11

The Riemann Solver of Roe

Perhaps, the most well–known of all approximate Riemann solvers today, is the one due to Roe, which was first presented in the open literature in 1981 [407]. Since then, the method has not only been refined, but it has also been applied to a very large variety of physical problems. Refinements to the Roe approach were introduced by Roe and Pike [416], whereby the computation of the necessary items of information does not explicitly require the Roe averaged Jacobian matrix. This second methodology appears to be simpler and is thus useful in solving the Riemann problem for new, complicated sets of hyperbolic conservations laws, or for conventional systems but for complex media. Glaister exploited the Roe–Pike approach to extend Roe's method to the time–dependent Euler equations with a general equation of state [208], [209]. The large body of experience accumulated by many workers over a considerable period of time has led to various improvements of the scheme. As originally presented the Roe scheme computes rarefaction shocks, thus violating the entropy condition. Harten and Hyman [243], Roe and Pike [416], Roe [414], Dubois and Mehlman [167] and others, have produced appropriate modifications to the scheme. Einfeldt et. al. [182] produced corrections to the basic Roe scheme to avoid the so–called *vacuum problem* near low–density flows; they also showed that in fact this anomaly afflicts *all linearised Riemann solvers*.

Ambitious applications of the Roe scheme were presented by Brio and Wu [80], who utilised Roe's method to solve the Magneto–Hydrodynamic equations (MHD). Clarke et. al. [118] applied the method in conjunction with adaptive gridding to the computation of two–dimensional unsteady detonation waves in solid materials. Giraud and Manzini [206] produced parallel implementions of the Roe scheme for two–dimensional Gas Dynamics. LeVeque and Shyue [313] have applied the Roe scheme in the context of *front tracking* in two space dimensions. Marx has applied the Roe scheme to solve the incompressible Navier–Stokes equations [345], [346] and the compressible Navier–Stokes equations [344] using *implicit* versions of the scheme; see also McNeil [348]. The method has also been applied to multiphase flows; Toro

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[501] solved reactive multi–phase problems in the context of propulsion systems via a phase–splitting procedure; recently, Sainsaulieu [421] has extended the Roe scheme to a class of multiphase flow problems without phase splitting.

The purpose of this chapter is to present the approximate Riemann solver of Roe as applied to the three–dimensional time dependent Euler equations. For the numerical methods considered here, we only need to derive the Riemann solver for the split three–dimensional equations. After a general introduction to the method, we present both the methodology of Roe and that of Roe and Pike. Both methodologies are suitably illustrated via the simpler isothermal equations. Useful background reading is found in Chaps. 2 to 6.

11.1 Bases of the Roe Approach

In this section we describe the Roe approach for a general system of m hyperbolic conservation laws. Detailed application of the scheme to the isothermal and Euler equations are given in subsequent sections.

11.1.1 The Exact Riemann Problem and the Godunov Flux

We are concerned with solving numerically the general Initial Boundary Value Problem (IBVP)

$$\left.\begin{array}{l} \text{PDEs} : \mathbf{U}_t + \mathbf{F}(\mathbf{U})_x = \mathbf{0} \ , \\ \text{ICs} \quad : \mathbf{U}(x,0) = \mathbf{U}^{(0)}(x) \ , \\ \text{BCs} \quad : \mathbf{U}(0,t) = \mathbf{U}_{\rm l}(t) \ , \ \mathbf{U}(L,t) = \mathbf{U}_{\rm r}(t) \ , \end{array}\right\} \tag{11.1}$$

in a domain $x_l \leq x \leq x_r$, utilising the explicit conservative formula

$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^n + \frac{\Delta t}{\Delta x}[\mathbf{F}_{i-\frac{1}{2}} - \mathbf{F}_{i+\frac{1}{2}}] \ . \tag{11.2}$$

We assume the solution of IBVP (11.1) exists. In Chap. 6 we defined the Godunov intercell numerical flux

$$\mathbf{F}_{i+\frac{1}{2}} = \mathbf{F}(\mathbf{U}_{i+\frac{1}{2}}(0)) \ , \tag{11.3}$$

in which $\mathbf{U}_{i+\frac{1}{2}}(0)$ is the exact similarity solution $\mathbf{U}_{i+\frac{1}{2}}(x/t)$ of the Riemann problem

$$\left.\begin{array}{l} \mathbf{U}_t + \mathbf{F}(\mathbf{U})_x = \mathbf{0} \ , \\ \mathbf{U}(x,0) = \left\{ \begin{array}{l} \mathbf{U}_{\rm L} \ if \ x < 0 \ , \\ \mathbf{U}_{\rm R} \ if \quad x > 0 \end{array} \right. \end{array}\right\} \tag{11.4}$$

evaluated at $x/t = 0$. Fig. 11.1 shows the structure of the exact solution of the Riemann problem for the x–split three dimensional Euler equations, for which the vectors of conserved variables and fluxes are

$$\mathbf{U} = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ E \end{bmatrix} \quad , \quad \mathbf{F} = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ \rho uw \\ u(E+p) \end{bmatrix} \quad . \tag{11.5}$$

The *Star Region* between the left and right waves contains the unknowns of the problem. The particular value at $x/t = 0$ corresponds to the t–axis and is the value required by the Godunov flux. See Chaps. 4 and 6 for details. The piece–wise constant initial data, in terms of primitive variables, is

$$\mathbf{W}_L = \begin{bmatrix} \rho_L \\ u_L \\ v_L \\ w_L \\ p_L \end{bmatrix} \quad , \quad \mathbf{W}_R = \begin{bmatrix} \rho_R \\ u_R \\ v_R \\ w_R \\ p_R \end{bmatrix} \quad . \tag{11.6}$$

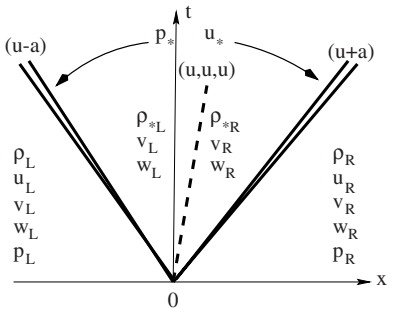

Fig. 11.1. Structure of the solution of the Riemann problem for the x–split three dimensional Euler equations

In Chap. 4 we provided an algorithm to compute the exact solution $\mathbf{U}_{i+\frac{1}{2}}(x/t)$ and in Chap. 6 we utilised this solution in the Godunov method. In Chap. 9 we provided approximations to the state $\mathbf{U}_{i+\frac{1}{2}}(x/t)$ and obtained a corresponding approximate Godunov method by evaluating the physical flux function $\mathbf{F}$ at this approximate state. The purpose of this chapter is to find *direct approximations to the flux function* $\mathbf{F}_{i+\frac{1}{2}}$ following the approach proposed by Roe [407] and Roe and Pike [416].

11.1.2 Approximate Conservation Laws

Roe [407] solved the Riemann problem (11.4) approximately. By introducing the Jacobian matrix

$$\mathbf{A}(\mathbf{U}) = \frac{\partial \mathbf{F}}{\partial \mathbf{U}} \tag{11.7}$$

and using the chain rule the conservation laws

$$\mathbf{U}_t + \mathbf{F}(\mathbf{U})_x = \mathbf{0}$$

in (11.4) may be written as

$$\mathbf{U}_t + \mathbf{A}(\mathbf{U})\mathbf{U}_x = \mathbf{0}\,. \tag{11.8}$$

Roe's approach replaces the Jacobian matrix $\mathbf{A}(\mathbf{U})$ in (11.8) by a *constant* Jacobian matrix

$$\tilde{\mathbf{A}} = \tilde{\mathbf{A}}(\mathbf{U}_L, \mathbf{U}_R)\,, \tag{11.9}$$

which is a function of the data states $\mathbf{U}_L$, $\mathbf{U}_R$. In this way the original PDEs in (11.4) are replaced by

$$\mathbf{U}_t + \tilde{\mathbf{A}}\mathbf{U}_x = \mathbf{0}\,. \tag{11.10}$$

This is a linear system with constant coefficients. The original, Riemann problem (11.4) is then replaced by the *approximate Riemann problem*

$$\left.\begin{array}{l} \mathbf{U}_t + \tilde{\mathbf{A}}\mathbf{U}_x = \mathbf{0} \\ \mathbf{U}(x,0) = \left\{\begin{array}{l} \mathbf{U}_L\,,\ x<0 \\ \mathbf{U}_R\,,\ x>0 \end{array}\right. \end{array}\right\}\,, \tag{11.11}$$

which is then *solved exactly*. The approximate problem results from replacing the original non–linear conservation laws by a linearised system with constant coefficients but the initial data of the exact problem is retained.

For a general hyperbolic system of m conservation laws, the Roe Jacobian matrix $\tilde{\mathbf{A}}$ is required to satisfy the following properties:

Property (A): Hyperbolicity of the system. $\tilde{\mathbf{A}}$ is required to have real eigenvalues $\tilde{\lambda}_i = \tilde{\lambda}_i(\mathbf{U}_L, \mathbf{U}_R)$, which we choose to order as

$$\tilde{\lambda}_1 \le \tilde{\lambda}_2 \le \cdots \le \tilde{\lambda}_m\,, \tag{11.12}$$

and a complete set of linearly independent right eigenvectors

$$\tilde{\mathbf{K}}^{(1)}\,,\quad \tilde{\mathbf{K}}^{(2)}\,,\cdots\,,\quad \tilde{\mathbf{K}}^{(m)}\,. \tag{11.13}$$

Property (B): Consistency with the exact Jacobian

$$\tilde{\mathbf{A}}(\mathbf{U},\mathbf{U}) = \mathbf{A}(\mathbf{U})\,. \tag{11.14}$$

Property (C): Conservation across discontinuities

$$\mathbf{F}(\mathbf{U}_R) - \mathbf{F}(\mathbf{U}_L) = \tilde{\mathbf{A}}\,(\mathbf{U}_R - \mathbf{U}_L)\,. \tag{11.15}$$

Property (A) on hyperbolicity is an obvious requirement; the approximate problem should at the very least preserve the mathematical character of the

original non–linear system. Property (B) ensures consistency with the conservation laws. Property (C) ensures conservation. It also ensures exact recognition of isolated discontinuities; that is, if the data $\mathbf{U}_L$, $\mathbf{U}_R$ are connected by a *single, isolated* discontinuity, then the approximate Riemann solver recognises this wave exactly. Note however that this does not mean that the corresponding, approximate, Godunov method with the Roe approximate numerical flux will in general give exact solutions for isolated discontinuities.

The construction of matrices satisfying properties (A)–(C) for *general hyperbolic systems* can be very complicated and thus computationally unattractive. For the *specific* case of the Euler equations of Gas Dynamics Roe [407] proposed a relatively simple way of constructing a matrix $\tilde{\mathbf{A}}$. Later, Roe and Pike [416] proposed a simpler approach, where the explicit construction of $\tilde{\mathbf{A}}$ is actually avoided.

11.1.3 The Approximate Riemann Problem and the Intercell Flux

Once the matrix $\tilde{\mathbf{A}}(\mathbf{U}_L, \mathbf{U}_R)$, its eigenvalues $\tilde{\lambda}_i(\mathbf{U}_L, \mathbf{U}_R)$ and right eigenvectors $\tilde{\mathbf{K}}^{(i)}(\mathbf{U}_L, \mathbf{U}_R)$ are available, one solves the Riemann problem (11.11) by direct application of methods discussed in Sect. 2.3 of Chap. 2 and Sect. 5.4 of Chap. 5, for linear hyperbolic systems with constant coefficients. By projecting the data difference

$$\Delta\mathbf{U} = \mathbf{U}_R - \mathbf{U}_L$$

onto the right eigenvectors we write

$$\Delta\mathbf{U} = \mathbf{U}_R - \mathbf{U}_L = \sum_{i=1}^{m} \tilde{\alpha}_i \tilde{\mathbf{K}}^{(i)} \,, \tag{11.16}$$

from which one finds the *wave strengths* $\tilde{\alpha}_i = \tilde{\alpha}_i(\mathbf{U}_L, \mathbf{U}_R)$. The solution $\mathbf{U}_{i+\frac{1}{2}}(x/t)$ evaluated along the t–axis, $x/t = 0$, is given by

$$\mathbf{U}_{i+\frac{1}{2}}(0) = \mathbf{U}_L + \sum_{\tilde{\lambda}_i \le 0} \tilde{\alpha}_i \tilde{\mathbf{K}}^{(i)} \,, \tag{11.17}$$

or

$$\mathbf{U}_{i+\frac{1}{2}}(0) = \mathbf{U}_R - \sum_{\tilde{\lambda}_i \ge 0} \tilde{\alpha}_i \tilde{\mathbf{K}}^{(i)} \,. \tag{11.18}$$

We now find the corresponding numerical flux. Recall that we have replaced the original set of conservation laws in (11.4) by the constant coefficient linear system (11.10); this can be viewed as a modified system of conservation laws

$$\overline{\mathbf{U}}_t + \overline{\mathbf{F}}(\overline{\mathbf{U}})_x = \mathbf{0} \,, \tag{11.19}$$

with flux function

$$\overline{\mathbf{F}}(\overline{\mathbf{U}}) = \tilde{\mathbf{A}}\overline{\mathbf{U}} \; . \tag{11.20}$$

The corresponding numerical flux, see (11.3), *is not* the obvious choice

$$\mathbf{F}_{i+\frac{1}{2}} = \tilde{\mathbf{A}} \; \overline{\mathbf{U}}_{i+\frac{1}{2}}(0) \; ,$$

where $\overline{\mathbf{U}}_{i+\frac{1}{2}}(0)$ is given by any of (11.17)–(11.18). That this would be incorrect becomes obvious when, for instance, assuming right supersonic flow in (11.17) one would compute an intercell flux $\mathbf{F}_{i+\frac{1}{2}} \neq \mathbf{F}_L$. Instead, the correct expression for the corresponding numerical flux is obtained from any of the integral relations

$$\mathbf{F}_{0L} = \mathbf{F}_L - S_L\mathbf{U}_L - \frac{1}{T}\int_{TS_L}^{0} \mathbf{U}(x,T)dx \; , \tag{11.21}$$

$$\mathbf{F}_{0R} = \mathbf{F}_R - S_R\mathbf{U}_R + \frac{1}{T}\int_{0}^{TS_R} \mathbf{U}(x,T)dx \; , \tag{11.22}$$

derived in Sect. 10.2 of Chap. 10. Here S_L, S_R are the smallest and largest signal speeds in the exact solution of the Riemann problem with data $\mathbf{U}_L, \mathbf{U}_R$ and T is a positive time. If the integrand $\mathbf{U}(x,t)$ in (11.21) or (11.22) is replaced by some approximate solution, then equality of the fluxes $\mathbf{F}_{0L}$ and $\mathbf{F}_{0R}$ requires the approximate solution to satisfy a *Consistency Condition*, see Sect. 10.2 of Chap. 10.

If $\overline{\mathbf{U}}_{i+\frac{1}{2}}(x,t)$ is the solution of the Riemann problem for the modified conservation laws (11.19) with data $\mathbf{U}_L, \mathbf{U}_R$, then the integrals in (11.21) and (11.22) respectively, are

$$\int_{TS_L}^{0} \overline{\mathbf{U}}_{i+\frac{1}{2}}(x,T)dx = T[\overline{\mathbf{F}}(\mathbf{U}_L) - \overline{\mathbf{F}}(\overline{\mathbf{U}}_{i+\frac{1}{2}}(0))] - TS_L\mathbf{U}_L \tag{11.23}$$

and

$$\int_{0}^{TS_R} \overline{\mathbf{U}}_{i+\frac{1}{2}}(x,T)dx = T[\overline{\mathbf{F}}(\overline{\mathbf{U}}_{i+\frac{1}{2}}(0)) - \overline{\mathbf{F}}(\mathbf{U}_R)] + TS_R\mathbf{U}_R \; . \tag{11.24}$$

Substitution of (11.23) and (11.24) into (11.21) and (11.22) gives

$$\mathbf{F}_{0L} = \overline{\mathbf{F}}(\overline{\mathbf{U}}_{i+\frac{1}{2}}(0)) + \mathbf{F}(\mathbf{U}_L) - \overline{\mathbf{F}}(\mathbf{U}_L) \tag{11.25}$$

and

$$\mathbf{F}_{0R} = \overline{\mathbf{F}}(\overline{\mathbf{U}}_{i+\frac{1}{2}}(0)) + \mathbf{F}(\mathbf{U}_R) - \overline{\mathbf{F}}(\mathbf{U}_R) \; . \tag{11.26}$$

Finally, by using $\overline{\mathbf{U}}_{i+\frac{1}{2}}(0)$ as given by (11.17) or (11.18) and the definition of the flux $\overline{\mathbf{F}} = \tilde{\mathbf{A}} \; \overline{\mathbf{U}}$ we obtain the numerical flux as

$$\mathbf{F}_{i+\frac{1}{2}} = \mathbf{F}_L + \sum_{\tilde{\lambda}_i \leq 0} \tilde{\alpha}_i \tilde{\lambda}_i \tilde{\mathbf{K}}^{(i)} \; , \tag{11.27}$$

or

$$\mathbf{F}_{i+\frac{1}{2}} = \mathbf{F}_R - \sum_{\tilde{\lambda}_i \geq 0} \tilde{\alpha}_i \tilde{\lambda}_i \tilde{\mathbf{K}}^{(i)} \ . \tag{11.28}$$

Alternatively, we may also write

$$\mathbf{F}_{i+\frac{1}{2}} = \frac{1}{2}(\mathbf{F}_L + \mathbf{F}_R) - \frac{1}{2}\sum_{i=1}^{m} \tilde{\alpha}_i |\tilde{\lambda}_i| \tilde{\mathbf{K}}^{(i)} \ . \tag{11.29}$$

We remark that all previous relations (11.19)–(11.29) are valid for any hyperbolic system and any linearisation of it. In order to compute Roe's numerical flux for a particular system of hyperbolic conservation laws, one requires expressions for the wave strengths $\tilde{\alpha}_i$, the eigenvalues $\tilde{\lambda}_i$ and the right eigenvectors $\tilde{\mathbf{K}}^{(i)}$ in any of the flux expressions (11.27)–(11.29). Note that the Jacobian matrix $\tilde{\mathbf{A}}(\mathbf{U}_L, \mathbf{U}_R)$ is not explicitly required by the numerical flux. In the next two sections we give details on methodologies to find $\tilde{\alpha}_i$, $\tilde{\lambda}_i$ and $\tilde{\mathbf{K}}^{(i)}$. There are two approaches, namely the original approach presented by Roe in 1981 [407] and the Roe–Pike approach [416].

11.2 The Original Roe Method

In order for the approximate Godunov method based on (11.2) with the Roe–type numerical flux (11.27)–(11.29) to be completely determined, we need to find the average eigenvalues $\tilde{\lambda}_i$, the corresponding averaged right eigenvectors $\tilde{\mathbf{K}}^{(i)}$ and averaged wave strengths $\tilde{\alpha}_i$. In his original paper [407] Roe finds an averaged Jacobian matrix $\tilde{\mathbf{A}}$, the Roe matrix, from which $\tilde{\lambda}_i$, $\tilde{\mathbf{K}}^{(i)}$ and $\tilde{\alpha}_i$ follow directly. In constructing a matrix $\tilde{\mathbf{A}}$ the properties (A)–(C), equations (11.12)–(11.15), are enforced. It is not difficult to think of candidates $\tilde{\mathbf{A}}$ that satisfy the first two properties. Property C is crucial and is the one that narrows down the choices. Roe showed that the existence of a matrix $\tilde{\mathbf{A}}$ satisfying Property C is assured by the mean value theorem. An early line of attack in constructing a matrix $\tilde{\mathbf{A}}$ satisfying all desirable properties is reported by Sells [440]. Roe identifies some disadvantages of this approach; it is argued, for instance, that the construction is far from unique and that the resulting schemes are too complicated.

A breakthrough in constructing $\tilde{\mathbf{A}}$ resulted from Roe's ingenious idea of introducing a *parameter vector* $\mathbf{Q}$, such that both the vector of conserved variables $\mathbf{U}$ and the flux vector $\mathbf{F}(\mathbf{U})$ could be expressed in terms of $\mathbf{Q}$. That is

$$\mathbf{U} = \mathbf{U}(\mathbf{Q}) \ , \quad \mathbf{F} = \mathbf{F}(\mathbf{Q}) \ . \tag{11.30}$$

Two important steps then follow. First, the changes

$$\Delta\mathbf{U} = \mathbf{U}_R - \mathbf{U}_L \ , \quad \Delta\mathbf{F} = \mathbf{F}(\mathbf{U}_R) - \mathbf{F}(\mathbf{U}_L) \tag{11.31}$$

can be expressed in terms of the change $\Delta\mathbf{Q} = \mathbf{Q}_R - \mathbf{Q}_L$. Then, averages are obtained in terms of *simple arithmetic means* of $\mathbf{Q}$. Next, we illustrate the technique as applied to a simple set of conservation laws.

11.2.1 The Isothermal Equations

Consider the isothermal equations

$$\left.\begin{aligned} \mathbf{U}_t + \mathbf{F}(\mathbf{U})_x &= \mathbf{0}\,, \\ \mathbf{U} \equiv \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \equiv \begin{bmatrix} \rho \\ \rho u \end{bmatrix}\;;\quad \mathbf{F} &\equiv \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \equiv \begin{bmatrix} \rho u \\ \rho u^2 + a^2\rho \end{bmatrix}\,, \end{aligned}\right\} \tag{11.32}$$

where a is a constant sound speed. See Sect. 1.6.2 of Chap. 1. See also Sect. 2.4.1 of Chap. 2, where the eigenstructure of the equations is given. The exact Jacobian, eigenvalues and corresponding right eigenvectors are

$$\left.\begin{aligned} \mathbf{A}(\mathbf{U}) &= \begin{bmatrix} 0 & 1 \\ a^2 - u^2 & 2u \end{bmatrix}\,, \\ \lambda_1 = u - a\,,\quad \lambda_2 &= u + a\,, \\ \mathbf{K}^{(1)} = \begin{bmatrix} 1 \\ u - a \end{bmatrix}\,,\quad \mathbf{K}^{(2)} &= \begin{bmatrix} 1 \\ u + a \end{bmatrix}\,. \end{aligned}\right\} \tag{11.33}$$

Choose the *parameter vector*

$$\mathbf{Q} \equiv \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \equiv \frac{\mathbf{U}}{\sqrt{\rho}} = \begin{bmatrix} \sqrt{\rho} \\ \sqrt{\rho}u \end{bmatrix}\,. \tag{11.34}$$

Then $\mathbf{U}$ and $\mathbf{F}$ can be expressed in terms of the components q_1, q_2 of $\mathbf{Q}$, namely

$$\mathbf{U} \equiv \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \equiv q_1\mathbf{Q} = \begin{bmatrix} q_1^2 \\ q_1 q_2 \end{bmatrix} \tag{11.35}$$

and

$$\mathbf{F} \equiv \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \equiv \begin{bmatrix} q_1 q_2 \\ q_2^2 + a^2 q_1^2 \end{bmatrix}\,. \tag{11.36}$$

One now looks for an averaged vector $\tilde{\mathbf{Q}} = (\tilde{q}_1, \tilde{q}_2)^T$. This is found by simple *arithmetic averaging*

$$\tilde{\mathbf{Q}} = \begin{bmatrix} \tilde{q}_1 \\ \tilde{q}_2 \end{bmatrix} = \frac{1}{2}(\mathbf{Q}_L + \mathbf{Q}_R) = \frac{1}{2}\begin{bmatrix} \sqrt{\rho_L} + \sqrt{\rho_R} \\ \sqrt{\rho_L}u_L + \sqrt{\rho_R}u_R \end{bmatrix}\,. \tag{11.37}$$

Then two matrices $\tilde{\mathbf{B}} = \tilde{\mathbf{B}}(\tilde{\mathbf{Q}})$ and $\tilde{\mathbf{C}} = \tilde{\mathbf{C}}(\tilde{\mathbf{Q}})$ are found, such that the jumps $\Delta\mathbf{U}$ and $\Delta\mathbf{F}$ in (11.31) can be expressed in terms of the jump $\Delta\mathbf{Q}$, namely

$$\Delta\mathbf{U} = \tilde{\mathbf{B}}\Delta\mathbf{Q}\;;\quad \Delta\mathbf{F} = \tilde{\mathbf{C}}\Delta\mathbf{Q}\,. \tag{11.38}$$

Use of these two expressions produces

$$\Delta\mathbf{F} = (\tilde{\mathbf{C}}\tilde{\mathbf{B}}^{-1})\Delta\mathbf{U} \,, \tag{11.39}$$

which if compared with condition (C), equation (11.15), produces the Roe averaged matrix

$$\tilde{\mathbf{A}} = \tilde{\mathbf{C}}\tilde{\mathbf{B}}^{-1} \,. \tag{11.40}$$

Matrices $\tilde{\mathbf{B}}$ and $\tilde{\mathbf{C}}$ satisfying (11.38) are

$$\tilde{\mathbf{B}} = \begin{bmatrix} 2\tilde{q}_1 & 0 \\ \tilde{q}_2 & \tilde{q}_1 \end{bmatrix} \;;\; \tilde{\mathbf{C}} = \begin{bmatrix} \tilde{q}_2 & \tilde{q}_1 \\ 2a^2\tilde{q}_1 & 2\tilde{q}_2^2 \end{bmatrix} \,, \tag{11.41}$$

which the reader can easily verify. The sought Roe matrix is then

$$\tilde{\mathbf{A}} = \begin{bmatrix} 0 & 1 \\ a^2 - \tilde{u}^2 & 2\tilde{u} \end{bmatrix} \,, \tag{11.42}$$

where $\tilde{u}$ is the *Roe averaged velocity* and is given by

$$\tilde{u} = \frac{\sqrt{\rho_L}u_L + \sqrt{\rho_R}u_R}{\sqrt{\rho_L} + \sqrt{\rho_R}} \,. \tag{11.43}$$

Compare (11.42) with the matrix in (11.33). As the sound speed a is constant, no averaged $\tilde{\rho}$ is required.

Having found $\tilde{\mathbf{A}}$ one computes the averaged eigenvalues, eigenvectors and wave strengths. The eigenvalues of $\tilde{\mathbf{A}}$ are

$$\tilde{\lambda}_1 = \tilde{u} - a \;;\;\; \tilde{\lambda}_2 = \tilde{u} + a \tag{11.44}$$

and are all real. The corresponding averaged right eigenvectors are

$$\tilde{\mathbf{K}}^{(1)} = \begin{bmatrix} 1 \\ \tilde{u} - a \end{bmatrix} \;;\; \tilde{\mathbf{K}}^{(2)} = \begin{bmatrix} 1 \\ \tilde{u} + a \end{bmatrix} \tag{11.45}$$

and are easily seen to be linearly independent. Thus condition (A) is satisfied. To find the wave strengths $\tilde{\alpha}_i$ we solve the 2×2 linear system, see (11.16),

$$\Delta\mathbf{U} \equiv \begin{bmatrix} \Delta u_1 \\ \Delta u_2 \end{bmatrix} = \sum_{i=1}^{2} \tilde{\alpha}_i \tilde{\mathbf{K}}^{(i)} \,.$$

The solution is easily verified to be

$$\left.\begin{aligned} \tilde{\alpha}_1 &= \frac{\Delta u_1(\tilde{u} + a) - \Delta u_2}{2a} \,, \\ \tilde{\alpha}_2 &= \frac{-\Delta u_1(\tilde{u} - a) + \Delta u_2}{2a} \,, \end{aligned}\right\} \tag{11.46}$$

with the obvious definitions $\Delta u_1 \equiv \rho_R - \rho_L$, $\Delta u_2 \equiv \rho_R u_R - \rho_L u_L$. The corresponding Roe numerical flux $\mathbf{F}_{i+\frac{1}{2}}$ now follows from using (11.43)–(11.46) into any of the expressions (11.27)–(11.29).

11.2.2 The Euler Equations

Here we present the Roe Riemann solver as applied to the Riemann problem (11.4)–(11.5) for the x–split three dimensional time dependent Euler equations for ideal gases. Details of the Euler equations are found in Sect. 1.1 and Sect. 1.2 of Chap. 1; mathematical properties of the Euler equations are studied in Chap. 3.

The exact, x–direction Jacobian matrix $\mathbf{A}(\mathbf{U})$ is

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ \hat{\gamma}H - u^2 - a^2 & (3-\gamma)u & -\hat{\gamma}v & -\hat{\gamma}w & \hat{\gamma} \\ -uv & v & u & 0 & 0 \\ -uw & w & 0 & u & 0 \\ \frac{1}{2}u[(\gamma-3)H - a^2] & H - \hat{\gamma}u^2 & -\hat{\gamma}uv & -\hat{\gamma}uw & \gamma u \end{bmatrix} , \tag{11.47}$$

where $\hat{\gamma} = \gamma - 1$. The eigenvalues are

$$\lambda_1 = u - a \ , \quad \lambda_2 = \lambda_3 = \lambda_4 = u \ , \quad \lambda_5 = u + a \ , \tag{11.48}$$

where $a = \sqrt{\gamma p/\rho}$ is the sound speed. The corresponding right eigenvectors are

$$\left.\begin{aligned} \mathbf{K}^{(1)} = \begin{bmatrix} 1 \\ u-a \\ v \\ w \\ H - ua \end{bmatrix} \ ; \quad \mathbf{K}^{(2)} = \begin{bmatrix} 1 \\ u \\ v \\ w \\ \frac{1}{2}V^2 \end{bmatrix} \ ; \quad \mathbf{K}^{(3)} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ v \end{bmatrix} \\ \mathbf{K}^{(4)} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ w \end{bmatrix} \ ; \quad \mathbf{K}^{(5)} = \begin{bmatrix} 1 \\ u+a \\ v \\ w \\ H + ua \end{bmatrix} . \end{aligned}\right\} \tag{11.49}$$

Here H is the total enthalpy

$$H = \frac{E + p}{\rho} \tag{11.50}$$

and E is the total energy per unit volume

$$E = \frac{1}{2}\rho \mathbf{V}^2 + \rho e \ , \tag{11.51}$$

with

$$\mathbf{V}^2 = u^2 + v^2 + w^2 \tag{11.52}$$

and e denoting the specific internal energy, which for ideal gases, see Sect. 1.2 of Chap. 1, is

$$e = \frac{p}{(\gamma - 1)\rho} \, . \tag{11.53}$$

Roe chooses the *parameter vector*

$$\mathbf{Q} \equiv \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \end{bmatrix} \equiv \sqrt{\rho} \begin{bmatrix} 1 \\ u \\ v \\ w \\ H \end{bmatrix} \, , \tag{11.54}$$

which has the property that every component u_i of $\mathbf{U}$ and every component f_i of $\mathbf{F}(\mathbf{U})$ in (11.4)–(11.5) is a quadratic in the components q_i of $\mathbf{Q}$. For instance $u_1 = q_1^2$ and $f_1 = q_1 q_2$, etc. Actually, the property is also valid for the components of the $\mathbf{G}$ and $\mathbf{H}$ fluxes for the full three–dimensional Euler equations.

As done for the isothermal equations, see equations (11.38), one can express the jumps $\Delta\mathbf{U}$ and $\Delta\mathbf{F}$ in terms of the jump $\Delta\mathbf{Q}$ via two matrices $\tilde{\mathbf{B}}$ and $\tilde{\mathbf{C}}$. Roe [407] gives the following expressions

$$\tilde{\mathbf{B}} = \begin{pmatrix} 2\tilde{q}_1 & 0 & 0 & 0 & 0 \\ \tilde{q}_2 & \tilde{q}_1 & 0 & 0 & 0 \\ \tilde{q}_3 & 0 & \tilde{q}_1 & 0 & 0 \\ \tilde{q}_4 & 0 & 0 & \tilde{q}_1 & 0 \\ \dfrac{\tilde{q}_5}{\gamma} & \dfrac{\gamma-1}{\gamma}\tilde{q}_2 & \dfrac{\gamma-1}{\gamma}\tilde{q}_3 & \dfrac{\gamma-1}{\gamma}\tilde{q}_4 & \dfrac{\tilde{q}_1}{\gamma} \end{pmatrix} \tag{11.55}$$

and

$$\tilde{\mathbf{C}} = \begin{pmatrix} \tilde{q}_2 & \tilde{q}_1 & 0 & 0 & 0 \\ \dfrac{\gamma-1}{\gamma}\tilde{q}_5 & \dfrac{\gamma+1}{\gamma}\tilde{q}_2 & -\dfrac{\gamma-1}{\gamma}\tilde{q}_3 & -\dfrac{\gamma-1}{\gamma}\tilde{q}_4 & \dfrac{\gamma-1}{\gamma}\tilde{q}_1 \\ 0 & \tilde{q}_3 & \tilde{q}_2 & 0 & 0 \\ 0 & \tilde{q}_4 & 0 & \tilde{q}_2 & 0 \\ 0 & \tilde{q}_5 & 0 & 0 & \tilde{q}_2 \end{pmatrix} \, . \tag{11.56}$$

The sought Roe matrix is then given by

$$\tilde{\mathbf{A}} = \tilde{\mathbf{B}}\tilde{\mathbf{C}}^{-1} \, . \tag{11.57}$$

The eigenvalues of $\tilde{\mathbf{A}}$ are

$$\tilde{\lambda}_1 = \tilde{u} - \tilde{a} \, , \quad \tilde{\lambda}_2 = \tilde{\lambda}_3 = \tilde{\lambda}_4 = \tilde{u} \, , \quad \tilde{\lambda}_5 = \tilde{u} + \tilde{a} \tag{11.58}$$

and the corresponding right eigenvectors are

$$\left.\begin{aligned} \tilde{\mathbf{K}}^{(1)} &= \begin{bmatrix} 1 \\ \tilde{u}-\tilde{a} \\ \tilde{v} \\ \tilde{w} \\ \tilde{H}-\tilde{u}\tilde{a} \end{bmatrix} \;;\; \tilde{\mathbf{K}}^{(2)} = \begin{bmatrix} 1 \\ \tilde{u} \\ \tilde{v} \\ \tilde{w} \\ \frac{1}{2}\tilde{V}^2 \end{bmatrix} \;;\; \tilde{\mathbf{K}}^{(3)} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \tilde{v} \end{bmatrix} \\ \tilde{\mathbf{K}}^{(4)} &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ \tilde{w} \end{bmatrix} \;;\; \tilde{\mathbf{K}}^{(5)} = \begin{bmatrix} 1 \\ \tilde{u}+\tilde{a} \\ \tilde{v} \\ \tilde{w} \\ \tilde{H}+\tilde{u}\tilde{a} \end{bmatrix} . \end{aligned}\right\} \tag{11.59}$$

The symbol $\tilde{r}$ in (11.58), (11.59) denotes a Roe average for a variable r. The relevant averages are given as follows

$$\left.\begin{aligned} \tilde{u} &= \frac{\sqrt{\rho_L}u_L + \sqrt{\rho_R}u_R}{\sqrt{\rho_L}+\sqrt{\rho_R}} , \\ \tilde{v} &= \frac{\sqrt{\rho_L}v_L + \sqrt{\rho_R}v_R}{\sqrt{\rho_L}+\sqrt{\rho_R}} , \\ \tilde{w} &= \frac{\sqrt{\rho_L}w_L + \sqrt{\rho_R}w_R}{\sqrt{\rho_L}+\sqrt{\rho_R}} , \\ \tilde{H} &= \frac{\sqrt{\rho_L}H_L + \sqrt{\rho_R}H_R}{\sqrt{\rho_L}+\sqrt{\rho_R}} , \\ \tilde{a} &= \left((\gamma-1)(\tilde{H}-\tfrac{1}{2}\tilde{\mathbf{V}}^2)\right)^{\frac{1}{2}} , \end{aligned}\right\} \tag{11.60}$$

where $\tilde{\mathbf{V}}^2 = \tilde{u}^2 + \tilde{v}^2 + \tilde{w}^2$.

In order to determine completely the Roe numerical flux $\mathbf{F}_{i+\frac{1}{2}}$ we need, in addition, the wave strengths $\tilde{\alpha}_i$. These are obtained by projecting the jump $\Delta\mathbf{U}$ onto the right, averaged eigenvectors (11.59), namely

$$\Delta\mathbf{U} = \sum_{i=1}^{5} \tilde{\alpha}_i \tilde{\mathbf{K}}^{(i)} . \tag{11.61}$$

When written in full these equations read

$$\tilde{\alpha}_1 + \tilde{\alpha}_2 + \tilde{\alpha}_5 = \Delta u_1 , \tag{11.62}$$

$$\tilde{\alpha}_1(\tilde{u}-\tilde{a}) + \tilde{\alpha}_2\tilde{u} + \tilde{\alpha}_5(\tilde{u}+\tilde{a}) = \Delta u_2 , \tag{11.63}$$

$$\tilde{\alpha}_1\tilde{v} + \tilde{\alpha}_2\tilde{v} + \tilde{\alpha}_3 + \tilde{\alpha}_5\tilde{v} = \Delta u_3 , \tag{11.64}$$

$$\tilde{\alpha}_1\tilde{w} + \tilde{\alpha}_2\tilde{w} + \tilde{\alpha}_4 + \tilde{\alpha}_5\tilde{w} = \Delta u_4 , \tag{11.65}$$

$$\tilde{\alpha}_1(\tilde{H} - \tilde{u}\tilde{a}) + \frac{1}{2}\tilde{\mathbf{V}}^2\tilde{\alpha}_2 + \tilde{\alpha}_3\tilde{v} + \tilde{\alpha}_4\tilde{w} + \tilde{\alpha}_5(\tilde{H} + \tilde{u}\tilde{a}) = \Delta u_5 \ . \tag{11.66}$$

Here the right–hand side terms of equations (11.62)–(11.66) are known: they are jumps Δu_i in the conserved quantity u_i, namely

$$\Delta u_i = (u_i)_R - (u_i)_L \ .$$

Before solving these equations we note that in the purely one–dimensional case

$$\tilde{v} = \tilde{w} = 0 \ , \quad \tilde{\alpha}_3 = \tilde{\alpha}_4 = 0 \ , \quad \tilde{\mathbf{K}}^{(3)} = \tilde{\mathbf{K}}^{(4)} = \mathbf{0} \tag{11.67}$$

and the problem reduces to solving (11.62), (11.63) and (11.66) for $\tilde{\alpha}_1, \tilde{\alpha}_2$ and $\tilde{\alpha}_5$, with terms involving $\tilde{\alpha}_3$ and $\tilde{\alpha}_4$ being absent.

For the x–split three dimensional problem the system (11.62)–(11.66) may be viewed in exactly the same manner as for the one–dimensional case. Use of equation (11.62) into (11.64) and (11.65) gives directly

$$\tilde{\alpha}_3 = \Delta u_3 - \tilde{v}\Delta u_1 \ ; \quad \tilde{\alpha}_4 = \Delta u_4 - \tilde{w}\Delta u_1 \ . \tag{11.68}$$

Then one solves (11.62), (11.63) and (11.66) for $\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_5$. Computationally, it is convenient to arrange the solution as follows

$$\left.\begin{aligned} \tilde{\alpha}_2 &= \frac{\gamma - 1}{\tilde{a}^2}\left[\Delta u_1(\tilde{H} - \tilde{u}^2) + \tilde{u}\Delta u_2 - \overline{\Delta u}_5\right] \ , \\ \tilde{\alpha}_1 &= \frac{1}{2\tilde{a}}\left[\Delta u_1(\tilde{u} + \tilde{a}) - \Delta u_2 - \tilde{a}\tilde{\alpha}_2\right] \ , \\ \tilde{\alpha}_5 &= \Delta u_1 - (\tilde{\alpha}_1 + \tilde{\alpha}_2) \ , \end{aligned}\right\} \tag{11.69}$$

where

$$\overline{\Delta u}_5 = \Delta u_5 - (\Delta u_3 - \tilde{v}\Delta u_1)\tilde{v} - (\Delta u_4 - \tilde{w}\Delta u_1)\tilde{w} \ . \tag{11.70}$$

An Algorithm

To compute the Roe numerical flux $\mathbf{F}_{i+\frac{1}{2}}$ according to any of the formulae (11.27)–(11.29) we do the following:

(1) Compute the Roe average values for $\tilde{u}, \tilde{v}, \tilde{w}, \tilde{H}$ and $\tilde{a}$ according to (11.60).
(2) Compute the averaged eigenvalues $\tilde{\lambda}_i$ according to (11.58).
(3) Compute the averaged right eigenvectors $\tilde{\mathbf{K}}^{(i)}$ according to (11.59).
(4) Compute the wave strengths $\tilde{\alpha}_i$ according to (11.68)–(11.70).
(5) Use all of the above quantities to compute $\mathbf{F}_{i+\frac{1}{2}}$, according to any of the formulae (11.27)–(11.29).

For the pure one–dimensional case, virtually all the required information for the application of the above algorithm is contained in this Chapter. An entropy fix is given in Sect. 11.4. The remaining items such as choosing the time step size and boundary conditions are found in Chap. 6. For two and three dimensional applications the reader requires the additional information provided in Chap. 16.

11.3 The Roe–Pike Method

Recall that solving the Riemann problem (11.4) approximately using Roe's method means finding *averaged* eigenvalues $\tilde{\lambda}_i$, right eigenvectors $\tilde{\mathbf{K}}^{(i)}$ and wave strengths $\tilde{\alpha}_i$, so that the Roe numerical flux may be evaluated by any of the formulae (11.27)–(11.29). In the previous section this task was carried out by following the original Roe approach, where the averaged Jacobian matrix $\tilde{\mathbf{A}}$ is first sought. In this section we present a different approach, due to Roe and Pike [416], whereby the construction of $\tilde{\mathbf{A}}$ is avoided; instead, one seeks directly averages of a set of scalar quantities that can then be used to evaluate the eigenvalues, right eigenvectors and wave strengths needed in formulae (11.27)–(11.29).

11.3.1 The Approach

The approach assumes, of course, that the appropriate original system is hyperbolic and that analytical expressions for the eigenvalues λ_i and the set of linearly independent right eigenvectors $\mathbf{K}^{(i)}$ are available. Analytical expressions $\hat{\alpha}_i$ for the wave strengths require extra work via an *extra linearisation*. One then selects a suitable vector of scalar quantities, typically the vector $\mathbf{W}$ of primitive variables in (11.6) or variations of it, for which an average $\tilde{\mathbf{W}}$ is to be found. The values of $\tilde{\lambda}_i$, $\tilde{\mathbf{K}}^{(i)}$ and $\tilde{\alpha}_i$ are then found by direct evaluation of the analytical expressions for λ_i, $\mathbf{K}^{(i)}$ and $\hat{\alpha}_i$ at the state $\tilde{\mathbf{W}}$. There are two distinct steps in the Roe–Pike approach.

Linearisation about a Reference State

To find analytical expressions for the wave strengths α_i Roe and Pike assume a linearised form of the governing equations based on the assumption that the data states $\mathbf{U}_L$ and $\mathbf{U}_R$ are close to a reference state $\hat{\mathbf{U}}$, to order $O(\Delta^2)$. Linearisation of the conservation laws in (11.4) about this state $\hat{\mathbf{U}}$ gives

$$\mathbf{U}_t + \mathbf{F}(\mathbf{U})_x \equiv \mathbf{U}_t + \left(\frac{\partial \mathbf{F}}{\partial \mathbf{U}}\right)\mathbf{U}_x \approx \mathbf{U}_t + \hat{\mathbf{A}}\mathbf{U}_x \ ,$$

where

$$\mathbf{U}_t + \hat{\mathbf{A}}\mathbf{U}_x = \mathbf{0} \tag{11.71}$$

is an approximation to the original conservation laws. Here $\hat{\mathbf{A}}$ is the Jacobian matrix, assumed available, computed at the reference state $\hat{\mathbf{U}}$. Eigenvalues and right eigenvectors follow. Analytical expressions for the wave strengths $\hat{\alpha}_i$ in the solution of the linear Riemann problem

$$\left.\begin{array}{l} \mathbf{U}_t + \hat{\mathbf{A}}\mathbf{U}_x = \mathbf{0} \ , \\ \mathbf{U}(x,0) = \left\{ \begin{array}{l} \mathbf{U}_{\mathrm{L}} \ if \ x < 0 \ , \\ \mathbf{U}_{\mathrm{R}} \ if \ x > 0 \ , \end{array} \right. \end{array}\right\} \tag{11.72}$$

are found by decomposing the data jump $\Delta\mathbf{U}$ onto the right eigenvectors, in the usual way; see Sect. 2.3 of Chap. 2 and Sect. 5.4 of Chap. 5. That is we solve

$$\Delta\mathbf{U} = \mathbf{U}_R - \mathbf{U}_L = \sum_{k=1}^{m} \hat{\alpha}_k \hat{\mathbf{K}}^{(k)} \ . \tag{11.73}$$

Before proceeding, we note that this linearisation *is not* the Roe linearisation resulting from the Roe matrix $\tilde{\mathbf{A}}$; it is merely a step to find some sufficiently simple analytical expressions for the wave strengths, which can then be evaluated at the unknown *Roe–Pike average state* $\tilde{\mathbf{W}}$, yet to be found.

The Algebraic Problem for the Average State

The sought Roe–Pike average vector $\tilde{\mathbf{W}}$ is then found by first setting

$$\tilde{\alpha}_i = \hat{\alpha}_i(\tilde{\mathbf{W}}) \ , \quad \tilde{\lambda}_i = \lambda_i(\tilde{\mathbf{W}}) \ , \quad \tilde{\mathbf{K}}^{(i)} = \mathbf{K}^{(i)}(\tilde{\mathbf{W}}) \ ; \tag{11.74}$$

the analytical expressions for λ_i, $\mathbf{K}^{(i)}$ and $\hat{\alpha}_i$ are evaluated at the unknown average state $\tilde{\mathbf{W}}$. Then $\tilde{\mathbf{W}}$ is found by solving the algebraic problem posed by the following two sets of equations

$$\Delta\mathbf{U} = \mathbf{U}_R - \mathbf{U}_L = \sum_{k=1}^{m} \tilde{\alpha}_k \tilde{\mathbf{K}}^{(k)} \tag{11.75}$$

and

$$\Delta\mathbf{F} = \mathbf{F}_R - \mathbf{F}_L = \sum_{k=1}^{m} \tilde{\alpha}_k \tilde{\lambda}_k \tilde{\mathbf{K}}^{(k)} \ . \tag{11.76}$$

In the following section we illustrate the Roe–Pike approach in terms of a simple system of conservation laws.

11.3.2 The Isothermal Equations

We solve the Riemann problem

$$\left.\begin{array}{l} \mathbf{U}_t + \mathbf{F}(\mathbf{U})_x = \mathbf{0} \ , \\ \mathbf{U}(x,0) = \left\{ \begin{array}{l} \mathbf{U}_{\mathrm{L}} \ if \ x < 0 \ , \\ \mathbf{U}_{\mathrm{R}} \ if \ x > 0 \ , \end{array} \right. \end{array}\right\} \tag{11.77}$$

for the isothermal equations using the Roe–Pike approach; the vectors $\mathbf{U}$ and $\mathbf{F}$ are given in (11.32). The exact Jacobian matrix, eigenvalues and right eigenvectors are

$$\left.\begin{array}{c} \mathbf{A}(\mathbf{U}) = \begin{bmatrix} 0 & 1 \\ a^2 - u^2 & 2u \end{bmatrix} \ , \\ \lambda_1 = u - a \ , \ \lambda_2 = u + a \ , \\ \mathbf{K}^{(1)} = \begin{bmatrix} 1 \\ u - a \end{bmatrix} \ , \quad \mathbf{K}^{(2)} = \begin{bmatrix} 1 \\ u + a \end{bmatrix} \ . \end{array}\right\} \tag{11.78}$$

Linearisation about a Reference State

Assume that the data states $\mathbf{U}_L$ and $\mathbf{U}_R$ are close to a state $\hat{\mathbf{U}}$ to order $O(\Delta^2)$. Linearisation of the conservation laws in (11.77) about this state $\hat{\mathbf{U}}$ gives linear Riemann problem

$$\left.\begin{array}{l} \mathbf{U}_t + \hat{\mathbf{A}}\mathbf{U}_x = \mathbf{0}\ , \\ \mathbf{U}(x,0) = \left\{ \begin{array}{l} \mathbf{U}_{\mathrm{L}}\ if\ x < 0\ , \\ \mathbf{U}_{\mathrm{R}}\ if\ x > 0\ . \end{array} \right. \end{array} \right\} \tag{11.79}$$

Here $\hat{\mathbf{A}}$ is the Jacobian $\mathbf{A}$ evaluated at the reference state $\hat{\mathbf{U}}$, which in terms of primitive variables is denoted by $\hat{\mathbf{W}} = (\hat{\rho}, \hat{u})^T$. The complete eigenstructure is

$$\left.\begin{array}{c} \mathbf{A}(\mathbf{U}) = \begin{bmatrix} 0 & 1 \\ a^2 - \hat{u}^2 & 2\hat{u} \end{bmatrix}\ , \\ \hat{\lambda}_1 = \hat{u} - a\ ,\ \hat{\lambda}_2 = \hat{u} + a\ , \\ \hat{\mathbf{K}}^{(1)} = \begin{bmatrix} 1 \\ \hat{u} - a \end{bmatrix}\ ,\ \ \hat{\mathbf{K}}^{(2)} = \begin{bmatrix} 1 \\ \hat{u} + a \end{bmatrix}\ . \end{array} \right\} \tag{11.80}$$

Recall that the sound speed a is constant. We look for solutions of (11.79). The system is linear with constant coefficients. One can therefore deploy appropriate techniques studied in Sect. 2.3 of Chap. 2 and Sect. 5.4 of Chap. 5. We decompose the data jump $\Delta\mathbf{U}$ onto the right eigenvectors as follows

$$\Delta\mathbf{U} = \mathbf{U}_R - \mathbf{U}_L = \sum_{k=1}^{2} \hat{\alpha}_k \hat{\mathbf{K}}^{(k)} = \hat{\alpha}_1 \hat{\mathbf{K}}^{(1)} + \hat{\alpha}_2 \hat{\mathbf{K}}^{(2)}\ , \tag{11.81}$$

where analytical expressions for the coefficients $\hat{\alpha}_1$, $\hat{\alpha}_2$ are to be found. Writing (11.81) in full gives

$$\Delta\rho = \rho_R - \rho_L = \hat{\alpha}_1 + \hat{\alpha}_2\ , \tag{11.82}$$

$$\Delta(\rho u) = (\rho u)_R - (\rho u)_L = \hat{\alpha}_1(\hat{u} - a) + \hat{\alpha}_2(\hat{u} + a)\ . \tag{11.83}$$

It can easily be shown that

$$\Delta(\rho u) = \hat{\rho}\Delta u + \hat{u}\Delta\rho + O(\Delta^2)\ , \tag{11.84}$$

where the leading term in $O(\Delta^2)$ is

$$(\rho_R - \hat{\rho})(u_R - \hat{u}) - (\rho_L - \hat{\rho})(u_L - \hat{u})\ .$$

By neglecting $O(\Delta^2)$, (11.83) becomes

$$\hat{\rho}\Delta u + \hat{u}\Delta\rho = \hat{\alpha}_1(\hat{u} - a) + \hat{\alpha}_2(\hat{u} + a)\ . \tag{11.85}$$

Solving equations (11.82) and (11.85) gives the sought analytical expressions for $\hat{\alpha}_1$ and $\hat{\alpha}_2$, namely

$$\hat{\alpha}_1 = \frac{1}{2}\left[\Delta\rho - \hat{\rho}\frac{\Delta u}{a}\right] , \quad \hat{\alpha}_2 = \frac{1}{2}\left[\Delta\rho + \hat{\rho}\frac{\Delta u}{a}\right] . \tag{11.86}$$

Compare these with expressions (11.46). The reader may easily verify that, to within $O(\Delta^2)$, the following two sets of equations are identically satisfied

$$\Delta\mathbf{U} = \mathbf{U}_R - \mathbf{U}_L = \sum_{k=1}^{2} \hat{\alpha}_k \hat{\mathbf{K}}^{(k)} , \quad \Delta\mathbf{F} = \mathbf{F}_R - \mathbf{F}_L = \sum_{k=1}^{2} \hat{\alpha}_k \hat{\lambda}_k \hat{\mathbf{K}}^{(k)} . \tag{11.87}$$

Here we give details for the second set. In full, these equations read

$$\Delta(\rho u) = \hat{\alpha}_1 \hat{\lambda}_1 + \hat{\alpha}_2 \hat{\lambda}_2 , \tag{11.88}$$

$$\Delta(\rho u^2 + \rho a^2) = \hat{\alpha}_1 \hat{\lambda}_1 (\hat{u} - a) + \hat{\alpha}_2 \hat{\lambda}_2 (\hat{u} + a) . \tag{11.89}$$

Equation (11.88) may be written as

$$\hat{\rho}\Delta u + \hat{u}\Delta\rho = \hat{u}(\hat{\alpha}_1 + \hat{\alpha}_2) + a(\hat{\alpha}_2 - \hat{\alpha}_1) ,$$

which after using (11.86) becomes an identity. To prove (11.89) we first expand its left–hand side

$$\Delta(\rho u^2 + \rho a^2) = 2\hat{\rho}\hat{u}\Delta u + \hat{u}^2\Delta\rho + a^2\Delta\rho .$$

The right–hand side of (11.89) can be expressed as

$$(\hat{\alpha}_1 + \hat{\alpha}_2)(\hat{u}^2 + a^2) + 2\hat{u}a(\hat{\alpha}_2 - \hat{\alpha}_1) .$$

Therefore, after use of (11.86), equation (11.89) becomes an identity and thus the second set of equations in (11.87), to order $O(\Delta^2)$, is identically satisfied.

The Algebraic Problem for the Average State

For the general case in which the data states $\mathbf{U}_L$ and $\mathbf{U}_R$ are not necessarily close, the Roe–Pike approach proposes the algebraic problem of finding the Roe–Pike averages $\tilde{\rho}$ and $\tilde{u}$ such that the two conditions (11.75) and (11.76) are valid, namely

$$\Delta\mathbf{U} = \sum_{k=1}^{2} \tilde{\alpha}_k \tilde{\mathbf{K}}^{(k)} , \quad \Delta\mathbf{F} = \sum_{k=1}^{2} \tilde{\alpha}_k \tilde{\lambda}_k \tilde{\mathbf{K}}^{(k)} . \tag{11.90}$$

Here, according to (11.74), $\tilde{\alpha}_k, \tilde{\lambda}_k$ and $\tilde{\mathbf{K}}^{(k)}$ are obtained by evaluating the available analytical expressions at the sought averages $\tilde{\rho}, \tilde{u}$. For the wave strengths these are given by (11.86). For the eigenvalues and right eigenvectors they are given by (11.78). We then set

$$\tilde{\alpha}_1 = \frac{1}{2}\left[\Delta\rho - \tilde{\rho}\frac{\Delta u}{a}\right] , \quad \tilde{\alpha}_2 = \frac{1}{2}\left[\Delta\rho + \tilde{\rho}\frac{\Delta u}{a}\right] , \tag{11.91}$$

$$\tilde{\lambda}_1 = \tilde{u} - a \ , \quad \tilde{\lambda}_2 = \tilde{u} + a \ , \tag{11.92}$$

$$\tilde{\mathbf{K}}^{(1)} = \begin{bmatrix} 1 \\ \tilde{u} - a \end{bmatrix} \ , \quad \tilde{\mathbf{K}}^{(2)} = \begin{bmatrix} 1 \\ \tilde{u} + a \end{bmatrix} \ . \tag{11.93}$$

Writing conditions (11.90) in full produces

$$\Delta\rho = \tilde{\alpha}_1 + \tilde{\alpha}_2 \ , \tag{11.94}$$

$$\Delta(\rho u) = \tilde{\alpha}_1(\tilde{u} - a) + \tilde{\alpha}_2(\tilde{u} + a) \ , \tag{11.95}$$

$$\Delta(\rho u) = \tilde{\lambda}_1\tilde{\alpha}_1 + \tilde{\lambda}_2\tilde{\alpha}_2 \ , \tag{11.96}$$

$$\Delta(\rho u^2 + a^2\rho) = \tilde{\lambda}_1\tilde{\alpha}_1(\tilde{u} - a) + \tilde{\lambda}_2\tilde{\alpha}_2(\tilde{u} + a) \ . \tag{11.97}$$

These are a set of four non–linear algebraic equations for the two unknowns $\tilde{\rho}$ and $\tilde{u}$. Note however that, by virtue of (11.91), (11.94) is an identity, for any average value $\tilde{\rho}$. Also, (11.95) is identical to (11.96) and thus we work with (11.96) and (11.97) only. From equation (11.96) one obtains

$$\Delta(\rho u) = \tilde{u}(\tilde{\alpha}_1 + \tilde{\alpha}_2) + a(\tilde{\alpha}_2 - \tilde{\alpha}_1).$$

Use of (11.91) here leads to

$$\Delta(\rho u) = \tilde{\rho}\Delta u + \tilde{u}\Delta\rho \ . \tag{11.98}$$

From (11.97) we write

$$\Delta(\rho u^2 + \rho a^2) = (\tilde{\alpha}_1 + \tilde{\alpha}_2)(\tilde{u}^2 + a^2) + 2a\tilde{u}(\tilde{\alpha}_2 - \tilde{\alpha}_1) \ ,$$

which after using (11.91) and the exact relation

$$\Delta(\rho u^2 + \rho a^2) = \Delta(\rho u^2) + a^2\Delta\rho$$

leads to the result

$$\Delta(\rho u^2) = 2\tilde{u}\tilde{\rho}\Delta u + \tilde{u}^2\Delta\rho \ . \tag{11.99}$$

Elimination of $\tilde{\rho}$ from (11.98) and (11.99) leads to a quadratic equation for $\tilde{u}$, namely

$$\Delta\rho\tilde{u}^2 - 2\Delta(\rho u)\tilde{u} + \Delta(\rho u^2) = 0 \ . \tag{11.100}$$

This equation has two solutions, namely

$$\tilde{u} = \frac{\Delta(\rho u) \pm \sqrt{[\Delta(\rho u)]^2 - \Delta\rho\Delta(\rho u^2)}}{\Delta\rho} \ . \tag{11.101}$$

After using the definition $\Delta r = r_R - r_L$ the discriminant is found to be

$$\rho_L\rho_R(\Delta u)^2 \ ,$$

which simplifies (11.101) to

$$\tilde{u} = \frac{\Delta(\rho u) \pm \Delta u \sqrt{\rho_L \rho_R}}{\Delta \rho} \ . \tag{11.102}$$

The root obtained by taking the *negative* sign in (11.102) produces the Roe–averaged velocity

$$\tilde{u} = \frac{\sqrt{\rho_L} u_L + \sqrt{\rho_R} u_R}{\sqrt{\rho_L} + \sqrt{\rho_R}} \ . \tag{11.103}$$

Compare (11.103) with (11.43). From (11.98) we obtain

$$\tilde{\rho} = \sqrt{\rho_L \rho_R} \ . \tag{11.104}$$

We have thus found algebraic expressions for the sought Roe–Pike averages $\tilde{\rho}$ and $\tilde{u}$. We observe that the second root obtained by taking the *positive* sign in (11.102) leads to the *spurious* solution

$$\tilde{u} = \frac{\sqrt{\rho_R} u_R - \sqrt{\rho_L} u_L}{\sqrt{\rho_R} - \sqrt{\rho_L}} \ . \tag{11.105}$$

There is a very good reason for rejecting this as a *useful solution*; in the trivial case $\rho_L = \rho_R$, $u_L \neq u_R$ the solution $\tilde{u}$ is not even defined.

Having found the Roe–Pike averages $\tilde{\rho}$ and $\tilde{u}$ we can then compute the wave strengths $\tilde{\alpha}_k$, the eigenvalues $\tilde{\lambda}_k$ and the right eigenvectors $\tilde{\mathbf{K}}^{(k)}$ according to expressions (11.91)–(11.93). The Roe numerical flux $\mathbf{F}_{i+\frac{1}{2}}$ to be used in the conservative formula (11.2) can now be obtained from any of the relations (11.27)–(11.29).

11.3.3 The Euler Equations

We solve the Riemann problem (11.4) for the x–split, three dimensional Euler equations using the Roe–Pike method. Assuming the analytical expressions (11.48)–(11.49) for the eigenvalues and eigenvectors, one then linearises the equations about a state $\hat{\mathbf{U}}$ to find analytical expressions for the wave strengths; this is done under the assumption that both data states $\mathbf{U}_L, \mathbf{U}_R$ are close to $\hat{\mathbf{U}}$ to $O(\Delta^2)$. This leads to the linear system

$$\left.\begin{aligned} &\mathbf{U}_t + \hat{\mathbf{A}} \mathbf{U}_x = \mathbf{0} \ , \\ &\mathbf{U}(x,t) = \begin{cases} \mathbf{U}_L \ , \ x < 0 \ , \\ \mathbf{U}_R \ , \ x > 0 \ . \end{cases} \end{aligned}\right\} \tag{11.106}$$

The Jacobian matrix $\hat{\mathbf{A}}$ is obtained by evaluating the exact Jacobian matrix (11.47) at the state $\hat{\mathbf{U}}$; the eigenvalues $\hat{\lambda}_i$ are

$$\hat{\lambda}_1 = \hat{u} - \hat{a} \ , \quad \hat{\lambda}_2 = \hat{\lambda}_3 = \hat{\lambda}_4 = \hat{u} \ , \quad \hat{\lambda}_5 = \hat{u} + \hat{a} \tag{11.107}$$

and the right eigenvectors $\hat{\mathbf{K}}^{(i)}$ are

$$\left.\begin{aligned}
\hat{\mathbf{K}}^{(1)} &= \begin{bmatrix} 1 \\ \hat{u}-\hat{a} \\ \hat{v} \\ \hat{w} \\ \hat{H}-\hat{u}\hat{a} \end{bmatrix} \;;\; \hat{\mathbf{K}}^{(2)} = \begin{bmatrix} 1 \\ \hat{u} \\ \hat{v} \\ \hat{w} \\ \frac{1}{2}\hat{V}^2 \end{bmatrix} \;;\; \hat{\mathbf{K}}^{(3)} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \hat{v} \end{bmatrix} \\
\hat{\mathbf{K}}^{(4)} &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ \hat{w} \end{bmatrix} \;;\; \hat{\mathbf{K}}^{(5)} = \begin{bmatrix} 1 \\ \hat{u}+\hat{a} \\ \hat{v} \\ \hat{w} \\ \hat{H}+\hat{u}\hat{a} \end{bmatrix} .
\end{aligned}\right\} \tag{11.108}$$

By expanding the data jump $\Delta\mathbf{U}$ onto the right eigenvectors we write

$$\Delta\mathbf{U} = \sum_{i=1}^{5} \hat{\alpha}_i \hat{\mathbf{K}}^{(i)} \tag{11.109}$$

The solution of this 5×5 linear system will provide analytical expressions for the wave strengths $\hat{\alpha}_i$. As a matter of fact we can use the solution for the wave strengths obtained in the Roe original method, (11.68)–(11.70), and reinterpret the solution appropriately. These are

$$\left.\begin{aligned}
\hat{\alpha}_3 &= \Delta u_3 - \hat{v}\Delta u_1 \;, \\
\hat{\alpha}_4 &= \Delta u_4 - \hat{w}\Delta u_1 \;, \\
\hat{\alpha}_2 &= \tfrac{\gamma-1}{\hat{a}^2}[\Delta u_1(\hat{H}-\hat{u}^2) + \hat{u}\Delta u_2 - \overline{\Delta u}_5] \;, \\
\hat{\alpha}_1 &= \tfrac{1}{2\hat{a}}[\Delta u_1(\hat{u}+\hat{a}) - \Delta u_2 - \hat{a}\hat{\alpha}_2] \;, \\
\hat{\alpha}_5 &= \Delta u_1 - (\hat{\alpha}_1 + \hat{\alpha}_2) \;,
\end{aligned}\right\} \tag{11.110}$$

where

$$\overline{\Delta u}_5 = \Delta u_5 - (\Delta u_3 - \hat{v}\Delta u_1)\hat{v} - (\Delta u_4 - \hat{w}\Delta u_1)\hat{w} \;. \tag{11.111}$$

By applying the operator

$$\Delta(rs) = \hat{r}\Delta s + \hat{s}\Delta r + O(\Delta^2) \tag{11.112}$$

and neglecting $O(\Delta^2)$ we arrive at the following solution:

$$\left.\begin{aligned}
\hat{\alpha}_1 &= \frac{1}{2\hat{a}^2}\left[\Delta p - \hat{\rho}\hat{a}\Delta u\right] \;, \\
\hat{\alpha}_2 &= \Delta\rho - \Delta p/\hat{a}^2 \;, \\
\hat{\alpha}_3 &= \hat{\rho}\Delta v \;, \\
\hat{\alpha}_4 &= \hat{\rho}\Delta w \;, \\
\hat{\alpha}_5 &= \frac{1}{2\hat{a}^2}\left[\Delta p + \hat{\rho}\hat{a}\Delta u\right]
\end{aligned}\right\} \tag{11.113}$$

The second step in the Roe–Pike method is to find an average state

$$\tilde{\mathbf{W}} = (\tilde{\rho}, \tilde{u}, \tilde{v}, \tilde{w}, \tilde{a})^T \,, \tag{11.114}$$

such that the algebraic problem posed by the following two sets of equations

$$\Delta\mathbf{U} = \sum_{i=1}^{5} \tilde{\alpha}_i \tilde{\mathbf{K}}^{(i)} \,, \tag{11.115}$$

$$\Delta\mathbf{F} = \sum_{i=1}^{5} \tilde{\alpha}_i \tilde{\lambda}_i \tilde{\mathbf{K}}^{(i)} \,, \tag{11.116}$$

is satisfied, where

$$\tilde{\alpha}_i = \hat{\alpha}_i(\tilde{\mathbf{W}}) \,, \quad \tilde{\lambda}_i = \lambda_i(\tilde{\mathbf{W}}) \,, \quad \tilde{\mathbf{K}}^{(i)} = \mathbf{K}^{(i)}(\tilde{\mathbf{W}}) \,, \tag{11.117}$$

with λ_i and $\mathbf{K}^{(i)}$ given by (11.48)–(11.49) and $\hat{\alpha}_i$ given by (11.113). Details of the algebra for the one–dimensional case are given by Roe and Pike [416]. For the x–split three dimensional case the solution for the average vector $\tilde{\mathbf{W}}$ is

$$\left.\begin{aligned}
\tilde{\rho} &= \sqrt{\rho_L \rho_R} \,, \\
\tilde{u} &= \frac{\sqrt{\rho_L} u_L + \sqrt{\rho_R} u_R}{\sqrt{\rho_L} + \sqrt{\rho_R}} \,, \\
\tilde{v} &= \frac{\sqrt{\rho_L} v_L + \sqrt{\rho_R} v_R}{\sqrt{\rho_L} + \sqrt{\rho_R}} \,, \\
\tilde{w} &= \frac{\sqrt{\rho_L} w_L + \sqrt{\rho_R} w_R}{\sqrt{\rho_L} + \sqrt{\rho_R}} \,, \\
\tilde{H} &= \frac{\sqrt{\rho_L} H_L + \sqrt{\rho_R} H_R}{\sqrt{\rho_L} + \sqrt{\rho_R}} \,, \\
\tilde{a} &= \left((\gamma - 1)(\tilde{H} - \tfrac{1}{2}\tilde{\mathbf{V}}^2) \right)^{\frac{1}{2}} \,,
\end{aligned}\right\} \tag{11.118}$$

where $\tilde{\mathbf{V}}^2 = \tilde{u}^2 + \tilde{v}^2 + \tilde{w}^2$. These are identical to the Roe averages obtained by the original Roe method, see (11.60). Now $\tilde{\alpha}_i, \tilde{\lambda}_i$ and $\tilde{\mathbf{K}}^{(i)}$ are computed according to (11.117) and then the Roe intercell flux $\mathbf{F}_{i+\frac{1}{2}}$ follows from any of the formulae (11.27)–(11.29).

An Algorithm

To compute the Roe numerical flux $\mathbf{F}_{i+\frac{1}{2}}$ according to any of the formulae (11.27)–(11.29) we do the following:

(1) Compute the Roe average values according to (11.118).
(2) Compute the eigenvalues $\tilde{\lambda}_i$ using the analytical expressions (11.107) evaluated on the averages (11.118).
(3) Compute the right eigenvectors using the analytical expressions (11.108) evaluated on the averages (11.118).
(4) Compute the wave strengths using the analytical expressions (11.113) evaluated on the averages (11.118).
(5) Use all of the above quantities to compute $\mathbf{F}_{i+\frac{1}{2}}$, according to any of the formulae (11.27)–(11.29).

Before applying the scheme as described to practical problems, a modification to handle *sonic flow* correctly is required. This is the subject of the next section.

11.4 An Entropy Fix

Linearised Riemann problem solutions consist of discontinuous jumps only. See Sect. 2.3 of Chap. 2. This can be a good approximation for contacts and shocks, in that the discontinuous character of the wave is correct, although the size of the jump may not be correctly approximated by the linearised solution. Rarefaction waves, on the other hand, carry a continuous change in flow variables, and as time increases, they tend to spread; that is spatial gradients tend to decay. Quite clearly then, the linearised approximation via discontinuous jumps is grossly incorrect. In a practical computational set up however, it is only in the case in which the rarefaction wave is *transonic*, or *sonic*, where linearised approximations encounter difficulties; these show up in the form of unphysical, entropy violating discontinuous waves, sometimes called *rarefaction shocks*.

11.4.1 The Entropy Problem

Consider the Riemann problem whose initial data is that of Test 1 in Table 11.1. The structure of the exact solution of this problem, depicted in Fig. 11.2, contains a left *sonic rarefaction*, a contact discontinuity of speed u_* and a right shock wave. As the left rarefaction is *sonic* the eigenvalue $\lambda_1 = u - a$ changes from negative to positive, as the wave is crossed from left to right. There is a point at which $\lambda_1 = u - a = 0$, giving the sonic flow condition $u = a$.

$$\lambda_1(\mathbf{U}_L) = S_{HL} = u_L - a_L < 0$$

is the speed of the *head* of the rarefaction and

$$\lambda_1(\mathbf{U}_{*L}) = S_{TL} = u_* - a_{*L} > 0$$

is the speed of the *tail*. Fig. 11.4 shows the numerical (symbols) and exact (line) solutions of this problem, where the numerical solution is obtained by

Roe's method as described so far. The numerical solution within the rarefaction exhibits a discontinuity within the wave; this discontinuity is unphysical, it violates the entropy condition. See Sect. 2.4.2 of Chap. 2. Recall that a physically admissible discontinuity of speed S requires $S_b \geq S \geq S_a$ where S_b and S_a are characteristic speeds behind and ahead of the wave respectively. That is, characteristics move into the discontinuity; the limiting case of parallel characteristic speeds is that of a contact discontinuity. For the example above, the opposite happens. See Sect. 2.4.2 of Chap. 2, for a discussion on entropy–violating solutions.

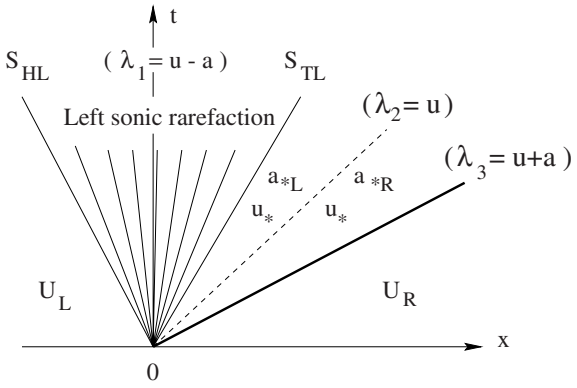

Fig. 11.2. Left transonic rarefaction wave. Left eigenvalue $\lambda_1 = u - a$ changes sign as the wave is crossed from left to right

Roe's solver can be modified so as to avoid entropy violating solutions. This is usually referred to as an *entropy fix*. Harten and Hyman [243] suggested an entropy fix for Roe's method, which has widespread use. Other ways of correcting the scheme have been discussed by Roe and Pike [416], Roe [414], Sweby [469] and Dubois and Mehlman [167], amongst others. Here we present the details of the Harten–Hyman approach.

11.4.2 The Harten–Hyman Entropy Fix

The general approach is presented in the original paper of Harten and Hyman of 1983 [243]. A description can also be found in [308]. The presentation here is tailored specifically to the time–dependent Euler equations, for which we only need to consider the left and right non–linear waves associated with the eigenvalues $\lambda_1 = u - a$ and $\lambda_5 = u + a$ respectively. Our version of the Harten–Hyman entropy fix relies on estimates for particle velocity u_* and sound speeds a_{*L}, a_{*R} in the *Star Region*; see Figs. 11.1 and 11.2. Various ways of finding these are given in Sect. 11.4.3.

Left Transonic Rarefaction

Consider the situation depicted in Fig. 11.2. Assuming u_* and a_{*L} are available, we compute the speeds

$$\lambda_1^L = u_L - a_L \; ; \quad \lambda_1^R = u_* - a_{*L} \; . \tag{11.119}$$

If

$$\lambda_1^L < 0 < \lambda_1^R \; , \tag{11.120}$$

then the left wave is a *transonic, or sonic, rarefaction wave*. In these circumstances the entropy fix is required and is enforced as follows. The single jump

$$\mathbf{U}_{*L} - \mathbf{U}_L = \tilde{\alpha}_1 \tilde{\mathbf{K}}^{(1)} \tag{11.121}$$

travelling with speed $\tilde{\lambda}_1$ is split into two smaller jumps $\mathbf{U}_{SL} - \mathbf{U}_L$ and $\mathbf{U}_{*L} - \mathbf{U}_{SL}$ travelling respectively at speeds λ_1^L and λ_1^R, where $\mathbf{U}_{SL}$ is a *transonic state* yet to be found; see Fig. 11.3. Application of the integral form of the conservation laws, see Chaps. 3 and 10, gives

$$\lambda_1^R(\mathbf{U}_{SL} - \mathbf{U}_{*L}) + \lambda_1^L(\mathbf{U}_L - \mathbf{U}_{SL}) = \tilde{\lambda}_1(\mathbf{U}_L - \mathbf{U}_{*L}) \; , \tag{11.122}$$

from which we obtain

$$\mathbf{U}_{SL} = \frac{(\tilde{\lambda}_1 - \lambda_1^L)\mathbf{U}_L + (\lambda_1^R - \tilde{\lambda}_1)\mathbf{U}_{*L}}{\lambda_1^R - \lambda_1^L} \; . \tag{11.123}$$

To compute the Roe intercell flux we adopt the one–sided formulae (11.27), namely

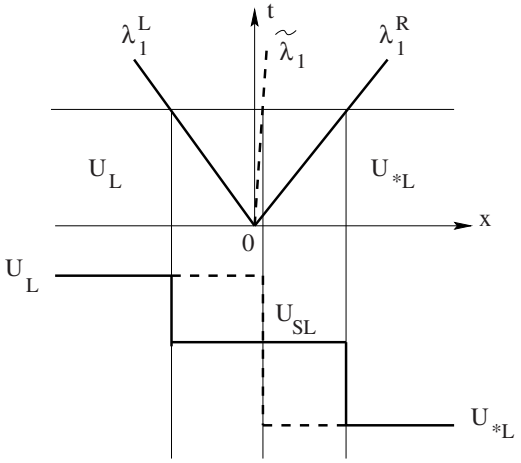

Fig. 11.3. Entropy fix for left transonic rarefaction wave. Single jump $\mathbf{U}_{*L} - \mathbf{U}_L$ travelling with speed $\tilde{\lambda}_1$ is split into the two jumps $\mathbf{U}_{SL} - \mathbf{U}_L$ and $\mathbf{U}_{*L} - \mathbf{U}_{SL}$ travelling with speeds λ_1^L and λ_1^R. Profile shown is a representation a single variable

$$\mathbf{F}_{i+\frac{1}{2}} = \mathbf{F}_L + \sum_{\tilde{\lambda}_k \le 0} \tilde{\lambda}_k \tilde{\alpha}_k \tilde{\mathbf{K}}^{(k)} , \tag{11.124}$$

where in the present case the summation applies to the single jump $\mathbf{U}_{SL} - \mathbf{U}_L$ travelling with speed $\lambda_1^L < 0$; in view of (11.122) the jump is

$$\mathbf{U}_{SL} - \mathbf{U}_L = \frac{(\lambda_1^R - \tilde{\lambda}_1)}{(\lambda_1^R - \lambda_1^L)} (\mathbf{U}_{*L} - \mathbf{U}_L) . \tag{11.125}$$

But the Roe approximation gives

$$\mathbf{U}_{*L} - \mathbf{U}_L = \tilde{\alpha}_1 \tilde{\mathbf{K}}^{(1)} \tag{11.126}$$

and thus the flux jump $(\Delta \mathbf{F})_1^L$ across the wave of speed λ_1^L is

$$(\Delta \mathbf{F})_1^L = \lambda_1^L \left(\frac{\lambda_1^R - \tilde{\lambda}_1}{\lambda_1^R - \lambda_1^L} \right) \tilde{\alpha}_1 \tilde{\mathbf{K}}^{(1)} . \tag{11.127}$$

By defining the new wave speed

$$\overline{\lambda}_1 = \lambda_1^L \left(\frac{\lambda_1^R - \tilde{\lambda}_1}{\lambda_1^R - \lambda_1^L} \right) , \tag{11.128}$$

the intercell flux (11.27) becomes

$$\mathbf{F}_{i+\frac{1}{2}} = \mathbf{F}_L + \overline{\lambda}_1 \tilde{\alpha}_1 \tilde{\mathbf{K}}^{(1)} . \tag{11.129}$$

Right Transonic Rarefaction

For a right transonic rarefaction, the entropy fix procedure is entirely analogous to the left rarefaction case. Assuming the speeds u_* and a_{*R} are available, we compute the two wave speeds

$$\lambda_5^L = u_* + a_{*R} , \quad \lambda_5^R = u_R + a_R . \tag{11.130}$$

If

$$\lambda_5^L < 0 < \lambda_5^R \tag{11.131}$$

then the right wave is a *transonic rarefaction wave*. The transonic state $\mathbf{U}_{SR}$ is defined between the waves of speeds λ_5^L and λ_5^R and is given by

$$\mathbf{U}_{SR} = \frac{(\lambda_5^R - \tilde{\lambda}_5)\mathbf{U}_R + (\tilde{\lambda}_5 - \lambda_5^L)\mathbf{U}_{*R}}{\lambda_5^R - \lambda_5^L} . \tag{11.132}$$

Next we define the new wave speed

$$\overline{\lambda}_5 = \lambda_5^R \left(\frac{\tilde{\lambda}_5 - \lambda_5^L}{\lambda_5^R - \lambda_5^L} \right) \tag{11.133}$$

and then use the one–sided flux formula (11.28) to compute the numerical flux. The resulting Roe numerical flux is

$$\mathbf{F}_{i+\frac{1}{2}} = \mathbf{F}_R - \overline{\lambda}_5 \tilde{\alpha}_5 \tilde{\mathbf{K}}^{(5)} \ . \tag{11.134}$$

In the present version of the Harten–Hyman entropy fix we have used the one–sided flux formulae (11.27) and (11.28). The procedure can be easily adapted for use in conjunction with the centred formulae (11.29), if desired.

Next we discuss ways of finding the speeds u_*, a_{*L} and a_{*R} needed to implement the entropy fix.

11.4.3 The Speeds u_*, a_{*L}, a_{*R}

The *star* states $\mathbf{U}_{*L}$, $\mathbf{U}_{*R}$ are required in order to obtain the speeds u_*, a_{*L}, a_{*R} and thus the characteristic speeds in (11.119) and (11.130). We present various possible alternatives.

The Roe–Averaged States

Given the Roe–averaged $\tilde{\alpha}_i$ and $\tilde{\mathbf{K}}^{(i)}$ one can find the state $\mathbf{U}_{*L}$ as

$$\mathbf{U}_{*L} = \mathbf{U}_L + \tilde{\alpha}_1 \tilde{\mathbf{K}}^{(1)} \ , \tag{11.135}$$

which leads to

$$\left.\begin{aligned} &\rho_{*L} = \rho_L + \tilde{\alpha}_1 \ , \quad u_* = \frac{\rho_L u_L + \tilde{\alpha}_1(\tilde{u} - \tilde{a})}{\rho_L + \tilde{\alpha}_1} \ , \\ &p_* = (\gamma - 1)\left[E_L + \tilde{\alpha}_1(\tilde{H} - \tilde{u}\tilde{a}) - \tfrac{1}{2}\rho_{*L} u_*^2\right] \ . \end{aligned}\right\} \tag{11.136}$$

Then we compute the sound speed

$$a_{*L} = \sqrt{\frac{\gamma p_*}{\rho_{*L}}} \tag{11.137}$$

and thus the speeds λ_1^L and λ_1^R in Eq. (11.119) follow. For the right wave one has

$$\mathbf{U}_{*R} = \mathbf{U}_R - \tilde{\alpha}_5 \tilde{\mathbf{K}}^{(5)} \ , \tag{11.138}$$

which produces

$$\left.\begin{aligned} &\rho_{*R} = \rho_R - \tilde{\alpha}_5 \ , \quad u_* = \frac{\rho_R u_R - \tilde{\alpha}_5(\tilde{u} + \tilde{a})}{\rho_R - \tilde{\alpha}_5} \ , \\ &p_* = (\gamma - 1)\left[E_R - \tilde{\alpha}_5(\tilde{H} + \tilde{u}\tilde{a}) - \tfrac{1}{2}\rho_{*R} u_*^2\right] \ . \end{aligned}\right\} \tag{11.139}$$

The sound speed follows as $a_{*R} = \sqrt{\frac{\gamma p_*}{\rho_{*R}}}$ and thus the wave speeds λ_5^L and λ_5^R in Eq. (11.130) are determined.

The PVRS Approximation

Another way of estimating the required wave speeds is by using the Primitive–Variable Riemann Solver (PVRS) of Toro [502] presented in Sect. 9.3 of Chap. 9. The relevant solution values are

$$\left.\begin{array}{rcll} p_* &=& \frac{1}{2}(p_L + p_R) &+ \frac{1}{2}(u_L - u_R)\bar{\rho}\bar{a}\ , \\ u_* &=& \frac{1}{2}(u_L + u_R) &+ \frac{1}{2}(p_L - p_R)/(\bar{\rho}\bar{a})\ , \\ \rho_{*L} &=& \rho_L &+ (u_L - u_*)\bar{\rho}/\bar{a}\ , \\ \rho_{*R} &=& \rho_R &+ (u_* - u_R)\bar{\rho}/\bar{a}\ , \end{array}\right\} \tag{11.140}$$

with

$$\bar{\rho} = \frac{1}{2}(\rho_L + \rho_R)\ ,\ \ \bar{a} = \frac{1}{2}(a_L + a_R). \tag{11.141}$$

In order to avoid negative pressures we recommend replacing the linearised solution p_* by $\max\{0, p_*\}$. The sound speeds a_{*L}, a_{*R} are then computed in the usual way.

TRRS Approximation

Another possibility is to use the Two–Rarefaction Riemann Solver (TRRS) discussed in Chap. 9, Sect. 9.4.1. The pressure p_* is given by

$$p_* = \left[\frac{a_L + a_R - \frac{\gamma-1}{2}(u_R - u_L)}{a_L/p_L^z + a_R/p_R^z}\right]^{\frac{1}{z}}\ , \tag{11.142}$$

with $z = \frac{\gamma-1}{2\gamma}$. For the left non–linear wave the sound speed and particle velocity follow directly as

$$a_{*L} = a_L(p_*/p_L)^z\ ,\ \ u_* = u_L + \frac{2}{(\gamma-1)}(a_L - a_{*L})\ . \tag{11.143}$$

For the right non–linear wave we have

$$a_{*R} = a_R(p_*/p_R)^z\ ,\ \ u_* = u_R + \frac{2}{(\gamma-1)}(a_{*R} - a_R)\ . \tag{11.144}$$

Hence speeds (11.119) and (11.130) are determined.

Other Alternatives

Both the PVRS and the Roe linearised solutions for the speeds u_*, a_{*L}, a_{*R} may fail in the vicinity of low density flow [182]. The TRRS approximation presented above would not suffer from such difficulties; in fact, in the case in which both non–linear waves are rarefactions such an approximation would be exact. But as seen in equations (11.142)–(11.144) there are four fractional powers to be computed in each case, which makes this approximation rather expensive to use. A robust and yet more efficient scheme is the Two–Shock Riemann Solver (TSRS) [509] of Sect. 9.4.2, Chap. 9. Even better is the adaptive Riemann solver scheme of Sect. 9.5.2, Chap. 9.

11.5 Numerical Results and Discussion

Here we illustrate the performance of Godunov's first–order upwind method used in conjunction with the Roe approximate Riemann solver, discuss the results and point directions for extending the method.

11.5.1 The Tests

We select five test problems for the one–dimensional, time dependent Euler equations for ideal gases with $\gamma = 1.4$; these have exact solutions. In all chosen tests, data consists of two constant states $\mathbf{W}_{\mathrm{L}} = [\rho_{\mathrm{L}}, u_{\mathrm{L}}, p_{\mathrm{L}}]^T$ and $\mathbf{W}_{\mathrm{R}} = [\rho_{\mathrm{R}}, u_{\mathrm{R}}, p_{\mathrm{R}}]^T$, separated by a discontinuity at a position $x = x_0$. The states $\mathbf{W}_{\mathrm{L}}$ and $\mathbf{W}_{\mathrm{R}}$ are given in Table 11.1. The exact and numerical solutions are found in the spatial domain $0 \leq x \leq 1$. The numerical solution is computed with $M = 100$ cells and the CFL condition is as for all previous computations, see Chap. 6; the chosen Courant number coefficient is $C_{\mathrm{cfl}} = 0.9$; boundary conditions are transmissive.

The exact solutions were found by running the code HE-E1RPEXACT of the library *NUMERICA* [519] and the numerical solutions were obtained by running the code HE-E1GODFLUX of *NUMERICA*.

Test	ρ_{L}	u_{L}	p_{L}	ρ_{R}	u_{R}	p_{R}
1	1.0	0.75	1.0	0.125	0.0	0.1
2	1.0	-2.0	0.4	1.0	2.0	0.4
3	1.0	0.0	1000.0	1.0	0.0	0.01
4	5.99924	19.5975	460.894	5.99242	-6.19633	46.0950
5	1.0	-19.59745	1000.0	1.0	-19.59745	0.01

Table 11.1. Data for five test problems with exact solution, for the time–dependent, one dimensional Euler equations

Test 1 is a *modified* version of Sod's problem [453]; the solution has a right shock wave, a right travelling contact wave and a left *sonic* rarefaction wave; this test is useful in assessing the *entropy satisfaction* property of numerical methods. The solution of Test 2 consists of two symmetric rarefaction waves and a trivial contact wave; the *Star Region* between the non–linear waves is close to vacuum, which makes this problem a suitable test for assessing the performance of numerical methods for low–density flows. Test 3 is designed to assess the robustness and accuracy of numerical methods; its solution consists of a strong shock wave of shock Mach number 198, a contact surface and a left rarefaction wave. Test 4 is also a very severe test, its solution consists of three strong discontinuities travelling to the right. A detailed discussion on the exact solution of the test problems is found in Sect. 4.3.3 of Chap. 4. Test 5 is also designed to test the robustness of numerical methods but the main reason for devising this test is to assess the ability of numerical methods to resolve

slowly–moving contact discontinuities. The exact solution of Test 5 consists of a left rarefaction wave, a right–travelling shock wave and a *stationary* contact discontinuity. For each test problem we select a convenient position x_0 of the initial discontinuity and the output time. These are stated in the legend of each figure displaying computational results.

11.5.2 The Results

The computed results for Tests 1 to 5 using the Godunov first–order method in conjunction with the Roe approximate Riemann solver are shown in Figs. 11.4–11.8, where the numerical solution is shown by the symbols and the full line denotes the exact solution. As discussed earlier, Fig. 11.4 shows the results obtained from the Roe Riemann solver without the entropy fix and, as expected, the computed solution is obviously incorrect. Fig. 11.5 shows the corresponding results from the modified scheme using the Harten–Hyman entropy fix presented in the previous section. These results are, to plotting accuracy, almost indistinguishable from those obtained by the Godunov method in conjunction with the exact Riemann solver; see Fig. 6.8, Chap. 6. As a matter of fact, near the sonic point, the modified Roe solution looks slightly better; it also looks better than the Flux Vector Splitting solution, with the van Leer splitting, see Fig. 8.4 of Chap. 8. The HLL and HLLC solutions of Chap. 10, still seem to be the most accurate near sonic points. Compare also with the Osher results of Chap. 12. As anticipated, the Roe solver will fail near low–density flows; Test 2 contains two strong rarefactions with a low density and low pressure region in the middle and the Roe method, as described, does actually fail on this test. To compute successfully this kind of flows one must modify the Roe solver following the methodology of Einfeldt et. al. [182]. The results for Tests 3 and 4 are virtually identical to those of Godunov's method with the exact Riemann solver, as the reader can verify by comparing Figs. 11.6 and 11.7 with Figs. 6.10 and 6.11 of Chap. 6. The results for Test 5 are also very similar to those obtained from the Godunov method with the exact Riemann solver; note however that the (non–isolated) stationary contact is not as sharply resolved as with the approximate HLLC Riemann solver of Chapt. 10, see Fig. 10.9. As expected of course, the resolution of the stationary contact is better than that of the Flux Vector Splitting Method with the Steger–Warming splitting and that with the van Leer splitting, see Figs. 8.14 and 8.15 of Chap. 8.

11.6 Extensions

The Roe approximate Riemann solver, following the original method of Roe and that of Roe and Pike, has been presented and illustrated via the isothermal equations of gas dynamics and the split three–dimensional, time dependent Euler equations. Details of the Roe solver for the three–dimensional

steady supersonic Euler equations are found in the original paper of Roe [407]. For one–dimensional applications all the required information is contained in this chapter and Chap. 6. Second–order Total Variation Diminishing (TVD) extensions of the schemes are presented in Chap. 13 for scalar problems and in Chap. 14 for non–linear one dimensional systems. In chap. 15 we present techniques that allow the extension of these schemes to solve problems with source terms. In Chap. 16 we study techniques to extend the methods of this chapter to three–dimensional problems.

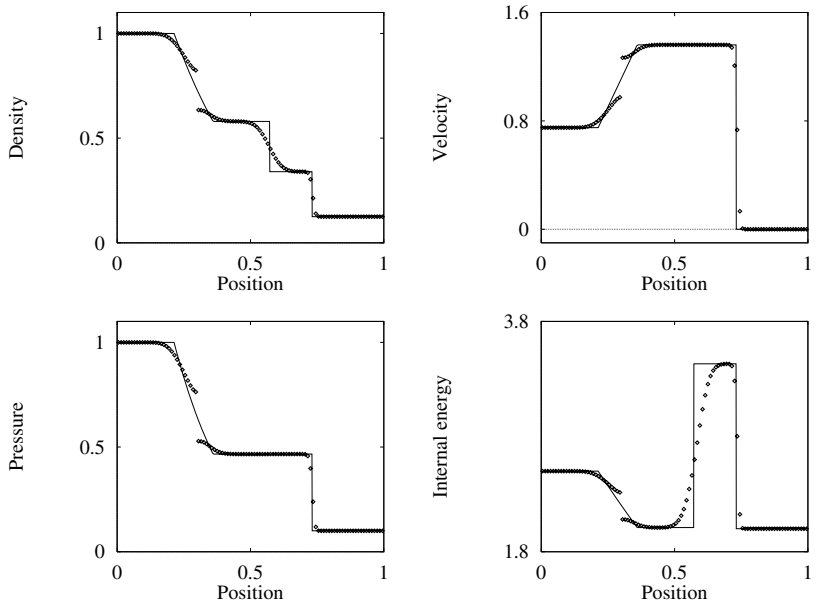

Fig. 11.4. Godunov's method with Roe's Riemann solver (no entropy fix) for Test 1, $x_0 = 0.3$. Numerical (symbol) and exact (line) solutions compared at time 0.2

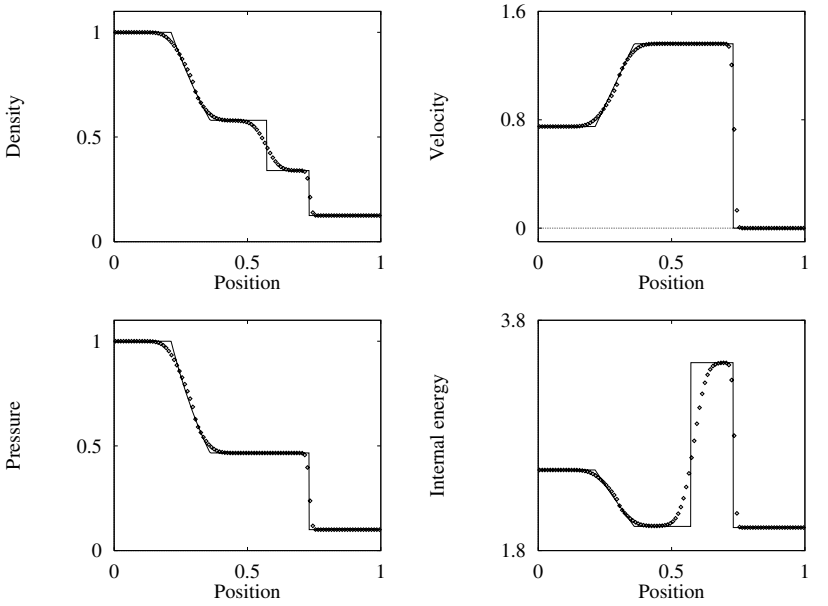

Fig. 11.5. Godunov's method with Roe's Riemann solver applied to Test 1, with $x_0 = 0.3$. Numerical (symbol) and exact (line) solutions are compared at time 0.2

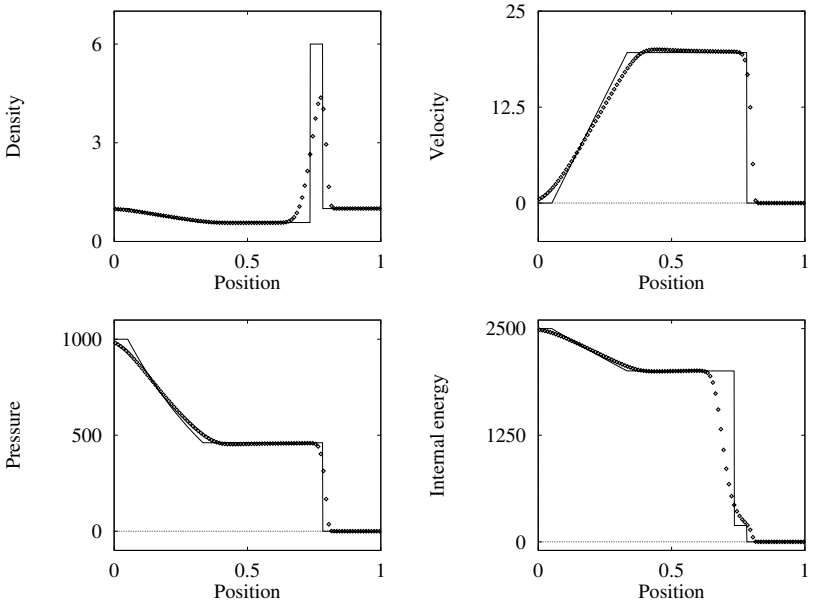

Fig. 11.6. Godunov's method with Roe's Riemann solver applied to Test 3, with $x_0 = 0.5$. Numerical (symbol) and exact (line) solutions are compared at time 0.012

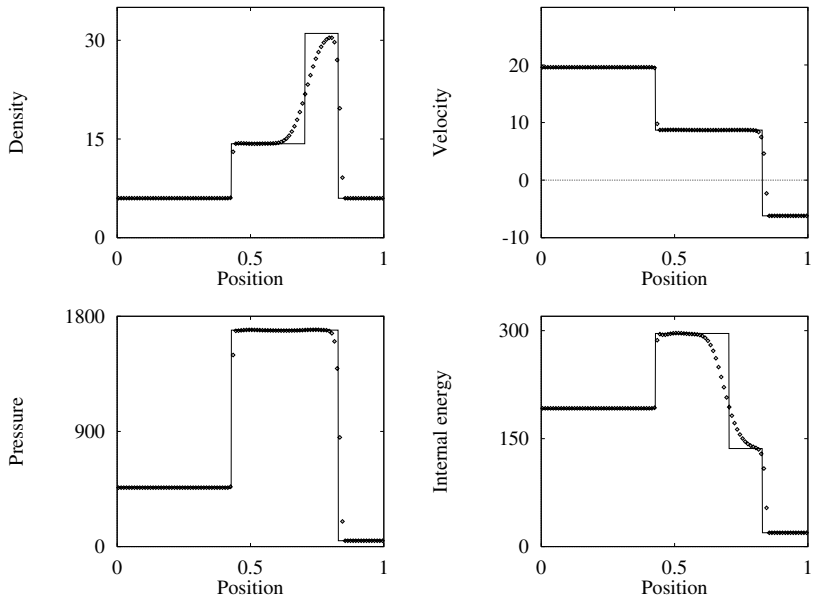

Fig. 11.7. Godunov's method with Roe's Riemann solver applied to Test 4, with $x_0 = 0.4$. Numerical (symbol) and exact (line) solutions are compared at time 0.035

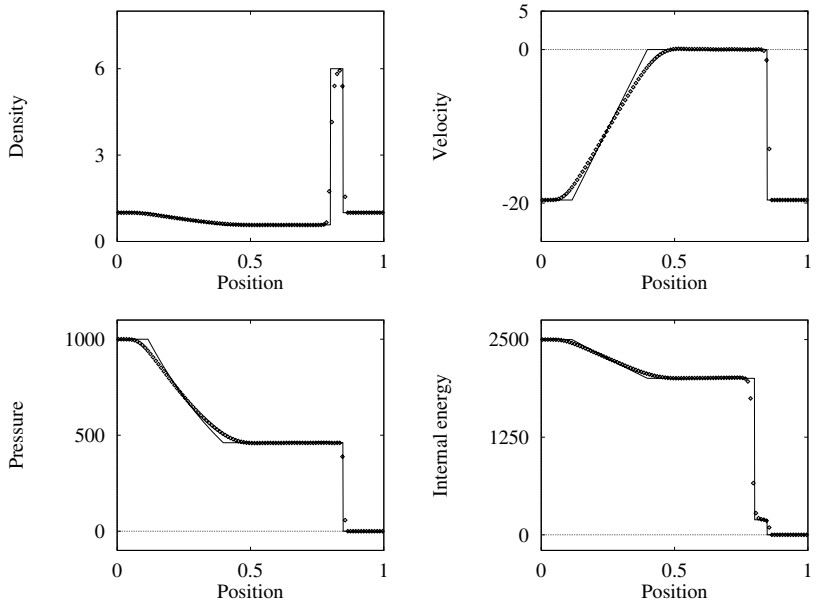

Fig. 11.8. Godunov's method with Roe's Riemann solver applied to Test 5, with $x_0 = 0.8$. Numerical (symbol) and exact (line) solutions are compared at time 0.012