^{(0)}(x) \right].$$

Each equation involves a single independent variable and is a linear advection equation of the form (2.48). The solution for w_1 and w_2 in terms of their initial data $w_1^{(0)}, w_2^{(0)}$, according to (2.50) is

$$w_1(x, t) = w_1^{(0)}(x + at), \quad w_2(x, t) = w_2^{(0)}(x - at),$$

or in full

$$w_1(x, t) = \frac{1}{2a\rho_0} \left[au_1^{(0)}(x + at) - \rho_0 u_2^{(0)}(x + at) \right],$$

$$w_2(x, t) = \frac{1}{2a\rho_0} \left[au_1^{(0)}(x - at) + \rho_0 u_2^{(0)}(x - at) \right].$$

This is the solution in terms of the characteristic variables. In order to obtain the solution to the original problem we transform back using $\mathbf{U} = \mathbf{KW}$. This gives the final solution as

$$\left. \begin{aligned} u_1(x, t) &= \frac{1}{2a} \left[au_1^{(0)}(x + at) - \rho_0 u_2^{(0)}(x + at) \right] \\ &\quad + \frac{1}{2a} \left[au_1^{(0)}(x - at) + \rho_0 u_2^{(0)}(x - at) \right], \end{aligned} \right\}$$

$$\left. \begin{aligned} u_2(x, t) &= -\frac{1}{2\rho_0} \left[au_1^{(0)}(x + at) - \rho_0 u_2^{(0)}(x + at) \right] \\ &\quad + \frac{1}{2\rho_0} \left[au_1^{(0)}(x - at) + \rho_0 u_2^{(0)}(x - at) \right]. \end{aligned} \right\}$$

Exercise 2.11. Find the solution of the general IVP for the Small Perturbation Equations (2.24) using the above methodology.

Solution 2.12. (Left to the reader).

We return to the expression $\mathbf{U} = \mathbf{KW}$ in (2.45) used to recover the solution to the original problem. When written in full this expression becomes

$$\begin{aligned} u_1 &= w_1 k_1^{(1)} + w_2 k_1^{(2)} + \dots + w_m k_1^{(m)}, \\ u_i &= w_1 k_i^{(1)} + w_2 k_i^{(2)} + \dots + w_m k_i^{(m)}, \\ u_m &= w_1 k_m^{(1)} + w_2 k_m^{(2)} + \dots + w_m k_m^{(m)}, \end{aligned}$$

or

$$\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} = w_1 \begin{bmatrix} k_1^{(1)} \\ k_2^{(1)} \\ \vdots \\ k_m^{(1)} \end{bmatrix} + w_2 \begin{bmatrix} k_1^{(2)} \\ k_2^{(2)} \\ \vdots \\ k_m^{(2)} \end{bmatrix} + \dots + w_m \begin{bmatrix} k_1^{(m)} \\ k_2^{(m)} \\ \vdots \\ k_m^{(m)} \end{bmatrix}, \tag{2.51}$$

or more succinctly

$$\mathbf{U}(x, t) = \sum_{i=1}^m w_i(x, t) \mathbf{K}^{(i)}. \tag{2.52}$$

This means that the function $w_i(x, t)$ is the coefficient of $\mathbf{K}^{(i)}$ in an *eigenvector expansion* of the vector \mathbf{U} . But according to (2.50), $w_i(x, t) = w_i^{(0)}(x - \lambda_i t)$ and hence

$$\mathbf{U}(x, t) = \sum_{i=1}^m w_i^{(0)}(x - \lambda_i t) \mathbf{K}^{(i)}. \tag{2.53}$$

Thus, given a point (x, t) in the $x-t$ plane, the solution $\mathbf{U}(x, t)$ at this point depends only on the initial data at the m points $x_0^{(i)} = x - \lambda_i t$. These are the

intersections of the characteristics of speed λ_i with the x -axis. The solution (2.53) for \mathbf{U} can be seen as the superposition of m waves, each of which is advected independently without change in shape. The i -th wave has shape $w_i^{(0)}(x)\mathbf{K}^{(i)}$ and propagates with speed λ_i .

2.3.3 The Riemann Problem

We study the Riemann problem for the hyperbolic, constant coefficient system (2.42). This is the special IVP

$$\left. \begin{aligned} \text{PDEs: } & \mathbf{U}_t + \mathbf{A}\mathbf{U}_x = \mathbf{0}, \quad -\infty < x < \infty, \quad t > 0, \\ \text{IC: } & \mathbf{U}(x, 0) = \mathbf{U}^{(0)}(x) = \begin{cases} \mathbf{U}_L & x < 0, \\ \mathbf{U}_R & x > 0 \end{cases} \end{aligned} \right\} \quad (2.54)$$

and is a generalisation of the IVP (2.32). We assume that the system is *strictly hyperbolic* and we order the real and distinct eigenvalues as

$$\lambda_1 < \lambda_2 < \dots < \lambda_m. \quad (2.55)$$

The General Solution

The structure of the solution of the Riemann problem (2.54) in the x - t plane is depicted in Fig. 2.4. It consists of m waves emanating from the origin, one for each eigenvalue λ_i . Each wave i carries a *jump discontinuity* in \mathbf{U} propagating with speed λ_i . Naturally, the solution to the left of the λ_1 -wave is simply the initial data \mathbf{U}_L and to the right of the λ_m -wave is \mathbf{U}_R . The task at hand is to find the solution in the wedge between the λ_1 and λ_m waves. As the eigenvectors $\mathbf{K}^{(1)}, \dots, \mathbf{K}^{(m)}$ are *linearly independent*, we

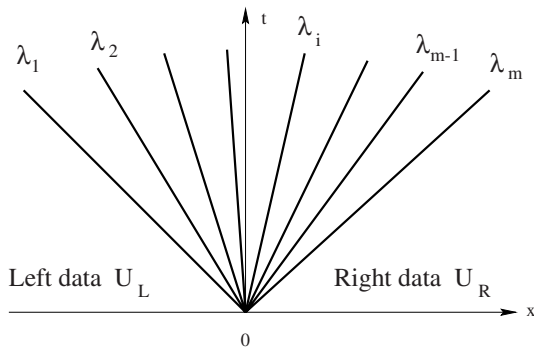


Fig. 2.4. Structure of the solution of the Riemann problem for a general $m \times m$ linear hyperbolic system with constant coefficients

can expand the data \mathbf{U}_L , constant left state, and \mathbf{U}_R , constant right state, as linear combinations of the set $\mathbf{K}^{(1)}, \dots, \mathbf{K}^{(m)}$, that is

$$\mathbf{U}_L = \sum_{i=1}^m \alpha_i \mathbf{K}^{(i)}, \quad \mathbf{U}_R = \sum_{i=1}^m \beta_i \mathbf{K}^{(i)}, \quad (2.56)$$

with constant coefficients α_i, β_i , for $i = 1, \dots, m$. Formally, the solution of the IVP (2.54) is given by (2.53) in terms of the initial data $w_i^{(0)}(x)$ for the characteristic variables and the right eigenvectors $\mathbf{K}^{(i)}$. Note that each of the expansions in (2.56) is a special case of (2.53). In terms of the characteristic variables we have m scalar Riemann problems for the PDEs

$$\frac{\partial w_i}{\partial t} + \lambda_i \frac{\partial w_i}{\partial x} = 0, \quad (2.57)$$

with initial data obtained by comparing (2.56) with (2.53), that is

$$w_i^{(0)}(x) = \begin{cases} \alpha_i & \text{if } x < 0, \\ \beta_i & \text{if } x > 0, \end{cases} \quad (2.58)$$

for $i = 1, \dots, m$. From the previous results, see equation (2.50), we know that the solutions of these scalar Riemann problems are given by

$$w_i(x, t) = w_i^{(0)}(x - \lambda_i t) = \begin{cases} \alpha_i & \text{if } x - \lambda_i t < 0, \\ \beta_i & \text{if } x - \lambda_i t > 0. \end{cases} \quad (2.59)$$

For a given point (x, t) there is an eigenvalue λ_I such that $\lambda_I < \frac{x}{t} < \lambda_{I+1}$, that is $x - \lambda_i t > 0 \forall i$ such that $i \leq I$. We can thus write the final solution to the Riemann problem (2.54) in terms of the original variables as

$$\mathbf{U}(x, t) = \sum_{i=I+1}^m \alpha_i \mathbf{K}^{(i)} + \sum_{i=1}^I \beta_i \mathbf{K}^{(i)}, \quad (2.60)$$

where the integer $I = I(x, t)$ is the maximum value of the sub-index i for which $x - \lambda_i t > 0$.

The Solution for a 2×2 System

As an example consider the Riemann problem for a general 2×2 linear system. From the origin $(0, 0)$ in the (x, t) plane there will be two waves travelling with speeds that are equal to the characteristic speeds λ_1 and λ_2 ($\lambda_1 < \lambda_2$); see Fig. 2.5. The solution to the left of $dx/dt = \lambda_1$ is simply the data state $\mathbf{U}_L = \alpha_1 \mathbf{K}^{(1)} + \alpha_2 \mathbf{K}^{(2)}$ and to the right of $dx/dt = \lambda_2$ the solution is the constant data state $\mathbf{U}_R = \beta_1 \mathbf{K}^{(1)} + \beta_2 \mathbf{K}^{(2)}$. The wedge between the λ_1 and λ_2 waves is usually called the *Star Region* and the solution there is denoted by \mathbf{U}^* ; its value is *due to the passage of two waves emerging from*

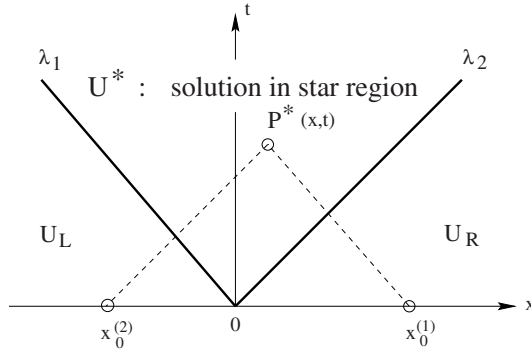


Fig. 2.5. Structure of the solution of the Riemann problem for a 2×2 linear system with constant coefficients

the origin of the initial discontinuity. From the point $P^*(x, t)$ we trace back the characteristics with speeds λ_1 and λ_2 . These are parallel to those passing through the origin. The characteristics through P^* pass through the *initial points* $x_0^{(2)} = x - \lambda_2 t$ and $x_0^{(1)} = x - \lambda_1 t$. The coefficients in the expansion (2.60) for $\mathbf{U}(x, t)$ are thus determined. The solution at a point P^* has the form (2.60). It is a question of choosing the correct coefficients α_i or β_i . Select a time t^* and a point x_L to the left of the slowest wave so $\mathbf{U}(x_L, t^*) = \mathbf{U}_L$, see Fig. 2.6. The solution at the starting point (x_L, t^*) is obviously

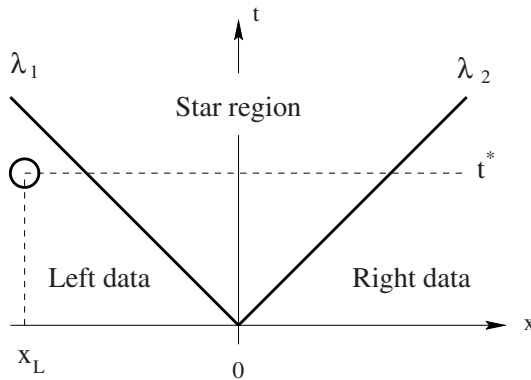


Fig. 2.6. The Riemann problem solution found by travelling along dashed horizontal line $t = t^*$

$$\mathbf{U}_L = \sum_{i=1}^2 \alpha_i \mathbf{K}_i = \alpha_1 \mathbf{K}^{(1)} + \alpha_2 \mathbf{K}^{(2)},$$

i.e. all coefficients are α 's, that is, the point (x_L, t^*) lies to the left of every wave. As we move to the right of (x_L, t^*) on the horizontal line $t = t^*$ we cross the wave $dx/dt = \lambda_1$, hence $x - \lambda_1 t$ changes from negative to positive, see (2.59), and therefore the coefficient α_1 above changes to β_1 . Thus the solution in the entire *Star Region*, between the λ_1 and λ_2 waves, is

$$\mathbf{U}^*(x, t) = \beta_1 \mathbf{K}^{(1)} + \alpha_2 \mathbf{K}^{(2)}. \quad (2.61)$$

As we continue moving right and cross the λ_2 wave the value $x - \lambda_2 t$ changes from negative to positive and hence the coefficient α_2 in (2.60) and (2.61) changes to β_2 , i.e the solution to the right of the fastest wave of speed λ_2 is, trivially,

$$\mathbf{U}_R = \beta_1 \mathbf{K}^{(1)} + \beta_2 \mathbf{K}^{(2)}.$$

Remark 2.13. From equation (2.56) it is easy to see that the jump in \mathbf{U} across the whole wave structure in the solution of the Riemann problem is

$$\Delta \mathbf{U} = \mathbf{U}_R - \mathbf{U}_L = (\beta_1 - \alpha_1) \mathbf{K}^{(1)} + \dots + (\beta_m - \alpha_m) \mathbf{K}^{(m)}. \quad (2.62)$$

It is an eigenvector expansion with coefficients that are the strengths of the waves present in the Riemann problem. The wave strength of wave i is $\beta_i - \alpha_i$ and the jump in \mathbf{U} across wave i , denoted by $(\Delta \mathbf{U})_i$, is

$$(\Delta \mathbf{U})_i = (\beta_i - \alpha_i) \mathbf{K}^{(i)}. \quad (2.63)$$

When solving the Riemann problem, sometimes it is more useful to expand the total jump $\Delta \mathbf{U} = \mathbf{U}_R - \mathbf{U}_L$ in terms of the eigenvectors and unknown wave strengths $\delta_i = \beta_i - \alpha_i$.

2.3.4 The Riemann Problem for Linearised Gas Dynamics

As an illustrative example we apply the methodology described in the previous section to solve the Riemann problem for the linearised equations of Gas Dynamics (2.12)

$$\mathbf{U}_t + \mathbf{A} \mathbf{U}_x = \mathbf{0},$$

with

$$\mathbf{U} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \equiv \begin{bmatrix} \rho \\ u \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 0 & \rho_0 \\ a^2/\rho_0 & 0 \end{bmatrix}.$$

The eigenvalues of the system are

$$\lambda_1 = -a, \quad \lambda_2 = +a,$$

and the corresponding right eigenvectors are

$$\mathbf{K}^{(1)} = \begin{bmatrix} \rho_0 \\ -a \end{bmatrix}, \quad \mathbf{K}^{(2)} = \begin{bmatrix} \rho_0 \\ a \end{bmatrix}.$$

First we decompose the left data state $\mathbf{U}_L = [\rho_L, u_L]^T$ in terms of the right eigenvectors according to equation (2.56), namely

$$\mathbf{U}_L = \begin{bmatrix} \rho_L \\ u_L \end{bmatrix} = \alpha_1 \begin{bmatrix} \rho_0 \\ -a \end{bmatrix} + \alpha_2 \begin{bmatrix} \rho_0 \\ a \end{bmatrix} .$$

Solving for the unknown coefficients α_1 and α_2 we obtain

$$\alpha_1 = \frac{a\rho_L - \rho_0 u_L}{2a\rho_0} , \quad \alpha_2 = \frac{a\rho_L + \rho_0 u_L}{2a\rho_0} .$$

Similarly, by expanding the right-hand data $\mathbf{U}_R = [\rho_R, u_R]^T$ in terms of the eigenvectors and solving for the coefficients β_1 and β_2 we obtain

$$\beta_1 = \frac{a\rho_R - \rho_0 u_R}{2a\rho_0} , \quad \beta_2 = \frac{a\rho_R + \rho_0 u_R}{2a\rho_0} .$$

Now by using equation (2.61) we find the solution in the star region as

$$\mathbf{U}^* = \begin{bmatrix} \rho^* \\ u^* \end{bmatrix} = \beta_1 \begin{bmatrix} \rho_0 \\ -a \end{bmatrix} + \beta_2 \begin{bmatrix} \rho_0 \\ a \end{bmatrix} .$$

After some algebraic manipulations we obtain the solution explicitly as

$$\left. \begin{aligned} \rho_* &= \frac{1}{2}(\rho_L + \rho_R) - \frac{1}{2}(u_R - u_L)\rho_0/a , \\ u_* &= \frac{1}{2}(u_L + u_R) - \frac{1}{2}(\rho_R - \rho_L)a/\rho_0 . \end{aligned} \right\} \quad (2.64)$$

Fig. 2.7 illustrates the solution for $\rho(x, t)$ and $u(x, t)$ at time $t = 1$ for the parameter values $\rho_0 = 1, a = 1$ and initial data $\rho_L = 1, u_L = 0, \rho_R = \frac{1}{2}$ and $u_R = 0$. The two symmetric waves that emerge from the initial position of the discontinuity carry a discontinuous jump in both density ρ and velocity u .

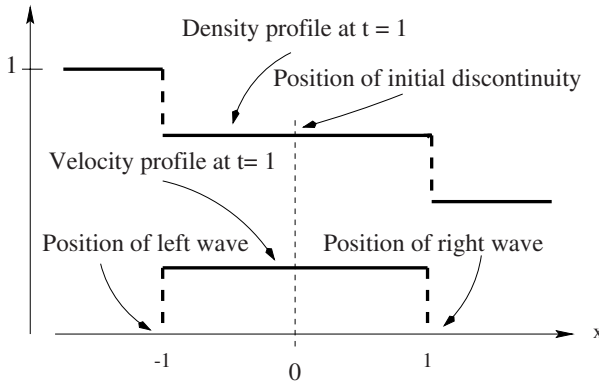


Fig. 2.7. Density and velocity solution profiles at time $t=1$

Remark 2.14. The exact solution (2.64) can be very useful in testing numerical methods for systems with discontinuous solutions.

2.3.5 Some Useful Definitions

Next we recall some standard definitions associated with hyperbolic systems.

Definition 2.15 (Domain of Dependence). *Recall that for the linear advection equation the solution at a given point $P = (x^*, t^*)$ depends solely on the initial data at a single point x_0 on the x -axis. This point is obtained by tracing back the characteristic passing through the point $P = (x^*, t^*)$. As a matter of fact, the solution at $P = (x^*, t^*)$ is identical to the value of the initial data $u_0(x)$ at the point x_0 . One says that the domain of dependence of the point $P = (x^*, t^*)$ is the point x_0 . For a 2×2 system the domain of dependence is an interval $[x_L, x_R]$ on the x -axis that is subtended by the characteristics passing through the point $P = (x^*, t^*)$.*

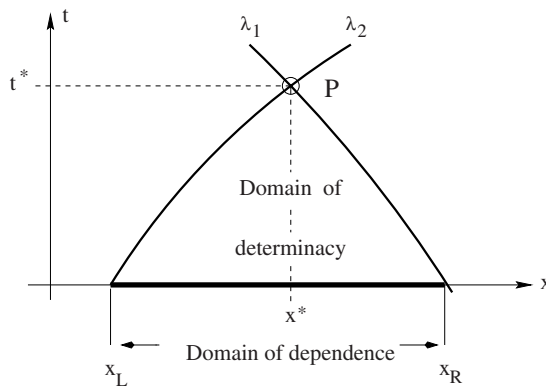


Fig. 2.8. Domain of dependence of point P and corresponding domain of determinacy, for a 2×2 system

Fig. 2.8 illustrates the domain of dependence for a 2×2 system with characteristic speeds λ_1 and λ_2 , with $\lambda_1 < \lambda_2$. In general, the characteristics of a hyperbolic system are curved. For a larger system the domain of dependence is determined by the slowest and fastest characteristics and is always a bounded interval, as the characteristic speeds for hyperbolic systems are always finite.

Definition 2.16 (Domain of Determinacy). *For a given domain of dependence $[x_L, x_R]$, the domain of determinacy is the set of points (x, t) , within the domain of existence of the solution $\mathbf{U}(x, t)$, in which $\mathbf{U}(x, t)$ is solely determined by initial data on $[x_L, x_R]$.*

In Fig. 2.8 we illustrate the domain of determinacy of an interval $[x_L, x_R]$ for the case of a 2×2 system with characteristic speeds λ_1 and λ_2 , with $\lambda_1 < \lambda_2$.

Definition 2.17 (Range of Influence). *Another useful concept is that of the range of influence of a point $Q = (x_0, 0)$ on the x -axis. It is defined as the set of points (x, t) in the x - t plane in which the solution $\mathbf{U}(x, t)$ is influenced by initial data at the point $Q = (x_0, 0)$.*

Fig. 2.9 illustrates the range of influence of a point $Q = (x_0, 0)$ for the case of a 2×2 system with characteristic speeds λ_1 and λ_2 , with $\lambda_1 < \lambda_2$.

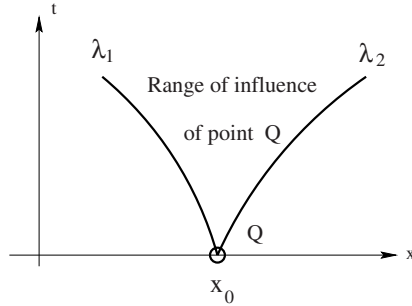


Fig. 2.9. Range of influence of point Q for a 2 by 2 system

2.4 Conservation Laws

The purpose of this section is to provide the reader with a succinct presentation of some mathematical properties of hyperbolic conservation laws. We restrict our attention to those properties thought to be essential to the development and application of numerical methods for conservation laws. In Chap. 1 we applied the physical principles of conservation of mass, momentum and energy to derive time-dependent, multidimensional non-linear systems of conservation laws. In this section we restrict ourselves to simple model problems. In Sect. 2.1 we advanced the formal definition of a system of m conservation laws

$$\mathbf{U}_t + \mathbf{F}(\mathbf{U})_x = \mathbf{0}, \tag{2.65}$$

where \mathbf{U} is the vector of conserved variables and $\mathbf{F}(\mathbf{U})$ is the vector of fluxes. This system is hyperbolic if the Jacobian matrix

$$\mathbf{A}(\mathbf{U}) = \frac{\partial \mathbf{F}}{\partial \mathbf{U}}$$

has real eigenvalues $\lambda_i(\mathbf{U})$ and a complete set of linearly independent eigenvectors $\mathbf{K}^{(i)}(\mathbf{U})$, $i = 1, \dots, m$, which we assume to be ordered as

$$\lambda_1(\mathbf{U}) < \lambda_2(\mathbf{U}) < \dots < \lambda_m(\mathbf{U}),$$

$$\mathbf{K}^{(1)}(\mathbf{U}), \mathbf{K}^{(2)}(\mathbf{U}), \dots, \mathbf{K}^{(m)}(\mathbf{U}).$$

It is important to note that now eigenvalues and eigenvectors depend on \mathbf{U} , although sometimes we shall omit the argument \mathbf{U} .

2.4.1 Integral Forms of Conservation Laws

As discussed in Sect. 1.5 of Chap. 1, conservation laws may be expressed in differential and integral form. There are two good reasons for considering the integral form (s) of the conservation laws: (i) the derivation of the governing equations is based on physical conservation principles expressed as integral relations on control volumes, (ii) the integral formulation requires less smoothness of the solution, which paves the way to extending the class of admissible solutions to include discontinuous solutions.

The integral form has variants that are worth studying in detail. Consider a one-dimensional time dependent system, such as the Euler equations introduced in Sect. 1.1 of Chap. 1. Choose a control volume $V = [x_L, x_R] \times [t_1, t_2]$ on the x - t plane as shown in Fig. 2.10. The integral form, see Sect. 1.5, of the

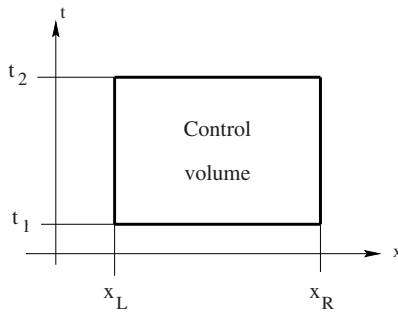


Fig. 2.10. A control volume $V = [x_L, x_R] \times [t_1, t_2]$ on x - t plane

equation for conservation of mass in one space dimension is

$$\frac{d}{dt} \int_{x_L}^{x_R} \rho(x, t) dx = f(x_L, t) - f(x_R, t),$$

where $f = \rho u$ is the flux. For the complete system we have

$$\frac{d}{dt} \int_{x_L}^{x_R} \mathbf{U}(x, t) dx = \mathbf{F}(\mathbf{U}(x_L, t)) - \mathbf{F}(\mathbf{U}(x_R, t)), \quad (2.66)$$

where $\mathbf{F}(\mathbf{U})$ is the flux vector. This is one version of the integral form of the conservation laws: *Integral Form I*. The corresponding differential form reads as (2.65). Another version of the integral form of the conservation laws is obtained by integrating (2.66) in time between t_1 and t_2 , with $t_1 \leq t_2$. See Fig. 2.10. Clearly,

$$\int_{t_1}^{t_2} \left[\frac{d}{dt} \int_{x_L}^{x_R} \mathbf{U}(x, t) dx \right] dt = \int_{x_L}^{x_R} \mathbf{U}(x, t_2) dx - \int_{x_L}^{x_R} \mathbf{U}(x, t_1) dx$$

and thus (2.66) becomes

$$\left. \begin{aligned} \int_{x_L}^{x_R} \mathbf{U}(x, t_2) dx &= \int_{x_L}^{x_R} \mathbf{U}(x, t_1) dx + \int_{t_1}^{t_2} \mathbf{F}(\mathbf{U}(x_L, t)) dt \\ &\quad - \int_{t_1}^{t_2} \mathbf{F}(\mathbf{U}(x_R, t)) dt, \end{aligned} \right\} \quad (2.67)$$

which we call: *Integral Form II* of the conservation laws.

Another version of the integral form of the conservation laws is obtained by integrating (2.65) in any domain V in x - t space and using Green's theorem. The result is

$$\oint [\mathbf{U} dx - \mathbf{F}(\mathbf{U}) dt] = \mathbf{0}, \quad (2.68)$$

where the line integration is performed along the boundary of the domain, in an anticlockwise manner. We call this version *Integral Form III* of the conservation laws. Note that *Integral Form II* of the conservation laws is a special case of *Integral Form III*, in which the control volume V is the rectangle $[x_L, x_R] \times [t_1, t_2]$.

A fourth integral form results from adopting a more mathematical approach for extending the concept of solution of (2.65) to include discontinuities. See Chorin and Marsden [112]. A *weak* or *generalized* solution \mathbf{U} is required to satisfy the integral relation

$$\int_0^{+\infty} \int_{-\infty}^{+\infty} [\phi_t \mathbf{U} + \phi_x \mathbf{F}(\mathbf{U})] dx dt = - \int_{-\infty}^{+\infty} \phi(x, 0) \mathbf{U}(x, 0) dx, \quad (2.69)$$

for all *test functions* $\phi(x, t)$ that are continuously differentiable and have *compact support*. A function $\phi(x, t)$ has compact support if it vanishes outside some bounded set. Note that in (2.69) the derivatives of $\mathbf{U}(x, t)$ and $\mathbf{F}(\mathbf{U})$ have been passed on to the test function $\phi(x, t)$, which is sufficiently smooth to admit these derivatives.

Remark 2.18. The integral forms (2.66)–(2.69) corresponding to (2.65) are valid for any system (2.65), not just for the Euler equations.

Examples of Conservation Laws

Scalar conservation laws ($m = 1$) in differential form read

$$u_t + f(u)_x = 0, \quad f(u) : \text{flux function.} \quad (2.70)$$

To be able to solve for the conserved variable $u(x, t)$ the flux function $f(u)$ must be a completely determined algebraic function of $u(x, t)$, and possibly some extra parameters of the problem. As seen in Sect. 2.2 the linear advection

equation is the simplest example, in which the flux function is $f(u) = au$, a linear function of u .

The **inviscid Burgers's equation** has flux $f(u) = \frac{1}{2}u^2$, a quadratic function of u . Another example of a conservation law is the **traffic flow equation**

$$\rho_t + f(\rho)_x = 0, \quad f(\rho) = u_m \left(1 - \frac{\rho}{\rho_m}\right) \rho. \quad (2.71)$$

Here the conserved variable $\rho(x, t)$ is a density function (density of motor vehicles), u_m and ρ_m are parameters of the problem, namely the maximum speed of vehicles and the maximum density, both positive constants. For details on the traffic flow equation see Whitham [582], Zachmanoglou and Thoe [596], Toro [528] and Haberman [232]. An example of practical interest in oil–reservoir simulation is the **Buckley-Leverett equation**

$$u_t + f(u)_x = 0, \quad f(u) = \frac{u^2}{u^2 + b(1 - u)^2}, \quad (2.72)$$

where b is a parameter of the problem. More details of this equation are found in LeVeque [308].

Systems of conservation laws are constructed, as obvious examples, from linear systems

$$\mathbf{U}_t + \mathbf{A}\mathbf{U}_x = \mathbf{0},$$

with constant coefficient matrix \mathbf{A} . The required *conservation-law form* is obtained by defining the flux function as the product of the coefficient matrix \mathbf{A} and the vector \mathbf{U} , namely

$$\mathbf{U}_t + \mathbf{F}(\mathbf{U})_x = \mathbf{0}, \quad \mathbf{F}(\mathbf{U}) = \mathbf{A}\mathbf{U}. \quad (2.73)$$

Trivially, the Jacobian matrix is \mathbf{A} .

Example 2.19 (Isothermal Gas Dynamics). The isothermal equations of Gas Dynamics, see Sect. 1.6.2 of Chap. 1, are one example of a non-linear system of conservation laws. These are

$$\left. \begin{aligned} &\mathbf{U}_t + \mathbf{F}(\mathbf{U})_x = \mathbf{0}, \\ &\mathbf{U} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \equiv \begin{bmatrix} \rho \\ \rho u \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \equiv \begin{bmatrix} \rho u \\ \rho u^2 + a^2 \rho \end{bmatrix}, \end{aligned} \right\} \quad (2.74)$$

where a is positive, constant speed of sound. The Jacobian matrix is found by first expressing \mathbf{F} in terms of the components $u_1 \equiv \rho$ and $u_2 \equiv \rho u$ of the vector \mathbf{U} of conserved variables, namely

$$\mathbf{F}(\mathbf{U}) = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \equiv \begin{bmatrix} u_2 \\ u_2^2/u_1 + a^2 u_1 \end{bmatrix}.$$

According to (2.8) the Jacobian matrix is

$$\mathbf{A}(\mathbf{U}) = \frac{\partial \mathbf{F}}{\partial \mathbf{U}} = \begin{bmatrix} 0 & 1 \\ -(u_2/u_1)^2 + a^2 & 2u_2/u_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ a^2 - u^2 & 2u \end{bmatrix}.$$

It is left to the reader to verify that the eigenvalues of \mathbf{A} are

$$\lambda_1 = u - a, \quad \lambda_2 = u + a \tag{2.75}$$

and that the right eigenvectors are

$$\mathbf{K}^{(1)} = \begin{bmatrix} 1 \\ u - a \end{bmatrix}, \quad \mathbf{K}^{(2)} = \begin{bmatrix} 1 \\ u + a \end{bmatrix}, \tag{2.76}$$

where the scaling factors for $\mathbf{K}^{(1)}$ and $\mathbf{K}^{(2)}$ have been taken to be unity. The isothermal equations of Gas Dynamics are thus hyperbolic.

Example 2.20 (Isentropic Gas Dynamics). Another non-linear example of a system of conservation laws are the isentropic equations of Gas Dynamics

$$\left. \begin{aligned} &\mathbf{U}_t + \mathbf{F}(\mathbf{U})_x = \mathbf{0}, \\ &\mathbf{U} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \equiv \begin{bmatrix} \rho \\ \rho u \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \equiv \begin{bmatrix} \rho u \\ \rho u^2 + p \end{bmatrix}, \end{aligned} \right\} \tag{2.77}$$

together with the closure condition, or equation of state (EOS),

$$p = C\rho^\gamma, \quad C = \text{constant}. \tag{2.78}$$

See Sect. 1.6.2 of Chap. 1.

Exercise 2.21. (i) Find the Jacobian matrix, the eigenvalues and the right eigenvectors for the isentropic equations (2.77)–(2.78). (ii) Show that for a generalized isentropic EOS, $p = p(\rho)$, the system is hyperbolic if and only if $p'(\rho) > 0$, that is, the pressure must be a monotone increasing function of ρ . (iii) Show that the sound speed has the general form

$$a = \sqrt{p'(\rho)}.$$

Solution 2.22. The eigenvalues are

$$\lambda_1 = u - a, \quad \lambda_2 = u + a, \tag{2.79}$$

and the right eigenvectors are

$$\mathbf{K}^{(1)} = \begin{bmatrix} 1 \\ u - a \end{bmatrix}, \quad \mathbf{K}^{(2)} = \begin{bmatrix} 1 \\ u + a \end{bmatrix}, \tag{2.80}$$

with the sound speed a as claimed.

2.4.2 Non-Linearities and Shock Formation

Here we study some distinguishing features of non-linear hyperbolic conservation laws, such as wave steepening and shock formation. We restrict our attention to the initial-value problem for scalar non-linear conservation laws, namely

$$u_t + f(u)_x = 0, \quad u(x, 0) = u_0(x). \quad (2.81)$$

A corresponding integral form of the conservation law is

$$\frac{d}{dt} \int_{x_L}^{x_R} u(x, t) dx = f(u(x_L, t)) - f(u(x_R, t)). \quad (2.82)$$

The flux function f is assumed to be a function of u only, which under certain circumstances is an inadequate representation of the physical problem being modelled. Relevant physical phenomena of our interest are shock waves in compressible media. These have viscous dissipation and heat conduction, in addition to pure advection. A more appropriate flux function for a model conservation law would also include a dependence on u_x , so that the modified conservation law would read

$$u_t + f(u)_x = \alpha u_{xx}, \quad (2.83)$$

with α a positive coefficient of viscosity. The conservation law in (2.81) may be rewritten as

$$u_t + \lambda(u)u_x = 0, \quad (2.84)$$

where

$$\lambda(u) = \frac{df}{du} = f'(u) \quad (2.85)$$

is the *characteristic speed*. In the system case this corresponds to the eigenvalues of the Jacobian matrix. For the linear advection equation $\lambda(u) = a$, constant. For the inviscid Burgers equation $\lambda(u) = u$, that is, the characteristic speed depends on the solution and is in fact identical to the conserved variable. For the traffic flow equation $\lambda(u) = u_m(1 - \frac{2u}{\rho_m})$.

The behaviour of the flux function $f(u)$ has profound consequences on the behaviour of the solution $u(x, t)$ of the conservation law itself. A crucial property is *monotonicity* of the characteristic speed $\lambda(u)$. There are essentially three possibilities:

- $\lambda(u)$ is a monotone *increasing* function of u , i.e.

$$\frac{d\lambda(u)}{du} = \lambda'(u) = f''(u) > 0 \text{ (convex flux)}$$

- $\lambda(u)$ is a monotone *decreasing* function of u , i.e.

$$\frac{d\lambda(u)}{du} = \lambda'(u) = f''(u) < 0 \text{ (concave flux)}$$

- $\lambda(u)$ has extrema, for some u , i.e.

$$\frac{d\lambda(u)}{du} = \lambda'(u) = f''(u) = 0 \text{ (non-convex, non-concave flux) .}$$

In the case of non-linear systems of conservation laws the character of the flux function is determined by the Equation of State. One speaks of convex, or otherwise, equations of state. See the review paper by Menikoff and Plohr [349]. For the inviscid Burgers equation $\lambda'(u) = f''(u) = 1 > 0$, the flux is convex. For the traffic flow equation $\lambda'(u) = f''(u) = -2u_m/\rho_m < 0$, the flux is concave.

Exercise 2.23. Analyse the character of the flux function for the Buckley–Leverett equation and show that it is non-convex, non-concave.

Solution 2.24. (Left to the reader).

We study the *inviscid* IVP (2.81) and for the moment we assume that the initial data $u(x, 0) = u_0(x)$ is smooth. For some finite time the solution $u(x, t)$ will remain smooth. We rewrite the IVP as

$$\left. \begin{aligned} u_t + \lambda(u)u_x &= 0, \quad \lambda(u) = f'(u), \\ u(x, 0) &= u_0(x). \end{aligned} \right\} \tag{2.86}$$

Note that the PDE in (2.86) is a non-linear extension of the linear advection equation in (2.32) in which the characteristic speed is $\lambda(u) = a = \text{constant}$. We construct solutions to IVP (2.86) following characteristic curves, in much the same way as performed for the linear advection equation.

Construction of Solutions on Characteristics

Consider characteristic curves $x = x(t)$ satisfying the IVP

$$\frac{dx}{dt} = \lambda(u), \quad x(0) = x_0. \tag{2.87}$$

Then, by regarding both u and x to be functions of t we find the total derivative of u along the curve $x(t)$, namely

$$\frac{du}{dt} = u_t + \lambda(u)u_x = 0. \tag{2.88}$$

That is, u is constant along the characteristic curve satisfying the IVP (2.87) and therefore the slope $\lambda(u)$ is constant along the characteristic. Hence the characteristic curves are straight lines. The value of u along each curve is the value of u at the initial point $x(0) = x_0$ and we write

$$u(x, t) = u_0(x_0). \tag{2.89}$$

Fig. 2.11 shows a typical characteristic curve emanating from the initial point x_0 on the x -axis. The slope $\lambda(u)$ of the characteristic may then be evaluated at x_0 so that the solution characteristics curves of IVP (2.87) are

$$x = x_0 + \lambda(u_0(x_0))t . \tag{2.90}$$

Relations (2.89) and (2.90) may be regarded as the analytical solution of IVP (2.86). Note that the point x_0 depends on the given point (x, t) , see Fig. 2.11, and thus $x_0 = x_0(x, t)$. The solution given by (2.89) and (2.90) is implicit, which is more apparent if we substitute x_0 from (2.90) into (2.89) to obtain

$$u(x, t) = u_0(x - \lambda(u_0(x_0))t) . \tag{2.91}$$

Note that this solution is identical in form to the solution (2.39) of the linear advection equation in (2.32).

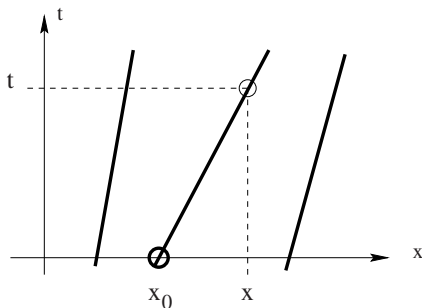


Fig. 2.11. Typical characteristic curves for a non-linear hyperbolic conservation law

Next we verify that relations (2.89) and (2.90) actually define the solution. From (2.89) we obtain the t and x derivatives

$$u_t = u'_0(x_0) \frac{\partial x_0}{\partial t} , \quad u_x = u'_0(x_0) \frac{\partial x_0}{\partial x} . \tag{2.92}$$

From (2.90) the t and x derivatives are found to be

$$\left. \begin{aligned} \lambda(u_0(x_0)) + [1 + \lambda'(u_0(x_0))u'_0(x_0)t] \frac{\partial x_0}{\partial t} &= 0 , \\ [1 + \lambda'(u_0(x_0))u'_0(x_0)t] \frac{\partial x_0}{\partial x} &= 1 . \end{aligned} \right\} \tag{2.93}$$

From (2.93) we obtain

$$\frac{\partial x_0}{\partial t} = - \frac{\lambda(u_0(x_0))}{1 + \lambda'(u_0(x_0))u'_0(x_0)t} \tag{2.94}$$

and

$$\frac{\partial x_0}{\partial x} = \frac{1}{1 + \lambda'(u_0(x_0))u'_0(x_0)t}. \tag{2.95}$$

Substitution of (2.94)–(2.95) into (2.92) verifies that u_t and u_x satisfy the PDE in (2.86).

Wave Steepening

Recall that in the case of the linear advection equation, in which the characteristic speed is $\lambda(u) = a = \text{constant}$, the solution consists of the initial data $u_0(x)$ translated with speed a *without distortion*. In the non-linear case the characteristic speed $\lambda(u)$ is a function of the solution itself. Distortions are therefore produced; this is a distinguishing feature of non-linear problems.

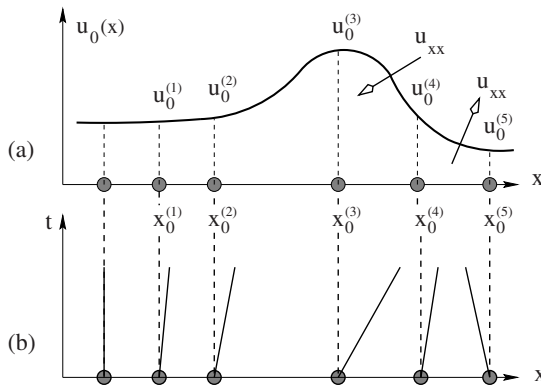


Fig. 2.12. Wave steepening in a convex, non-linear hyperbolic conservation law: (a) initial condition and (b) corresponding picture of characteristics

To explain the wave distortion phenomenon we consider initial data $u_0(x)$ as shown in Fig. 2.12. A smooth initial profile is shown in Fig. 2.12a along with five initial points $x_0^{(i)}$ and their corresponding initial data values $u_0^{(i)} = u_0(x_0^{(i)})$. For the moment let us assume that the flux function $f(u)$ is convex, that is $\lambda'(u) = f''(u) > 0$. In this case the characteristic speed is an increasing function of u . Fig. 2.12b shows the characteristics $x^{(i)}(t)$ emanating from the initial points $x_0^{(i)}$ and carrying the constant initial values $u_0^{(i)}$ along them. Given the assumed convex character of the flux, higher values of $u_0(x)$ will travel faster than lower values of $u_0(x)$. There are two intervals on the x -axis where distortions are most evident. These are the intervals $I_E = [x_0^{(1)}, x_0^{(3)}]$ and $I_C = [x_0^{(3)}, x_0^{(5)}]$. In I_E the value $u_0^{(3)}$ will propagate faster than $u_0^{(2)}$ and this in turn will propagate faster than $u_0^{(1)}$. The orientation of the respective characteristics in Fig. 2.12b makes this situation clear. At a later time the initial data in I_E will have been transformed into a broader

and flatter profile. We say that I_E is an *expansive* region. In the expansive region the characteristic speed increases as x increases, that is $\lambda_x > 0$. By contrast the interval I_C is *compressive* and $\lambda_x < 0$; the value $u_0^{(3)}$ will propagate faster than $u_0^{(4)}$ and this in turn will propagate faster than $u_0^{(5)}$, as shown by the orientation of the respective characteristics in Fig. 2.12b. The compressive region will tend to get steeper and narrower as time evolves. The wave steepening mechanism will eventually produce folding over of the solution profile, with corresponding crossing of characteristics, and triple-valued solutions. Note that the compressive and expansive character of the data just described reverses for the case of a concave flux, $\lambda'(u) = f''(u) < 0$. Before crossing of characteristics the single-valued solution may be found following characteristics, as described previously. When characteristics first intersect we say that the wave breaks; the derivative u_x becomes infinite and this happens at a precise breaking time t_b given by

$$t_b = \frac{-1}{\lambda_x(x_0)}. \quad (2.96)$$

This is confirmed by equations (2.94)–(2.95). Breaking first occurs on the characteristic emanating from $x = x_0$ for which $\lambda_x(x_0)$ is negative and $|\lambda_x(x_0)|$ is a maximum. For details see Whitham [582].

This is an anomalous situation that may be rescued by going back to the physical origins of the equations and questioning the adequacy of the model furnished by (2.81). The improved model equation (2.83) says that the time rate of change of u is not just due to the advection term $f(u)_x$ but is a competing balance between advection and the diffusion term αu_{xx} . As shown in Fig. 2.12a in the interval $[x_0^{(3)}, x_0^{(4)}]$ the *wave steepening* effect of $f(u)_x$ is opposed by the *wave-easing* effect of αu_{xx} , which is negative there. In the interval $[x_0^{(4)}, x_0^{(5)}]$ the role of these contradictory effects is reversed. The more complete description of the physics does not allow folding over of the solution. But rather than working with the more complete, and therefore more complex, viscous description of the problem, it is actually possible to insist on using the inviscid model (2.81) by allowing discontinuities to be formed as a *process of increasing compression*, namely shock waves. Further details are found in Lax [301], Whitham [582] and Smoller [451].

Shock Waves

Shock waves in air are small transition layers of very rapid changes of physical quantities such as pressure, density and temperature. The transition layer for a strong shock is of the same order of magnitude as the mean-free path of the molecules, that is about 10^{-7} m. Therefore replacing these waves as mathematical discontinuities is a reasonable approximation. Very weak shock waves such as sonic booms, are an exception; the discontinuous approximation

here can be very inaccurate indeed, see Whitham [582]. For a discussion on shock thickness see Landau and Lifshitz [297], pp. 337–341.

We therefore insist on using the simplified model (2.81) but in its integral form, e.g. (2.82). Consider a solution $u(x, t)$ such that $u(x, t)$, $f(u)$ and their derivatives are continuous everywhere except on a line $s = s(t)$ on the x - t plane across which $u(x, t)$ has a jump discontinuity. Select two fixed points x_L and x_R on the x -axis such that $x_L < s(t) < x_R$. Enforcing the conservation law in integral form (2.82) on the control volume $[x_L, x_R]$ leads to

$$f(u(x_L, t)) - f(u(x_R, t)) = \frac{d}{dt} \int_{x_L}^{s(t)} u(x, t) dx + \frac{d}{dt} \int_{s(t)}^{x_R} u(x, t) dx .$$

Direct use of formula (1.68) of Chap. 1 yields

$$\left. \begin{aligned} f(u(x_L, t)) - f(u(x_R, t)) &= [u(s_L, t) - u(s_R, t)] S \\ &+ \int_{x_L}^{s(t)} u_t(x, t) dx + \int_{s(t)}^{x_R} u_t(x, t) dx , \end{aligned} \right\}$$

where $u(s_L, t)$ is the limit of $u(s(t), t)$ as x tends to $s(t)$ from the left, $u(s_R, t)$ is the limit of $u(s(t), t)$ as x tends to $s(t)$ from the right and $S = ds/dt$ is the speed of the discontinuity. As $u_t(x, t)$ is bounded the integrals vanish identically as $s(t)$ is approached from left and right and we obtain

$$f(u(x_L, t)) - f(u(x_R, t)) = [u(s_L, t) - u(s_R, t)] S . \tag{2.97}$$

This algebraic expression relating the jumps $\Delta f = f(u(x_R, t)) - f(u(x_L, t))$, $\Delta u = u(x_R, t) - u(x_L, t)$ and the speed S of the discontinuity is called the *Rankine–Hugoniot condition* and is usually expressed as

$$\Delta f = S \Delta u . \tag{2.98}$$

For the scalar case considered here one can solve for the speed S as

$$S = \frac{\Delta f}{\Delta u} . \tag{2.99}$$

Therefore, in order to admit discontinuous solutions we may formulate the problem in terms of PDEs, which are valid in smooth parts of the solution, and the Rankine–Hugoniot conditions across discontinuities.

Two Examples of Discontinuous Solutions

Consider the following initial–value problem for the inviscid Burgers equation

$$\left. \begin{aligned} u_t + f(u)_x &= 0 , \quad f(u) = \frac{1}{2}u^2 , \\ u(x, 0) &= u_0(x) = \begin{cases} u_L & \text{if } x < 0 , \\ u_R & \text{if } x > 0 . \end{cases} \end{aligned} \right\} \tag{2.100}$$

First *assume that* $u_L > u_R$. As the flux is convex $\lambda'(u) = f''(u) > 0$ the characteristic speeds on the left are greater than those on the right, that is $\lambda_L \equiv \lambda(u_L) > \lambda_R \equiv \lambda(u_R)$. Based on the discussion about Fig. 2.12 the initial data in IVP (2.100) is the extreme case of compressive data. Crossing of characteristics takes place immediately, as illustrated in Fig. 2.13b. The

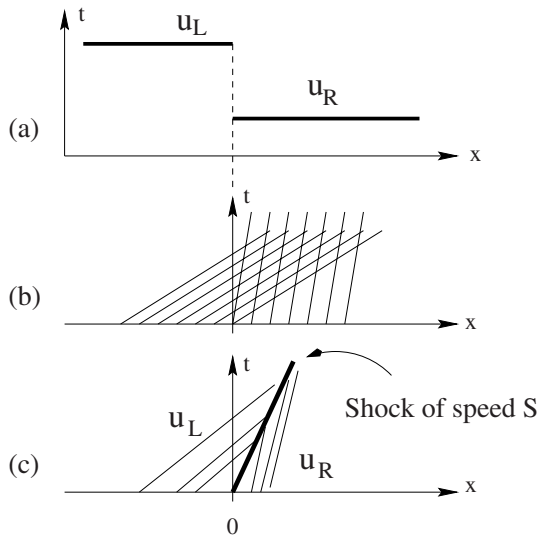


Fig. 2.13. (a) Compressive discontinuous initial data (b) picture of characteristics and (c) solution on x - t plane

discontinuous solution of the IVP is

$$u(x, t) = \begin{cases} u_L & \text{if } x - St < 0, \\ u_R & \text{if } x - St > 0, \end{cases} \quad (2.101)$$

where the speed of the discontinuity is found from (2.99) as

$$S = \frac{1}{2}(u_L + u_R). \quad (2.102)$$

This discontinuous solution is a shock wave and is compressive in nature as discussed previously and as observed in Fig. 2.13a; it satisfies the following condition

$$\lambda(u_L) > S > \lambda(u_R), \quad (2.103)$$

which is called the *entropy condition*. More details are found in Chorin and Marsden [112], LeVeque [308], Smoller [451], Whitham [582].

Now we *assume that* $u_L < u_R$ in the IVP (2.100). This data is the extreme case of *expansive data*, for convex $f(u)$. A possible *mathematical* solution has identical form as solution (2.101)–(2.102) for the *compressive* data case. See

Fig. 2.14. However, this solution is physically incorrect. The discontinuity has not arisen as the result of compression, $\lambda_L < \lambda_R$; the characteristics diverge from the discontinuity. This solution is called a *rarefaction shock*, or *entropy-violating shock*, and does not satisfy the entropy condition (2.103); it is therefore rejected as a physical solution. Compare Figs. 2.13 and 2.14; in the compressive case characteristics run into the discontinuity. Given the

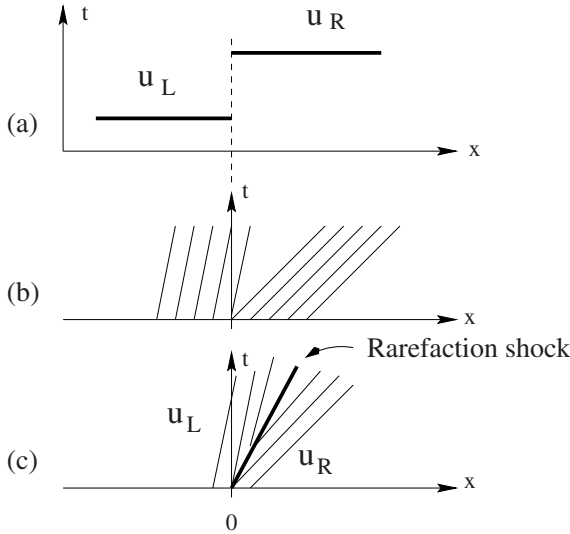


Fig. 2.14. (a) Expansive discontinuous initial data (b) picture of characteristics and (c) rarefaction shock solution on $x-t$ plane

expansive character of the data and based on the discussion on Fig. 2.12, it would be more reasonable to expect the initial data to break up immediately and to broaden with time. This actually gives another solution to be discussed next.

Rarefaction Waves

Reconsider the IVP (2.100) with general convex flux function $f(u)$

$$\left. \begin{aligned} u_t + f(u)_x &= 0, \\ u(x, 0) = u_0(x) &= \begin{cases} u_L & \text{if } x < 0, \\ u_R & \text{if } x > 0, \end{cases} \end{aligned} \right\} \quad (2.104)$$

and *expansive* initial data, $u_L < u_R$. As discussed previously, the entropy-violating solution to this problem is

$$u(x, t) = \left. \begin{cases} u_L & \text{if } x - St < 0, \\ u_R & \text{if } x - St > 0, \end{cases} \right\} \quad (2.105)$$

$$S = \frac{\Delta f}{\Delta u}.$$

Amongst the various other reasons for rejecting this solution as a physical solution, instability stands out as a prominent argument. By instability it is meant that *small perturbations of the initial data lead to large changes in the solution*. As a matter of fact, under small perturbations, the whole character of the solution changes completely, as we shall see.

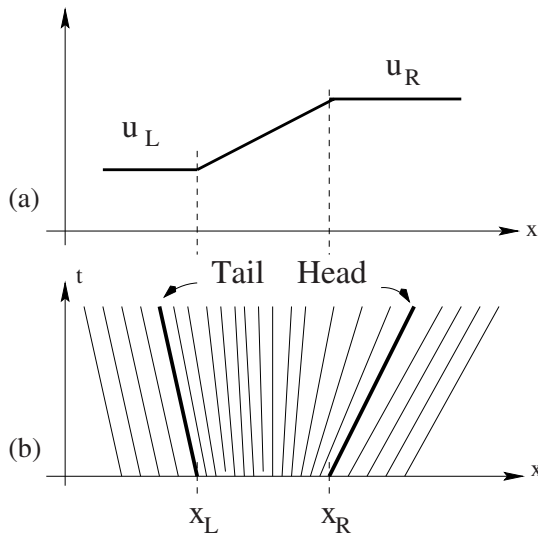


Fig. 2.15. Non-centred rarefaction wave: (a) expansive smooth initial data, (b) picture of characteristics on $x-t$ plane

Let us modify the initial data in (2.104) by replacing the discontinuous change from u_L to u_R by a linear variation of $u_0(x)$ between two fixed points $x_L < 0$ and $x_R > 0$. Now the initial data reads

$$u_0(x) = \begin{cases} u_L & \text{if } x \leq x_L, \\ u_L + \frac{u_R - u_L}{x_R - x_L}(x - x_L) & \text{if } x_L < x < x_R, \\ u_R & \text{if } x \geq x_R, \end{cases} \quad (2.106)$$

and is illustrated in Fig. 2.15a. The corresponding picture of characteristics emanating from the initial time $t = 0$ is shown in Fig. 2.15b. The solution $u(x, t)$ to this problem is found by following characteristics, as discussed previously, and consists of two constant states, u_L and u_R , separated by a region of *smooth transition* between the data values u_L and u_R . This is called a *rarefaction wave*. The right edge of the wave is given by the characteristic emanating

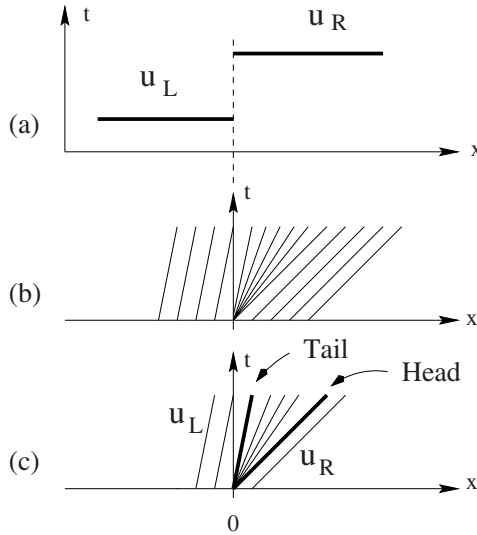


Fig. 2.16. Centred rarefaction wave: (a) expansive discontinuous initial data (b) picture of characteristics (c) entropy satisfying (rarefaction) solution on $x-t$ plane

from x_R

$$x = x_R + \lambda(u_R)t \tag{2.107}$$

and is called the *Head* of the rarefaction. It carries the value $u_0(x_R) = u_R$. The left edge of the wave is given by the characteristic emanating from x_L

$$x = x_L + \lambda(u_L)t \tag{2.108}$$

and is called the *Tail* of the rarefaction. It carries the value $u_0(x_L) = u_L$.

As we assume convexity, $\lambda'(u) = f''(u) > 0$, larger values of $u_0(x)$ propagate faster than lower values and thus the wave spreads and flattens as time evolves. The spreading of waves is a typical non-linear phenomenon not seen in the study of linear hyperbolic systems with constant coefficients. The entire solution is

$$\left. \begin{aligned} u(x, t) &= u_L & \text{if } \frac{x-x_L}{t} \leq \lambda_L, \\ \lambda(u) &= \frac{x-x_L}{t} & \text{if } \lambda_L < \frac{x-x_L}{t} < \lambda_R, \\ u(x, t) &= u_R & \text{if } \frac{x-x_R}{t} \geq \lambda_R. \end{aligned} \right\} \tag{2.109}$$

No matter how small the size $\Delta x = x_R - x_L$ of the interval over which the discontinuous data in IVP (2.104) has been spread over, the structure of the above rarefaction solution remains unaltered and is entirely different from the rarefaction shock solution (2.105), for which small changes to the data lead to large changes in the solution. Thus the rarefaction shock solution is unstable. From the above construction the rarefaction solution is stable and as x_L and x_R approach zero from below and above respectively, the discontinuous data at $x = 0$ in IVP (2.104) is reproduced. Therefore, the limiting case is to be

interpreted as follows: $u_0(x)$ takes on all the values between u_L and u_R at $x = 0$ and consequently $\lambda(u_0(x))$ takes on all the values between λ_L and λ_R at $x = 0$. As higher values propagate faster than lower values the initial data disintegrates immediately giving rise to a rarefaction solution. This limiting rarefaction in which all characteristics of the wave emanate from a single point is called a *centred rarefaction wave*. The solution is

$$\left. \begin{aligned} u(x, t) &= u_L && \text{if } \frac{x}{t} \leq \lambda_L, \\ \lambda(u) &= \frac{x}{t} && \text{if } \lambda_L < \frac{x}{t} < \lambda_R, \\ u(x, t) &= u_R && \text{if } \frac{x}{t} \geq \lambda_R, \end{aligned} \right\} \quad (2.110)$$

and is illustrated in Fig. 2.16.

Now we have at least two solutions to the IVP (2.104). Thus, having extended the concept of solution to include discontinuities, extra spurious solutions are now part of this extended class. The question is how to distinguish between a physically correct solution and a spurious solution. The anticipated answer is that a physical discontinuity, in addition to the Rankine–Hugoniot condition (2.98), also satisfies the entropy condition (2.103).

The Riemann Problem for the Inviscid Burgers Equation

We finalise this section by giving the solution of the Riemann problem for the inviscid Burgers equation, namely

$$\left. \begin{aligned} \text{PDE : } &u_t + \left(\frac{u^2}{2}\right)_x = 0, \\ \text{IC : } &u(x, 0) = \begin{cases} u_L, & x < 0, \\ u_R, & x > 0. \end{cases} \end{aligned} \right\} \quad (2.111)$$

From the previous discussion the exact solution is a *single wave* emanating from the origin as shown in Fig. 2.17a. In view of the entropy condition this wave is either a shock wave, when $u_L > u_R$, or a rarefaction wave, when $u_L \leq u_R$. The complete solution is

$$\left. \begin{aligned} u(x, t) &= \begin{cases} u_L & \text{if } x - St < 0 \\ u_R & \text{if } x - St > 0 \end{cases} && \text{if } u_L > u_R, \\ S &= \frac{1}{2}(u_L + u_R) \end{aligned} \right\} \\ \left. \begin{aligned} u(x, t) &= \begin{cases} u_L & \text{if } \frac{x}{t} \leq u_L \\ \frac{x}{t} & \text{if } u_L < x/t < u_R \\ u_R & \text{if } x/t \geq u_R \end{cases} && \text{if } u_L \leq u_R. \end{aligned} \right\} \quad (2.112)$$

Fig. 2.17 shows the solution of the Riemann problem for the inviscid Burgers equation. Fig. 2.17a depicts the structure of the general solution and consists of a single wave, Fig. 2.17b shows the case in which the solution is a shock wave and Fig. 2.17c shows the case in which it is a rarefaction wave.

Some of the studied notions for scalar conservations laws extend quite directly to systems of hyperbolic conservations laws, as we see in the next section.

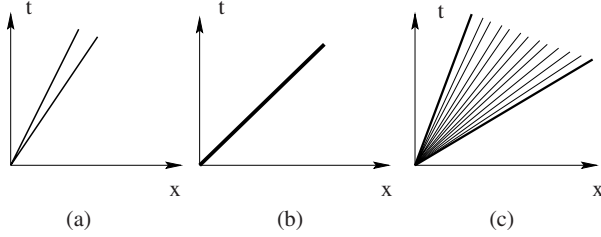


Fig. 2.17. Solution of the Riemann problem for the inviscid Burgers equation: (a) structure of general solution (single wave, shock or rarefaction), (b) solution is a shock wave and (c) solution is a rarefaction wave

2.4.3 Characteristic Fields

Consider a hyperbolic system of conservation laws of the form (2.65) with real eigenvalues $\lambda_i(\mathbf{U})$ and corresponding right eigenvectors $\mathbf{K}^{(i)}(\mathbf{U})$. The characteristic speed $\lambda_i(\mathbf{U})$ defines a *characteristic field*, the λ_i -field. Sometimes one also speaks of the $\mathbf{K}^{(i)}$ -field, that is the characteristic field defined by the eigenvector $\mathbf{K}^{(i)}$.

Definition 2.25 (Linearly degenerate fields). A λ_i -characteristic field is said to be linearly degenerate if

$$\nabla \lambda_i(\mathbf{U}) \cdot \mathbf{K}^{(i)}(\mathbf{U}) = 0, \quad \forall \mathbf{U} \in \mathfrak{R}^m, \tag{2.113}$$

where \mathfrak{R}^m is the set of real-valued vectors of m components.

Definition 2.26 (Genuinely nonlinear fields). A λ_i -characteristic field is said to be genuinely nonlinear if

$$\nabla \lambda_i(\mathbf{U}) \cdot \mathbf{K}^{(i)}(\mathbf{U}) \neq 0, \quad \forall \mathbf{U} \in \mathfrak{R}^m. \tag{2.114}$$

The symbol ‘ \cdot ’ denotes the dot product in *phase space*. $\nabla \lambda_i(\mathbf{U})$ is the gradient of the eigenvalue $\lambda_i(\mathbf{U})$, namely

$$\nabla \lambda_i(\mathbf{U}) = \left(\frac{\partial}{\partial u_1} \lambda_i, \frac{\partial}{\partial u_2} \lambda_i, \dots, \frac{\partial}{\partial u_m} \lambda_i \right)^T.$$

The *phase space* is the space of vectors $\mathbf{U} = (u_1, \dots, u_m)$; for a 2×2 system we speak of the *phase plane* u_1 - u_2 . Note that for a linear system (2.42) the eigenvalues λ_i are constant and therefore $\nabla \lambda_i(\mathbf{U}) = \mathbf{0}$. Hence all characteristic fields of a linear hyperbolic system with constant coefficients are linearly degenerate.

Exercise 2.27. Show that both characteristic fields of the isothermal equations of Gas Dynamics (2.74) are genuinely non-linear.

Solution 2.28. First we write the eigenvalues (2.75) in terms of the conserved variables, namely

$$\lambda_1 = \frac{u_2}{u_1} - a, \quad \lambda_2 = \frac{u_2}{u_1} + a,$$

$$\nabla\lambda_1(\mathbf{U}) = \left(-\frac{u}{\rho}, \frac{1}{\rho}\right)^T, \quad \nabla\lambda_2(\mathbf{U}) = \left(-\frac{u}{\rho}, \frac{1}{\rho}\right)^T.$$

Therefore

$$\nabla\lambda_1(\mathbf{U}) \cdot \mathbf{K}^{(1)}(\mathbf{U}) = -\frac{a}{\rho} \neq 0,$$

$$\nabla\lambda_2(\mathbf{U}) \cdot \mathbf{K}^{(2)}(\mathbf{U}) = \frac{a}{\rho} \neq 0$$

and thus both characteristic fields are *genuinely non-linear*, as claimed.

Example 2.29 (Detonation Analogue). In the study of detonation waves in high energy solids it is found useful to devise mathematical objects that preserve some of the basic physical features of detonation phenomena but are simpler to analyse than more comprehensive models. Fickett [191] proposed a system that is essentially the inviscid Burgers equation plus a reaction model. He called the system *detonation analogue*. Clarke and colleagues [118] pointed out that this analogue is also exceedingly useful for numerical purposes. Writing the system in *conservation-law form* one has the *inhomogeneous system* with a source term, namely

$$\mathbf{U}_t + \mathbf{F}(\mathbf{U})_x = \mathbf{S}(\mathbf{U}), \quad (2.115)$$

$$\mathbf{U} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \equiv \begin{bmatrix} \rho \\ \alpha \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} \frac{1}{2}(\rho^2 + \alpha Q) \\ 0 \end{bmatrix}, \quad \mathbf{S} = \begin{bmatrix} 0 \\ 2\sqrt{1-\alpha} \end{bmatrix}. \quad (2.116)$$

The parameter Q plays the role of *heats of reaction* and α is a *reaction progress variable*. The mathematical character of the system is determined solely by the homogeneous part, $\mathbf{S} = \mathbf{0}$. The Jacobian matrix is

$$\mathbf{A}(\mathbf{U}) = \frac{\partial \mathbf{F}}{\partial \mathbf{U}} = \begin{bmatrix} u_1 & \frac{1}{2}Q \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \rho & \frac{1}{2}Q \\ 0 & 0 \end{bmatrix}.$$

Simple calculations show that the eigenvalues are

$$\lambda_1 = 0, \quad \lambda_2 = \rho \quad (2.117)$$

and the right eigenvectors are

$$\mathbf{K}^{(1)} = \begin{bmatrix} 1 \\ -2\rho/Q \end{bmatrix}, \quad \mathbf{K}^{(2)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \quad (2.118)$$

The detonation analogue is therefore hyperbolic.

Exercise 2.30. Check that the λ_1 -field is linearly degenerate and that the λ_2 -field is genuinely non-linear.

Solution 2.31. (Left to the reader).

Rankine-Hugoniot Conditions

Given a system of hyperbolic conservation laws

$$\mathbf{U}_t + \mathbf{F}(\mathbf{U})_x = \mathbf{0} \tag{2.119}$$

and a discontinuous wave solution of speed S_i associated with the λ_i -characteristic field, the Rankine-Hugoniot conditions state

$$\Delta \mathbf{F} = S_i \Delta \mathbf{U} , \tag{2.120}$$

with

$$\Delta \mathbf{U} \equiv \mathbf{U}_R - \mathbf{U}_L , \Delta \mathbf{F} \equiv \mathbf{F}_R - \mathbf{F}_L , \mathbf{F}_L = \mathbf{F}(\mathbf{U}_L) , \mathbf{F}_R = \mathbf{F}(\mathbf{U}_R) ,$$

where \mathbf{U}_L and \mathbf{U}_R are the respective states immediately to the left and right of the discontinuity. Fig. 2.18 illustrates the Rankine-Hugoniot conditions. Note that unlike the scalar case, see (2.99), it is generally not possible to

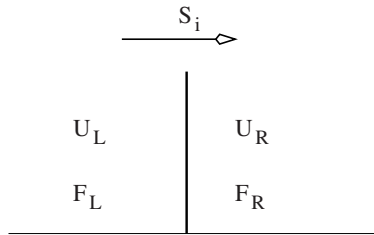


Fig. 2.18. Illustration of the Rankine-Hugoniot conditions for a single discontinuity of speed S_i connecting two constant states \mathbf{U}_L and \mathbf{U}_R via a system of conservation laws

solve for the speed S_i . For a linear system with constant coefficients

$$\mathbf{U}_t + \mathbf{A}\mathbf{U}_x = \mathbf{0} ,$$

with eigenvalues λ_i , for $i = 1, \dots, m$, the Rankine-Hugoniot conditions across the wave of speed $S_i \equiv \lambda_i$ read

$$\Delta \mathbf{F} = \mathbf{A} \Delta \mathbf{U} = \lambda_i (\Delta \mathbf{U})_i . \tag{2.121}$$

See (2.63). Actually, these conditions provide a technique for finding the solution of the Riemann problem for linear hyperbolic system with constant coefficients.

Exercise 2.32. Solve the Riemann problem for the linearised equations of Gas Dynamics using the Rankine-Hugoniot conditions across each wave.

Solution 2.33. The structure of the solution is depicted in Fig. 2.5. The unknowns are ρ_* and u_* in the *Star Region*. Recall that the vector \mathbf{U} and the coefficient matrix \mathbf{A} are given by

$$\mathbf{U} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \equiv \begin{bmatrix} \rho \\ u \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 0 & \rho_0 \\ a^2/\rho_0 & 0 \end{bmatrix}$$

and the eigenvalues are

$$\lambda_1 = -a, \quad \lambda_2 = +a.$$

Application of the Rankine–Hugoniot conditions across the λ_1 -wave of speed $S_1 = \lambda_1$ gives

$$\begin{bmatrix} 0 & \rho_0 \\ \frac{a^2}{\rho_0} & 0 \end{bmatrix} \begin{bmatrix} \rho_* - \rho_L \\ u_* - u_L \end{bmatrix} = -a \begin{bmatrix} \rho_* - \rho_L \\ u_* - u_L \end{bmatrix}.$$

Expanding and solving for u_* gives

$$u_* = u_L - (\rho_* - \rho_L) \frac{a}{\rho_0}.$$

For the λ_2 -wave of speed $S_2 = \lambda_2$ we obtain

$$u_* = u_R + (\rho_* - \rho_R) \frac{a}{\rho_0}.$$

The simultaneous solution of these two linear algebraic equations for the unknowns ρ_* and u_* is

$$\rho_* = \frac{1}{2}(\rho_L + \rho_R) - \frac{1}{2}(u_R - u_L)\rho_0/a,$$

$$u_* = \frac{1}{2}(u_L + u_R) - \frac{1}{2}(\rho_R - \rho_L)a/\rho_0,$$

which is the solution (2.64) obtained using a different technique based on eigenvector expansion of the initial data. The technique that makes use of the Rankine–Hugoniot conditions is more direct.

Generalised Riemann Invariants

For a general quasi-linear hyperbolic system

$$\mathbf{W}_t + \mathbf{A}(\mathbf{W})\mathbf{W}_x = \mathbf{0}, \tag{2.122}$$

with

$$\mathbf{W} = [w_1, w_2, \dots, w_m]^T,$$

we consider the wave associated with the i -characteristic field with eigenvalue λ_i and corresponding right eigenvector

$$\mathbf{K}^{(i)} = \left[k_1^{(i)}, k_2^{(i)}, \dots, k_m^{(i)} \right]^T .$$

The vector of dependent variables \mathbf{W} here is some suitable set, which may be the set conserved variables, for instance. Recall that any system of conservation laws may always be expressed in quasi-linear form via the Jacobian matrix, see (2.6) and (2.8).

The *Generalised Riemann Invariants* are relations that hold true, for certain waves, across the wave structure and lead the following $(m - 1)$ ordinary differential equations

$$\frac{dw_1}{k_1^{(i)}} = \frac{dw_2}{k_2^{(i)}} = \frac{dw_3}{k_3^{(i)}} = \dots = \frac{dw_m}{k_m^{(i)}} . \tag{2.123}$$

They relate ratios of changes dw_s of a quantity w_s to the respective component $k_s^{(i)}$ of the right eigenvector $\mathbf{K}^{(i)}$ corresponding to a λ_i -wave family . For a detailed discussion see the book by Jeffrey [269].

Example 2.34 (Linearised Gas Dynamics revisited). Here we find the Generalised Riemann Invariants for the linearised equations of Gas Dynamics. The dependent variables are

$$\mathbf{W} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \equiv \begin{bmatrix} \rho \\ u \end{bmatrix}$$

and the right eigenvectors are

$$\mathbf{K}^{(1)} = \begin{bmatrix} \rho_0 \\ -a \end{bmatrix} , \quad \mathbf{K}^{(2)} = \begin{bmatrix} \rho_0 \\ a \end{bmatrix} .$$

Across the λ_1 -wave we have

$$\frac{d\rho}{\rho_0} = \frac{du}{-a} ,$$

which leads to

$$du + \frac{a}{\rho_0} d\rho = 0 .$$

After integration this produces

$$I_L(\rho, u) = u + \frac{a}{\rho_0} \rho = \text{constant} . \tag{2.124}$$

The constant of integration is obtained by evaluating $I_L(\rho, u)$ at a reference state. Across the λ_2 -wave we have

$$\frac{d\rho}{\rho_0} = \frac{du}{a} ,$$

which leads to

$$I_R(\rho, u) = u - \frac{a}{\rho_0} \rho = \text{constant} . \tag{2.125}$$

Again the constant of integration is obtained by evaluating $I_R(\rho, u)$ at a reference state.

Exercise 2.35. Solve the Riemann problem for the linearised equations of Gas Dynamics using the Generalised Riemann Invariants.

Solution 2.36. Application of $I_L(\rho, u)$ across the left wave connecting the states \mathbf{W}_L and \mathbf{W}_* gives

$$u_* + \frac{a}{\rho_0} \rho_* = u_L + \frac{a}{\rho_0} \rho_L .$$

Similarly, application of $I_R(\rho, u)$ across the right wave connecting the states \mathbf{W}_R and \mathbf{W}_* gives

$$u_* - \frac{a}{\rho_0} \rho_* = u_R - \frac{a}{\rho_0} \rho_R$$

and the simultaneous solution for the unknowns ρ_* and u_* gives

$$\left. \begin{aligned} \rho_* &= \frac{1}{2}(\rho_L + \rho_R) - \frac{1}{2}(u_R - u_L)\rho_0/a , \\ u_* &= \frac{1}{2}(u_L + u_R) - \frac{1}{2}(\rho_R - \rho_L)a/\rho_0 , \end{aligned} \right\}$$

which is the same solution (2.64) obtained from applying other techniques.

Exercise 2.37. Solve the Riemann problem for the Small Perturbation Equations (2.24) using the following techniques:

- by expanding the initial data \mathbf{U}_L and \mathbf{U}_R in terms of the eigenvectors, see (2.56).
- by expanding the total jump $\Delta\mathbf{U}$ in terms of the eigenvectors, see (2.62).
- by using the Rankine-Hugoniot Conditions across each wave, see (2.121).
- by applying the Generalised Riemann Invariants, see (2.123).

Solution 2.38. Use of any of the suggested techniques will give the general solution

$$\left. \begin{aligned} u_* &= \frac{1}{2}(u_L + u_R) + \frac{1}{2}a(v_R - v_L) , \\ v_* &= \frac{1}{2}(v_L + v_R) + \frac{1}{2a}(u_R - u_L) . \end{aligned} \right\}$$

Example 2.39 (Isentropic Gas Dynamics Revisited). For this example the eigenvalues are $\lambda_1 = u - a$ and $\lambda_2 = u + a$, with $a = \sqrt{p'(\rho)} = \sqrt{\frac{\gamma p}{\rho}}$ defining the sound speed. The corresponding right eigenvectors are given by

$$\mathbf{K}^{(1)} = \begin{bmatrix} 1 \\ u - a \end{bmatrix}, \quad \mathbf{K}^{(2)} = \begin{bmatrix} 1 \\ u + a \end{bmatrix} .$$

Across the left $\lambda_1 = u - a$ wave we have

$$\frac{d\rho}{1} = \frac{d(\rho u)}{u - a},$$

which after expanding differentials yields

$$du + \frac{a}{\rho} d\rho = 0 .$$

On exact integration we obtain

$$I_L(\rho, \rho u) = u + \int \frac{a}{\rho} d\rho = \text{constant} . \tag{2.126}$$

Across the right $\lambda_2 = u + a$ wave we obtain

$$I_R(\rho, \rho u) = u - \int \frac{a}{\rho} d\rho = \text{constant} . \tag{2.127}$$

As as the sound speed a is a function of ρ alone we can evaluate the integral term above exactly as

$$\int \frac{a}{\rho} d\rho = \frac{2a}{\gamma - 1}$$

by first noting that

$$a = \sqrt{p'(\rho)} = \sqrt{C\gamma\rho^{\gamma-1}} = \sqrt{C\gamma}\rho^{\frac{\gamma-1}{2}} .$$

Then the *left* and *right* Riemann Invariants become

$$\left. \begin{aligned} I_L(\rho, \rho u) &= u + \frac{2a}{\gamma-1} = \text{constant across the } \lambda_1\text{-wave,} \\ I_R(\rho, \rho u) &= u - \frac{2a}{\gamma-1} = \text{constant across the } \lambda_2\text{-wave.} \end{aligned} \right\} \tag{2.128}$$

Generalised Riemann Invariants provide a powerful tool of analysis of hyperbolic conservation laws.

2.4.4 Elementary–Wave Solutions of the Riemann Problem

The Riemann problem for a general $m \times m$ non-linear hyperbolic system with data $\mathbf{U}_L, \mathbf{U}_R$ is the IVP

$$\left. \begin{aligned} \mathbf{U}_t + \mathbf{F}(\mathbf{U})_x &= \mathbf{0} , \\ \mathbf{U}(x, 0) &= \mathbf{U}^{(0)}(x) = \begin{cases} \mathbf{U}_L & \text{if } x < 0 , \\ \mathbf{U}_R & \text{if } x > 0 . \end{cases} \end{aligned} \right\} \tag{2.129}$$

The similarity solution $\mathbf{U}(x/t)$ of (2.129) consists of $m + 1$ constant states separated by m waves, as depicted by the x - t picture of Fig. 2.19. For each eigenvalue λ_i there is a wave family. For linear systems with constant coefficients each wave is a discontinuity of speed $S_i = \lambda_i$ and defines a linearly degenerate field.

For non-linear systems the waves may be discontinuities such as shock waves and contact waves, or smooth transition waves such as rarefactions. The possible types of waves present in the solution of the Riemann problem depends crucially on closure conditions. For the Euler equations we shall

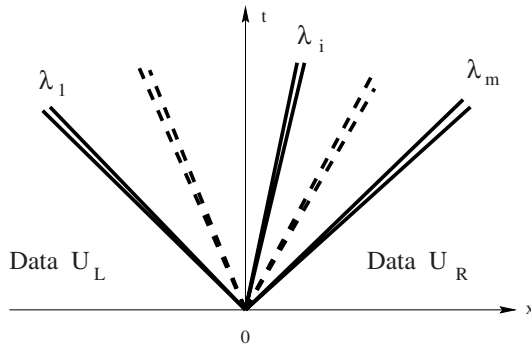


Fig. 2.19. Structure of the solution of the Riemann problem for a system of non-linear conservation laws

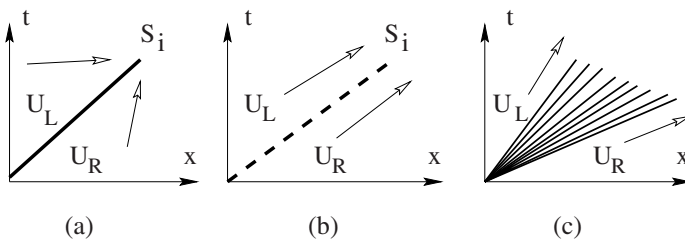


Fig. 2.20. Elementary wave solutions of the Riemann problem: (a) shock wave of speed S_i , (b) contact discontinuity of speed S_i and (c) rarefaction wave

only consider Equations of State such that the only waves present are shocks, contacts and rarefactions. Suppose that the initial data states \mathbf{U}_L , \mathbf{U}_R are connected by a single wave, that is, the solution of the Riemann problem consists of a single non-trivial wave; all other waves have zero strength. This assumption is entirely justified as we can always solve the Riemann problem with general data and then select the constant states on either side of a particular wave as the initial data for the Riemann problem. If the wave is a discontinuity then the wave is a shock wave or a contact wave.

Shock Wave

For a shock wave the two constant states \mathbf{U}_L and \mathbf{U}_R are connected through a single jump discontinuity in a *genuinely non-linear field* i and the following conditions apply

- the Rankine–Hugoniot conditions

$$\mathbf{F}(\mathbf{U}_R) - \mathbf{F}(\mathbf{U}_L) = S_i(\mathbf{U}_R - \mathbf{U}_L) . \tag{2.130}$$

- the entropy condition

$$\lambda_i(\mathbf{U}_L) > S_i > \lambda_i(\mathbf{U}_R) . \tag{2.131}$$

Fig. 2.20a depicts a shock wave of speed S_i . The characteristic $dx/dt = \lambda_i$ on both sides of the wave run into the shock wave, which illustrates the compressive character of a shock.

Contact Wave

For a *contact wave* the two data states \mathbf{U}_L and \mathbf{U}_R are connected through a single jump discontinuity of speed S_i in a *linearly degenerate field* i and the following conditions apply

- the Rankine–Hugoniot conditions

$$\mathbf{F}(\mathbf{U}_R) - \mathbf{F}(\mathbf{U}_L) = S_i(\mathbf{U}_R - \mathbf{U}_L) . \tag{2.132}$$

- constancy of the Generalised Riemann Invariants across the wave

$$\frac{dw_1}{k_1^{(i)}} = \frac{dw_2}{k_2^{(i)}} = \frac{dw_3}{k_3^{(i)}} = \dots = \frac{dw_m}{k_m^{(i)}} . \tag{2.133}$$

- the parallel characteristic condition

$$\lambda_i(\mathbf{U}_L) = \lambda_i(\mathbf{U}_R) = S_i . \tag{2.134}$$

Fig. 2.20b depicts a contact discontinuity. Characteristics on both sides of the wave run parallel to it.

Rarefaction Wave

For a rarefaction wave the two data states \mathbf{U}_L and \mathbf{U}_R are connected through a *smooth transition* in a *genuinely non-linear field* i and the following conditions are met

- constancy of the Generalised Riemann Invariants across the wave

$$\frac{dw_1}{k_1^{(i)}} = \frac{dw_2}{k_2^{(i)}} = \frac{dw_3}{k_3^{(i)}} = \dots = \frac{dw_m}{k_m^{(i)}} . \tag{2.135}$$

- divergence of characteristics

$$\lambda_i(\mathbf{U}_L) < \lambda_i(\mathbf{U}_R) . \tag{2.136}$$

Fig. 2.20c depicts a rarefaction wave. Characteristics on the left and right of the wave diverge as do characteristics inside the wave.

Remark 2.40. The solution of the general Riemann problem contains m waves of any of the above type, namely: shock waves, contact discontinuities and rarefaction waves. In solving the general Riemann problem we shall enforce these conditions by discriminating the particular type of wave present.

For further study we recommend the following references: Lax [301], Whitham [582], Chorin and Marsden [112], Courant and Friedrichs [143], Smoller [451] and LeVeque [308]. See also the papers by Lax [302] and [300].

The introductory concepts of this chapter will be used to analyse some of the properties of the Euler equations in Chap. 3. For the time-dependent one dimensional Euler equations we solve the Riemann problem exactly in Chap. 4, while in Chaps. 9 to 12 we present approximate Riemann solvers.