Analytical Dynamics

Theory and Applications

Mark D. Ardema

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Kluwer Academic / Plenum Publishers
New York, Boston, Dordrecht, London, Moscow

Library of Congress Cataloging-in-Publication Data

Ardema, Mark D.
Analytical dynamics: theory and applications/Mark D. Ardema.
p. cm.
Includes bibliographical references and index.
ISBN 0-306-48681-4
1. Dynamics. I. Title.

TA352.A85 2005
620.1'04—dc22

2004054841

ISBN: 0-306-48681-4
ISBN E-book: 0-306-48682-2

233 Spring Street, New York, New York 10013

http://www.wkap.nl/

10 9 8 7 6 5 4 3 2 1

A C.I.P. record for this book is available from the Library of Congress

Printed in the United States of America

may your forces be conservative,
your constraints holonomic, your coordinates ignorable,
and your principal function separable

Contents

Preface

In his great work, *Mecanique Analytique* (1788)[1] Lagrange used the term "analytical" to mean "non-geometrical." Indeed, Lagrange made the following boast:

> "No diagrams will be found in this work. The methods that I explain in it require neither constructions nor geometrical or mechanical arguments, but only the algebraic operations inherent to a regular and uniform process. Those who love Analysis will, with joy, see mechanics become a new branch of it and will be grateful to me for thus having extended its field."

This was in marked contrast to Newton's *Philosohiae Naturalis Principia Mathematica* (1687) which is full of elaborate geometrical constructions. It has been remarked that the classical Greeks would have understood some of the *Principia* but none of the *Mecanique Analytique.*

The term analytical dynamics has now come to mean the developments in dynamics from just after Newton to just before the advent of relativity theory and quantum mechanics, and it is this meaning of the term that is meant here. Frequent use will be made of diagrams to illustrate the theory and its applications, although it will be noted that as the book progresses and the material gets "more analytical", the number of figures per chapter tends to decrease, although not monotonically.

Dynamics is the oldest of the mathematical theories of physics. Its basic principles are few in number and relatively easily understood, and its consequences are very rich. It was of great interest to the Greeks in the classical period. Although the Greeks had some of the concepts of statics correct, their knowledge of dynamics was seriously flawed. It was not until Kepler, Galileo, Decartes, Huyghens, and others in the seventeenth century that the principles of dynamics were first understood. Then Newton united both terrestrial and celestial dynamics into one comprehensive theory in the *Principia.* To this day, the "proof" of a new result in classical dynamics consists in showing that it is consistent with Newton's three "Laws of Motion". Being the oldest and best established, dynamics has become the prototype of the branches of mathematical physics.

Among the branches of physics that have adapted the techniques of dynamics are the various theories of deformable continuous media, thermodynamics, electricity and magnetism, relativity theory, and quantum mechanics. The concept of a dynamic system has been abstracted and applied to fields as diverse as economics and biology. Branches of mathematics that have benefited from concepts that first arose in dynamics are algebra, differential geometry, non-Euclidean geometry, functional analysis, theory of groups and fields, and, especially, differential equations.

A few decades ago, the dynamics of interest in engineering applications was extremely simple, being mostly concerned with two-dimensional motion of rigid bodies in simple machines. This situation has now changed. The dynamics problems that arise in the fields of robotics, biomechanics, and space flight, to name just a few, are usually quite complicated, involving typically three-dimensional motion of collections of inter-connected bodies subject to constraints of various kinds. These problems require careful, sophisticated analysis, and this has sparked a renewed interest in the methods of analytical dynamics.

Essentially, the development of analytical dynamics begins with d' Alembert's *Traite de dynamique* of 1743 and ends with Appell's *Traite de Mecanique Rationelle* of 1896. Thus the time span is the 150 years covering the last half of the eighteenth and the entire nineteenth century. Many authors contributed to the theory, but two works stand out: First, the *Mecanique Analytique* of Lagrange, and second, the *Second Essay on a General Method in Dynamics* (1835) by Hamilton.

Over the years, the body of knowledge called analytical dynamics has coalesced into two parts, the first called Lagrangian and the second Hamiltonian dynamics. Although the division is somewhat artificial, it is a useful one. Both subjects are covered in this book.

There are two principal sources for this book. These are the books by Rosenberg (*Analytical Dynamics of Discrete Systems*, 1977) and by Pars (*A Treatise on Analytical Dynamics*, 1965). The scope of the first of these is more limited than the present book, being confined to Lagrangian Dynamics, and the second is much broader, covering much material of limited interest to engineering analysts. For the Lagrangian portion of the book, the excellent treatment of Rosenberg is the primary source. It should be remarked that one of the primary sources for Rosenberg is the book by Pars. Pars' book is a monumental work, covering the subject in a single comprehensive and rigorous volume. Disadvantages of these books are that Rosenberg is out of print and Pars has no student exercises. No specific references will be made to material adapted from these two books because to do so would make the manuscript cumbersome; the only exceptions are references to related material not covered in this book.

As would be expected of a subject as old and well-established as classical dynamics, there are hundreds of available books. I have listed in the Bibliography only those books familiar to me. Also listed in the Bibliography are the principal original sources and two books relating to the history of the subject.

There are two types of examples used in this book. The first type is intended to illustrate key results of the theoretical development, and these are deliberately kept as simple as possible. Thus frequent use is made of, for example, the simple pendulum, the one-dimensional harmonic oscillator (linear spring-mass system), and central gravitational attraction. The other type of example is included to show the application of the theoretical results to complex, real-life problems. These examples are often quite lengthy, comprising an entire chapter in some cases. Features involved include three-dimensional motion, rigid bodies, multi-body systems, and nonholonomic constraints.

Throughout the book there are historical footnotes and longer historical remarks describing the origins of the key concepts and the people who first discovered them. For readers less interested in the history of dynamics than I, this historical information may be skipped with no loss in continuity.[2]

Because most dynamics problems may be solved by Newton's laws, alternative methods must have relative advantages to warrant interest. In the case of Lagrangian dynamics this justification is easy, since, as will be shown in this book, many specific dynamics problems are easier to solve by the Lagrangian than by the Newtonian method. Because the Hamiltonian formalism generally requires the same effort to solve dynamics problems as the Lagrangian, the study of Hamiltonian dynamics is more difficult to justify.

One of the aims of Hamiltonian dynamics is to obtain not just the equations of motion of a dynamic system, but their *solution*; however, it must be confessed that the usefulness of the techniques developed is often limited. Another advantage lies in the mathematical elegance of the presentation, although this may be of limited importance to engineering analysts. The Hamiltonian approach to dynamics has had, and continues to have, a far-reaching impact on many fields of mathematical physics, and this is an important reason for its study. Perhaps Gauss said it best: "It is always interesting and instructive to regard the laws of nature from a new and advantageous point of view, so as to solve this or that problem more simply, or to obtain a more precise presentation".

One approach to dynamics is to develop it by the axiomatic method familiar from Euclidean geometry, and this has been done by many authors. It seems to me, however, that this method is inappropriate for a subject that is *experimentally* based.[3] Today we use the principles of classical dynamics because they give a sufficiently accurate *model* of physical phenomena. This is in marked contrast to the metaphysical view in Newton's time, which held that Newton's laws described how nature actually behaves.

A summary of the organization of the book is as follows. Chapter 1 is a review of Newtonian dynamics. This is not meant to be comprehensive but rather covers only concepts that are needed later. Chapters 2 – 4 cover the foundations of analytical dynamics that will be used throughout the rest of the book – constraints, virtual displacements, virtual work, and variational principles. (It may be somewhat frustrating to some students to spend so much time on preliminary material, but this effort will pay off in the long run.) Lagrangian dynamics is contained in Chapters 5 – 11. This includes the derivation of Lagrange's equations as well as numerous applications. The next three Chapters – on stability, impulsive motion, and the Gibbs-Appell equations – are outside the main development, and, although important topics, are not necessary to subsequent developments. The remaining Chapters, 15 – 18, concern the development of Hamiltonian dynamics and its applications.

Some comments on function notation used in the book are required. The symbol $F(x)$ will be used to mean both "F is a function of x" and "the value of the function F for a specific value of x"; when there is a chance of confusion, the distinction will be made. I will write $F = F(x_s)$ to mean that F is a function of the variables $x_1, \cdot\cdot, x_n$, at most, and no others. Similarly, $F \neq F(x_s)$ will mean that F is not (allowed to be) a function of the variables $x_1, \cdot\cdot, x_n$. The same symbol will denote sometimes two different functions. For example, if $F(x_s)$ and $x_s = f(q_r)$ then $F(q_r)$ will mean $F(f_1(q_1, \cdot\cdot, q_n), \cdot\cdot, f_n(q_1, \cdot\cdot, q_n))$, when it is clear what is meant. The symbolism $F(x_s) \in C^n$ will mean that $F(\cdot)$ is of class n, that is that it is continuous with continuous derivatives up to order n in all of its arguements $x_1, \cdot\cdot, x_n$.

This book is intended both as an advanced undergraduate or graduate text, and as a reference for engineering analysts. In my own graduate course, the material is covered in forty fifty-minute lectures. The background expected is an undergraduate understanding of Newtonian dynamics and of mathematics, expecially differential equations.

Finally, it is with great pleasure that I acknowledge the faculty at the University of California at Berkeley who first imparted to me the knowl-

edge and appreciation of dynamics – Professors Rosenberg, Leitmann, and Goldsmith. I am especially indebted to the late Professor Rosenberg who granted me permission to use freely material from his book.

Notes

1 Specific works referenced here, and in the rest of the book, are listed in the *Bibliography*.

2 Lagrange himself was deeply interested in the history of dynamics, devoting much space in *Mechnique Analytique* to the subject.

3 This was first clearly recognized by Carnot.

Chapter 1

Review of Newtonian Dynamics

1.1 Basic Concepts

Assumptions. Classical mechanics rests on three basic assumptions:

1. The physical world is a three dimensional Euclidean space. This implies that the Pythagorean theorem, vector addition by parallelograms, and all elementary geometry and trigonometry are valid.

2. There exist inertial (Galilean) reference frames in this space. An inertial frame is one in which Newton's three laws hold to a sufficient degree of accuracy. We generally will take reference frames fixed relative to the surface of the earth to be inertial.

3. The quantities mass and time are invariant, that is, they are measured as the same by all observers.

4. Physical objects are particles or collections of particles constituting rigid bodies.

Assumptions (1) – (3) were regarded as laws of physics at one time; now they are regarded as engineering approximations.[1] Assumption (4) is clearly an approximation; all known materials deform under forces, but this deformation is frequently negligible.

Newton's Laws. Let $\sum \underline{F}$ be the resultant (vector sum) of all the forces acting on a mass particle of mass m. Then Newton's Second Law states that:[2]

$$\sum \underline{F} = m\underline{a} \tag{1.1}$$

where $\underline{a} = d^2\underline{r}/dt^2 = \ddot{\underline{r}}$ is the acceleration of the mass particle, and where $\underline{r}$ is the position vector of the mass particle in an inertial frame of reference (Fig. 1-1) and $d(\ )/dt = (\dot{\ })$ is the time derivative in that frame. That is, force is proportional to acceleration with proportionality constant m.

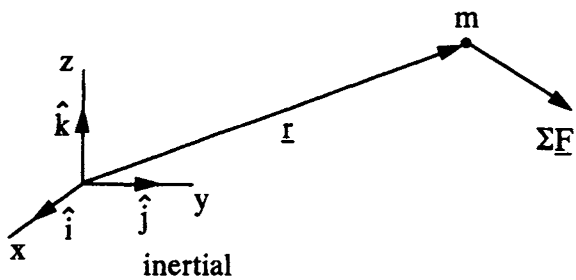


Fig. 1-1

Newton's Third Law states that given any two particles p_1 and p_2 with masses m_1 and m_2, the force exerted by p_1 on p_2, say $\underline{F}_{21}$, is equal and opposite to that exerted by p_2 on p_1, $\underline{F}_{12}$, and these forces act on a line adjoining the two particles (Fig. 1-2):

$$\underline{F}_{12} = |\underline{F}_{12}|\,\hat{e} = -\underline{F}_{21} \tag{1.2}$$

where $\hat{e}$ is a unit vector in direction $\underline{F}_{12}$.

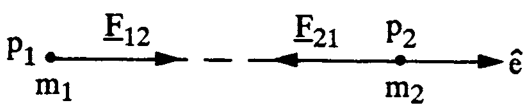


Fig. 1-2

Newton's First Law states that if $\sum \underline{F} = \underline{0}$ then $\underline{v}(t) = \underline{\text{constant}}$. This "Law" is therefore a consequence of the Second Law.

Newton stated these laws for a single particle, as we have just done; L. Euler and others generalized them to a rigid body, that is a collection of particles whose relative positions are fixed.

Definitions. The subject of mechanics is conveniently divided into branches as follows:

1. Statics is that branch of mechanics concerned with the special case $\underline{v}(t) = \underline{0}$. This implies that $\underline{a}(t) = \underline{0}$ and consequently $\sum \underline{F} = \underline{0}$.
2. Dynamics is that branch of mechanics concerned with $\underline{v}(t) \neq$ <u>constant</u>.
3. Kinematics is that branch of dynamics concerned with motion independent of the forces that produce the motion.
4. Kinetics is that branch of dynamics concerned with the connection between forces and motion, as defined by Newton's three laws.

Basic Problems in Kinetics. It is clear that there are two basic problems:

1. Given the forces, find the motion (that is, the position, velocity, and acceleration as a function of time). This is sometimes called the "forward" or "dynamics" problem.
2. Given the motion, find the forces that produced it (actually, one can usually only find the resultant force). Statics is a special case of this. This is sometimes called the "backward" or "controls" problem.

In practice, mixed problems frequently arise; for example, given the motion and some of the forces, what is the resultant of the remaining force(s)?

Reasons for Reviewing Newtonian Dynamics. In the rest of this chapter, we will review elementary Newtonian dynamics, for the following reasons:

1. Some of this material will be needed later.
2. It allows a chance in a familiar setting to get used to the approach and notation used throughout this book.
3. It gives insight that leads to other approachs to dynamics.
4. It provides a benchmark with which to measure the worth of these new approachs.

Many of the following results will be presented without proof.

1.2 Kinematics and Newtonian Particle Dynamics

Motion of a Point. Consider a point P moving along a curve C relative to a reference frame $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$. Denote the position vector of P at time t by $\underline{r}(t)$. Then the velocity of P is defined as (Fig. 1-3):

$$\underline{v}(t) = \frac{d\underline{r}}{dt} = \dot{\underline{r}} = \lim_{\Delta t \to 0} \frac{\underline{r}(t+\Delta t) - \underline{r}(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta \underline{r}}{\Delta t} \tag{1.3}$$

Similarly the acceleration of the point is defined as:

$$\underline{a}(t) = \frac{d\underline{v}}{dt} = \dot{\underline{v}} = \ddot{\underline{r}} = \lim_{\Delta t \to 0} \frac{\underline{v}(t+\Delta t) - \underline{v}(t)}{\Delta t} \tag{1.4}$$

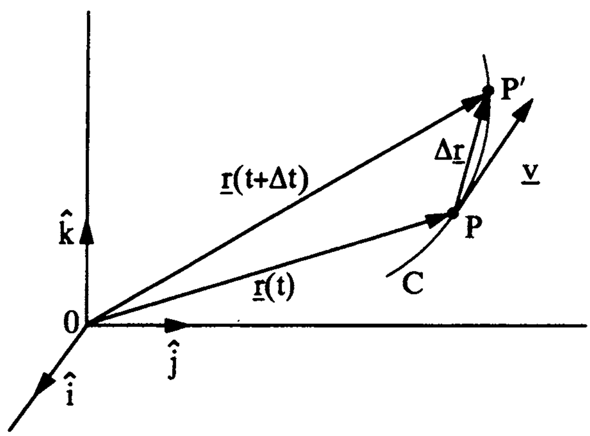


Fig. 1-3

Note that $\underline{v}(t)$ is tangent to the curve C. The magnitude of the velocity vector, $v(t) = |\underline{v}(t)|$, is called the speed of the point.

Rectangular Components. To obtain scalar equations of motion, the vectors of interest are written in components. In rectangular components (Fig. 1-4), the position vector is given by:

$$\underline{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k} \tag{1.5}$$

From Eqns. (1.3) and (1.4) $\underline{v}$ and $\underline{a}$ are:

$$\underline{v} = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath} + \dot{z}\hat{k} \tag{1.6}$$
$$\underline{a} = \ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath} + \ddot{z}\hat{k} \tag{1.7}$$

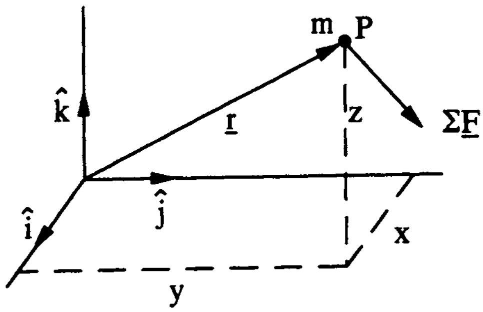


Fig. 1-4

We call (x, y, z) the rectangular components of position (or the rectangular coordinates) of point P. Similarly, $(\dot{x}, \dot{y}, \dot{z})$ and $(\ddot{x}, \ddot{y}, \ddot{z})$ are the rectangular components of velocity and acceleration, respectively. The distance of P from the origin and the speed of P are given by:

$$r = |\underline{r}| = \sqrt{x^2 + y^2 + z^2}\,, \qquad v = |\underline{v}| = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} \tag{1.8}$$

Expressing the resultant force on the mass m at point P in rectangular components

$$\sum \underline{F} = \sum F_x \hat{\imath} + \sum F_y \hat{\jmath} + \sum F_z \hat{k} \tag{1.9}$$

and combining this with Eqns. (1.1) and (1.7) gives three scalar equations of motion:

$$\sum F_x = m\ddot{x}\,, \qquad \sum F_y = m\ddot{y}\,, \qquad \sum F_z = m\ddot{z} \tag{1.10}$$

This is a sixth order system of ordinary differential equations.

Any vector in three dimensions can be written as a linear combination of any three linearly independent vectors, called basis vectors. In this book, all basis vectors will be triads of mutually-orthogonal, right-handed unit vectors and will be denoted by "hats".

Normal – Tangential Components. It is possible, and frequently desirable, to express $\underline{r}$, $\underline{v}$, and $\underline{a}$ in components along directions other than $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$. For planar motion (take this to be in the (x, y) plane), normal-tangential components are frequently useful. Introduce unit vectors tangent and normal to $\underline{v}$ as shown in Fig. 1-5.

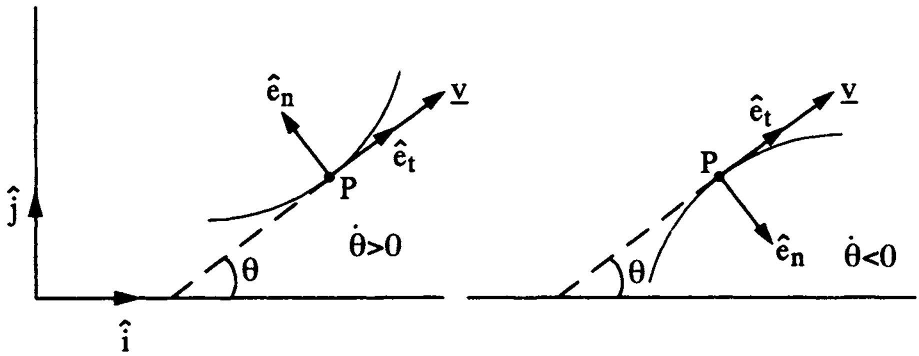


Fig. 1-5

The velocity and acceleration vectors expressed in these components are:

$$\underline{v} = v\hat{e}_t \tag{1.11}$$

$$\underline{a} = \frac{d\underline{v}}{dt} = \dot{v}\hat{e}_t + v\frac{d\hat{e}_t}{dt} \tag{1.12}$$

It is clear that in general $\hat{e}_t$ will vary with time and thus $d\hat{e}_t/dt \neq \underline{0}$. We must consider the two cases shown on Fig. 1-5 separately. First, for $\dot{\theta} > 0$ (Fig. 1-6):

$$\hat{e}_t = \cos\theta\hat{\imath} + \sin\theta\hat{\jmath}$$

$$\hat{e}_n = -\sin\theta\hat{\imath} + \cos\theta\hat{\jmath}$$

$$\frac{d\hat{e}_t}{dt} = -\dot{\theta}\sin\theta\hat{\imath} + \dot{\theta}\cos\theta\hat{\jmath} = \dot{\theta}\hat{e}_n$$

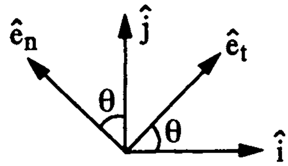


Fig. 1-6

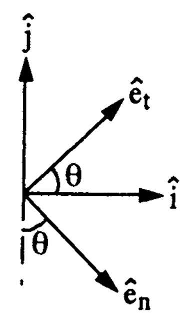


Fig. 1-7

Next for $\dot{\theta} < 0$ (Fig. 1-7):

$$\hat{e}_t = \cos\theta\hat{\imath} + \sin\theta\hat{\jmath}$$

$$\hat{e}_n = \sin\theta\hat{\imath} - \cos\theta\hat{\jmath}$$

$$\frac{d\hat{e}_t}{dt} = -\dot{\theta}\sin\theta\hat{\imath} + \dot{\theta}\cos\theta\hat{\jmath} = -\dot{\theta}\hat{e}_n$$

Thus for both cases:

$$\frac{d\hat{e}_t}{dt} = |\dot{\theta}|\hat{e}_n \tag{1.13}$$

so that, from Eqn. (1.12),

$$\underline{a} = \dot{v}\hat{e}_t + v|\dot{\theta}|\hat{e}_n \tag{1.14}$$

We call $(\dot{v},\ v|\dot{\theta}|)$ the tangential and normal components of acceleration.

Now let $\rho = \frac{v}{|\dot{\theta}|} \geq 0$. Suppose the motion is on a circle of radius R (Fig. 1-8); then

$$S = R\theta \Longrightarrow \dot{S} = R\dot{\theta} = v \Longrightarrow R = \frac{v}{\dot{\theta}}$$

Thus we call in general ρ the *radius of curvature.* Hence we may write

$$\underline{a} = \dot{v}\hat{e}_t + \frac{v^2}{\rho}\hat{e}_n \tag{1.15}$$

Note that $\hat{e}_n$ is undefined and $\rho = \infty$ for $\dot{\theta} = 0$, i.e. for rectilinear motion or at a point of inflection (Fig. 1-9). If the resultant force acting on a particle is expressed in normal-tangential components, $\sum \underline{F} = \sum F_t\hat{e}_t + \sum F_n\hat{e}_n$, then Eqns. (1.1) and (1.15) give the scalar equations of motion:

$$\sum F_t = m\dot{v}\,, \qquad \sum F_n = m\frac{v^2}{\rho} \tag{1.16}$$

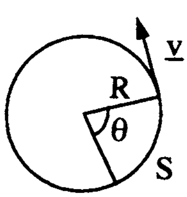


Fig. 1-8

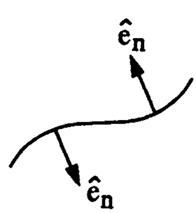


Fig. 1-9

Cylindrical and Spherical Coordinates and Components. In three-dimensional (3-D) motion it is often advantageous to resolve the velocity and acceleration into cylindrical or spherical components; only the velocity components will be given here. The cylindrical coordinates (r, ϕ, z) are shown on Fig. 1-10. From the geometry, the cylindrical and rectangular coordinates are related by

$$
\begin{aligned}
x &= r\cos\phi \\
y &= r\sin\phi \\
z &= z
\end{aligned}
\tag{1.17}
$$

The velocity expressed in cylindrical components is

$$
\underline{v} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta + \dot{z}\hat{k} \tag{1.18}
$$

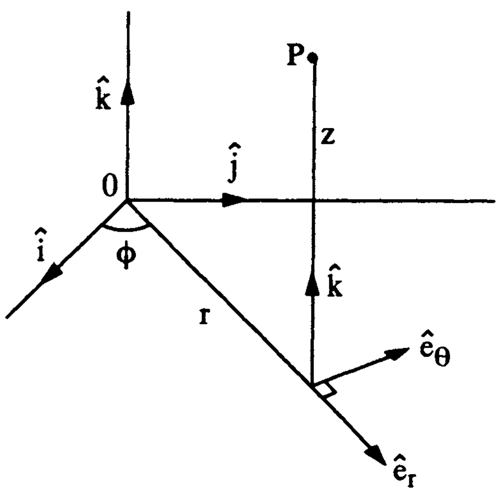


Fig. 1-10

Note that if $z =$ constant, (r, θ) are just the familiar plane polar coordinates.

The spherical coordinates (r, θ, ϕ) are shown on Fig. 1-11. The relation to rectangular coordinates is given by

$$
\begin{aligned}
x &= r\sin\theta\cos\phi \\
y &= r\sin\theta\sin\phi \\
z &= r\cos\theta
\end{aligned}
\tag{1.19}
$$

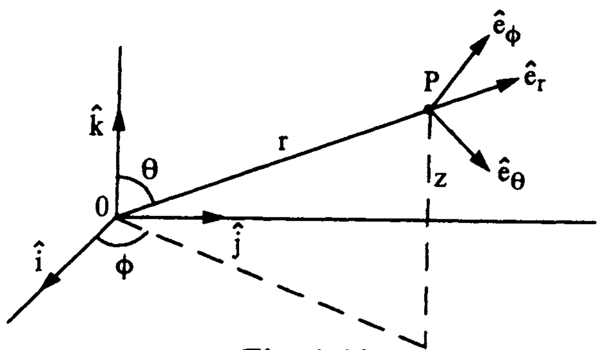


Fig. 1-11

and the velocity in spherical components is

$$\underline{v} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta + r\dot{\phi}\sin\theta\hat{e}_\phi \tag{1.20}$$

Spherical coordinates and components are particularly advantageous for central force motion (Chapter 10).

Relative Velocity. It is sometimes necessary to relate the motion of a point as measured in one reference frame to the motion of the same point as measured in another frame moving with respect to (w.r.t.) the first one. First consider two reference frames moving with respect to each other such that one axis, say z, is always aligned (Fig. 1-12). Define the *angular velocity* and *angular acceleration* of frame $\{\hat{i}, \hat{j}\}$ w.r.t. frame $\{\hat{I}, \hat{J}\}$ by:

$$\underline{\omega} = \dot{\theta}\hat{k} = \dot{\theta}\hat{K} \tag{1.21}$$

$$\underline{\alpha} = \underline{\dot{\omega}} = \ddot{\theta}\hat{k} = \ddot{\theta}\hat{K} \tag{1.22}$$

where we have used the fact that $\hat{k} = \hat{K}$ is a constant vector.

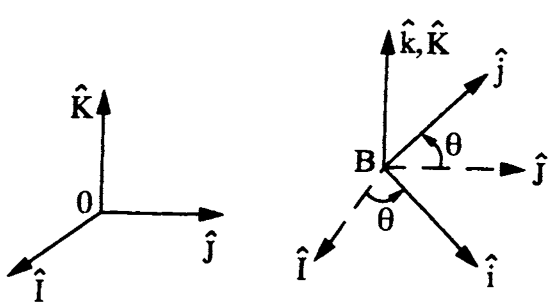


Fig. 1-12

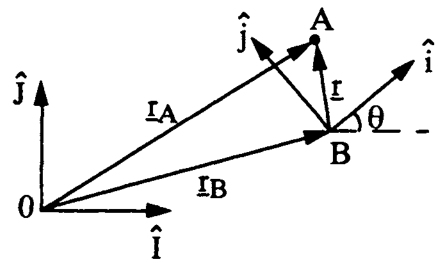


Fig. 1-13

Figure 1-13 shows a point A moving in a plane w.r.t. to two frames which are also moving w.r.t. each other.

Let

$$\frac{D}{Dt} = \text{time derivative w.r.t.}\{\hat{I}, \hat{J}\}$$
$$\frac{d}{dt} = \text{time derivative w.r.t.}\{\hat{\imath}, \hat{\jmath}\}$$

For a scalar Q, $DQ/Dt = dQ/dt$, but for a vector $\underline{Q}$, $D\underline{Q}/Dt \neq d\underline{Q}/dt$, in general. The relation between the two is given by the *basic kinematic equation*

$$\frac{D\underline{Q}}{Dt} = \frac{d\underline{Q}}{dt} + \underline{\omega} \times \underline{Q} \tag{1.23}$$

which holds for any vector $\underline{Q}$. We are now ready to derive the relative velocity equation. From Fig. 1-13:

$$\underline{r}_A = \underline{r}_B + \underline{r} \tag{1.24}$$

Differentiating and applying Eqn. (1.23):

$$\frac{D\underline{r}_A}{Dt} = \frac{D\underline{r}_B}{Dt} + \frac{D\underline{r}}{Dt}$$

$$\underline{v}_A = \underline{v}_B + \frac{d\underline{r}}{dt} + \underline{\omega} \times \underline{r}$$
$$\underline{v}_A = \underline{v}_B + \underline{v}_r + \underline{\omega} \times \underline{r} \tag{1.25}$$

where

$$\underline{v}_A = D\underline{r}_A/Dt = \underline{\text{velocity}} \text{ of } A \text{ w.r.t. } \{\hat{I}, \hat{J}\}$$
$$\underline{v}_B = D\underline{r}_B/Dt = \underline{\text{velocity}} \text{ of } B \text{ w.r.t. } \{\hat{I}, \hat{J}\}$$
$$\underline{v}_r = d\underline{r}/dt = \underline{\text{velocity}} \text{ of } A \text{ w.r.t. } \{\hat{\imath}, \hat{\jmath}\}$$

These results also apply to general 3-D motion provided that the angular velocity is suitably defined. This is most conveniently done using Euler's Theorem. This theorem states that any displacement of one reference frame relative to another may be replaced by a simple rotation about some line. The motion of the one frame w.r.t. the other may then be thought of as a sequence of such rotations. At any instant, $\underline{\omega}$ is defined as the vector whose direction is the axis of rotation and whose magnitude is the rotation rate. With this definition of $\underline{\omega}$, Eqns. (1.23) and (1.25) are valid for 3-D motion. For a full discussion of 3-D kinematics see Ardema, *Newton-Euler Dynamics*.

Example. Car B is rounding a curve of radius R with speed v_B (Fig. 1-14). Car A is traveling toward car B at speed v_A and is distance x from car B at the instant shown. We want the velocity of car A as seen by car B. The cars are modelled as points.

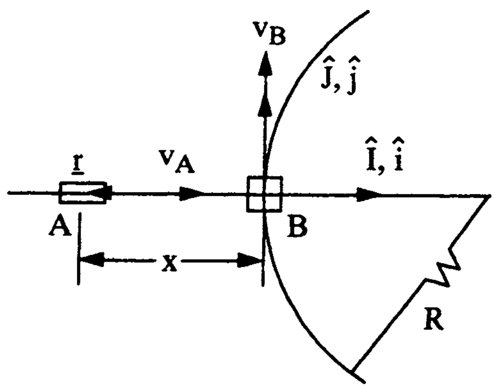


Fig. 1-14

Introduce reference frames:

$\{\hat{I}, \hat{J}\}$ fixed in ground (the data is given in this frame)

$\{\hat{\imath}, \hat{\jmath}\}$ fixed in car B (the answer is required in this frame)

Applying Eqn. (1.25):

$$\underline{v}_A = \underline{v}_B + \underline{v}_r + \underline{\omega} \times \underline{r} \qquad \underline{v}_A = v_A\hat{I}$$

$$\underline{v}_B = v_B\hat{J}$$

$$\underline{v}_r = \underline{v}_A - \underline{v}_B - \underline{\omega} \times \underline{r} \qquad \underline{r} = -x\hat{I}$$

$$= v_A\hat{I} - v_B\hat{J} - \left(-\frac{v_B}{R}\hat{K}\right) \times (-x\hat{I}) \qquad \underline{\omega} = -\frac{v_B}{R}\hat{K}$$

$$\underline{v}_r = v_A\hat{I} - \left(v_B + \frac{v_B x}{R}\right)\hat{J}$$

1.3 Work and Energy

Definitions. Suppose a force $\underline{F}$ acts on a particle of mass m as it moves along curve C (Fig. 1-15). Define the *work done* by $\underline{F}$ during the displacement of m from $\underline{r}_0$ to $\underline{r}_1$ along C by

$$U_{0,1} = \int_{\underline{r}_0}^{\underline{r}_1} \underline{F} \cdot d\underline{r} \tag{1.26}$$

Since $\underline{v} = \dfrac{d\underline{r}}{dt}$, this may be written as

$$U_{0,1} = \int_{t_0}^{t_1} \underline{F} \cdot \underline{v}\, dt = \int_{t_0}^{t_1} P\, dt \tag{1.27}$$

where $P = \underline{F} \cdot \underline{v}$ is called the power.

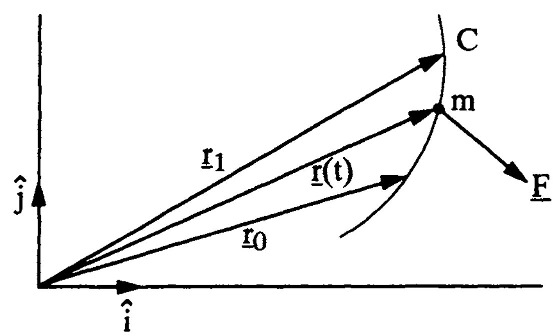


Fig. 1-15

Now suppose $\underline{F}$ is the resultant of all forces and $\{\hat{\imath}, \hat{\jmath}\}$ is an inertial frame; then, Newton's Second Law, Eqn. (1.1), holds:

$$m\, \frac{d\underline{v}}{dt} = \underline{F}$$

Taking the scalar product of both sides with $\underline{v}$ and inserting the result in Eqn. (1.27):

$$m\, \frac{d\underline{v}}{dt} \cdot \underline{v} = \underline{F} \cdot \underline{v}$$

$$\begin{aligned} U_{0,1} &= \int_{t_0}^{t_1} m \frac{d\underline{v}}{dt} \cdot \underline{v}\, dt = m \int_{t_0}^{t_1} \underline{v} \cdot d\underline{v} = \frac{1}{2} m \left(v_1^2 - v_0^2\right) \\ &= T_1 - T_0 = \Delta T_{0,1} \end{aligned} \tag{1.28}$$

where the *kinetic energy* of the particle is defined as:

$$T = \frac{1}{2} m v^2 \tag{1.29}$$

In words, Eqn. (1.28) states that the change in kinetic energy from position 0 to position 1 is equal to the work done by the resultant force from 0 to 1.

Potential Energy. In rectangular coordinates,

$$\begin{aligned} &\underline{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k} \; ; \\ &d\underline{r} = dx\hat{\imath} + dy\hat{\jmath} + dz\hat{k} \; ; \\ &\underline{F} = F_x\hat{\imath} + F_y\hat{\jmath} + F_z\hat{k} \end{aligned}$$

and if $\underline{F}$ is a function only of position $\underline{r}$, Eqn. (1.26) gives

$$U_{0,1} = \int_{\underline{r}_0}^{\underline{r}_1} \underline{F} \cdot d\underline{r} = \int_{x_0}^{x_1} F_x dx + \int_{y_0}^{y_1} F_y dy + \int_{z_0}^{z_1} F_z dz \tag{1.30}$$

Generally, this integration will depend on path C, and not just the end points.

Recall that the *gradient* of a scalar function of a vector argument $V(\underline{r})$ in rectangular coordinates is

$$\text{grad } V(\underline{r}) = \frac{\partial V}{\partial x}\hat{\imath} + \frac{\partial V}{\partial y}\hat{\jmath} + \frac{\partial V}{\partial z}\hat{k} \tag{1.31}$$

Suppose that $\underline{F}(\underline{r})$ is such that there exists a function $V(\underline{r})$ such that

$$\underline{F}(\underline{r}) = -\text{grad } V(\underline{r}) = F_x\hat{\imath} + F_y\hat{\jmath} + F_z\hat{k} \tag{1.32}$$

Then, comparing Eqns. (1.31) and (1.32), and writing $V = V(x, y, z)$,

$$F_x = -\frac{\partial V}{\partial x}, \qquad F_y = -\frac{\partial V}{\partial y}, \qquad F_z = -\frac{\partial V}{\partial z} \tag{1.33}$$

so that

$$\underline{F} \cdot d\underline{r} = -\frac{\partial V}{\partial x} dx - \frac{\partial V}{\partial y} dy - \frac{\partial V}{\partial z} dz = -dV \tag{1.34}$$

Therefore, from Eqn. (1.26),

$$U_{0,1} = \int_{V_0}^{V_1} (-dV) = -(V_1 - V_0) = -\Delta V_{0,1} \tag{1.35}$$

This shows that now the work done by $\underline{F}$ depends only on the endpoints and *not* on the path C.

$V(\underline{r})$ is called a *potential energy function* and $\underline{F}(\underline{r})$ with this property is called a *conservative force.*

Gravitation. Consider two masses with the only force acting on them being their mutual gravitation (Fig. 1-16). If m_e (the earth, for example) $\gg m$ (an earth satellite, for example), we may take m_e as fixed in an inertial frame. If the two bodies are spherically symmetric they can be regarded as particles for the purpose of determing the gravitational force.

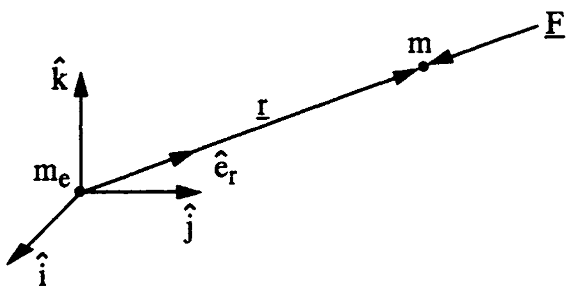


Fig. 1-16

Newton's law of gravitation gives the force acting on mass m as

$$\underline{F} = -\frac{Km_e m}{r^2}\hat{e}_r \tag{1.36}$$

where $K = 6.673 \times 10^{-11}$ m^3/(Kg$\cdot$ sec^2) is the universal gravitational constant.

Because

$$\underline{r} = r\hat{e}_r$$
$$\underline{r} = x\hat{i} + y\hat{j} + z\hat{k}$$
$$r = (x^2 + y^2 + z^2)^{1/2}$$

we have

$$\underline{F} = -\frac{Km_e m}{r^3}(x\hat{i} + y\hat{j} + z\hat{k}) \tag{1.37}$$

so that

$$
\begin{aligned}
F_x &= -Km_em\frac{x}{(x^2+y^2+z^2)^{3/2}} \\
F_y &= -Km_em\frac{y}{(x^2+y^2+z^2)^{3/2}} \\
F_z &= -Km_em\frac{z}{(x^2+y^2+z^2)^{3/2}}
\end{aligned}
\tag{1.38}
$$

Thus the gravitational force is conservative with potential energy function given by

$$V = -\frac{Km_em}{(x^2+y^2+z^2)^{1/2}} \tag{1.39}$$

This is verified by observing that Eqns. (1.33) are satisfied for this function.

In central force motion, as mentioned earlier, it is usually best to use spherical coordinates. Writing $V = V(r, \theta, \phi)$, the gradient of V in spherical components is

$$\text{grad } V(\underline{r}) = \frac{\partial V}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\hat{e}_\phi \tag{1.40}$$

Thus the gravitational potential function is

$$V = -\frac{Km_em}{r} \tag{1.41}$$

which could have been obtained directly from Eqn. (1.39).

For motion over short distances on or near the surface of the earth it is usually sufficient to take the gravitational force as a constant in both magnitude and direction (Fig. 1-17). The force acting on the particle is

$$\underline{F} = F_x\hat{\imath} = -mg\hat{\imath}$$

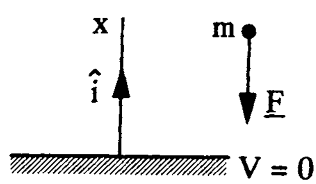


Fig. 1-17

Therefore the gravitational potential energy function is

$$V(x) = mgx \tag{1.42}$$

because $-\dfrac{\partial V}{\partial x} = -mg = F_x$.

Energy Equation. Suppose a number of forces act on m, some conservative and some not. Then

$$\underline{F}_i^c = -\text{grad } V_i \tag{1.43}$$

for each conservative force. The resultant force is

$$\underline{F} = \sum_{i=1}^{n_c} \underline{F}_i^c + \sum_{j=1}^{n_{nc}} \underline{F}_j^{nc} = \sum_i (-\text{grad } V_i) + \sum_j \underline{F}_j^{nc}$$

The work done is

$$U_{0,1} = \int_{\underline{r}_0}^{\underline{r}_1} \underline{F} \cdot d\underline{r} = -\sum_i \Big[V_i(\underline{r}_1) - V_i(\underline{r}_0) \Big] + \int_{\underline{r}_0}^{\underline{r}_1} \sum_j \underline{F}_j^{nc} \cdot d\underline{r}$$

Using $U_{0,1} = \Delta T_{0,1}$, this becomes

$$\Delta T_{0,1} = -\Delta V_{0,1} + U_{0,1}^{nc} \tag{1.44}$$

where V is the sum of all potential energies and U^{nc} is the work done by all nonconservative forces.

Let the *total mechanical energy* be defined by:

$$E = T + V \tag{1.45}$$

Then Eqn. (1.44) may be written

$$\Delta E_{0,1} = U_{0,1}^{nc} \tag{1.46}$$

and, in particular if all forces are conservative and accounted for in V,

$$\Delta E_{0,1} = 0 \tag{1.47}$$

that is, energy is conserved.

Remarks:

1. U is defined over an interval of motion but T, V, and E are defined at an instant.

2. U, T, V, and E are all scalars. Therefore, the energy equation gives only one piece of information.
3. The energy equation is a once-integrated form of Newton's Second Law; it is a relation among speeds, not accelerations.
4. The energy equation is most useful when a combination of the following factors is present: the problem is of low dimension, forces are not needed to be determined, and energy is conserved.
5. The energy equation involves only *changes* in T and V between two positions; thus adding a constant to either one does not change the equation.

1.4 Eulerian Rigid Body Dynamics

Kinetics of Particle System. First consider a collection of particles (not necessarily rigid), Fig. 1-18. Let $\{\hat{\imath}, \hat{\jmath}\}$ be an inertial reference frame and:

$\underline{F}_i^e$ = sum of all external forces on particle i.

$\underline{F}_{ij}$ = (internal) force exerted by particle j on particle i.

The *center of mass* is a position, labeled G, whose position vector is given by

$$\bar{\underline{r}} = \frac{1}{m} \sum_i m_i \underline{r}_i \tag{1.48}$$

where $m = \sum_i m_i$ is the total mass.

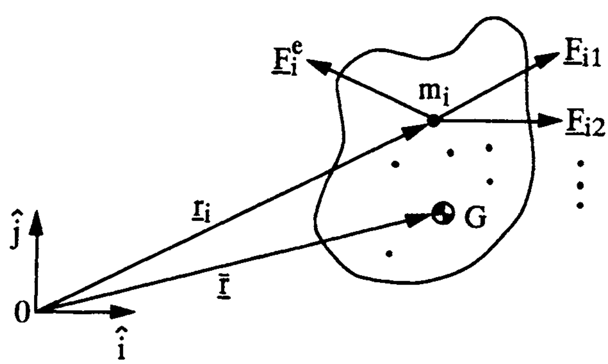


Fig. 1-18

Newton's Second Law for particle i is:

$$\underline{F}_i^e + \sum_j \underline{F}_{ij} = m_i \frac{d^2\underline{r}_i}{dt^2} = m_i \underline{a}_i$$

Sum these equations for all particles and recall from Newton's Third Law that $\underline{F}_{ij} = -\underline{F}_{ji}$ for all i, j:

$$\sum_i \underline{F}_i^e + \sum_i \sum_j \underline{F}_{ij} = \sum_i m_i \frac{d^2\underline{r}_i}{dt^2}$$

$$\sum_i \underline{F}_i^e + \underline{0} = \frac{d^2}{dt^2}\left(\sum_i m_i \underline{r}_i\right) = \frac{d^2}{dt^2}(m\overline{\underline{r}})$$

$$\sum_i \underline{F}_i^e = m \frac{d^2\overline{\underline{r}}}{dt^2} = m\overline{\underline{a}} \tag{1.49}$$

Therefore, for a system of constant mass, including a rigid body, the sum of all *external* forces equals the total mass times acceleration of the center of mass. This means that if the same force $\underline{F}$ is applied to two dissimilar rigid bodies, each having the same mass, the accelerations of their centers of mass will be the same (Fig. 1-19).

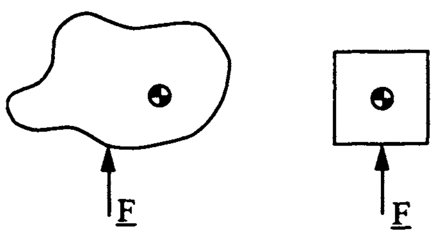


Fig. 1-19

Rigid Body. A *rigid body* is a collection of particles such that there exists a reference frame in which all particles have fixed positions in this reference frame. The *angular velocity of a rigid body* relative to a given reference frame is just the angular velocity of any body fixed reference frame w.r.t. the other frame.[3]

Planar Rigid Body Kinetics. Here we consider the 2-D motion of a rigid body, which we take to be in the $\{\hat{\imath}, \hat{\jmath}\}$ plane. Let $\{\hat{\imath}', \hat{\jmath}'\}$ be a body fixed frame and let the position of a particle in the rigid body be given by $\underline{d}_i = x_i\hat{\imath}' + y_i\hat{\jmath}'$ (Fig. 1-20).[4] The quantities $\underline{d}_i$, x_i, and y_i are all

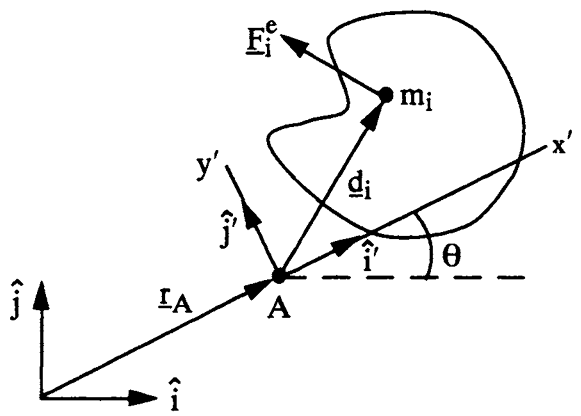


Fig. 1-20

constants. Also let the position of A, the origin of $\{\hat{\imath}', \hat{\jmath}'\}$, w.r.t. the inertial frame $\{\hat{\imath}, \hat{\jmath}\}$ be given by $\underline{r}_A = x\hat{\imath} + y\hat{\jmath}$. It is clear that the location of all the particles of the rigid body are known if the values of x, y and θ are known. We say that the body has three degrees of freedom.

The first two degrees of freedom are accounted for by Eqn. (1.49); in rectangular components:

$$\sum_i F_{x_i}^e = m\ddot{x}\ ; \qquad \sum_i F_{y_i}^e = m\ddot{y} \tag{1.50}$$

The third degree of freedom comes from relating the rate of change of angular momentum to the sum of the *external* force moments. The result is in general quite complicated.

A special case is

$$\sum_i M_{A_i}^e \hat{k} = I_A \alpha \hat{k} \Longrightarrow \sum_i M_{A_i}^e = I_A \alpha \tag{1.51}$$

Here, the sum of the moments of all external forces about the body fixed point A is (Fig. 1-21)

$$\sum_i \underline{M}_{A_i}^e = \sum_i \underline{d}_i \times \underline{F}_i^e \tag{1.52}$$

and

$$I_A = \sum_i \left(x_i'^2 + y_i'^2\right) m_i = \int_m (x'^2 + y'^2) dm \tag{1.53}$$

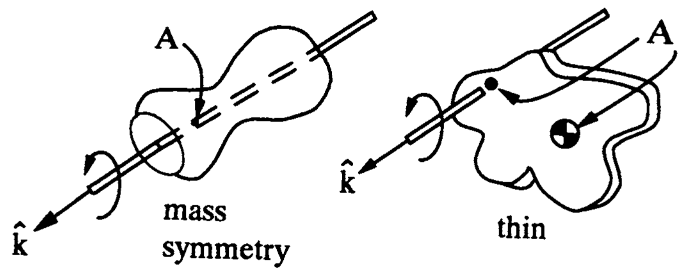


Fig. 1-21

is the mass moment of inertia of the body about the axis through A parallel to $\hat{k}$, and $\alpha = \dot{\omega} = \ddot{\theta}$ is the angular acceleration of the body. The first representation of I_A in Eqn. (1.53) considers the rigid body to be a finite collection of mass particles and the second is the limit as the number of particles tends to infinity, that is the body is regarded as a continuum.

Equation (1.51) is only valid if [5](see Fig. 1-21):

1. Point A is either G, the center of mass, or moves with constant velocity in the inertial frame; and
2. Axis $\hat{k}$ is an axis of rotational mass symmetry or the body is "thin".

Concepts of 3-D kinetics will be introduced as needed in future Chapters.

Work and Energy for Rigid Body. As before, Eqn. (1.44) applies, repeated here:

$$\Delta T_{0,1} + \Delta V_{0,1} = U_{0,1}^{np} \tag{1.54}$$

For most of the approaches to dynamics developed later in this book, an essential step in deriving the equations of motion of a dynamic system is the determination of the system's kinetic energy. If a body is considered as a collection of mass particles, its kinetic energy is, by definition,

$$T = \frac{1}{2}\sum_r m_r v_r^2 \tag{1.55}$$

Use of this formula is seldom convenient, however, and in what follows we present several alternative methods for obtaining T for rigid bodies.

First, for 2-D motion T can be obtained from

$$T = \frac{1}{2}m\bar{v}^2 + \frac{1}{2}\bar{I}\omega^2 \tag{1.56}$$

where $\overline{v}$ is the speed of the center of mass and $\overline{I}$ is the moment of inertia about an axis passing through the center of mass and parallel to the axis of rotation. As an alternative for 2-D problems

$$T = \frac{1}{2} I_A \omega^2 \tag{1.57}$$

where I_A is the moment of inertia about an axis passing through a body-fixed point that is also fixed in an inertial frame.

A result that is sometimes useful is Koenig's theorem. Consider a rigid body moving w.r.t. an inertial frame $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$, as shown on Fig. 1-22. Introduce a frame $\{\hat{\imath}', \hat{\jmath}', \hat{k}'\}$ with origin at the body's center of mass that moves in such a way that it's axes always remain parallel to those of the inertial frame (thus this frame is not body fixed). Let the coordinates of mass particle r (with mass m_r) be (x_r, y_r, z_r) and (ζ_r, η_r, ν_r) in the inertial and the other frame, respectively. Then

$$x_r = \overline{x} + \zeta_r \ , \qquad y_r = \overline{y} + \eta_r \ , \qquad z_r = \overline{z} + \nu_r$$

Koenig's theorem states that the kinetic energy of the body is given by

$$T = \frac{1}{2} m \left(\dot{\overline{x}}^2 + \dot{\overline{y}}^2 + \dot{\overline{z}}^2 \right) + \frac{1}{2} \sum_r m_r \left(\dot{\zeta}_r^2 + \dot{\eta}_r^2 + \dot{\nu}_r^2 \right) \tag{1.58}$$

where $m = \sum_r m_r$ is the mass of the body. Note that, like Eqn. (1.56), this equation divides T into a translational and a rotational part.

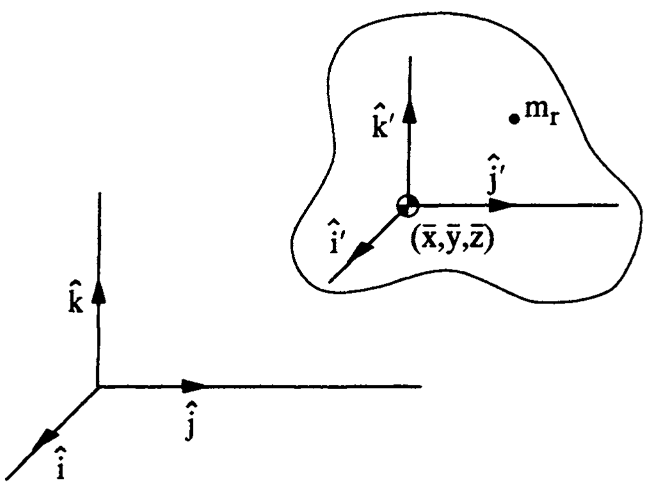


Fig. 1-22

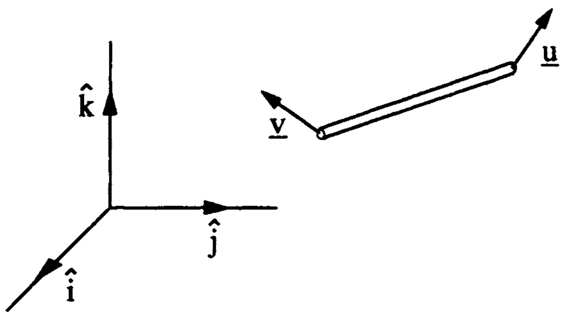


Fig. 1-23

A method of obtaining T for a rigid body for the special case in which one point of the body remains fixed in an inertial frame is given in Section 11.1.

One final result is useful for a special case. Suppose a rigid body may be idealized as a homogeneous, thin, straight rod and suppose that at some instant the velocities of the ends of the rod are $\underline{u}$ and $\underline{v}$ (Fig. 1-23). Then T of the rod is

$$T = \frac{1}{6}m(\underline{u} \cdot \underline{u} + \underline{u} \cdot \underline{v} + \underline{v} \cdot \underline{v}) \tag{1.59}$$

System of Rigid Bodies. Consider a system of constant mass, not necessarily rigid, but consisting of a number of rigid bodies. Then for the system Eqn. (1.54) applies where now T is the sum of all the kinetic energies of all the rigid bodies, V is the total potential energy of the system, and U^{nc} is the work done by all the nonconservative forces.

1.5 Examples

Simple Pendulum. A bob of mass m is suspended by an inextensible, weightless cord and moves in the (x, y) plane (Fig. 1-24). We obtain the equation of motion by three methods:

(a) $\sum \underline{F} = m\underline{a}$ in rectangular components (Eqns. 1.10):

$$\sum F_x = m\ddot{x} \quad \Longrightarrow \quad -T\sin\theta = m\ddot{x}$$

$$\sum F_y = m\ddot{y} \quad \Longrightarrow \quad T\cos\theta - mg = m\ddot{y}$$

eliminate T to get: $\quad -\frac{\cos\theta}{\sin\theta}\ddot{x} - g = \ddot{y}$

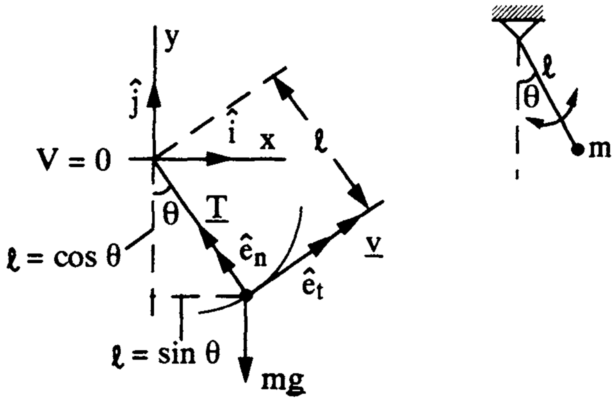


Fig. 1-24

but

$$x = \ell \sin\theta \quad \Longrightarrow \quad \ddot{x} = \ell\cos\ddot{\theta} - \ell\sin\theta\dot{\theta}^2$$

$$y = -\ell\cos\theta \quad \Longrightarrow \quad \ddot{y} = \ell\sin\theta\ddot{\theta} + \ell\cos\theta\dot{\theta}^2$$

Therefore

$$\ddot{\theta} + \frac{g}{\ell}\sin\theta = 0 \tag{1.60}$$

(b) $\sum \underline{F} = m\underline{a}$ in normal-tangential components (Eqn. 1.16):

$$\sum F_t = m\dot{v} \quad \Longrightarrow \quad -mg\sin\theta = m\ell\ddot{\theta}$$

$$\sum F_n = m\frac{v^2}{\rho} \quad \Longrightarrow \quad T - mg\cos\theta = m\tfrac{v^2}{\ell}$$

The first equation provides the equation of motion:

$$\ddot{\theta} + \frac{g}{\ell}\sin\theta = 0$$

and the second gives the force T.

(c) The work-energy relation (Eqn. 1.46): Since the chord tension $\underline{T}$ is perpendicular to $\underline{v}$, $U = \int \underline{T}\cdot\underline{v}dt = 0$ and $\underline{T}$ does no work. The only force doing work is weight, mg, and this force is conservative; therefore energy is conserved and Eqn. (1.47) applies:

$$E = T + V = \frac{1}{2} m \underbrace{v^2}_{\ell^2\dot{\theta}^2} - mg\ell\cos\theta = \text{constant}$$

$$\dot{E} = 2\frac{1}{2} m\ell^2\dot{\theta}\ddot{\theta} + mg\ell\sin\theta\dot{\theta} = 0$$

$$\ddot{\theta} + \frac{g}{\ell}\sin\theta = 0$$

Clearly, method (a) requires the most work and (c) the least. Note that method (c) does not give force T, which may be of interest, and gives a once integrated form of the equation of motion.

Robot Link. A rigid body moves in the (x, y) plane such that one of its points, say B, remains fixed in an inertial frame. An external moment $\underline{M} = M\hat{k}$ is applied at B (Fig. 1-25). We obtain the equation of motion by two methods.

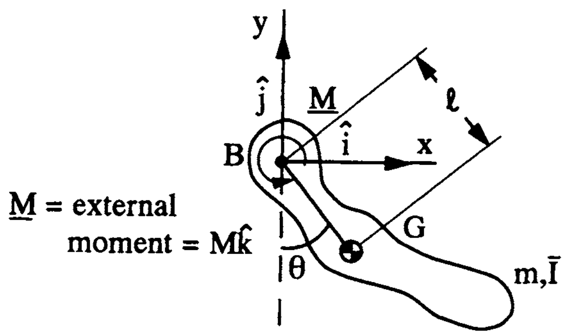


Fig. 1-25

(a) Equation (1.51) relative to point B:

$$\sum M_B = I_B\alpha \Longrightarrow M = (\bar{I} + m\ell^2)\ddot{\theta}$$

$$[I_B = \bar{I} + m\ell^2 \text{ by the } \textit{parallel axis theorem}]$$

$$\ddot{\theta} - \frac{1}{(\bar{I} + m\ell^2)} M = 0$$

(b) Equations (1.50) and (1.51) relative to point A:

$$F_x = m\ddot{\bar{x}}\,, \quad F_y = m\ddot{\bar{y}}\,, \quad M - F_y\ell\sin\theta - F_x\ell\cos\theta = \bar{I}\ddot{\theta}$$

$$\bar{x} = \ell \sin\theta\,, \qquad \bar{y} = -\ell\cos\theta$$
$$M - m\ell^2\ddot{\theta} = \bar{I}\ddot{\theta}$$
$$\ddot{\theta} - \frac{1}{(\bar{I} + m\ell^2)}M = 0 \qquad (1.61)$$

Physical Pendulum. This may be treated as a special case of the robot link with the gravitational force providing the moment (Fig. 1-26). The equation of motion is obtained by three methods.

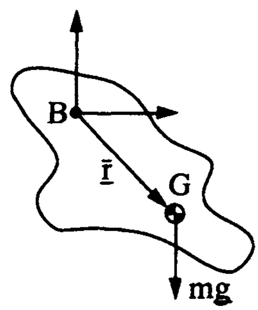


Fig. 1-26

(a) Equation (1.61):

$$\underline{M} = \underline{\bar{r}} \times m\underline{g}$$
$$M = -mg\ell\sin\theta$$
$$\ddot{\theta} + \frac{mg\ell}{(\bar{I} + m\ell^2)}\sin\theta = 0$$

(b) Conservation of energy, Eqn. (1.47), with T computed from Eqn. (1.57):

$$E = T + V = \frac{1}{2}I_B\omega^2 - mg\ell\cos\theta = \text{constant}$$
$$\dot{E} = 2\frac{1}{2}(\bar{I} + m\ell^2)\dot{\theta}\ddot{\theta} + mg\ell\sin\theta\dot{\theta} = 0$$
$$\ddot{\theta} + \frac{mg\ell}{(\bar{I} + m\ell^2)}\sin\theta = 0$$

(c) Conservation of energy with T computed from Eqn. (1.56):

$$E = T + V = \frac{1}{2}\bar{I}\underbrace{\omega^2}_{\dot{\theta}^2} + \frac{1}{2}m\underbrace{\bar{v}^2}_{\ell^2\dot{\theta}^2} - mg\ell\cos\theta = \text{constant}$$

$$E = \tfrac{1}{2}(\bar{I} + m\ell^2)\dot{\theta}^2 - mg\ell\cos\theta = \text{constant}$$

The simple pendulum, Eqn. (1.60), is the special case of $\bar{I} = 0$.

Car Accelerating Up a Hill. A car of mass m has acceleration $\underline{\bar{a}}$ and velocity $\underline{\bar{v}}$ up an incline of angle θ (Fig. 1-27). The wind resistance is $\underline{D}$ and each of the four wheels has a moment of inertia $\bar{I}$ and a radius r. Friction is sufficient to prevent wheel slipping. We want to find the power, P_e, delivered by the engine to the wheels.

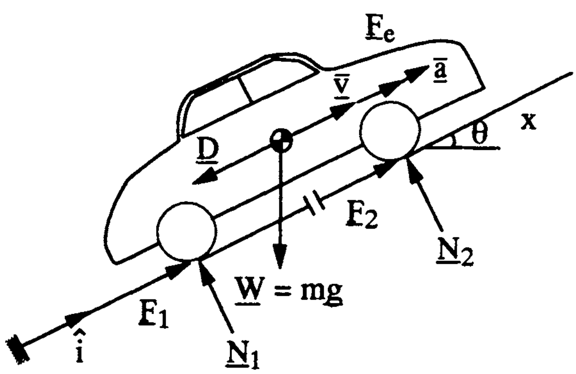


Fig. 1-27

Since the wheels roll without slipping, there is no velocity at the point of contact with the road. Consequently, from Eqn. (1.27), $\underline{F}_1$ and $\underline{F}_2$ do no work. Also, $\underline{N}_1$ and $\underline{N}_2$ do no work because they are normal to any possible velocity, even if there is slipping. The kinetic and potential energies are found from Eqns. (1.56) and (1.42) as (see Fig. 1-28):

$$\begin{aligned} T &= \frac{1}{2}m\bar{v}^2 + 4\frac{1}{2}\bar{I}\left(\frac{\bar{v}}{r}\right)^2 \\ V &= mg\sin\theta x \end{aligned}$$

The work done by force D is

$$U^D = \int \underline{D}\cdot\underline{\bar{v}}dt = -\int D\bar{v}dt$$

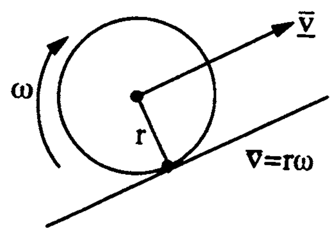


Fig. 1-28

Now differentiate the energy equation, Eqn. (1.44), and substitute the above relations:

$$U_{0,1}^{nc} = \Delta T_{0,1} + \Delta V_{0,1}$$

$$\int_{t_0}^{t_1} \underline{F}^{nc} \cdot \underline{v} dt = T_1 - T_0 + V_1 - V_0$$

$$\int_{t_0}^{t} \underline{F}^{nc} \cdot \underline{v} dt = T - T_0 + V - V_0$$

$$F^{nc}\overline{v} = \dot{T} + \dot{V}$$

$$F_e\overline{v} - D\overline{v} = 2\frac{1}{2}m\overline{v}\,\overline{a} + 4\frac{1}{2}\overline{I}\frac{2\overline{v}\,\overline{a}}{r^2} + mg\overline{v}\sin\theta$$

$$P_e = F_e\overline{v} = m\overline{v}\,\overline{a} + 4\frac{\overline{I}\overline{v}\,\overline{a}}{r^2} + mg\sin\theta\overline{v} + D\overline{v}$$

This shows that the power produced by the engine is used in four separate ways: (1) to accelerate the car, (2) to spin-up the wheels, (3) to gain altitude, and (4) to overcome air resistance.

1.6 Motivation for Analytical Dynamics

Lessons from Newtonian Dynamics. This chapter has revealed the following:

1. There are differences in forces. Forces may be classified as external or internal (the latter cancel out in a system of particles). They may be conservative or nonconservative (the former can be accounted for by potential energy functions). And finally, some forces do work while others do not.

2. Components along the coordinate axes are not always the best parameters to describe the motion; for example, θ is the best parameter for the simple pendulum.

3. The energy method is simple when appropriate. The advantages are: (i) Unit vectors, free-body diagrams, and coordinate systems are not needed, (ii) It gives an integral of the motion; that is, it gives a relation of speeds, not accelerations. The disadvantages are: (i) It gives only one equation, and thus possibly doesn't give the information desired, (ii) It is cumbersome when forces do work and are nonconservative.

These observations motivate the search for new approachs to dynamics.

Purposes of Analytical Dynamics. The main goals of the rest of this book are to:

1. Obtain scalar equations of motion invariant to coordinate transformation in the minimum number of variables.

2. Eliminate constraint forces and treat conservative forces via potential energy functions.

3. Obtain solutions of the equations of motion.

Notes

1 About 100 years ago, experiments showed that in certain situations Newton's Laws were significantly inaccurate. Relativity theory was developed to account for these discrepancies, and is now the most accurate description of dynamics. In relativity theory, the physical world is a four dimensional, non-Eulclidean space, there are no inertial reference frames, and the values of mass and time are different for different observers.

2 Vectors will be underlined here and throughout most of the book; the exception is unit vectors, which will get "hats".

3 It may be shown (Ardema, *Newton-Euler Dynamics*) that all body-fixed frames have the same angular velocity w.r.t. any other frame.

4 Note that the origin of the body-fixed frame need not be "in" the body.

5 If these conditions are not met, then the equation contains products as well as moments of inertia. See Ardema, *Newton-Euler Dynamics*, for a complete discussion of Eulerian dynamics of a rigid body.

PROBLEMS

1/1. A point moves at a constant speed of 5 ft/s along a path given by $y = 10e^{-2x}$, where x and y are in ft. Find the acceleration of the point when $x = 2$ ft.

1/2. A point moves at constant speed v along a curve defined by $r = A\theta$, where A is a constant. Find the normal and tangential components of acceleration.

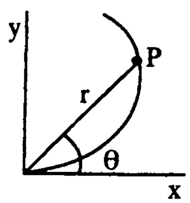


Fig. 1/2

Problem 1/2

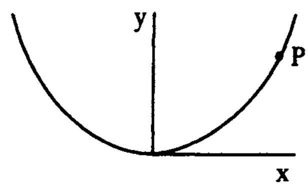
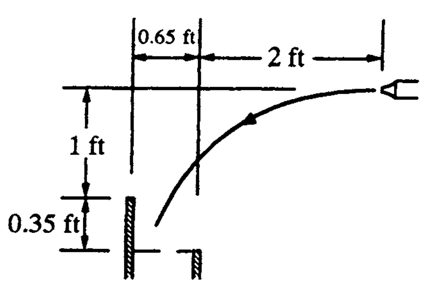


Fig. 1/3

Problem 1/3

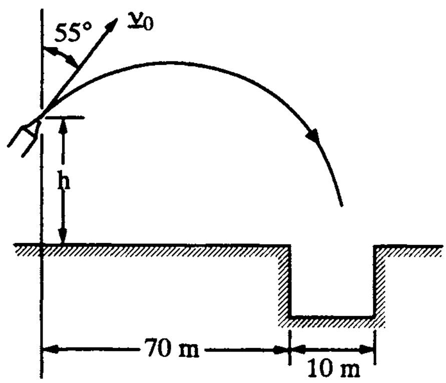
1/3. A point travels along a parabola $y = kx^2$, k a constant, such that the horizontal component of velocity, $\dot{x}$, remains a constant. Determine the acceleration of the point as a function of position.

1/4. Grain is being discharged from a nozzle into a vertical shute with an initial horizontal velocity $\underline{v}_0$. Determine the range of values of v_0 for which the grain will enter the shute.



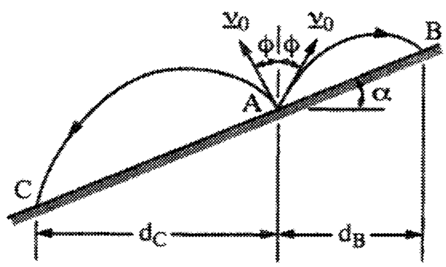
Problem 1/4

1/5. A nozzle is located at a height h above the ground and discharges water at a speed $v_0 = 25$ m/s at an angle of 55° with the vertical. Determine the range of values of h for which the water enters the trench in the ground.

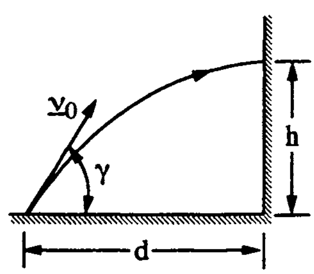


Problem 1/5

1/6. A rotating water sprinkler is positioned at point A on a lawn inclined at an angle $\alpha = 10°$ relative to the horizontal. The water is discharged with a speed of $v_0 = 8$ ft/s at an angle of $\phi = 40°$ to the vertical. Determine the horizontal distances d_C and d_B where the water lands.



Problem 1/6



Problem 1/7

1/7. A ball is thrown with velocity $\underline{v}_0$ against a vertical wall a distance d away. Determine the maximum height h at which the ball can strike the wall and the corresponding angle γ, in terms of v_0, d, and g.

1/8. In Problem 1/7, is the ball ascending or descending when it strikes the wall? What minimum speed v_0 is needed to strike the wall at all?

1/9. A condition of "weightlessness" may be obtained by an airplane flying a curved path in the vertical plane as shown. If the plane's speed is $v = 800$ km/h, what must be the rate of rotation of the airplane $\dot{\gamma}$ to obtain this condition at the top of its loop?

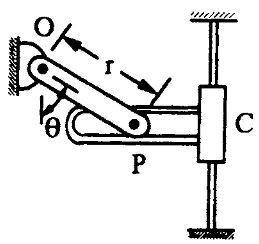
1/10. The speed of a car is increasing at a constant rate from 60 mi/h to 75 mi/h over a distance of 600 ft along a curve of 800 ft radius. What is the magnitude of the total acceleration of the car after it has traveled 400 ft along the turn?

1/11. Consider the situation of Problem 1/5. Determine the radius of curvature of the stream both as it leaves the nozzle and at its maximum height.

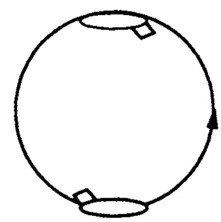
1/12. Consider again the situation of Problem 1/5. It was observed that the radius of curvature of the stream of water as it left the nozzle was 35 ft. Find the speed v_0 with which the water left the nozzle, and the radius of curvature of the stream when it reaches its maximum height, for $\theta = 36.87°$.

1/13. The velocity of a point at a certain instant is $\underline{v} = 3\hat{\imath} + 4\hat{\jmath}$ ft/s, and the radius of curvature of its path is 6.25 ft. The speed of the point is decreasing at the rate of 2 ft/s^2. Express the velocity and acceleration of the point in tangential-normal components.

1/14. Link OP rotates about O, and pin P slides in the slot attached to collar C. Determine the velocity and acceleration of collar C as a function of θ for the following cases:
(i) $\dot{\theta} = \omega$ and $\ddot{\theta} = 0$,
(ii) $\dot{\theta} = 0$ and $\ddot{\theta} = \alpha$.



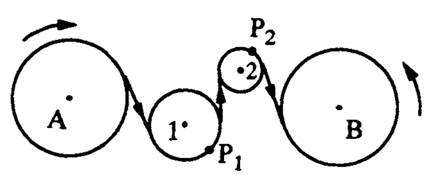
Problem 1/14



Problem 1/15

1/15. At the bottom A of a vertical inside loop, the magnitude of the total acceleration of the airplane is 3g. If the airspeed is 800 mph and is increasing at the rate of 20 mph per second, determine the radius of curvature of the path at A.

1/16. Tape is being transferred from drum A to drum B via two pulleys. The radius of pulley 1 is 1.0 in and that of pulley 2 is 0.5 in. At P_1, a point on pulley 1, the normal component of acceleration is 4 in/s^2 and at P_2, a point on pulley 2, the tangential component of acceleration is 3 in/s^2. At this instant, compute the speed of the tape, the magnitude of the total acceleration at P_1, and the magnitude of the total acceleration at P_2.



Problem 1/16

1/17. The shape of the stationary cam is that of a limacon, defined by $r = b - c\cos\theta$, $b > c$. Determine the magnitude of the total acceleration as a function of θ if the slotted arm rotates with a constant angular rate $\omega = \dot{\theta}$ in the counter clockwise direction.

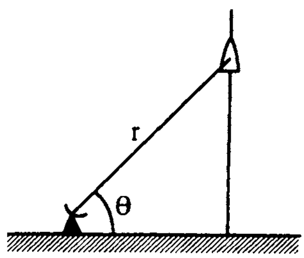
1/18. A radar used to track rocket launches is capable of measuring r, $\dot{r}$, $\dot{\theta}$, and $\ddot{\theta}$. The radar is in the vertical plane of the rocket's flight path. At a certain time, the measurements of a rocket are

$r = 35,000$ m, $\dot{r} = 1600$ m/s, $\dot{\theta} = 0$, and $\ddot{\theta} = -0.0072$ rad/s^2. What direction is the rocket heading relative to the radar at this time? What is the radius of curvature of its path?

1/19. A collar A slides on a thin rod OB such that $r = 60t^2 - 20t^3$, with r in meters and t in seconds. The rod rotates according to $\theta = 2t^2$, with θ in radius. Determine the velocity and total acceleration of the collar when $t = 1s$, using radial-transverse components.

1/20. Consider the same situation as in Problem 1/19, but with $r = 1.25t^2 - 0.9t^3$ and $\theta = \frac{1}{2}\pi(4t - 3t^3)$. Answer the same questions.

1/21. A vertically ascending rocket is tracked by radar as shown. When $\theta = 60°$, measurements give $r = 30,000$ ft, $\ddot{r} = 70$ ft/s^2, and $\dot{\theta} = 0.02$ rad/s. Determine the magnitudes of the velocity and the acceleration of the rocket at this instant.

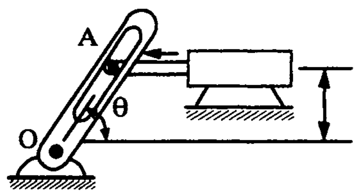


Problem 1/21

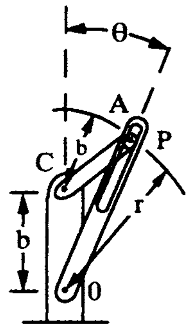
1/22. The path of fluid particles in a certain centrifugal pump is closely approximated by $r = r_0 e^{n\theta}$ where r_0 and n are constants. If the pump turns at a constant rate $\dot{\theta} = \omega$, determine the expression for the magnitude of the acceleration of a fluid particle when $r = R$.

1/23. The pin A at the end of the piston of the hydraulic cylinder has a constant speed 3 m/s in the direction shown. For the instant when $\theta = 60°$, determine $\dot{r}, \ddot{r}, \dot{\theta}$ and $\ddot{\theta}$, where $r = \overline{OA}$.

1/24. Slotted arm OA oscillates about O and drives crank P via the pin at P. For an interval of time, $\dot{\theta} = \omega =$ constant. During this time, determine the magnitude of the acceleration of P as a function of θ. Also, show that the magnitudes of the velocity and acceleration of P are constant during this time interval.

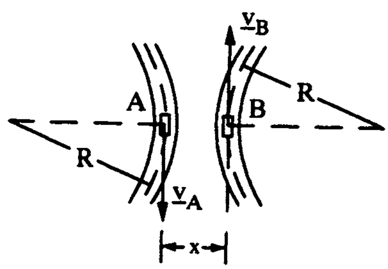


Problem 1/23



Problem 1/24

1/25. Two cars, labeled A and B, are traveling on curves with constant equal speeds of 72 km/hr. The curves both have radius $R = 100$m and their point of closest approach is $x = 30$m. Find the velocity of B relative to the occupants of A at the point of closest approach.

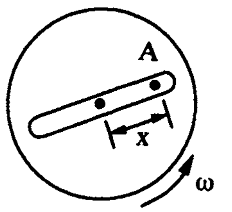


Problem 1/25

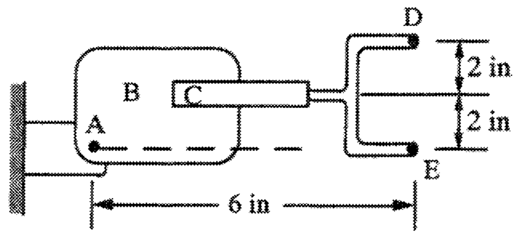
1/26. For the same conditions of Problem 1/25, find the acceleration of B relative to A.

1/27. For the same conditions of Problem 1/25, find the acceleration of B relative to A if A is speeding up at the rate of 3 m/s^2 and B is slowing down at the rate of 6 m/s^2.

1/28. At a certain instant, the disk is rotating with an angular speed of $\omega = 15$ rad/s and the speed is increasing at a rate of 20 rad/s^2. The slider moves in the slot in the disk at the constant rate $\dot{x} = 120$ in/s and at the same instant is at the center of the disk. Obtain the acceleration and velocity of the slider at this instant.



Problem 1/28

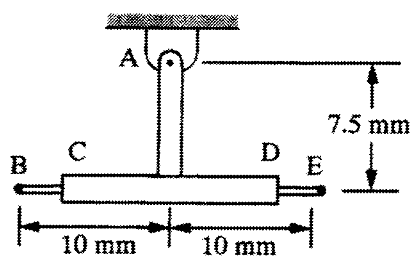


Problem 1/29

1/29. Shown is an automated welding device. Plate B rotates about point A, and the welding bracket with tips D and E moves in a cylinder C attached to B. At a certain instant, bracket DE is moving to the right with respect to plate B at a constant rate of 3 in/s and B is rotating counter clockwise about A at a constant rate of 1.6 rad/s. Determine the velocity and acceleration of tip E at that instant.

1/30. For the same situation as in Problem 1/29, determine the velocity and acceleration of tip D.

1/31. Bracket ACD is rotating clockwise about A at the constant rate of 2.4 rad/s. When in the position shown, rod BE is moving to the right relative to the bracket at the constant rate of 15 mm/s. Find the velocity and acceleration of point B.



Problem 1/31

1/32. Same as Problem 1/31, except that the rotation of the bracket is speeding up at a rate of 0.3 rad/s^2.

1/33. Same as Problem 1/31, except that the rod is slowing down at the rate of 2 mm/s^2.

1/34. Find the velocity and acceleration of point E for the situation in Problem 1/31.

1/35. Prove Eqn. (1.56).

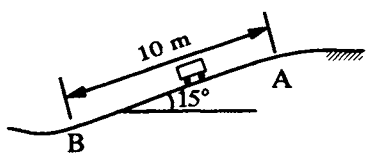
1/36. Prove Eqn. (1.57).

1/37. Prove Koenig's theorem, Eqn. (1.58).

1/38. Show that Eqn. (1.58) reduces to Eqn. (1.57) for the case of 2-D motion.

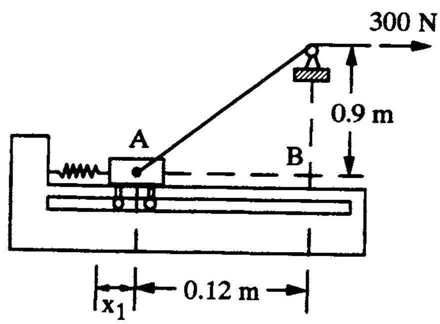
1/39. Prove Eqn. (1.59).

1/40. A 50 kg cart slides down an incline from A to B as shown. What is the speed of the cart at the bottom at B if it starts at the top at A with a speed of 4 m/s? The coefficient of kinetic friction is 0.30.



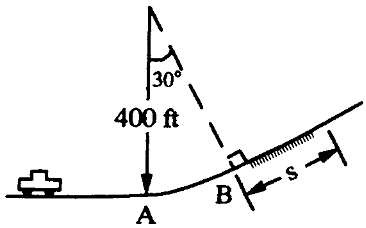
Problem 1/40

1/41. A 50 kg block slides without friction as shown. There is a constant force of 300 N in the cable and the spring attached to the block has stiffness 80 N/m. If the block is released from rest at a position A in which the spring is stretched by amount $x_1 = 0.233$ m, what is the speed when the block reaches position B.

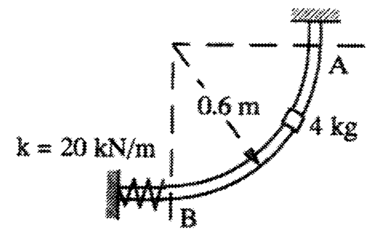


Problem 1/41

1/42. A 4000 lb car travels up a hill as shown. The car starts from rest at A and the engine exerts a constant force in the direction of travel of 1000 lb until position B is reached, at which time the engine is shut off. How far does the car roll up the hill before stopping? Neglect all friction and air resistance.



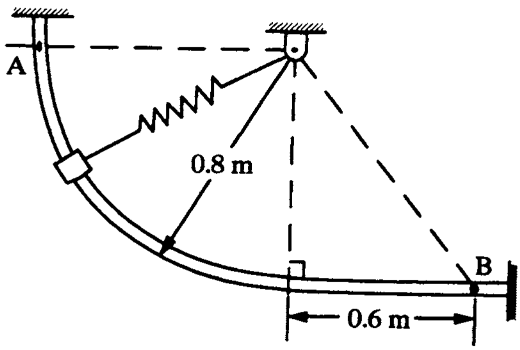
Problem 1/42



Problem 1/43

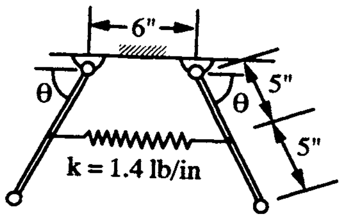
1/43. The small 4 kg collar is released from rest at A and slides down the circular rod in the vertical plane. Find the speed of the collar as it reaches the bottom at B and the maximum compression of the spring. Neglect friction.

1/44. The small 3 kg collar is released from rest at A and slides in the vertical plane to B. The attached spring has stiffness 200 N/m and an unstretched length of 0.4 m. What is the speed of the collar at B? Neglect friction.



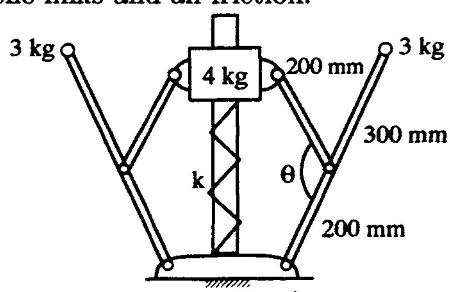
Problem 1/44

1/45. The identical links are released simultaneously from rest at $\theta = 30°$ and rotate in the vertical plane. Find the speed of each 2 lb mass when $\theta = 90°$. The spring is unstretched when $\theta = 90°$. Ignore the mass of the links and model the masses as particles.



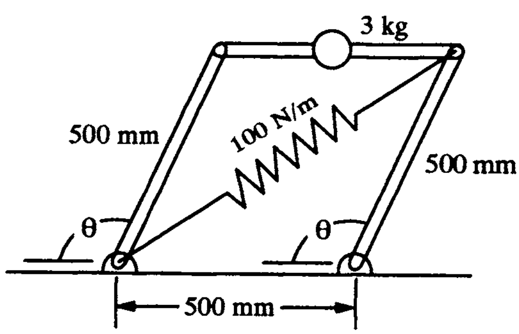
Problem 1/45

1/46. The device shown is released from rest with $\theta = 180°$ and moves in the vertical plane. The spring has stiffness 900 N/m and is just touching the underside of the collar when $\theta = 180°$. Determine the angle θ when the spring reaches maximum compression. Neglect the masses of the links and all friction.



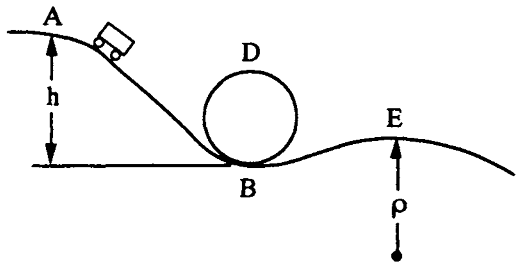
Problem 1/46

1/47. Shown is a frame of negligible weight and friction that rotates in the vertical plane and carries a 3 kg mass. The spring is unstretched when $\theta = 90°$. If the frame is released from rest at $\theta = 90°$, determine the speed of the mass when $\theta = 135°$ is passed.



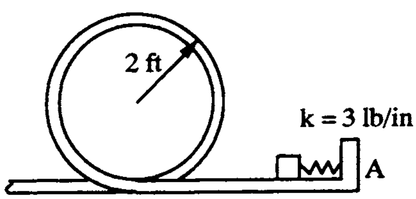
Problem 1/47

1/48. A roller coaster car starts from rest at A, rolls down the track to B, transits a circular loop of 40 ft diameter, and then moves over the hump at E. If $h = 60$ ft, determine (a) the force exerted by his or her seat on a 160 lb rider at both B and D, and (b) the minimum value of the radius of curvature of E if the car is not to leave the track at that point. Neglect all friction and air resistance.

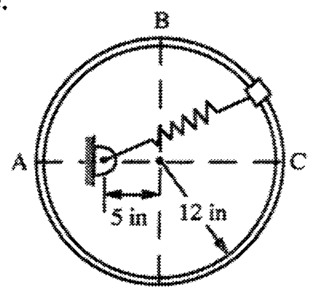


Problem 1/48

1/49. The 0.5 lb pellet is pushed against the spring at A and released from rest. Determine the smallest deflection of the spring for which the pellet will remain in contact with the circular loop at all times. Neglect friction and air resistance.



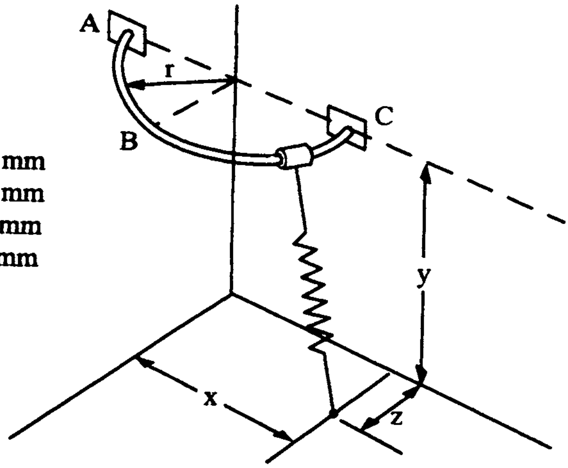
Problem 1/49



Problem 1/50

1/50. A 3 lb collar is attached to a spring and slides without friction on a circular hoop in a horizontal plane. The spring constant is 1.5 lb/in and is undeformed when the collar is at A. If the collar is released at C with speed 6 ft/s, find the speeds of the collar as it passes through points B and A.

1/51. A 600 g collar slides without friction on a horizontal semicircular rod ABC of radius 200 mm and is attached to a spring of spring constant 135 N/m and undeformed length 250 mm. If the collar is



Problem 1/51

released from rest at A, what are the speeds of the collar at B and C?

1/52. Prove that a force is conservative if and only if the following relations are satisfied:

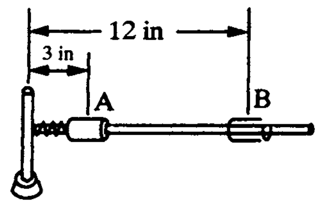
$$\frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x}, \qquad \frac{\partial F_y}{\partial z} = \frac{\partial F_z}{\partial y}, \qquad \frac{\partial F_z}{\partial x} = \frac{\partial F_x}{\partial z}$$

1/53. Show that the force

$$\underline{F} = (x\hat{\imath} + y\hat{\jmath} + z\hat{k})(x^2 + y^2 + z^2)$$

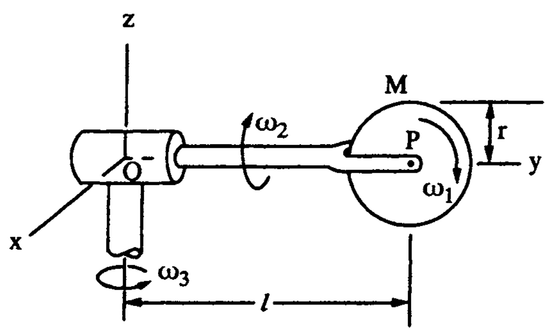
is conservative by applying the results of Problem 1/52. Also find the potential energy function $V(x, y, z)$ associated with $\underline{F}$.

1/54. A 1/2 lb collar slides without friction on a horizontal rod which rotates about a vertical shaft. The collar is initially held in position A against a spring of spring constant 2.5 lb/ft and unstretched length 9 in. As the rod is rotating at angular speed 12 rad/s the cord is cut, releasing the collar to slide along the rod. The spring is attached to the collar and the rod. Find the angular speed of the rod and the radial and transverse components of the velocity of the collar as the rod passes postion B. Also find the maximum distance from the vertical shaft that the collar will reach.



Problem 1/54

1/55. Determine the kinetic energy of the uniform circular disk of mass M at the instant shown.



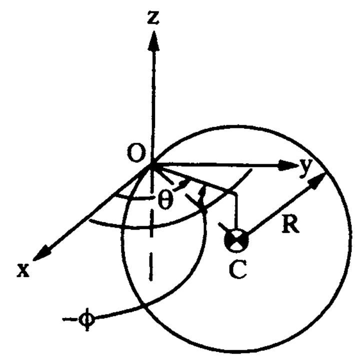
Problem 1/55

1/56. Find the kinetic energy of a homogeneous solid disk of mass m and radius r that rolls without slipping along a straight line. The center of the disk moves with constant velocity v.

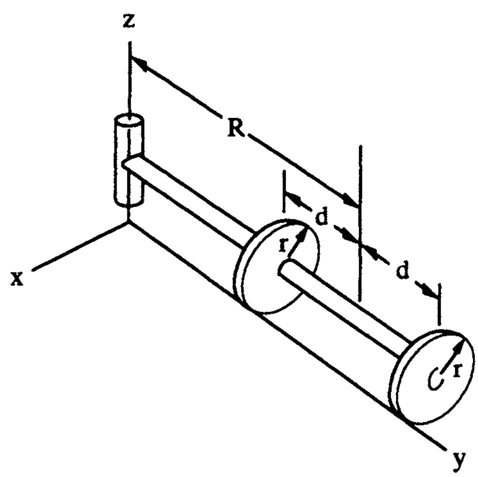
1/57. A homogeneous solid sphere of mass M and radius R is fixed at a point O on its surface by a ball joint. Find the kinetic energy of the sphere for general motion.

1/58. Two uniform circular disks, each of mass M and radius r, are mounted on the same shaft as shown. The shaft turns about the z-axis, while the two disks roll on the xy-plane without slipping. Prove that the ratio of the kinetic energies of the two disks is

$$\frac{6(R+d)^2+r^2}{6(R-d)^2+r^2}$$



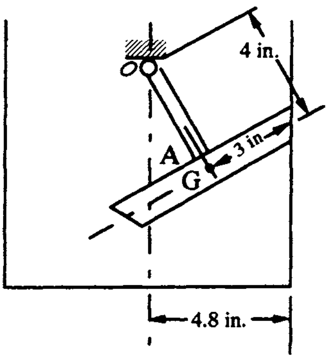
Problem 1/57



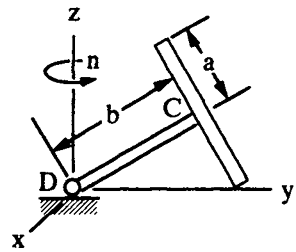
Problem 1/58

1/59. A disk with arm OA is attached to a socket joint at O. The moment of inertia of the disk and arm about axis OA is I and the total mass is M, with the center of gravity at G. The disk rolls inside a cylinder whose radius is 4.8 in. Find the kinetic energy of the disk when the line of contact turns around the cylinder at 10 cycles per second.

1/60. A uniform circular disk of radius a and mass M is mounted on a weightless shaft CD of length b. The shaft is normal to the disk at its center C. The disk rolls on the xy-plane without slipping, with point D remaining at the origin. Determine the kinetic energy of the disk if shaft CD rotates about the z-axis with constant angular speed n.

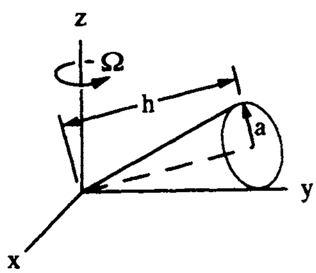


Problem 1/59

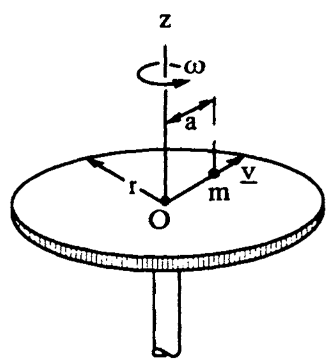


Problem 1/60

1/61. A homogeneous solid right circular cone rolls on a plane without slipping. The line of contact turns at constant angular speed Ω about the z-axis. Find the kinetic energy of the cone.



Problem 1/61

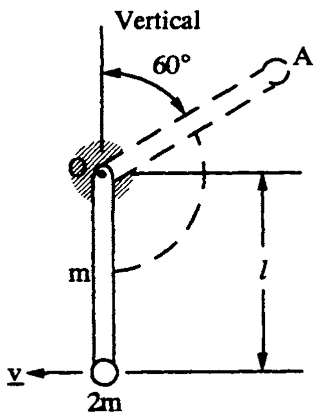


Problem 1/62

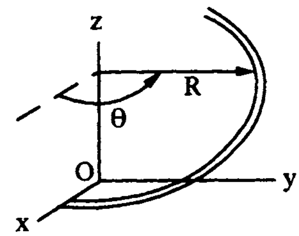
1/62. A particle of mass m slides along one radius of a circular platform of mass M. At the instant shown, the platform has an angular velocity ω and the particle has a velocity v relative to the platform. Determine the kinetic energy and the angular momentum of the system about point O.

1/63. A pendulum consists of a uniform rod of mass m and a bob of mass $2m$. The pendulum is released from rest at position A as shown. What is the kinetic energy of the system at the lowest position? What is the velocity of the bob at the lowest position?

1/64. A particle of mass m is attracted toward the origin by a force with magnitude $(mK)/r^2$ where K is a constant and r is the distance



Problem 1/63



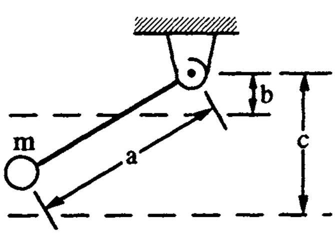
Problem 1/64

between the particle and the center of attraction. The particle is constrained to move in a frictionless tube which lies along the space curve given by

$$\left.\begin{aligned} z &= 5\theta \\ R &= 1 + \frac{1}{2}\theta \end{aligned}\right\} \quad \text{in cylindrical coordinates}$$

If the particle was at rest when $z = 10$, what is the velocity of the particle at $z = 0$?

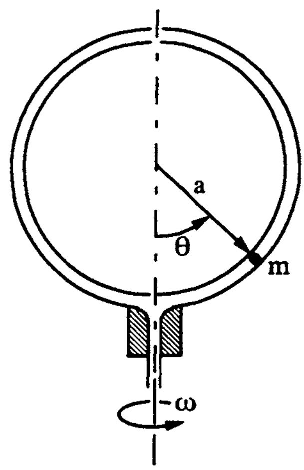
1/65. A spherical pendulum of mass m and length a oscillates between levels b and c, located below the support. Find the expression for the total energy of the system in terms of a, b, c, m, and g, taking the horizontal plane passing through the support as the zero potential level.



Problem 1/65

1/66. A spherical pendulum, consisting of a massless rod and a bob of mass m, is initially held at rest in the horizontal plane. A horizontal velocity v_0 is imparted to the bob normal to the rod. In the resulting motion, what is the angle between the rod and the horizontal plane when the bob is at its lowest position?

1/67. A particle of mass m is placed inside a frictionless tube of negligible mass. The tube is bent into a circular ring with the lowest point left open as shown. The ring is given an initial angular velocity ω about the vertical axis passing through the diameter containing the opening, and simultaneously the particle is released from rest (relative to the tube) at $\theta = \pi/2$. In the subsequent motion, will the particle drop through the opening?



Problem 1/67

Chapter 2

Motion and Constraints

2.1 Newton's Second Law

Vector Form. Consider a system of n mass particles of masses m_1, $m_2, \ldots, m_n$. (Occasionally we will call a system of particles a *dynamic system.*) Let the position of particle r in an inertial reference frame be denoted $\underline{x}^r(t)$, as shown on Fig. 2-1. Let the resultant forces on the particles be bounded functions of the particles' positions, velocities, and time. Then Eqn. (1.1) gives

$$\begin{aligned}
m_1\ddot{\underline{x}}^1(t) &= \sum \underline{F}^1(\underline{x}^1,\ \cdot\cdot,\ \underline{x}^n,\ \dot{\underline{x}}^1,\ \cdot\cdot\ \dot{\underline{x}}^n,\ t) \\
m_2\ddot{\underline{x}}^2(t) &= \sum \underline{F}^2(\underline{x}^1,\ \cdot\cdot,\ \underline{x}^n,\ \dot{\underline{x}}^1,\ \cdot\cdot\ \dot{\underline{x}}^n,\ t) \\
&\ \vdots \\
m_n\ddot{\underline{x}}^n(t) &= \sum \underline{F}^n(\underline{x}^1,\ \cdot\cdot,\ \underline{x}^n,\ \dot{\underline{x}}^1,\ \cdot\cdot\ \dot{\underline{x}}^n,\ t)
\end{aligned} \tag{2.1}$$

or

$$m_r\ddot{\underline{x}}^r(t) = \sum \underline{F}^r(\underline{x}^1,\ \cdot\cdot,\ \underline{x}^n,\ \dot{\underline{x}}^1,\ \cdot\cdot,\ \dot{\underline{x}}^n,\ t)\ ; \qquad r = 1, 2,\ \cdot\cdot,\ n \tag{2.2}$$

If none of the forces depend explicitly on time, we say the system is *autonomous.* Note that forces are not allowed to be functions of the particles' accelerations.[1]

In the "Newtonian" problem, unbounded forces are allowed provided they are measurable, that is if $\int \underline{F}^r dt$; $r = 1, \cdot\cdot, n$ are always bounded. A force which is unbounded but measurable is called an impulsive force. In the "strictly Newtonian" problem, all forces are bounded. For most

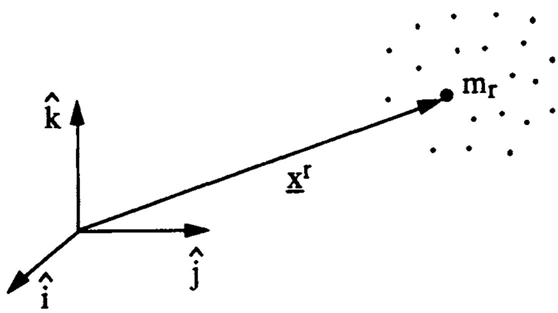


Fig. 2-1

of this book we restrict our attention to the strictly Newtonian problem. The exception is Chapter 13 where impulsive forces will be considered.

Recall that there are two ways Newton's Second Law can be used. One way (problem of the second kind) is to determine the forces acting on a system when the motion of the system is given. This is typically the situation at the design stage. For example, when designing a space launch system the motion is known (transition from earth surface to orbit location and speed) and Newton's Second Law can be used to predict the propulsive forces required, and hence the size of the vehicle. The second way is to determine the motion when the forces are given (problem of the first kind). This situation typically arises in the performance estimation of an existing system. For example, it may be of interest to determine the range of orbits accessible by an existing launch vehicle. In this book we approach dynamics as a problem of the first kind, although all the results obtained apply equally to either type of problem. Thus it is characteristic that the equations of motion of a particle system give the accelerations of the particles in terms of their positions, velocities, and time.

Component Form. Now introduce linearly independent unit vectors $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$. Then, if $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ are fixed in the inertial frame,

$$\begin{aligned} \underline{x}^r(t) &= x_1^r(t)\hat{e}_1 + x_2^r(t)\hat{e}_2 + x_3^r(t)\hat{e}_3 \\ \ddot{\underline{x}}^r(t) &= \ddot{x}_1^r(t)\hat{e}_1 + \ddot{x}_2^r(t)\hat{e}_2 + \ddot{x}_3^r(t)\hat{e}_3 \end{aligned} \tag{2.3}$$

Label the components of $\underline{x}^1$, $\underline{x}^2$, $\cdot\cdot$, $\underline{x}^n$ as follows

$$u_1 = x_1^1 \ , \ \ u_2 = x_2^1 \ , \ \ u_3 = x_3^1 \ , \ \ u_4 = x_1^2 \ ,$$

$$u_5 = x_2^2\,,\ \cdot\cdot,\ u_N = x_3^n\,;\quad N = 3n \tag{2.4}$$

Then we can write the Second Law as

$$\begin{aligned}
m_1\ddot{u}_1 &= \sum F_1^1(u_1, u_2,\ \cdot\cdot,\ u_N, \dot{u}_1, \dot{u}_2,\ \cdot\cdot,\ \dot{u}_N,\ t)\\
m_1\ddot{u}_2 &= \sum F_2^1(u_1, u_2,\ \cdot\cdot,\ u_N, \dot{u}_1, \dot{u}_2,\ \cdot\cdot,\ \dot{u}_N,\ t)\\
m_1\ddot{u}_3 &= \sum F_3^1(u_1, u_2,\ \cdot\cdot,\ u_N, \dot{u}_1, \dot{u}_2,\ \cdot\cdot,\ \dot{u}_N,\ t)\\
m_2\ddot{u}_4 &= \sum F_1^2(u_1, u_2,\ \cdot\cdot,\ u_N, \dot{u}_1, \dot{u}_2,\ \cdot\cdot,\ \dot{u}_N,\ t)\\
&\vdots\\
m_n\ddot{u}_N &= \sum F_3^n(u_1, u_2,\ \cdot\cdot,\ u_N, \dot{u}_1, \dot{u}_2,\ \cdot\cdot,\ \dot{u}_N,\ t)
\end{aligned} \tag{2.5}$$

or

$$m_s\ddot{u}_s = \sum F_s(u_1,\ \cdot\cdot,\ u_N,\ \dot{u}_1,\ \cdot\cdot,\ \dot{u}_N,\ t)\,;\quad s = 1,\ \cdot\cdot,\ N \tag{2.6}$$

Note the interpretations of m_s and F_s; for example, m_1, m_2, and m_3 are all the mass of the first particle and F_1, F_2, and F_3 are the three components of the resultant force acting on the first particle.

2.2 Motion Representation

Configuration Space. By the correspondence between n-tuples and vectors in Euclidean spaces, the components of displacement, $u_1,\ \cdot\cdot,\ u_N$ can be thought of as forming a vector in a subset of $\mathbb{E}^N$, the N-dimensional Euclidean space:

$$\underline{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{pmatrix} \in C \subset \mathbb{E}^N \tag{2.7}$$

We call C the *configuration space.* As the motion of the system proceeds, a path is generated in this space called a C trajectory.

Event Space. The combination of the configuration components and time is called an event; an event is a vector in the *event space* E:

$$(\underline{u}, t) = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \\ t \end{pmatrix} \in E \subset \mathbb{E}^{N+1} \tag{2.8}$$

Paths in this space are called E trajectories.

State Space. The combination of the configuration components and the components of velocity defines a point in *state space* S:

$$(\underline{u}, \underline{\dot{u}}) = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \\ \dot{u}_1 \\ \dot{u}_2 \\ \vdots \\ \dot{u}_N \end{pmatrix} \in S \subset \mathbb{E}^{2N} \tag{2.9}$$

Paths in this space are S trajectories.

State-Time Space. The combination of states and time gives a point in *state-time space*

$$(\underline{u}, \underline{\dot{u}}, t) = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \\ \dot{u}_1 \\ \dot{u}_2 \\ \vdots \\ \dot{u}_N \\ t \end{pmatrix} \in T \subset \mathbb{E}^{2N+1} \tag{2.10}$$

2.3 Holonomic Constraints

Introduction. The motion of a particle system is frequently subject to constraints. As an example, suppose the motion of a single particle is constrained to be on a surface, as shown on Fig. 2-2. Note that now only two of the coordinates are independent; the third, say z, is determined by the constraint.

A special case is motion in the (x, y) plane, for which the constraint is:

$$f(x, y, z) = z = 0 \Longrightarrow \dot{z} = 0, \quad \ddot{z} = 0$$

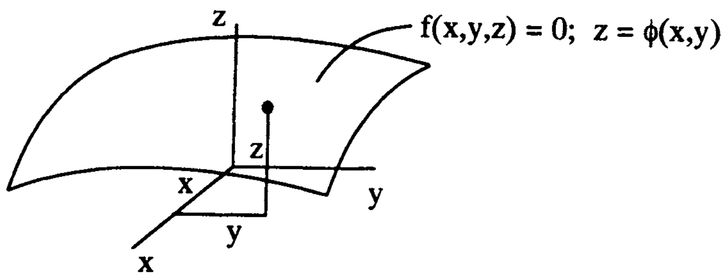


Fig. 2-2

Equations (2.6) become, for this case:

$$\begin{aligned} m\ddot{x} &= F_x(x, y, 0, \dot{x}, \dot{y}, 0, t) \\ m\ddot{y} &= F_y(x, y, 0, \dot{x}, \dot{y}, 0, t) \\ 0 &= F_z(x, y, 0, \dot{x}, \dot{y}, 0, t) \end{aligned}$$

These are the equations of planar motion as expected.

Of course constraints also may be prescribed functions of time, for example:

$$f(x, y, z, t) = 0$$

Definitions. Consider a system of n particles; such a system has a $N = 3n$ dimensional configuration space C. A *holonomic* constraint on the motion of the particles is one that can be expressed in the form

$$f(u_1, u_2, \cdots, u_N, t) = 0 \tag{2.11}$$

Otherwise the constraint is *nonholonomic.* As a special case, if a holonomic constraint can be expressed as

$$f(u_1, u_2, \cdots, u_N) = 0 \tag{2.12}$$

then it is *scleronomic*; otherwise it is *rheonomic.* If all constraints are holonomic we say the system is holonomic, and if all holonomic constraints are scleronomic the system is scleronomic.

The constraint equation is an $N - 1$ dimensional surface in the configuration space $C \subset \mathbb{E}^N$; the C trajectories must lie on this surface. Of course there may be several such constraints.

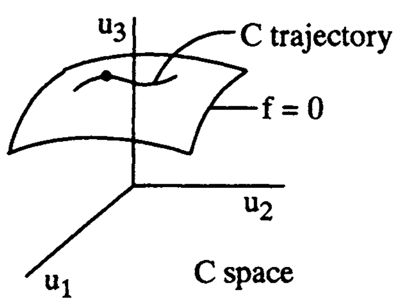


Fig. 2-3

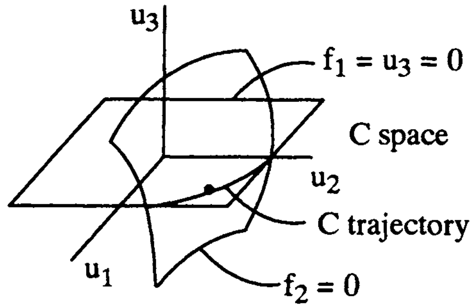


Fig. 2-4

Figures 2-3 and 2-4 show the cases of a single particle subject to one and two constraints, respectively. In E space, for the case of two components, the E trajectories lie on a right cylindrical surface, if the constraints are scleronomic (Fig. 2-5).

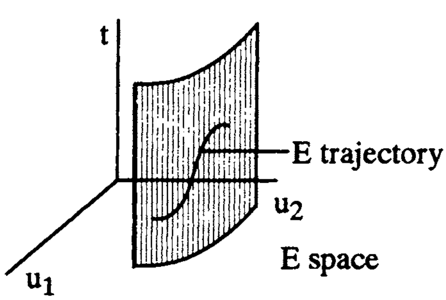


Fig. 2-5

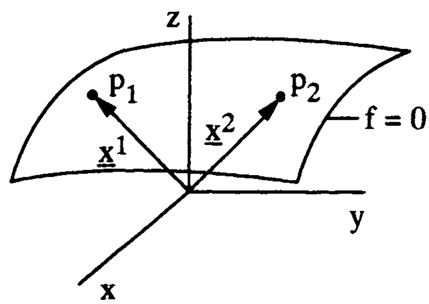


Fig. 2-6

Next consider two particles constrained to move on a single surface (Fig. 2-6). The position vectors of the particles resolved into components are:

$$\underline{x}^1 = x_1^1\hat{e}_1 + x_2^1\hat{e}_2 + x_3^1\hat{e}_3$$
$$\underline{x}^2 = x_1^2\hat{e}_1 + x_2^2\hat{e}_2 + x_3^2\hat{e}_3$$

Relabeling to put in component form:

$$u_1 = x_1^1\,, \quad u_2 = x_2^1\,, \quad u_3 = x_3^1$$
$$u_4 = x_1^2\,, \quad u_5 = x_2^2\,, \quad u_6 = x_3^2$$

If the surface is given by $f(x, y, z) = 0$, then there are *two* constraints in configuration space:

$$f_1 = f(u_1, u_2, u_3) = 0 ; \qquad f_2 = f(u_4, u_5, u_6) = 0$$

The unconstrained particles had a total of six independent components, but the two constraints have reduced this number to four.

Every C trajectory satisfying all constraints is a *possible motion*, but is not necessarily an *actual motion* because actual motions also obey Newton's Laws.

Degrees of Freedom. Given a system of $n = N/3$ particles suppose there are L independent constraints. The number of *degrees of freedom* of the system is then

$$DOF = N - L > 0 \tag{2.13}$$

If $N = L$ the system is fixed in space if all constraints are scleronomic and moves with prescribed motion if at least one is rheonomic.

For a single particle, if there are no constraints $DOF = N - L = 3 - 0 = 3$ and it takes three independent parameters to specify position in configuration space (Fig. 2-7). If there is one holonomic constraint, $DOF = 2$ and it takes two (motion on a surface). If there are two holonomic constraints, $DOF = 1$ and it takes one (motion on a line) and if there are three holonomic constraints, $DOF = 0$ and the particle is fixed.

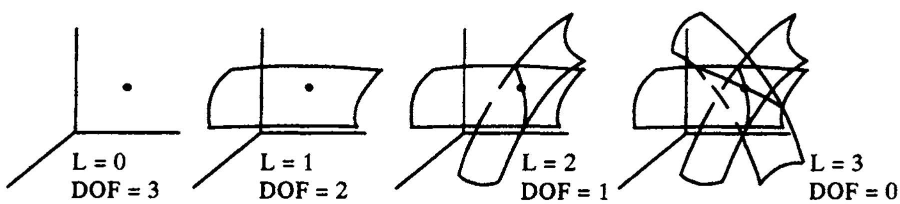


Fig. 2-7

The situation for a rigid body is more difficult. We establish the number of DOF of a rigid body in 3-D unconstrained motion in two different ways. First, fix a body-fixed reference frame with axes ζ, η, ν at the body's center of mass and let the coordinates of the origin of this frame be $\bar{x}, \bar{y}, \bar{z}$ with respect to a non-body-fixed frame with axes x, y, z (Fig. 2-8). Let the direction cosines of ζ, η, ν relative to x, y, z be

$$\begin{pmatrix} \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \\ \ell_3 & m_3 & n_3 \end{pmatrix}$$

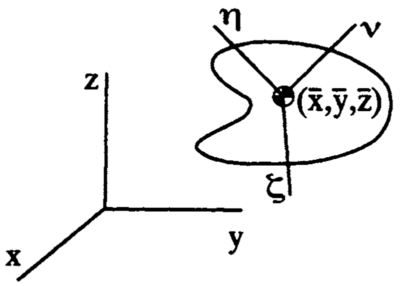


Fig. 2-8

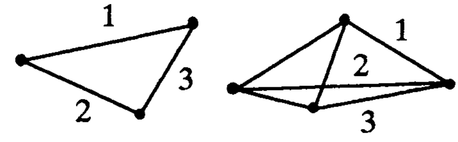


Fig. 2-9

(For example, ℓ_1 is the cosine of the angle between ζ and x). The coordinates of the r^{th} particle of the rigid body are then

$$\begin{aligned} x_r &= \bar{x} + \ell_1\zeta_r + \ell_2\eta_r + \ell_3\nu_r \\ y_r &= \bar{y} + m_1\zeta_r + m_2\eta_r + m_3\nu_r \\ z_r &= \bar{z} + n_1\zeta_r + n_2\eta_r + n_3\nu_r \end{aligned}$$

Due to the orthogonality of the direction cosine matrix, the direction cosines are functions of three independent angles, say θ_1, θ_2, θ_3. Thus the location of all the points of the rigid body are specified by $(\bar{x}, \bar{y}, \bar{z}, \theta_1, \theta_2, \theta_3)$ relative to the other frame, and therefore the body has 6 DOF.

Alternatively, we may view the rigid body as a system of constrained particles. It is clear that the first three particles take 3 constraints, and that each additional particle takes 3 more (Fig. 2-9). Thus if the rigid body has n particles, the total number of constraints is $3+3(n-3)$. But since a particle without constraints has 3 DOF, the DOF for the rigid body is

$$DOF = 3n - [3 + 3(n-3)] = 6$$

Infinitesimal Displacements. The constraint as specified by Eqn. (2.11) or (2.12) is for arbitrarily large displacements. We now derive local conditions, that is, conditions on small displacements. Suppose we have L holonomic constraints and let $u_s = u_s(\alpha)$ and $t = t(\alpha)$ where α is a parameter:

$$f_r[u_1(\alpha),\ u_2(\alpha),\ \cdots,\ u_N(\alpha),\ t] = 0\ ; \qquad r = 1,\ 2,\ \cdot\cdot,\ L \tag{2.14}$$

Differentiating w.r.t. α:

$$\sum_{s=1}^{N} \frac{\partial f_r}{\partial u_s}\frac{du_s}{d\alpha} + \frac{\partial f_r}{\partial t}\frac{dt}{d\alpha} = 0\ ; \qquad r = 1,\ 2,\ \cdot\cdot,\ L \tag{2.15}$$

An important special case is $\alpha = t$:

$$\sum_{s=1}^{N} \frac{\partial f_r}{\partial u_s} \dot{u}_s + \frac{\partial f_r}{\partial t} = 0 ; \qquad r = 1,\ 2,\ \cdot\cdot,\ L \tag{2.16}$$

In differential form

$$\sum_{s=1}^{N} \frac{\partial f_r}{\partial u_s} du_s + \frac{\partial f_r}{\partial t} dt = 0 ; \qquad r = 1,\ 2,\ \cdot\cdot,\ L \tag{2.17}$$

For the special case of *all* constraints scleronomic, Eqns. (2.16) and (2.17) become

$$\sum_{s=1}^{N} \frac{\partial f_r}{\partial u_s} \dot{u}_s = 0 ; \qquad r = 1,\ 2,\ \cdot\cdot,\ L \tag{2.18}$$

$$\sum_{s=1}^{N} \frac{\partial f_r}{\partial u_s} du_s = 0 ; \qquad r = 1,\ 2,\ \cdot\cdot,\ L \tag{2.19}$$

The first of these is a local condition on velocities and displacements and the latter is a local condition that small displacements must remain in the tangent plane of the constraint. If u_s^* is a position on the constraint, then infinitesimal displacements du_s satisfying

$$\sum_{s=1}^{N} \left.\frac{\partial f}{\partial u_s}\right|_{u_s^*} du_s = 0$$

are in the tangent plane (Fig. 2-10) of the constraint at u_s^*.

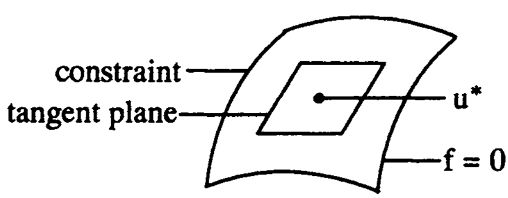


Fig. 2-10

2.4 Nonholonomic Constraints

Configuration Constraints. A nonholonomic constraint is one that is not holonomic. This can happen in several ways and we discuss two.

If a constraint can be reduced to an *inequality* in the configuration space,

$$f(u_1,\ u_2,\ \cdot\cdot,\ u_N,\ t) \leq 0 \tag{2.20}$$

it is called a *configuration* constraint. Such a constraint may depend explicitly on t (rheonomic), or not (scleronomic). An example of a configuration constraint is the requirement that an object must stay on or above a plane surface (Fig. 2-11).

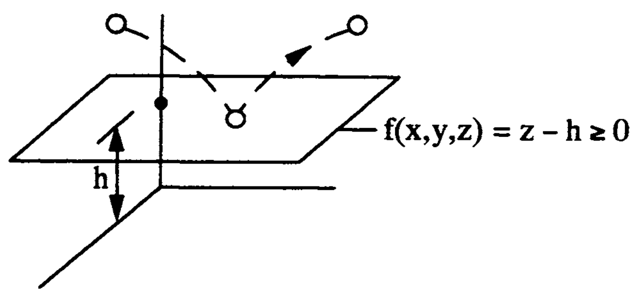


Fig. 2-11

Equality Constraints. These are differential relations among the $u_1,\ u_2,\ \cdot\cdot,\ u_N,\ t$ of the form

$$\sum_{s=1}^{N} A_{rs} du_s + A_r dt = 0\ ; \qquad r = 1,\ 2,\ \cdot\cdot,\ L \tag{2.21}$$

that are not integrable; that is, we cannot use this to get a relation between finite displacements.

Recall that starting with a holonomic constraint

$$f_r(u_1, u_2,\ \cdots,\ u_N, t) = 0\ ;$$

we differentiated to get

$$\sum_{s=1}^{N} \frac{\partial f_r}{\partial u_s}\, du_s + \frac{\partial f_r}{dt}\, dt = 0$$

Integrating, we can go back to the finite form. With a nonholonomic constraint of the type of Eqn. (2.21) this cannot be done.

From Eqn. (2.16), we see that one way nonholonomic constraints can occur is as constraints on the velocity components, and that such constraints are restricted to those in which this dependence is linear. The most common situations in which such constraints arise involve bodies rolling on other bodies without slipping; several example of this will be analyzed later.

Pfaffian Form. The differential form (whether integrable or not) of a constraint, Eqn. (2.21), is called the *Pfaffian form* and it is the most general type of constraint we will consider in this book. Thus the DOF of a system is the number of velocity components that can be given arbitrary values.

2.5 Catastatic Constraints

Definitions. Consider a system of constraints in Pfaffian form

$$\sum_{s=1}^{N} A_{rs} du_s + A_r dt = 0 \, ; \qquad r = 1, \cdot\cdot, L \tag{2.22}$$

Each constraint may be either holonomic (integrable) or nonholonomic. Each constraint may be either scleronomic ($A_{rs} \neq A_{rs}(t)$ and $A_r = 0$) or rheonomic. We make an additional distinction. A constraint is *catastatic* if $A_r = 0$ and *acatastatic* otherwise; if all constraints are catastatic, we say the system is catastatic. Note that the A_{rs} may be functions of t in a catastatic constraint.

Equation (2.22) implies that the condition of static equilibrium, $\dot{u}_s = 0$, $s = 1, \cdot\cdot, n$, is possible if and only if the system is catastatic.

2.6 Determination of Holonomic Constraints

Remarks. Holonomic constraints usually come in integrated form. However, sometimes they come in Pfaffian form. We must then be able to distinguish between holonomic and nonholonomic. If the Pfaffian form is an exact differential, then the constraint is integrable, but this is not necessary. The following theorem (without proof) is a very general result.

Theorem. Suppose an equation of independent variables $y_1, y_2, \cdot\cdot,$ y_M is given in differential form as:

$$\sum_{s=1}^{M} A_s(y_1, y_2, \cdot\cdot, y_M) dy_s = 0 \tag{2.23}$$

Then it is necessary and sufficient for the existence of an integral of this equation of the form

$$f(y_1, y_2, \cdot\cdot, y_M) = 0$$

that the equations

$$\begin{aligned} A_\gamma \left(\frac{\partial A_\beta}{\partial y_\alpha} - \frac{\partial A_\alpha}{\partial y_\beta} \right) + A_\beta \left(\frac{\partial A_\alpha}{\partial y_\gamma} - \frac{\partial A_\gamma}{\partial y_\alpha} \right) \\ + A_\alpha \left(\frac{\partial A_\gamma}{\partial y_\beta} - \frac{\partial A_\beta}{\partial y_\gamma} \right) = 0 \, ; \qquad \alpha, \beta, \gamma = 1, 2, \cdot\cdot, M \end{aligned} \tag{2.24}$$

be simultaneously and identically satisfied. There are $M(M-1)(M-2)/6$ such equations, of which $(M-1)(M-2)/2$ are independent.

For three variables, $M = 3$, Eqn. (2.23) is

$$A_1(y_1, y_2, y_3) dy_1 + A_2(y_1, y_2, y_3) dy_2 + A_3(y_1, y_2, y_3) dy_3 = 0 \tag{2.25}$$

and

$$\frac{M(M-1)(M-2)}{6} = 1 \, ; \qquad \frac{(M-1)(M-2)}{2} = 1$$

Thus for the equation to be integrable the only requirement is:

$$A_1 \left(\frac{\partial A_2}{\partial y_3} - \frac{\partial A_3}{\partial y_2} \right) + A_2 \left(\frac{\partial A_3}{\partial y_1} - \frac{\partial A_1}{\partial y_3} \right) + A_3 \left(\frac{\partial A_1}{\partial y_2} - \frac{\partial A_2}{\partial y_1} \right) = 0 \tag{2.26}$$

In the application of this theorem to the constraints on a dynamical system, the y_r may be displacement components, velocity components, or time.

As an example, suppose a particle moves in the (x, y) plane such that the slope of it's path is proportional to time, $dy/dx = Kt$. In Pfaffian form, this constraint is

$$Kt \, dx - dy = 0$$

This is a differential constraint of three variables; thus we take $y_1 = x$, $y_2 = y$, and $y_3 = t$ so that $A_1 = Ky_3$, $A_2 = -1$, and $A_3 = 0$. The left-hand side of Eqn. (2.26) equals K, so that the constraint is nonholonomic.

One consequence of the theorem is that any differential relationship between only two variables is always integrable. Indeed, in this case the constraint may be written

$$\frac{dy_2}{dy_1} = -\frac{A_1(y_1, y_2)}{A_2(y_1, y_2)} \tag{2.27}$$

This may be integrated under mild assumptions on the functions $A_1(\cdot)$ and $A_2(\cdot)$, if not analytically, then numerically. This means for example, that *any* time-independent constraint on the position of a particle in 2-D motion is holonomic.

2.7 Accessibility of Configuration Space

Definition. Recall that a holonomic constraint reduces the number of quantities required to define a point in configuration space. If there are n particles and L holonomic constraints this number is $N - L$, where $N = 3n$. We say that there is an *$N-L$ fold ∞ of motion* or, alternatively, that the *dimensionality of the space of accessible configurations (DSAC)* is $N - L$.

Nonholonomic equality constraints *do not* reduce the DSAC; consequently in general the DSAC is given by

$$DSAC = N - L' \tag{2.28}$$

where L' is the number of holonomic constraints ($L' \leq L$).[2] The fact that nonholonomic constraints do not reduce the DSAC will be shown by some of the following examples.

2.8 Examples

Example. Suppose a constraint on the motion of a particle is $z = dy/dx$. In Pfaffian form,

$$dy - z\,dx = 0$$

Relabeling:

$$\begin{array}{lll} y_1 = x\,, & y_2 = y\,, & y_3 = z \\ A_1 = -z\,, & A_2 = 1\,, & A_3 = 0 \end{array}$$

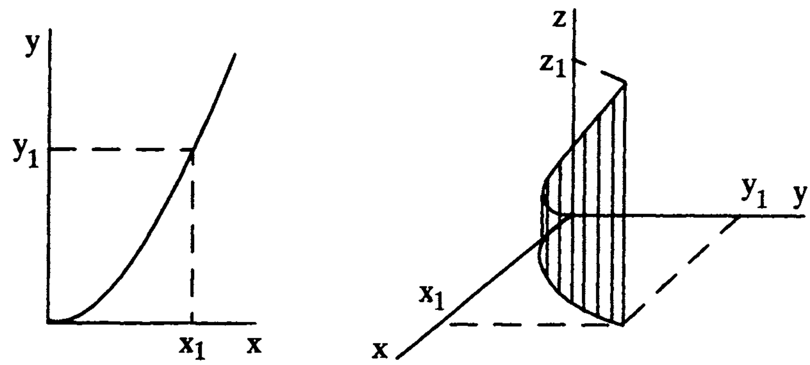


Fig. 2-12

The left-hand-side of Eqn. (2.26) is equal to one so that the constraint is nonholonomic and therefore the DSAC is not reduced. We now show this directly. We need to show that there is at least one path from the origin to any arbitrary fixed point (x_1, y_1, z_1) that satisfies the constraint. Consider any path such that (Fig. 2-12):

$$y = f(x) \,, \qquad z = \frac{df}{dx} \,, \qquad f(0) = f'(0) = 0 \,,$$

$$f(x_1) = y_1 \,, \quad f'(x_1) = z_1$$

The following shows that the constraint is satisfied and that the endpoint is reached.

$$dy - zdx = f'dx - f'dx = 0$$

$$\text{at } x = x_1 \,, \qquad y = y_1 \quad \text{and} \quad z = z_1$$

Example. Two particles p_1 and p_2 moving in the (x, y) plane are connected by a light rod of length a which changes as a prescribed function of time, $a(t) \in C^1$. Let the coordinates of the two particles be (x_1, y_2) and (x_2, y_2). Then the constraint is

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 = a^2$$

In Pfaffian and velocity forms, respectively,

$$(x_2 - x_1)(dx_2 - dx_1) + (y_2 - y_1)(dy_2 - dy_1) - a\dot{a}dt = 0$$

$$(x_2 - x_1)\left(\frac{dx_2}{dt} - \frac{dx_1}{dt}\right) + (y_2 - y_1)\left(\frac{dy_2}{dt} - \frac{dy_1}{dt}\right) - a\frac{da}{dt} = 0$$

It is clear that the constraint is holonomic rheonomic and that DOF = DSAC = 3.

Now suppose the rod has constant length a, but there is additionally the constraint that the velocity of p_1 is always directly along the rod. The two constraints are

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 = a^2$$

$$(y_2 - y_1)\frac{dx_1}{dt} - (x_2 - x_1)\frac{dy_1}{dt} = 0$$

or, in Pfaffian form,

$$(x_2 - x_1)(dx_2 - dx_1) + (y_2 - y_1)(dy_2 - dy_1) = 0$$

$$(y_2 - y_1)dx_1 - (x_2 - x_1)dy_1 = 0$$

The first of these is holonomic scleronomic and the second is nonholonomic. Thus DOF = 2 and DSAC = 3.

Example – Disk Rolling on Plane. A knife-edged disk rolls without slipping on a horizontal plane (Fig. 2-13). There are two constraints – (i) the edge remains in contact with the plane, and (ii) the no slipping condition. The first is *holonomic* and reduces the DSAC from six (the general number for a rigid body) to five, say the (x, y) coordinates of the contact point and three angles usually taken as Euler's angles. The second constraint is a relation between velocities (the contact point must have instantaneous zero velocity relative to the surface) and is in general *nonholonomic*.

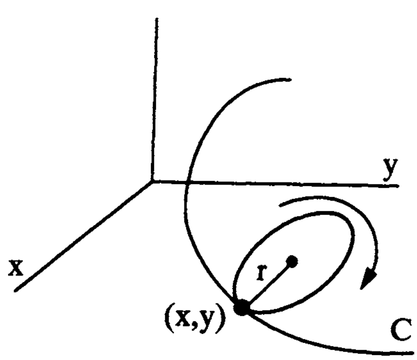
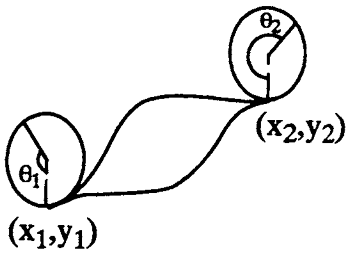
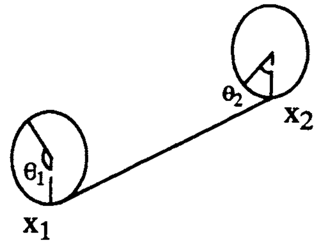


Fig. 2-13



rolling without slipping on a plane.

Fig. 2-14



rolling without slipping on a line.

Fig. 2-15

One of the Euler angles is the angle of rotation of some fixed line in the plane of the disk, say θ, Fig. 2-14. It is clear that there is no finite relation between θ and the coordinates of the contact point, (x, y), because the path can vary and still satisfy the constraint. If the rolling is confined to be on a line (Fig. 2-15), however, there is such a relationship, namely $(x_2 - x_1) = r(\theta_2 - \theta_1)$, and therefore the constraint is holonomic. In Chapter 7, we will return to the problem of a disk rolling on a plane and obtain the equations of motion.

Notes

1 Pars shows that otherwise the law of vector addition of forces would be violated.

2 It is clear that inequality constraints of the type of Eqn. (2.20) do not decrease the DSAC but rather restrict configurations to regions, but we are not considering this type of constraint.

PROBLEMS

2/1. Two particles having Cartesian coordinates $(x_1,\ y_1,\ z_1)$ and $(x_2,\ y_2,\ z_2)$, respectively, are attached to the extremities of a bar whose length $l(t)$ changes with time in a prescribed fashion. Give the equations of constraint on the finite and infinitesimal displacements of the Cartesian coordinates.

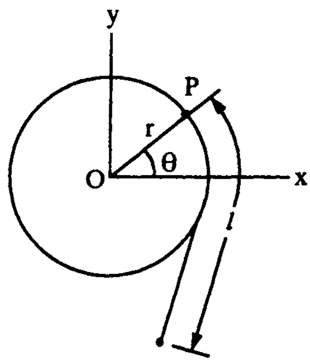
2/2. What are the equations of constraint on the finite and infinitesimal coordinates $(x_1,\ y_1,\ z_1)$ and $(x_2,\ y_2,\ z_2)$ of the bobs of a double spherical pendulum of lengths l_1 and l_2, respectively?

2/3. A thin bar of length $l < 2r$ can move in a plane in such a way that its endpoints are always in contact with a circle of radius r. If the Cartesian coordinates of its endpoints are (x_1, y_1) and (x_2, y_2), respectively, what constraints on finite and infinitesimal displacements must these coordinates satisfy?

2/4. Answer the same questions as in Problem 2/3 if the circle is replaced by an ellipse having major axis $2a$ and minor axis $2b$, and $l < 2b$. Are these constraints holonomic?

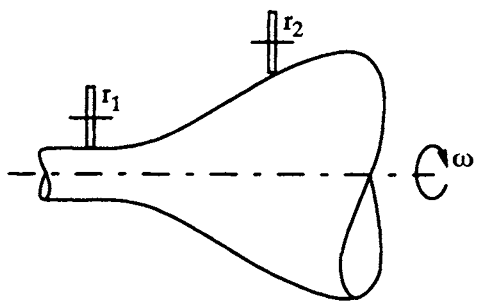
2/5. The motion of an otherwise unconstrained particle is subject to the conditions $\dot{z} = \dot{x}\dot{y}$. Discuss the constraint on the infinitesimal and finite displacements.

2/6. A particle moving in the xy plane is connected by an inextensible string of length l to a point P on the rim of a fixed disk of radius r, as shown. The line PO makes the angle θ with the x axis. What are the constraints on the finite and infinitesimal displacements of the point at the free end of the string having the position (x, y)?



Problem 2/6

2/7. A circular shaft of variable radius $r(x)$ rotates with angular velocity $\omega(t)$ about its centerline, as shown. The shaft is translated along its centerline in a prescribed fashion $f(t)$. Two disks of radii r_1 and r_2, respectively, roll without slipping on the shaft. A mechanism permits the disks to rise and fall in such a way that the disk rims never lose contact with the shaft. Show that the relation, free of ω, between the angular displacement ψ_1 and ψ_2 of the disks is in general nonholonomic. State the general condition that must be satisfied in the exceptional case that the constraint is holonomic and give an example.



Problem 2/7

2/8. A particle moving in the vertical plane is steered in such a way that the slope of its trajectory is proportional to its height. Formulate this constraint mathematically and classify it.

2/9. Write down and classify the equation of constraint of a particle moving in a plane if its slope is always proportional to the time.

2/10. A particle P moving in 3-space is steered in such a way that its velocity is directed for all time toward a point O which has a prescribed motion in space and time. Formulate and classify the equation(s) of constraint of the particle motion under the assumption that the positions of P and O never coincide.

2/11. A dynamic system is subject to the constraint

$$(\cos\theta)dx + (\sin\theta)dy + (y\cos\theta - x\sin\theta)d\theta = 0$$

Is this constraint holonomic? Prove your answer.

Chapter 3

Virtual Displacement and Virtual Work

3.1 D'Alembert's Principle

Introduction. Newton's Second Law tells us that the only way the motion of a particle system can be affected is by the application of forces. Thus it is the application of forces that restrict the motion so that all specified constraints are satisfied. It is natural to call such forces constraint forces. We have seen that in certain simple problems a hallmark of constraint forces is that they do no work, and we wish to exploit this fact to obtain a formulation of dynamics devoid of constraint forces. A problem arises, however, because some constraint forces do in fact do work. To circumvent this problem, we define a new type of displacements, called virtual displacements, and a new type of work, called virtual work, and define constraint forces as those that do no virtual work.

Problem of Dynamics. The problem to be addressed in most of the rest of this book is the strictly Newtonian problem of the first kind: Given bounded functions $\sum F_s(u_1, \cdot\cdot, u_N, \dot{u}_1, \cdot\cdot, \dot{u}_N, t)$ and initial conditions $u_s(0)$ and $\dot{u}_s(0)$ find the functions $u_s(t)$ that satisfy

$$m_s\ddot{u}_s = \sum F_s(u_1, \cdot\cdot, u_N, \dot{u}_1, \cdot\cdot, \dot{u}_N, t) \; ; \quad s = 1, \cdot\cdot, N \tag{3.1}$$

and the constraint equations

$$\sum_{s=1}^{N} A_{rs} du_s + A_r dt = 0 \; ; \quad r = 1, \cdot\cdot, L < N \tag{3.2}$$

Displacements. We now make the following definitions. (1) *Actual displacements* $u_s(t)$ are those that satisfy both Eqns. (3.1) and (3.2); (2) *Possible displacements* du_s satisfy Eqn. (3.2); and (3) *Virtual displacements* δu_s satisfy

$$\sum_{s=1}^{N} A_{rs}\delta u_s = 0\ ; \quad r = 1, \cdot\cdot, L < N \tag{3.3}$$

Thus the actual displacements are to be found among the possible displacements. Comparison of Eqns. (3.2) and (3.3) shows that for catastatic systems, virtual and possible displacements are the same. Recall that static equilibrium is possible only in catastatic systems.

Virtual Work. The *virtual work* done by a force $\underline{F} = (F_1, F_2, \cdot\cdot, F_N)$ in a virtual displacement $\delta\underline{u} = (\delta u_1, \delta u_2, \cdot\cdot, \delta u_N)$ is defined to be the inner product

$$\delta W = \underline{F} \cdot \delta\underline{u} = \sum_{s=1}^{N} F_s \delta u_s \tag{3.4}$$

Constraint Forces. A force $\underline{F}' = (F_1', \cdot\cdot, F_N')$ that does no virtual work, i.e. which is such that

$$\delta W = \underline{F}' \cdot \delta\underline{u} = \sum_{s=1}^{N} F_s' \delta u_s = 0 \tag{3.5}$$

is a *constraint force.* All forces that are not constraint forces are called *given forces.* Therefore we may write

$$\sum F_s = F_s + F_s'\ ; \quad s = 1, \cdot\cdot, N \tag{3.6}$$

where F_s and F_s' are the s components of the resultant given and constraint forces, respectively.

A constraint force is a force that ensures a constraint is satisfied. In the case of a holonomic constraint, the constraint force is normal to the constraint surface and its magnitude is such that the particle stays on the surface. Since this force may be in either direction, technically speaking the constraint must be regarded as two-sided (Fig. 3-1), but, in the interests of simplicity, holonomic constraints will continue to be depicted as one-sided.

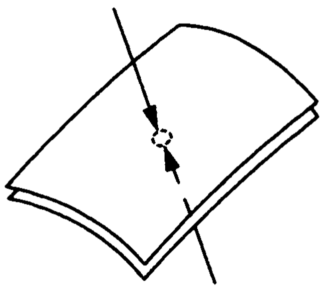


Fig. 3-1

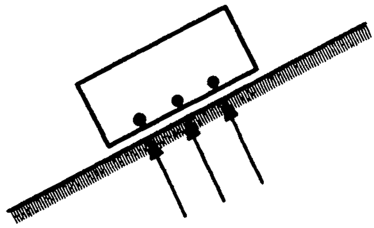


Fig. 3-2

These ideas are easily extended to a rigid body. Consider a rigid body sliding on a smooth (frictionless) surface. Considering the rigid body to be a collection of particles, each of the particles in contact with the surface is subjected to a constraint force normal to the surface (Fig. 3-2). The sum of all these constraint forces is the total constraint force on the body. This force is clearly normal to the surface and will do no virtual work (and no actual work if the constraint is scleronomic).

D'Alembert's Principle. Substituting Eqn. (3.6) in (3.1), taking the scalar product with $\delta \underline{u}$, and summing over all components gives

$$m_s \ddot{u}_s - F_s = F'_s \,; \qquad s = 1, 2, \cdots, N$$

$$(m_s \ddot{u}_s - F_s)\delta u_s = F'_s \delta u_s \,; \qquad s = 1, 2, \cdots, N$$

$$\sum_{s=1}^{N} (m_s \ddot{u}_s - F_s)\delta u_s = 0 \tag{3.7}$$

where Eqn. (3.5) was used. This is *D'Alembert's Principle*, also called the *fundamental equation* by Pars.[1] It is a *formulation of dynamics independent of constraint forces.* It is not too useful for writing the equations of motion of a specific system; rather, it leads to other equations that are.

Example – Single Constraint On a Single Particle. First suppose the constraint is holonomic scleronomic (Fig. 3-3) and given by

$$f(x, y, z) = 0 \tag{3.8}$$

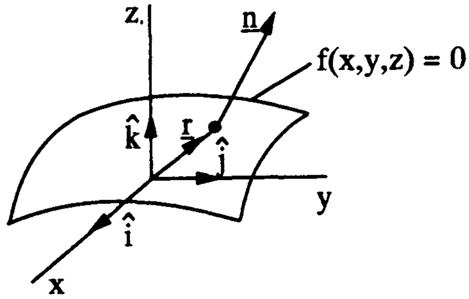


Fig. 3-3

The Pfaffian and velocity forms of this constraint are

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{df}{\partial z}dz = 0 \tag{3.9}$$

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\dot{x} + \frac{\partial f}{\partial y}\dot{y} + \frac{\partial f}{\partial z}\dot{z} = 0 \tag{3.10}$$

A vector normal to the constraint surface is given by[2]:

$$\underline{n} = \text{grad } f = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{k} \tag{3.11}$$

In order that the constraint force do no work, we take it to be

$$\underline{F} = \lambda\underline{n} \tag{3.12}$$

where λ is such that that particle stays on the constraint. Now consider an infinitesimal displacement consistent with the constraint given by

$$d\underline{r} = dx\hat{\imath} + dy\hat{\jmath} + dz\hat{k} \tag{3.13}$$

The work done in this displacement is

$$dW = \underline{F} \cdot d\underline{r} = \left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz\right)\lambda = 0 \tag{3.14}$$

where Eqns. (3.9) and (3.11)–(3.13) were used. Thus the constraint force does zero work. It is easy to see that it also does zero virtual work.

Next suppose that the constraint is holonomic rheonomic (Fig. 3-4):

$$f(x, y, z, t) = 0 \tag{3.15}$$

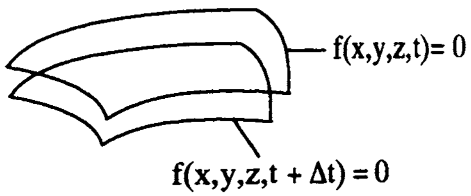


Fig. 3-4

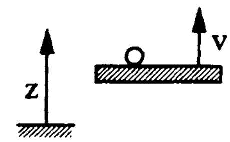


Fig. 3-5

The possible displacements now satisfy:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial t}dt = 0 \tag{3.16}$$

The force is still normal to the constraint at any given time; thus

$$dW = \underline{F} \cdot d\underline{r} = \left(-\frac{\partial f}{\partial t}dt\right)\lambda \tag{3.17}$$

But this is non-zero in general and therefore work is done. In order to maintain the idea that "constraint forces do no work", we must consider virtual displacements and virtual work; the virtual displacements satisfy:

$$\frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial z}\delta z = 0 \tag{3.18}$$

So that the virtual work done is

$$\delta W = \underline{F} \cdot \delta\underline{r} = 0 \tag{3.19}$$

where $\delta\underline{r} = \delta x\hat{\imath} + \delta y\hat{\jmath} + \delta z\hat{k}$ is a virtual displacement.

For example, consider a particle on an elevator floor (Fig. 3-5). From $v = dz/dt$, possible displacements satisfy $dz - vdt = 0$ and virtual displacements satisfy $\delta z = 0$. Thus real work is done (the constraint force changes the potential energy of the particle) but no virtual work. Note that the set of possible and virtual displacements are not only different, they have no member in common; the former have a verticle component and the latter do not.

Finally suppose that the constraint is nonholonomic acatastatic. Then possible and virtual displacements satisfy

$$a\,dx + b\,dy + c\,dz + p\,dt = 0 \tag{3.20}$$

$$a\,\delta x + b\,\delta y + c\,\delta z = 0 \tag{3.21}$$

respectively. The virtual displacements define a tangent plan; the constraint force is normal to this tangent plane and does no virtual work:

$$\underline{F} = \lambda(a\hat{i} + b\hat{j} + c\hat{k}) \tag{3.22}$$

$$\delta W = \underline{F} \cdot \delta \underline{r} = 0 \tag{3.23}$$

In summary, constraint forces sometimes do work and sometimes they do not, but, by definition, they never do virtual work.

3.2 Lagrange Multiplier Rule

Lagrange Multipliers. Recall that the Fundamental equation is Eqn. (3.7) where the δu_s are not arbitrary but are subject to Eqns. (3.3).

A common technique (due to Lagrange) in problems with constraints is to adjoin the constraints with multipliers:

$$\sum_{s=1}^{N}(m_s\ddot{u}_s - F_s)\delta u_s + \sum_{r=1}^{L}\lambda_r \sum_{s=1}^{N} A_{rs}\delta u_s = 0 \tag{3.24}$$

Note that there is one multiplier, λ_r, for each constraint.

Factoring out the δu_s,

$$\sum_{s=1}^{N}\left(m_s\ddot{u}_s - F_s + \sum_{r=1}^{L}\lambda_r A_{rs}\right)\delta u_s = 0 \tag{3.25}$$

The advantage of this approach is that now the δu_s are completely arbitrary.[3] Thus the only way Eqn. (3.25) can be valid is if each of the coefficients of the δu_s are zero:

$$m_s\ddot{u}_s - F_s + \sum_{r=1}^{L}\lambda_r A_{rs} = 0\ ; \qquad s = 1,\ \cdot\cdot,\ N \tag{3.26}$$

Since the F_s are the given force components, the constraint force components must be:

$$F_s' = -\sum_{r=1}^{L}\lambda_r A_{rs} \tag{3.27}$$

Thus the multipliers are directly related to the magnitudes of the constraint forces.

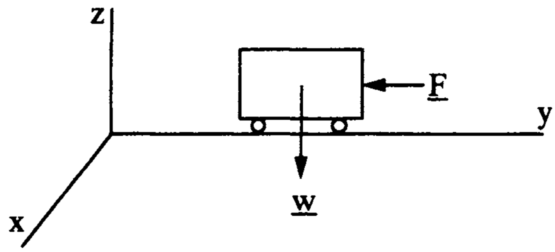


Fig. 3-6

Example. A cart is constrained to roll on a horizontal plane (Fig. 3-6). It starts at rest on the y-axis and is subject to a contant force F in the negative y-direction. The constraint is $z = 0$ so that $dz = 0$ and $\delta z = 0$ which is holonomic scleronomic. In this case $L = 1$, $N = 3$, and $A_{11} = A_{12} = 0$. The given forces are $F_x = 0$, $F_y = -F$, and $F_z = -w$. Equation (3.25) gives

$$m\ddot{x}\delta x + (m\ddot{y} + F)\delta y + (m\ddot{z} + w + \lambda)\delta z = 0$$

Since δx, δy, and δz are arbitrary, this implies

$$\ddot{x} = 0\,, \qquad m\ddot{y} = -F\,, \qquad \lambda = -w$$

with solution

$$x = 0\,; \qquad y = y_0 - \frac{1}{2}\frac{F}{m}t^2\,, \quad z = 0$$

Note that the constraint force is equal to w as expected.

Example. The point of suspension of a simple pendulum moves in a prescribed manner as shown on Fig. 3-7. The equation of constraint is

$$[x - f(t)]^2 + y^2 - \ell^2 = 0$$

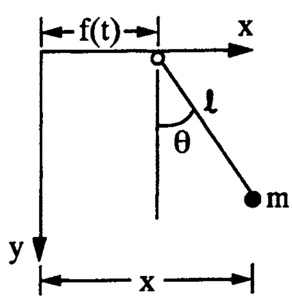


Fig. 3-7

This constraint is holonomic rheonomic (integrable and time dependent).

The Pfaffian form is

$$2[x - f(t)](dx - \dot{f}\ dt) + 2y\ dy = 0$$

$$(x - f)dx + y\ dy - (x - f)\dot{f}\ dt = 0$$

Therefore the constraint is acatastatic, so that virtual displacements are not the same as possible displacements. The virtual displacements satisfy

$$(x - f)\delta x + y\ \delta y = 0$$

The given forces are

$$F_x = 0\ , \qquad F_y = mg$$

Now apply Eqn. (3.25) with $L = 1$ and $N = 2$:

$$[m\ddot{x} + \lambda'(x - f)]\delta x + [m\ddot{y} - mg + \lambda' y]\delta y = 0$$

where λ' is the Lagrange multiplier. Let $\lambda' = -m\lambda$; since δx and δy are arbitrary:

$$\ddot{x} - \lambda(x - f) = 0\ , \qquad \ddot{y} - g - \lambda y = 0$$

These two equations plus the constraint equation give $x(t), y(t)$, and $\lambda(t)$.

These results may be expressed compactly in terms of the angle θ. From Fig. 3-7,

$$x = f + \ell \sin\theta \qquad y = \ell\cos\theta$$

$$\ddot{x} = \ddot{f} + \ell\ddot{\theta}\cos\theta - \ell\dot{\theta}^2\sin\theta \quad \ddot{y} = -\ell\ddot{\theta}\sin\theta - \ell\dot{\theta}^2\cos\theta$$

These relations satisfy the constraint equation; substituting them into the equations of motion and eliminating λ gives

$$\frac{\ddot{x}}{(x - f)} = \frac{\ddot{y} - g}{y}$$

$$\ell\cos\theta(\ddot{f} + \ell\ddot{\theta}\cos\theta - \ell\dot{\theta}^2\sin\theta) = (f + \ell\sin\theta - f)$$
$$(-\ell\ddot{\theta}\sin\theta - \ell\dot{\theta}^2\cos\theta - g)$$

$$\ddot{\theta} + \frac{g}{\ell}\sin\theta = -\frac{\ddot{f}}{\ell}\cos\theta$$

Example. Suppose a particle moves in 3-D space subject to two constraints, one holonomic rheonomic and one nonholonomic acatastatic:

$$f(x, y, z, t) = 0$$
$$a\dot{x} + b\dot{y} + c\dot{z} + d = 0$$

Changing to component form:

$$f(u_1, u_2, u_3, t) = 0$$
$$a\dot{u}_1 + b\dot{u}_2 + c\dot{u}_3 + d = 0$$

The Pfaffian forms of these constraints are

$$\frac{\partial f}{\partial u_1}\,du_1 + \frac{\partial f}{\partial u_2}\,du_2 + \frac{\partial f}{\partial u_3}\,du_3 + \frac{\partial f}{\partial t}\,dt = 0$$
$$a\,du_1 + b\,du_2 + c\,du_3 + d\,dt = 0$$

The general form of the constraints is

$$\sum_{s=1}^{3} A_{rs}\,du_s + A_r\,dt = 0\,; \quad r = 1, 2$$

$$A_{11}\,du_1 + A_{12}\,du_2 + A_{13}\,du_3 + A_1\,dt = 0$$
$$A_{21}\,du_1 + A_{22}\,du_2 + A_{23}\,du_3 + A_2\,dt = 0$$

Comparing:

$$A_{11} = \frac{\partial f}{\partial u_1}\,, \qquad A_{12} = \frac{\partial f}{\partial u_2}\,, \qquad A_{13} = \frac{\partial f}{\partial u_3}\,, \qquad A_1 = \frac{\partial f}{\partial t}$$
$$A_{21} = a\,, \qquad A_{22} = b\,, \qquad A_{23} = c\,, \qquad A_2 = d$$

The virtual displacements satisfy

$$\frac{\partial f}{\partial u_1}\,\delta u_1 + \frac{\partial f}{\partial u_2}\,\delta u_2 + \frac{\partial f}{\partial u_3}\,\delta u_3 = 0$$
$$a\,\delta u_1 + b\,\delta u_2 + c\,\delta u_3 = 0$$

Equation (3.25) gives

$$(m_1\ddot{u}_1 - F_1 + \lambda_1 A_{11} + \lambda_2 A_{21})\,\delta u_1 + (m_2\ddot{u}_2 - F_2 + \lambda_1 A_{12} + \lambda_2 A_{22})\,\delta u_2$$
$$+ (m_3\ddot{u}_3 - F_3 + \lambda_1 A_{13} + \lambda_2 A_{23})\,\delta u_3 = 0$$

or, changing back to the original variables,

$$\left(m\ddot{x} - F_x + \lambda_1 \frac{\partial f}{\partial x} + \lambda_2 a\right) \delta x + \left(m\ddot{y} - F_y + \lambda_1 \frac{\partial f}{\partial y} + \lambda_2 b\right) \delta y \\ + \left(m\ddot{z} - F_z + \lambda_1 \frac{\partial f}{\partial z} + \lambda_2 c\right) \delta z = 0$$

Since δx, δy, and δz are now completely arbitrary, we must have

$$m\ddot{x} - F_x + \lambda_1 \frac{\partial f}{\partial x} + \lambda_2 a = 0$$

$$m\ddot{y} - F_y + \lambda_1 \frac{\partial f}{\partial y} + \lambda_2 b = 0$$

$$m\ddot{z} - F_z + \lambda_1 \frac{\partial f}{\partial z} + \lambda_2 c = 0$$

where F_x, F_y and F_z are the rectangular components of the total given force. These three equations plus the two constraint equations give five equations for the five unknowns x, y, z, λ_1, and λ_2.

3.3 Virtual Velocity and Variations

Virtual Velocity. If the constraints are sufficiently smooth, we have

$$\frac{d}{dt}(\delta u) = \delta\left(\frac{du}{dt}\right) = \delta\dot{u} \tag{3.28}$$

This defines *virtual velocity*. Recall that the state of a system is the set of components of displacement and velocity

$$(\underline{u}, \dot{\underline{u}}) = (u_1, \cdot\cdot, u_N, \dot{u}_1, \cdot\cdot, \dot{u}_N)$$

A virtual charge of state is

$$(\underline{u} + \delta\underline{u},\ \dot{\underline{u}} + \delta\dot{\underline{u}}) = (u_1 + \delta u_1, \cdot\cdot, u_N + \delta u_N, \dot{u}_1 + \delta\dot{u}_1, \cdot\cdot, \dot{u}_N + \delta\dot{u}_N)$$

where $\delta\underline{u}$ is a virtual displacement and $\delta\dot{\underline{u}}$ is a virtual velocity. If these displacements and velocities satisfy

$$\sum_{s=1}^{N} A_{rs}\dot{u}_s + A_r = 0\,; \qquad r = 1, \cdot\cdot, L \tag{3.29}$$

they are possible states.

Theorem. Virtual changes from a possible state lead to another possible state if and only if the system is holonomic. (One might have thought the condition would be catastatic because in that case possible displacements are same as virtual displacements, but this is not the case.)

This is proved as follows. Suppose there is a nonintegrable constraint on the motion of a particle:

$$adx + bdy + cdz = 0$$

where $a, b, c, \in C'(x, y, z)$ are time independent. The derivative form is

$$a\dot{x} + b\dot{y} + c\dot{z} = 0$$

and the virtual displacements satisfy

$$a\delta x + b\delta y + c\delta z = 0$$

Assume that the varied path satisfies the constraint (that is, it is a possible motion). On the varied and actual paths

$$\delta(a\dot{x} + b\dot{y} + c\dot{z}) = 0$$

and on the actual path

$$\frac{d}{dt}(a\delta x + b\delta y + c\delta z) = 0$$

Substracting these equations,

$$(\dot{a}\delta x - \dot{x}\delta a) + (\dot{b}\delta y - \dot{y}\delta b) + (\dot{c}\delta z - \dot{z}\delta c) = 0$$

Using the chain rule for $\dot{a}$, $\dot{b}$, $\dot{c}$, δa, δb, and δc, this can be put into the form

$$a\left(\frac{\partial b}{\partial z} - \frac{\partial c}{\partial y}\right) + b\left(\frac{\partial c}{\partial x} - \frac{\partial a}{\partial z}\right) + c\left(\frac{\partial a}{\partial y} - \frac{\partial b}{\partial x}\right) = 0$$

But from Eqn. (2.26) this is just the necessary and sufficient condition for the constraint to be integrable, a contradiction. This proves the theorem.

Variation of a Function. Consider a function $f(x_1, \cdot\cdot, x_n, \dot{x}_1, \cdot\cdot, \dot{x}_n, t)$ where all $2n + 1$ arguments are regarded as independent of each other.

The *differential* of f is defined as

$$df = \sum_{r=1}^{n} \frac{\partial f}{\partial x_r} dx_r + \sum_{r=1}^{n} \frac{\partial f}{\partial \dot{x}_r} d\dot{x}_r + \frac{\partial f}{\partial t} dt \tag{3.30}$$

We now define the *variation* of f as

$$\delta f = \sum_{r=1}^{n} \frac{\partial f}{\partial x_r} \delta x_r + \sum_{r=1}^{n} \frac{\partial f}{\partial \dot{x}_r} \delta \dot{x}_r \tag{3.31}$$

that is, in the δ operation the x_r and $\dot{x}_r$ are varied but t is not.

Relation to Virtual Work. This definition causes ambiguity when applied to virtual work. Let $\underline{F}$ be a force acting on a particle; then, in general,

$$\underline{F} = \underline{F}(x_1, x_2, x_3, \dot{x}_1, \dot{x}_2, \dot{x}_3, t)$$

The work done by $\underline{F}$ during a displacement $d\underline{x}$ is by definition

$$\begin{aligned} dW &= \underline{F} \cdot d\underline{x} \\ W &= \int_c \underline{F} \cdot d\underline{x} \end{aligned}$$

where c is a possible curve. Thus $W = W(x_1, x_2, x_3, \dot{x}_1, \dot{x}_2, \dot{x}_3, t)$ in general. Consequently the variation of W is

$$\delta W = \sum_{r=1}^{3} \frac{\partial W}{\partial x_r} \delta x_r + \sum_{s=1}^{3} \frac{\partial W}{\partial \dot{x}_r} \delta \dot{x}_r$$

But virtual work is defined as

$$\delta W = \underline{F} \cdot \delta \underline{x} = \sum_{r=1}^{3} F_r \delta x_r$$

which contains no terms in $\delta \underline{\dot{x}}$. Therefore the virtual work δW is generally not the same as the variation of the work δW. When the symbol δW is used subsequently, it will denote virtual work, not the variation of W.

3.4 Forms of the Fundamental Equation

Possible Velocities. Recall that possible displacements du_s satisfy Eqn. (3.2) and that virtual displacements δu_s satisfy Eqn. (3.3). Recall also that possible velocities $\dot{u}_s$ satisfy Eqn. (3.29).

Now consider another possible velocity

$$\dot{u}_s + \Delta\dot{u}_s \; ; \qquad s = 1, \cdot\cdot, N \tag{3.32}$$

where the $\Delta\dot{u}_s$ are not necessarily small. Substituting this into Eqn. (3.29):

$$\sum_{s=1}^{N} A_{rs}(\dot{u}_s + \Delta\dot{u}_s) + A_r = 0 \; ; \qquad r = 1, \cdot\cdot, L \tag{3.33}$$

Subtracting Eqns. (3.29) from Eqns. (3.33) gives

$$\sum_{s=1}^{N} A_{rs}\Delta\dot{u}_s = 0 \; ; \qquad r = 1, \cdot\cdot, L \tag{3.34}$$

Comparing Eqns. (3.3) and (3.34) we see that *possible velocity changes satisfy the same constraints as virtual displacements.*

Possible Accelerations. Differentiate Eqn. (3.29) w.r.t. time to get

$$\sum_{s=1}^{N} \left(A_{rs}\ddot{u}_s + \frac{dA_{rs}}{dt}\dot{u}_s \right) + \frac{dA_r}{dt} = 0 \; ; \qquad r = 1, \cdot\cdot, L \tag{3.35}$$

If $\dot{u}_s$ is a possible velocity, then any $\ddot{u}_s$ satisfying Eqn. (3.35) is a *possible acceleration.* Now consider another possible acceleration:

$$\ddot{u}_s + \Delta\ddot{u}_s \; ; \qquad s = 1, \cdot\cdot, N \tag{3.36}$$

Substituting this in Eqn. (3.35) and subtracting the result from Eqn. (3.35) gives

$$\sum_{s=1}^{N} A_{rs}\Delta\ddot{u}_s = 0 \; ; \qquad r = 1, \cdot\cdot, L \tag{3.37}$$

Thus *possible acceleration changes also satisfy the same constraints as virtual displacements.*

Fundamental Equation. Recall the fundamental equation, Eqn. (3.7), where the δu_s must satisfy Eqn. (3.3). But any quantities satisfying Eqn. (3.3) are virtual displacements by definition; consequently either the $\Delta\dot{u}_s$ or the $\Delta\ddot{u}_s$ serve as virtual displacements and we obtain two other forms of the fundamental equation:

$$\sum_{s=1}^{N}(m_s\ddot{u}_s - F_s)\Delta\dot{u}_s = 0 \tag{3.38}$$

$$\sum_{s=1}^{N}(m_s\ddot{u}_s - F_s)\Delta\ddot{u}_s = 0 \tag{3.39}$$

Recall that the $\Delta\dot{u}_s$ and $\Delta\ddot{u}_s$ are not necessarily small. Equation (3.38) is useful in cases where there are impulsive forces, because in such cases there may be large finite instantaneous velocity changes; this will be the topic of Chapter 13. Equation (3.39) will be used in Chapter 14.

3.5 Given Forces

Conservative Forces. After the constraint forces are accounted for, the given forces remain; generally, their components are of the form

$$F_s(u_1,\ \cdot\cdot,\ u_N, \dot{u}_1,\ \cdot\cdot,\ \dot{u}_N, t)$$

Here we wish to single out a sub-set of these forces, called conservative.

Now suppose the force components of a given force depend only on the displacement components, $F_s = F_s(u_1, \cdot\cdot, u_N)$. If there exists a scalar function $V(u_1, \cdots, u_N)$ such that a force $\underline{F}$ satisfies

$$\underline{F} = -\text{grad } V \tag{3.40}$$

then $\underline{F}$ is a *conservative force* and V is a *potential energy function.* Every force for which this is not true is a *nonconservative force.* Note that conservative forces cannot be functions of velocity components or time. If all the given forces acting on a system are conservative, we say that the system is conservative.

Virtual Work Done by Conservative Forces. From Eqn. (3.40) the components of a conservative force are

$$F_s = -\frac{\partial V}{\partial u_s} \tag{3.41}$$

Forming the inner product of $\underline{F}$ and $\delta \underline{u}$:

$$\sum_{s=1}^{N} F_s \, \delta u_s = -\sum_{s=1}^{N} \frac{\partial V}{\partial u_s} \, \delta u_s$$

The term on the left-hand side is the virtual work done by F_s; it is not zero because F_s is a given force. The other term is the variation of V because V is not a function of velocity. Therefore we have[4]

$$\delta W = -\delta V \tag{3.42}$$

Thus the total virtual work done by a conservative force in going from $c_1 \in C$ to $c_2 \in C$ through a sequence of virtual displacements depends only on the endpoints and not on the path.

Example. Consider a block sliding on a rough surface (Fig. 3-8). Let's classify and discuss the forces acting on the block:

1. The normal force $\underline{N}$ and the gravitational force $\underline{W}$ are both constraint forces because they do no virtual work (and no real work either). It is clear that they can be ignored in determining the motion.

2. The friction force $\underline{F}_f$ does work in a virtual displacement and is therefore a given force.

3. If $\underline{F}_f$ depends on $|\underline{N}|$ (Coulomb friction), this force is technically outside the theory we are developing (forces must depend only on displacements, velocity, and time). From a practical standpoint, however, such forces usually may be included.

4. The forces $\underline{N}$ and $\underline{W}$ are different in that $\underline{N}$ vanishes if the constraint is removed; $\underline{W}$ does not vanish in this case but rather becomes a given force.

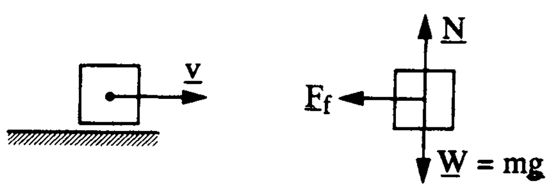


Fig. 3-8

Classification of Forces and Dynamic Systems. We have seen that there are differences among the forces acting on a system of particles, and that these differences can be exploited to simplify the equations of dynamics. The classification of forces is summarized in Fig. 3-9.

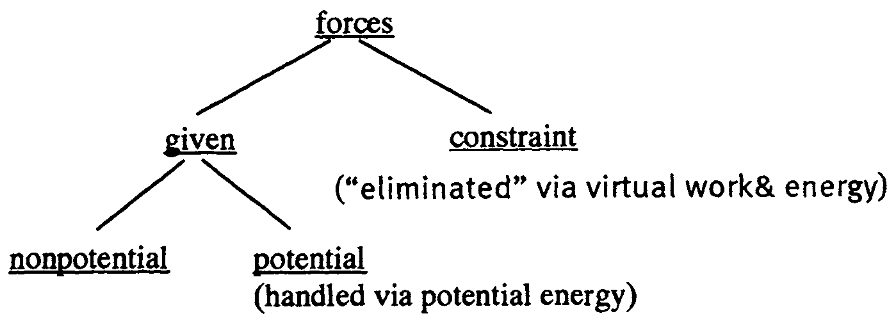


Fig. 3-9

Recall the following definitions:

1. A dynamic system is holonomic if all constraints on it's motion are holonomic.
2. A system is scleronomic if all holonomic constraints are scleronomic.
3. A system is catastatic if all constraints are catastatic.
4. A system is conservative if all given forces are conservative.

It will be convenient to make two other definitions.

1. A dynamic system is *closed* if it is catastatic and conservative.
2. A dynamic system is *natural* if it is holonomic, scleronomic, and conservative.

Thus a natural system is always closed, but not conversely.

Notes

1 D'Alembert's statement of his principle is somewhat difficult to comprehend for the modern reader. The simple and concise form of the principle stated here is due to Lagrange.

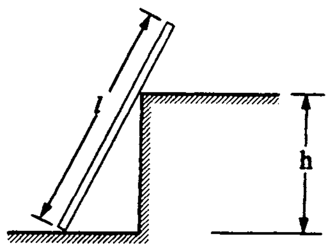
2 Provided the constraint function is sufficiently smooth, which we assume.

3 The Lagrange multiplier rule will be precisely stated and proved in Section 6.2.

4 This equation may not be true for more general systems than strictly mechanical, for example electro-mechanical ones.

PROBLEMS

3/1. A heavy, uniform, smooth ladder of length l stands on a horizontal floor and leans against a wall of height $h < l$ as shown. Let the coordinates of its lower extremity be (x_1, y_1) and those of its upper (x_2, y_2) with y vertical, positive in the up direction. What are the equations of constraint on finite, possible, and virtual displacements for the cases $y_2 > h$ and $h > y_2 > 0$?



Problem 3/1

3/2. A particle of mass m is attached to one end of a massless inextensible string of length l. The particle and string are placed on a smooth horizontal table so that the string is straight. At the time t_0, the free end of the string is set in motion with uniform velocity v_0 in the plane of the table and normal to the string; this velocity is maintained constant for all $t > t_0$. Give the equations of constraint on finite, possible, and virtual displacements of the particle position in Cartesian coordinates, write down the fundamental equation and the equations of motion and discuss the C trajectories.

3/3. Show that, when the virtual displacements satisfy the knife edge constraint $\sin\,\theta\;\delta x = \cos\,\theta\;\delta y$, virtual displacements from a possible state do not lead to another possible state.

3/4. Show that, when the virtual displacements satisfy $\sin\,\theta_0\;\delta x = \cos\,\theta_0\;\delta y$ where θ_0 is a nonzero constant, virtual displacements from a possible state do lead to another possible state.

3/5. A particle of mass m is subjected to a force whose Cartesian components are

$$X = -\frac{x^2 + y^2 - z^2 - a^2}{(x^2 + y^2)^{3/2}}x\,,$$

$$Y = -\frac{x^2 + y^2 - z^2 - a^2}{(x^2 + y^2)^{3/2}} y ,$$
$$Z = -\frac{2z}{(x^2 + y^2)^{1/2}} .$$

Suppose the particle is constrained to move on a smooth sphere centered at the origin of the Cartesian coordinate system. Write the fundamental equation and the equations of motion. Find the equilibrium positions.

3/6. A particle of mass m is constrained to move on the curve defined by

$$x = 2\sin^2 \theta , \qquad y = \cos 2\theta , \qquad z = 2\cos^2 \theta .$$

It is subjected to a force whose Cartesian components are

$$X = \frac{z^2 - y^2}{(x + y)^2} , \qquad Y = \frac{2y}{x + y} , \qquad Z = \frac{x^2 - y^2}{(x + z)^2} .$$

Calculate the work done by this force when the particle moves on the arc corresponding to $0 \le \theta \le \pi/2$.

3/7. An unconstrained particle is acted upon by a force whose Cartesian components are

$$X = \frac{x}{x^2 + y^2 + z^2} ,$$
$$Y = \frac{y}{x^2 + y^2 + z^2} ,$$
$$Z = \frac{z}{x^2 + y^2 + z^2} .$$

Find the equilibrium positions.

Chapter 4

Variational Principles

4.1 Energy Relations

Kinetic Energy. Suppose a particle p has mass m, position $\underline{x}$, and velocity $\dot{\underline{x}}$ relative to an inertial frame of reference. Then the *kinetic energy* of the particle is defined as

$$T = \frac{1}{2} m \dot{\underline{x}} \cdot \dot{\underline{x}} = \frac{1}{2} m |\dot{\underline{x}}|^2 = \frac{1}{2} m \dot{x}^2 \tag{4.1}$$

For a system of n particles, with $\underline{x}^r$ and $\dot{\underline{x}}^r$ the position and velocity of particle r with mass m_r, $r = 1, \cdot\cdot, n$,

$$T = \sum_{r=1}^{n} T^r = \frac{1}{2} \sum_{r=1}^{n} m_r \dot{\underline{x}}^r \cdot \dot{\underline{x}}^r = \frac{1}{2} \sum_{r=1}^{n} m_r (\dot{x}^r)^2 \tag{4.2}$$

where

$$\begin{aligned} \dot{\underline{x}}^r &= \dot{x}_1^r \hat{e}_1 + \dot{x}_2^r \hat{e}_2 + \dot{x}_3^r \hat{e}_3 \ ; \qquad r = 1, \cdot\cdot, n \\ (\dot{x}^r)^2 &= (\dot{x}_1^r)^2 + (\dot{x}_2^r)^2 + (\dot{x}_3^r)^2 \end{aligned}$$

Now change to component form:

$$\dot{u}_1 = \dot{x}_1^1 \ , \qquad \dot{u}_2 = \dot{x}_2^1 \ , \qquad \dot{u}_3 = \dot{x}_3^1 \ , \qquad \dot{u}_4 = \dot{x}_1^2 \ , \cdots \ ,$$

Thus Eqn. (4.2) gives[1]

$$T = \frac{1}{2} \sum_{s=1}^{N} m_s \dot{u}_s^2 \ ; \qquad N = 3n \tag{4.3}$$

Kinetic Energy in Catastatic System. Recall the fundamental equation, Eqn. (3.7):

$$\sum_{s=1}^{N}(m_s\ddot{u}_s - F_s)\delta u_s = 0$$

In a catastatic system, the constraints are (see Eqn. (3.2)):

$$\sum_{s=1}^{N} A_{rs} du_s = 0 \; ; \qquad r = 1, \cdot\cdot, L \tag{4.4}$$

The virtual displacements always satisfy Eqn. (3.3):

$$\sum_{s=1}^{N} A_{rs} \delta u_s = 0 \; ; \qquad r = 1, \cdot\cdot, L$$

Since the possible and the virtual displacements now satisfy the same equations, the fundamental equation may be written

$$\sum_{s=1}^{N}(m_s\ddot{u}_s - F_s)du_s = 0 \tag{4.5}$$

or as

$$\sum_{s=1}^{N} m_s\ddot{u}_s\dot{u}_s = \sum_{s=1}^{N} F_s\dot{u}_s \tag{4.6}$$

where the $\dot{u}_s$ satisfy

$$\sum_{s=1}^{N} A_{rs}\dot{u}_s = 0 \; ; \qquad r = 1, \cdot\cdot, L \tag{4.7}$$

Now differentiate T

$$\frac{dT}{dt} = \sum_{s=1}^{N} m_s\dot{u}_s\ddot{u}_s \tag{4.8}$$

Comparing Eqns. (4.6) and (4.8) gives:

$$\frac{dT}{dt} = \sum_{s=1}^{N} F_s\dot{u}_s \tag{4.9}$$

This states that in a catastatic system, the time rate of change of the kinetic energy equals the power of the given forces under possible velocities.

Energy Relations in Catastatic Systems. If the $\dot{u}_s$ are all continuous (as we are assuming because all forces are bounded) and if the number of particles is constant (as we are also assuming), we can integrate Eqn. (4.9) to get

$$T = \int \sum_{s=1}^{N} F_s \dot{u}_s dt + h = \int \sum_{s=1}^{N} F_s du_s + h \tag{4.10}$$

where h is a constant of integration.

Suppose that some of the given forces are conservative and some are not and let

$F_s^c = s$ component of resultant of all conservative forces $= -\partial V / \partial u_s$
$F_s^{nc} = s$ component of resultant of all nonconservative forces.

Then Eqn. (4.10) gives

$$\begin{aligned} T &= \int \sum_{s=1}^{N} F_s^c du_s + \int \sum_{s=1}^{N} F_s^{nc} du_s + h \\ &= -\int \sum_{s=1}^{N} \frac{\partial V}{\partial u_s} du_s + \int \sum_{s=1}^{N} F_s^{nc} du_s + h \end{aligned}$$

$$T + V = \int \sum_{s=1}^{N} F_s^{nc} du_s + h \tag{4.11}$$

If all forces are conservative and included in V, the total mechanical energy of the system is constant over time for actual motions:

$$T + V = h = \text{constant} \tag{4.12}$$

That is, in a closed system (catastatic and conservative), and only in a closed system, the total mechanical energy is a constant (is conserved). Note that these relations are true for all catastatic systems; they hold for holonomic or nonholonomic systems.

4.2 Central Principle

Central Principle. Now consider a general system. The fundamental equation, Eqn. (3.7), in vector form is

$$\sum_{r=1}^{n}(m_r\ddot{\underline{x}}^r - \underline{F}^r)\cdot\delta\underline{x}^r = 0 \tag{4.13}$$

where $\underline{F}^r$ are the resultants of the given forces. Consider

$$\frac{d}{dt}\left(\sum_{r=1}^{n} m_r\dot{\underline{x}}^r\cdot\delta\underline{x}^r\right) = \sum_{r=1}^{n} m_r\ddot{\underline{x}}^r\cdot\delta\underline{x}^r + \sum_{r=1}^{n} m_r\dot{\underline{x}}^r\cdot\frac{d}{dt}(\delta\underline{x}^r)$$

$$\sum_{r=1}^{n} m_r\ddot{\underline{x}}^r\cdot\delta\underline{x}^r = \frac{d}{dt}\left(\sum_{r=1}^{n} m_r\dot{\underline{x}}^r\cdot\delta\underline{x}^r\right) - \sum_{r=1}^{n} m_r\dot{\underline{x}}^r\cdot\delta\dot{\underline{x}}^r \tag{4.14}$$

From Eqn. (4.2) the variation of T is

$$\delta T = \underbrace{\sum_{r=1}^{n}\frac{\partial T}{\partial\underline{x}^r}\cdot\delta\underline{x}_r}_{0} + \sum_{r=1}^{n}\frac{\partial T}{\partial\dot{\underline{x}}^r}\cdot\delta\dot{\underline{x}}^r = \sum_{r=1}^{n} m_r\dot{\underline{x}}^r\cdot\delta\dot{\underline{x}}^r \tag{4.15}$$

Combining Eqns. (4.14) and (4.15) gives

$$\sum_{r=1}^{n} m_r\ddot{\underline{x}}^r\cdot\delta\underline{x}^r = \frac{d}{dt}\left(\sum_{r=1}^{n} m_r\dot{\underline{x}}^r\cdot\delta\underline{x}^r\right) - \delta T \tag{4.16}$$

Finally, using the fundamental equation, Eqn. (4.13),

$$\frac{d}{dt}\left(\sum_{r=1}^{n} m_r\dot{\underline{x}}^r\cdot\delta\underline{x}^r\right) - \delta T = \sum_{r=1}^{n}\underline{F}^r\cdot\delta\underline{x}^r = \delta W \tag{4.17}$$

This is called the *central principle* by Hamel.

4.3 Hamilton's Principle

First Form. We will derive several forms of Hamilton's principle, each more specialized. Integrating Eqn. (4.17) between times t_0 and t_1,

$$\left[\sum_{r=1}^{n} m_r\dot{\underline{x}}^r\cdot\delta\underline{x}^r\right]_{t_0}^{t_1} = \int_{t_0}^{t_1}(\delta T + \delta W)dt \tag{4.18}$$

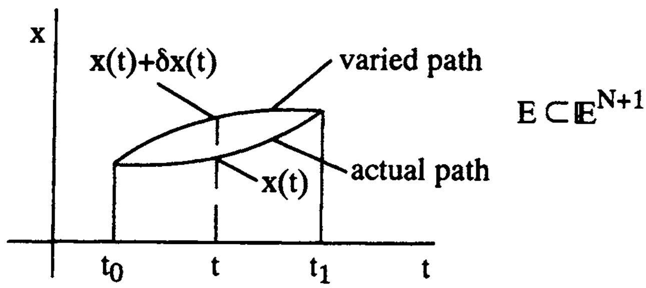


Fig. 4-1

Now consider virtual displacements from the actual motion satisfying

$$\delta\underline{x}^r(t_0) = \delta\underline{x}^r(t_1) = \underline{0}$$

Figure 4-1 shows the situation in the event space. Note that the variations take place with time fixed; thus they are called *contemporaneous.* Equation (4.18) therefore reduces to:

$$\int_{t_0}^{t_1} (\delta T + \delta W)dt = 0 \tag{4.19}$$

This is the first form of *Hamilton's principle*, known as the *extended* or *unrestricted* form, which states that "The time integral of the sum of the virtual work and the variation of the kinetic energy vanishes when virtual displacements are made from the actual motion with endpoints held fixed".

Second Form. If all given forces are conservative,[2] Eqn. (3.42) applies:

$$\delta W = -\delta V$$

Recall that δW is the virtual work, not the variation of W, but that δV is the variation of V. Therefore, in this case,

$$\delta T + \delta W = \delta T - \delta V = \delta(T - V) \tag{4.20}$$

which is the variation in $T - V$. Define the Lagrangian function

$$L = T - V \tag{4.21}$$

Then Hamilton's principle for a conservative system is:

$$\int_{t_0}^{t_1} \delta L \, dt = 0 \tag{4.22}$$

or "The time integral of the variation of the Lagrangian function vanishes for the actual motion".

Since, in general, virtual changes from a possible state do not lead to another possible state, in Hamilton's principle constraints are generally violated and this is not a problem of the calculus of variations. The exception, stated earlier, is a holonomic system.[3]

Third Form. Now suppose the system is conservative and holonomic. Then the variations satisfy the constraints and Hamilton's principle is

$$\delta \int_{t_0}^{t_1} L\, dt = 0 \tag{4.23}$$

or "The time integral of the Lagrangian is stationary along the actual path relative to other possible paths having the same endpoints and differing by virtual displacements". This equation is usually referred to as simply *Hamilton's Principle.*

Remark. The derivation of the various forms of Hamilton's principle given here are completely reversible; that is, starting from them we may derive the corresponding fundamental equations. Thus Hamilton's principle is necessary and sufficient for a motion to be an actual motion. It is precisely an integrated form of the fundamental equation.

Example. A particle moves on a smooth surface with gravity the only given force (Fig. 4-2). We have:

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

$$V = mgz$$

$$\dot{z} = f_x\dot{x} + f_y\dot{y}$$

$$L = T - V = \frac{1}{2}m\left[\dot{x}^2 + \dot{y}^2 + (f_x\dot{x} + f_y\dot{y})^2\right] - mgf$$

Since the only given force is conservative and the only constraint is holonomic, the third form of Hamilton's principle applies:

$$\delta \int_{t_0}^{t_1} L\, dt = \delta \int_{t_0}^{t_1} \frac{m}{2}\left\{\dot{x}^2 + \dot{y}^2 + (f_x\dot{x} + f_y\dot{y})^2 - 2gf\right\} dt = 0$$

We could carry out these variations to get the equation of motion; we will not do this for this problem, but will do it for the following one.

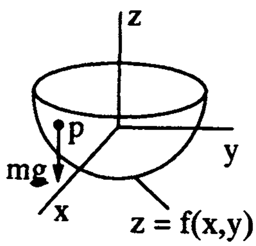


Fig. 4-2

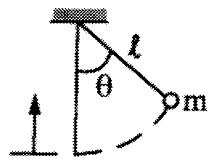


Fig. 4-3

Example. Consider again the simple pendulum (Fig. 4-3). We have:

$$T = \frac{1}{2}mv^2 = \frac{1}{2}(\ell\dot{\theta})^2$$

$$V = mg\ell(1 - \cos\theta)$$

$$L = T - V = \frac{1}{2}m\ell^2\dot{\theta}^2 - mg\ell(1 - \cos\theta)$$

The third form of Hamilton's principle applies:

$$\delta\int_{t_0}^{t_1} L\,dt = \delta\int_{t_0}^{t_1}\left[\frac{1}{2}m\ell^2\dot{\theta}^2 - mg\ell(1-\cos\theta)\right]dt = 0$$

We will now carry out the variation to get the equation of motion:

$$\int_{t_0}^{t_1}\left[m\ell^2\dot{\theta}\,\delta\dot{\theta} - mg\ell\sin\theta\,\delta\theta\right]dt = 0$$

Integration by parts gives[4]

$$\int_{t_0}^{t_1}\dot{\theta}\,\delta\dot{\theta}\,dt = \int_{t_0}^{t_1}\dot{\theta}\frac{d}{dt}(\delta\theta)dt = \underbrace{\dot{\theta}\delta\theta\Big|_{t_0}^{t_1}}_{=0} - \int_{t_0}^{t_1}\delta\theta\,\ddot{\theta}\,dt$$

Consequently

$$\int_{t_0}^{t_1}\left(-\ell\ddot{\theta}\,\delta\theta - g\sin\theta\,\delta\theta\right)dt = 0$$

$$\int_{t_0}^{t_1}\left(\ell\ddot{\theta} + g\sin\theta\right)\delta\theta\,dt = 0$$

But $\delta\theta$ is an arbitrary virtual displacement; therefore by the Fundamental Lemma of the calculus of variations (see next section) we must have

$$\ddot{\theta} + \frac{g}{\ell}\sin\theta = 0$$

We see that getting equations of motion from Hamilton's principle is somewhat cumbersome. Lagrange's Equations, to be derived shortly, essentially carry out this variation in general for all problems and are much easier to use.

4.4 Calculus of Variations

Statement of the Problem. Because of the close connection between the variational principles of dynamics and the calculus of variations, the latter will be briefly reviewed. Attention will be restricted to the "simplest problem" of the calculus of variations, stated as follows. We seek the function $x = x(t)$, $t \in [t_0, t_1]$, that renders the integral

$$J = \int_{t_0}^{t_1} f(x, \dot{x}, t)dt \tag{4.24}$$

a minimum subject to fixed endpoints $x(t_0) = x_0$ and $x(t_1) = x_1$.

Euler – Lagrange Equation. In ordinary calculus, necessary conditions for the minimum of a function are obtained by considering the first and second derivatives. Analogously, in the calculus of variations necessary conditions are obtained by considering the first and second *variations* of J. The most important result, obtained from setting the first variation, δJ, to zero, is that if $x(t)$ minimizes J then it must satisfy the Euler-Lagrange equation:

$$f_x - \frac{d}{dt}f_{\dot{x}} = 0 \tag{4.25}$$

where subscripts indicate partial derivatives. A function satisfying this equation is called an *extremal*; it is a candidate for the minimizing function. Carrying out the differentiation gives the long form of the Euler – Lagrange equation:

$$f_x - f_{\dot{x}t} - f_{\dot{x}x}\dot{x} - f_{\dot{x}\dot{x}}\ddot{x} = 0 \tag{4.26}$$

Two key lemmas are needed to establish this result. The Fundamental Lemma states that if $M(t)$ is a continuous function on $[t_0, t_1]$ and

$$\int_{t_0}^{t_1} M(t)\eta(t)dt = 0$$

for all $\eta(t) \in C^1$ with $\eta(t_0) = \eta(t_1) = 0$ then $M(t) = 0$ for all $t \in [t_0, t_1]$.

The Du Bois - Reymond Lemma states that if $N(t)$ is continuous on $[t_0, t_1]$ and if

$$\int_{t_0}^{t_1} \dot{\eta} N \, dt = 0$$

for all $\eta(t) \in C'$ with $\eta(t_0) = \eta(t_1) = 0$ then $N(t) =$ constant for all $t \in [t_0, t_1]$.

The other necessary conditions, arising from consideration of the second variation, will not be discussed here.

Application to Dynamics. Consider a holononomic, conservative system with a single coordinate, x. Hamilton's principle for such a system is Eqn. (4.23):

$$\delta \int_{t_0}^{t_1} L(x, \dot{x}, t) dt = 0 \tag{4.27}$$

Applying Eqn. (4.25),

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \tag{4.28}$$

This is in fact Lagrange's equation for the system. (Lagrange's equations for general systems will derived in Chapter 6.)

Inverse Problem. In the inverse problem of the calculus of variations, we are given a two parameter family of curves

$$x = g(t, \alpha, \beta) \tag{4.29}$$

and we want to find a function $f(x, \dot{x}, t)$ such that the family members are the extremals of

$$J = \int_{t_0}^{t_1} f(x, \dot{x}, t) dt \tag{4.30}$$

Differentiating Eqn. (4.29) twice,

$$\dot{x} = g_t(t, \alpha, \beta) \tag{4.31}$$

$$\ddot{x} = g_{tt}(t, \alpha, \beta) \tag{4.32}$$

Under general conditions, Eqns. (4.29) and (4.31) may in principle be solved for α and β,

$$\begin{aligned} \alpha &= \rho(x, \dot{x}, t) \\ \beta &= \psi(x, \dot{x}, t) \end{aligned}$$

and then substituted into Eqn. (4.32) to obtain an equation of the form

$$\ddot{x} = G(x, \dot{x}, t) \tag{4.33}$$

This equation must be the Euler-Lagrange equation; that is, it must be identical to Eqn. (4.26). Substitute Eqn. (4.33) into Eqn. (4.26) and differentiate the result with respect to $\dot{x}$ to get

$$f_{\dot{x}t\dot{x}} + \dot{x} f_{\dot{x}x\dot{x}} + G f_{\dot{x}\dot{x}\dot{x}} + G_{\dot{x}} f_{\dot{x}\dot{x}} = 0$$

Letting $M = f_{\dot{x}\dot{x}}$, this becomes

$$\frac{\partial M}{\partial t} + \dot{x}\frac{\partial M}{\partial x} + G\frac{\partial M}{\partial \dot{x}} + G_{\dot{x}} M = 0 \tag{4.34}$$

Now define the function

$$\theta(t, \alpha, \beta) = \exp\left[\int G_{\dot{x}}\left(t, g(t, \alpha, \beta), g_t(t, \alpha, \beta)\right) dt\right] \tag{4.35}$$

The solution of Eqn. (4.34) may be shown to be of the form

$$M = \frac{\Phi(\varphi(x, \dot{x}, t),\ \psi(x, \dot{x}, t))}{\theta(t,\ \varphi(x, \dot{x}, t),\ \psi(x, \dot{x}, t))} = f_{\dot{x}\dot{x}} \tag{4.36}$$

where Φ is an arbitrary but nonzero function of φ and ψ. Integrating Eqn. (4.36) twice gives an expression for f:

$$f = \int\int M\, d\dot{x} d\dot{x} + \dot{x}\lambda(x, t) + \mu(x, t) \tag{4.37}$$

where λ and μ must be chosen so that f satisfies Eqn. (4.26). Since Φ is arbitrary there are an infinity of such functions f and thus the solution to the inverse problem is not unique.

Example. Consider again a system with a single generalized coordinate subject to a conservative force. From Newton's Second Law we know that the equation of motion is

$$\ddot{x} = F = -\frac{dV(x)}{dx}$$

where $V(x)$ is the potential energy function of F. We want to find a function $f(x, \dot{x}, t)$ such that this differential equation is the Euler-Lagrange equation for

$$J = \int_{t_0}^{t_1} f(x, \dot{x}, t)dt$$

That is,

$$\delta \int_{t_0}^{t_1} f(x, \dot{x}, t)dt = 0$$

From Eqn. (4.33) we see that

$$G = \ddot{x} = F = -\frac{dV(x)}{dx}$$

so that $G_{\dot{x}} = 0$ and Eqn. (4.35) gives

$$\theta(t, \alpha, \beta) = \exp\left[\int G_{\dot{x}} dt\right] = 1$$

Therefore, from Eqns. (4.36) and (4.37),

$$\begin{aligned} M &= \Phi = f_{\dot{x}\dot{x}} \\ f &= \int\int \Phi(x, \dot{x}, t)d\dot{x}d\dot{x} + \dot{x}\lambda(x, t) + \mu(x, t) \end{aligned}$$

To get the "simplest case", select $\Phi = 1$ to obtain

$$f = \frac{1}{2}\dot{x}^2 + \dot{x}\lambda + \mu$$

Substituting into Eqn. (4.26),

$$f_x - f_{\dot{x}t} - \dot{x}f_{\dot{x}x} - \ddot{x}f_{\dot{x}\dot{x}} = 0$$

$$\frac{\partial\mu(x, t)}{\partial x} - \frac{\partial\lambda(x, t)}{\partial t} = -\frac{dV(x)}{dx}$$

Thus we must have $\lambda = 0$ and $\mu = -V(x)$, and f becomes

$$f = \frac{1}{2}\dot{x}^2 - V = T - V = L$$

Hence the "simplest" variational problem that leads to the correct equation of motion for this case is

$$\delta \int_{t_0}^{t_1} L\, dt = 0$$

which is, of course, Hamilton's principle. Choosing other functions Φ gives other variational principles.

4.5 Principle of Least Action

Historical Remarks. Although Newton's Second Law gives highly accurate results in most situations, it doesn't seem to emanate from any deeper philosophic or scientific principle, a matter of great concern to eighteenth century scientists. Variational principles arose initially to meet this perceived need. The idea was that of all the possible motions of a dynamic system, the one that is actually followed is the one that minimizes some fundamental quantity; in other words, nature acts in the way that is most efficient.

The first successful variational principle was Fermat's principle of minimum time in optics. He starts "from the principle that Nature always acts in the shortest ways". With this principle, Fermat was able to derive the laws of refraction.

Maurpertius stated the Principle of Least Action (PLA) in dynamics from analogy to Fermat's principle. Maurepertius' viewpoint was that "nature in the production of her works always acts in the most simple ways". He stated the principle in metaphysical terms and never proved the PLA in the sense of showing that it was equivalent to the established laws of dynamics. Immediately after statement of the principle, a controversy started. On the one hand, some claimed, most notably Koenig, that the principle was not valid or that Leibniz had discovered it previously, or both! Even the great Voltaire, who knew little of mathematics and science, got into the act, satirizing the PLA in some of his books.

Euler sided with Maupertuis and managed to prove (in the sense just stated) the PLA for a particle, thus putting the principle an a sound basis. Many years later, Mach remarked that "Euler, a truly great man, lent his reputation to the PLA and the glory of his invention to Maupertuis; but he made a new thing of the principle, practical and useful". (Euler was also the first to consider the inverse problem.)

We now have the PLA in two forms, associated with the names of Lagrange and Jacobi. The latter's version has path length as the independent variable and may be viewed as a geometrical statement. In this view, the principle becomes a problem of finding geodesics in a Riemann space.

The previous Section shows that it is possible to generate an infinite number of variational principles. The only requirement is that they be "valid", that is, that they lead to the same equations of motion as does Newton's Second Law. It is surprising that the principle that is perhaps the most straight-forward and useful, that of Hamilton, did not emerge until much later than the PLA.

Noncontemporaneous Variations. In Hamilton's principle, the variations from the actual path take place with time fixed and the variations are zero at the endpoints. Now we relax this restriction and consider noncontemporaneous variations, as shown on Fig. 4-4.

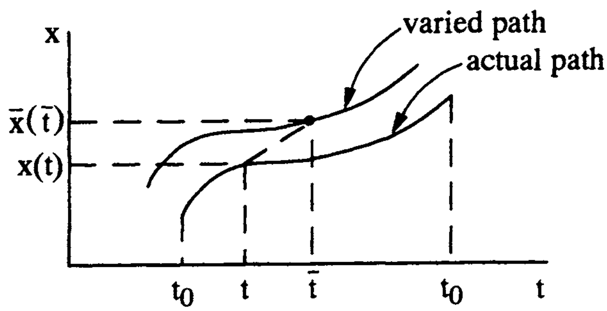


Fig. 4-4

Lagrange's Principle of Least Action. In the principle of least action we consider variations from the actual path with energy held fixed. We consider closed systems only, so that energy is conserved, and denote the noncontemporaneous variation operator by δ_t. Thus from Eqn. (4.12),

$$\delta_t\, T + \delta_t\, V = 0 \tag{4.38}$$

The relation between the operators δ and δ_t for a function $F(x,t)$ is given by

$$\delta_t\, F = \delta F + \frac{\partial F}{\partial t}\delta_t\, t \tag{4.39}$$

and is illustrated on Fig. 4-5.

Because the principle of least action is largely of historical interest only, the details of the derivation will be omitted and the results will be summarized.[5] The *action* is defined by

$$A = \int_{t_0}^{t_1} 2T\, dt \tag{4.40}$$

The Lagrange form of the principle of least action is

$$\delta_t\, A = 0 \tag{4.41}$$

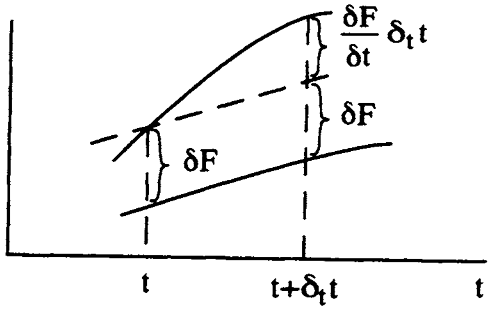


Fig. 4-5

In words, "the action is stationary for the actual path in comparison with neighboring paths having the same energy". The principle is both necessary and sufficient and thus it may be used to derive equations of motion. We note that the varied motion does not in general take the same time as the actual motion and in fact the varied motion is not in general a possible motion. Clearly, the factor 2 in Eqn. (4.40) may be omitted.

Jacobi's Principle of Least Action. Since energy is conserved, Eqn. (4.40) may be written

$$A' = \int_{t_0}^{t_1} 2\sqrt{T(h-V)}\,dt \tag{4.42}$$

However, from Eqn. (4.3) T is a quadratic function of the $\dot{u}_s$ and thus the integral in Eqn. (4.42) is homogeneous of degree one in the $\dot{u}_s$. This means that A depends only on the path in the configuration space and not in the event space.[6] Writing Eqn. (4.42) in terms of s, the path length in configuration space, gives

$$A' = \int \sqrt{2(h-V)}\,ds \tag{4.43}$$

Consequently, the Jacobi form of the principle is

$$\delta_t A' = 0 \tag{4.44}$$

This is a problem in the calculus of variations.

Notes

1 Strictly speaking, $T = T(\dot{x}_1^1, \dot{x}_2^1, \dot{x}_3^1, \dot{x}_1^2, \cdots, \dot{x}_3^n)$ and $T'(\dot{u}_1, \cdots, \dot{u}_N)$ are two different functions but we will use the same symbol, T, for both.

2 Hamilton's principle for a class of non-conservative systems may be found in "Some Remarks on Hamilton's Principle", G. Leitmann, *J. Appl. Mech.*, Dec. 1963.

3 This was proved in Section 3.3; an alternate proof will be given in Section 6.5.

4 Recall that $\int u\,dv = uv - \int v\,du$; here we take $u = \dot{\theta}$ and $dv = \frac{d}{dt}(\delta\theta)dt$.

5 See Rosenberg or Pars for the details.

6 See Pars

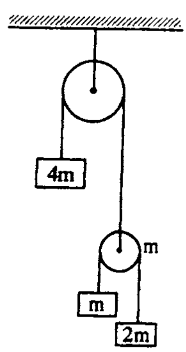
PROBLEMS

4/1. A weight of mass $4m$ is attached to a massless, inextensible string which passes over a frictionless, massless pulley, as shown on Fig. 4/1. The other end of this string is attached to the center of a frictionless, homogeneous pulley of mass m. A second massless inextensible string having masses m and $2m$ attached to its extremities passes over the pulley of mass m. Gravity is the only force acting on this system.

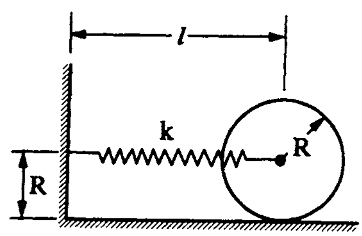
(a) Give the kinetic energy for this system;

(b) Give the energy integral, if one exists;

(c) Write down Hamilton's principle.



Problem 4/1

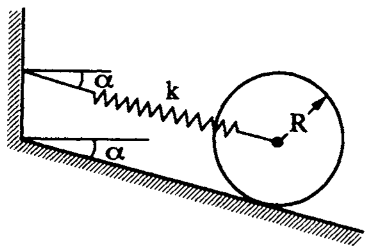


Problem 4/2

4/2. A homogeneous disk of mass M, constrained to remain in a vertical plane, rolls without sliding on a horizontal line as shown. A massless horizontal, linear spring of rate k is attached to the center of the disk and to a fixed point. If the free length of the spring is l, and the disk radius is R,

(a) Give the kinetic energy for this system;
(b) Give the energy integral, if one exists;
(c) Write down Hamilton's principle.

4/3. Give the same answers as in Problem 4/2 when the configuration is changed so that the line is inclined by the angle α to the horizontal, as shown.



Problem 4/3

4/4. Three particles of mass m_1, m_2, and m_3, respectively, are constrained to move so that they lie for all time on a straight line passing through a fixed point. For the force-free problem in Cartesian coordinates:

(a) Give the kinetic energy;
(b) Give the energy integral, if one exists;
(c) Write down Hamilton's principle.

4/5. A heavy, homogeneous inextensible string of given length remains for all time in a vertical plane. It lies in part on a smooth, horizontal table, and in part, it hangs vertically down over the table edge. What is Hamilton's principle?

4/6. A particle of mass m moves in the x, y plane under a force which is derivable from a potential energy function. The particle velocity is directed for all time toward a point P which moves along the x axis so that its distance from the origin is given by the prescribed function $\xi(t)$.

(a) How many degrees of freedom does the particle have?

(b) What is Hamilton's principle?

(c) Give the energy integral, if one exists.

4/7. One point of a rigid body is constrained to move on a smooth space curve defined by $f(x_0, y_0, z_0, t) = 0$. If the forces and moments acting on the body are conservative, give Hamilton's principle. Does an energy integral exist? If so, write it down. If none exists, explain why.

Chapter 5

Generalized Coordinates

5.1 Theory of Generalized Coordinates

Remarks. As revealed by the simple pendulum (Fig. 5-1), for example, use of Newton's Second Law in rectangular coordinates has the shortcoming that both the constraint force T and the gravitational force W appear explicitly. In the energy method, however, the first of these doesn't appear at all and the second appears via a potential energy function (see Fig. 3-9).

A second important shortcoming is that coordinates x and y are awkward and, more fundamentally, one is redundant; only one coordinate is needed and θ is the obvious choice. We now take up this second point and introduce "generalized coordinates", of which an example is θ for the pendulum.

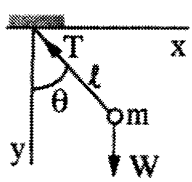


Fig. 5-1

Generalized Coordinates. Suppose a system of $N/3$ particles is constrained by L independent constraints, Eqns. (3.2):

$$\sum_{s=1}^{N} A_{rs} du_s + A_r dt = 0 \; ; \qquad r = 1, \cdot\cdot, L$$

Suppose further that $L'(< L)$ of these are holonomic and $L - L' = \ell$ are nonholonomic, i.e. if the integrated form of the holonomic constraints is

$$f_i(u_1, \cdot\cdot, u_N, t) = \alpha_i \; ; \qquad i = 1, \cdot\cdot, L' \tag{5.1}$$

then the Pfaffian form of the constraints is

$$\sum_{s=1}^{N} \frac{\partial f_i}{\partial u_s} du_s + \frac{\partial f_i}{\partial t} dt = 0 \; ; \qquad i = 1, \cdot\cdot, L' \tag{5.2}$$

$$\sum_{s=1}^{N} A_{js}\, du_s + A_j\, dt = 0 \; ; \qquad j = L' + 1, \cdot\cdot, L \tag{5.3}$$

Recall from Sections 2.3 and 2.7 that DOF $= N - L$ and DSAC $= N - L'$. From now on we will use the symbol n to denote the DSAC; that is $n = N - L'$ (note that, therefore, n will *not* denote the number of particles as before).

Now make the following definitions:

1. Any finite set of numbers $\{q_1, q_2, \cdot\cdot, q_{\bar{n}}\}$, $\bar{n} \geq n$, that completely defines the configuration of a system at a given instant is a set of *coordinates.*

2. Any set of numbers $\{q_1, \cdot\cdot, q_n\}$ is called a set of *generalized coordinates* where n is defined as above. Thus n is the least possible number of coordinates and excedes the DOF by the number of nonholonomic constraints.[1]

Transformation of Coordinates. We wish to transform from rectangular coordinates to generalized coordinates. Introduce transformation functions

$$q_s = \rho_s(u_1, \cdot\cdot, u_N, t) \; ; \qquad s = 1, \cdot\cdot, N \tag{5.4}$$

such that the first L' satisfy the holonomic constraints, i.e.

$$\rho_s(u_1, \cdot\cdot, u_N, t) = \alpha_s \; ; \qquad s = 1, \cdot\cdot, L' \tag{5.5}$$

where the α_s are constants, and the remaining $\rho_s(\cdot)$, $s = L' + 1, \cdot\cdot, N$ are

1. Single-valued.

2. Continuous with continuous derivatives.

3. Such that the Jacobian is not zero; that is,

$$J = \begin{vmatrix} \dfrac{\partial \rho_1}{\partial u_1} & \cdots & \dfrac{\partial \rho_1}{\partial u_N} \\ \vdots & & \vdots \\ \dfrac{\partial \rho_N}{\partial u_1} & \cdots & \dfrac{\partial \rho_N}{\partial u_N} \end{vmatrix} \neq 0 . \tag{5.6}$$

Under these restrictions, the transformation is one-to-one and onto (Fig. 5-2); therefore by the implicit function theorem the transformation can be inverted to give

$$\begin{aligned} u_s &= u_s(q_1, \cdot\cdot, q_N, t) \\ &= u_s(\alpha_1, \cdot\cdot, \alpha_{L'}, q_{L'+1}, \cdot\cdot, q_N, t) ; \qquad s = 1, \cdot\cdot, N \end{aligned} \tag{5.7}$$

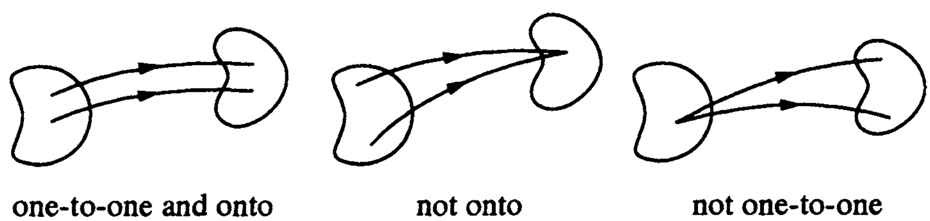


Fig. 5-2

Now re-label:

$$q_{L'+1} = q_1, \cdots, q_N = q_n$$

These then are the generalized coordinates; we have now

$$q_s = \rho_s(u_1, \cdot\cdot, u_N, t) ; \qquad s = 1, \cdot\cdot, n$$

$$u_s = u_s(q_1, \cdot\cdot, q_n, t) ; \qquad s = 1, \cdot\cdot, N \tag{5.8}$$

From the last of these, the differential displacements are related by

$$du_s = \sum_{k=1}^{n} \frac{\partial u_s}{\partial q_k} dq_k + \frac{\partial u_s}{\partial t} dt ; \qquad s = 1, \cdot\cdot, N \tag{5.9}$$

and therefore the virtual displacements are related by

$$\delta u_s = \sum_{k=1}^{n} \frac{\partial u_s}{\partial q_k}\delta q_k \, ; \qquad s = 1, \cdot\cdot, N \tag{5.10}$$

Possible and Virtual Displacements in Generalized Coordinates. The nonholonomic constraints are

$$\sum_{s=1}^{N} A_{rs} du_s + A_r dt = 0 \, ; \qquad r = 1, \cdot\cdot, \ell \tag{5.11}$$

where $\ell = L - L'$. The generalized coordinates are not subject to the holonomic constraints; the discarded L' coordinates have accomplished this. Substituting Eqn. (5.9) into (5.11):

$$\sum_{s=1}^{N} A_{rs} \left(\sum_{k=1}^{n} \frac{\partial u_s}{\partial q_k} dq_k + \frac{\partial u_s}{\partial t} dt \right) + A_r dt = 0 \, ; \qquad r = 1, \cdot\cdot, \ell$$

$$\sum_{k=1}^{n} \left(\sum_{s=1}^{N} A_{rs} \frac{\partial u_s}{\partial q_k} \right) dq_k + \left(\sum_{s=1}^{N} A_{rs} \frac{\partial u_s}{dt} + A_r \right) dt = 0$$

$$\sum_{k=1}^{n} B_{rk} dq_k + B_r dt = 0 \, ; \qquad r = 1, \cdot\cdot, \ell \tag{5.12}$$

where

$$\begin{aligned} B_{rk} &= \sum_{s=1}^{N} A_{rs} \frac{\partial u_s}{\partial q_k} \\ B_r &= \sum_{s=1}^{N} A_{rs} \frac{\partial u_s}{\partial t} + A_r \end{aligned} \tag{5.13}$$

In terms of velocity components, these are

$$\sum_{k=1}^{n} B_{rk} \dot{q}_k + B_r = 0 \, ; \qquad r = 1, \cdot\cdot, \ell \tag{5.14}$$

As before, these equations define *possible* displacements and velocities. *Virtual* displacements satisfy

$$\sum_{k=1}^{n} B_{rk} \delta q_k = 0 \, ; \qquad r = 1, \cdot\cdot, \ell \tag{5.15}$$

It is important to realize that generalized coordinates are *general*; they can be distances, angles, etc. and have any dimensions.

Variation Operator. Recall Eqn. (3.28):

$$\frac{d}{dt}(\delta u) = \delta\left(\frac{du}{dt}\right) = \delta\dot{u}$$

or,

$$d(\delta u) = \delta(du) \tag{5.16}$$

It may be shown that also

$$d(\delta q) = \delta(dq) \tag{5.17}$$

In words, the d and δ operators are communicative in generalized coordinates.

5.2 Examples

Simple Pendulum. Let the rectangular components of the bob be (x, y); then the constraint is (see Fig. 5-1)

$$\sqrt{x^2 + y^2} = \ell$$

which is of the form $f(x, y) = \alpha$ and which is holonomic, scleronomic, and catastatic. We have:

N = number of rectangular components = 2
L' = number of holonomic constraints = 1
ℓ = number of nonholonomic constraints = 0
L = number of constraints = $L' + \ell = 1$
DOF = degrees of freedom = $N - L = 1$
n = number of generalized coordinates = $N - L' = 1$

According to our approach, we transform to new variables such that first is equal to α and second is convenient:

$$\begin{aligned} q_1 &= \alpha = \ell = \rho_1(x, y, t) \\ q_2 &= \tan^{-1}\frac{y}{x} = \rho_2(x, y, t) \end{aligned}$$

The inverse transformation is

$$\begin{aligned} x &= q_1 \cos q_2 = \alpha_1 \cos q_2 = \ell \cos q_2 \\ y &= q_1 \sin q_2 = \alpha_1 \sin q_2 = \ell \sin q_2 \end{aligned}$$

The Jacobian of this transformation is

$$J = \begin{vmatrix} \dfrac{\partial x}{\partial q_1} & \dfrac{\partial x}{\partial q_2} \\ \dfrac{\partial y}{\partial q_1} & \dfrac{\partial y}{\partial q_2} \end{vmatrix} = \begin{vmatrix} \cos q_2 & -q_1 \sin q_2 \\ \sin q_2 & -q_1 \cos q_2 \end{vmatrix}$$

$$= q_1 \cos^2 q_2 + q_1 \sin^2 q_2 = q_1 = \ell \neq 0$$

Therefore the transformation satisfies all three requirements. Now relabel to get the generalized coordinate:

$$q_2 = q_1 = \theta$$

and the transformations are now

$$\theta = \tan^{-1} \frac{y}{x}$$

$$\begin{aligned} x &= \ell \cos \theta \\ y &= \ell \sin \theta \end{aligned}$$

The q_1 and q_2 in this problem are of course just the polar coordinates (Fig. 5-1) and the choice of θ could have been made by inspection.

In Section 2.3 it was shown that a rigid body in unconstrained motion has DOF $= 6$; thus it has 6 generalized coordinates. These are usually taken to be the coordinates of some body-fixed point, say the center of mass, and three angles defining the location of body fixed axes. A common choice of angles are the Euler angles; these are defined and used in Chapter 11.

Example. Three bars are hinged and lie in a plane such that one end is attached at 0 and the other end carries particle p (Fig. 5-2). Then either (x, y) or $(\theta_1, \theta_2, \theta_3)$ determine the location of p. This seems to imply that there is a relationship $f(\theta_1, \theta_2, \theta_3) =$ constant because it only takes two parameters to give the location of p. This, however, is not true because the transformations have the properties (see Fig. 5-3):

$$(\theta_1, \theta_2, \theta_3) \longrightarrow (x, y) \text{ is one-to-one, but not onto}$$

$$(x, y) \longrightarrow (\theta_1, \theta_2, \theta_3) \text{ is not one-to-one, but onto}$$

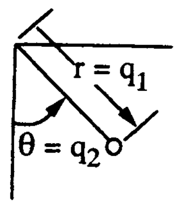


Fig. 5-3

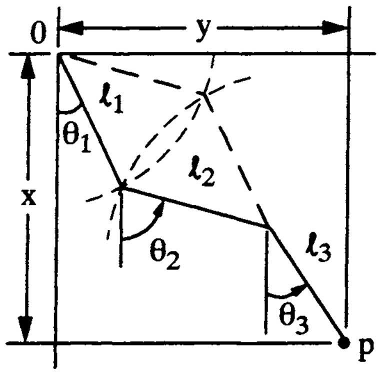


Fig. 5-4

A basic assumption is violated and the transformation theory does not apply.

Remarks

1. For specific problems, generalized coordinates usually suggest themselves from the geometry of the problem; the choice of θ for the simple pendulum is an example of this. The general theory of transformation is needed, however, because we desire to put the key equations of dynamics in terms of generalized coordinates.

2. In some cases there are isolated points in configuration space for which one or more of the three conditions on the coordinate transformation are not satisfied. In this case, the transformation is restricted to regions of the configuration space not containing these points.

Notes

1 In some texts, "generalized coordinates" are not necessarily minimal. Also, they are sometimes called Lagrangian coordinates.

PROBLEMS

5/1. A particle moves on the surface of a three-dimensional sphere.

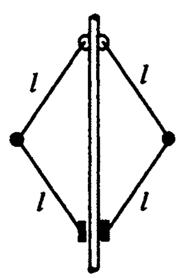
(a) Choose suitable generalized coordinates for the motion.

(b) What are the Eqns. (5.7) for this case?

(c) Examine the Jacobian.

5/2. A particle moves on the surface of a right circular cylinder whose radius expands according to the law $r = f(t)$ while its axis remains stationary. Answer the same questions as in Problem 5/1.

5/3. A centrifugal governor has the configuration shown. If unconstrained, six coordinates would be required to define the configurations of the flyballs. How many constraints must the Cartesian coordinates satisfy? What are they? Choose suitable generalized coordinates to describe the position of the flyballs. Construct Eqns. (5.7) for this problem.



Problem 5/3

Chapter 6

Lagrange's Equations

6.1 Fundamental Equation in Generalized Coordinates

Kinetic Energy. We now wish to put our previous results in terms of generalized coordinates, instead of rectangular coordinates. We begin with kinetic energy. Previously we've expressed the kinetic energy of a system of particles in component form:

$$T = \frac{1}{2}\sum_{r=1}^{N} m_r\, \dot{u}_r^2 \tag{4.3}$$

From Eqns. (5.9) the $\dot{u}_r$ in terms of the $\dot{q}_r$ are given by:

$$\dot{u}_r = \sum_{k=1}^{n} \frac{\partial u_r}{\partial q_k}\dot{q}_k + \frac{\partial u_r}{\partial t}\,; \qquad r = 1, \cdot\cdot, N \tag{6.1}$$

Therefore[1]

$$T = \frac{1}{2}\sum_{r=1}^{N} m_r \left[\sum_{k=1}^{n} \frac{\partial u_r}{\partial q_k}\dot{q}_k + \frac{\partial u_r}{\partial t}\right]^2 \tag{6.2}$$

Expanding,

$$T = \frac{1}{2}\sum_{\alpha}\sum_{\beta} a_{\alpha\beta}\dot{q}_\alpha\dot{q}_\beta + \sum_{\alpha} b_\alpha\dot{q}_\alpha + c \tag{6.3}$$

where

$$a_{\alpha\beta} = \overline{\sum_r} m_r \frac{\partial u_r}{\partial q_\alpha} \frac{\partial u_r}{\partial q_\beta} = a_{\beta\alpha}$$

$$b_\alpha = \overline{\sum_r} m_r \frac{\partial u_r}{\partial q_\alpha} \frac{\partial u_r}{\partial t} \tag{6.4}$$

$$c = \frac{1}{2} \overline{\sum_r} m_r \left(\frac{\partial u_r}{\partial t}\right)^2 \geq 0$$

and where the following notation has been adopted

$$\sum_i = \sum_{i=1}^{n}; \qquad \overline{\sum_i} = \sum_{i=1}^{N} \tag{6.5}$$

As a special case, for $\partial u_s/\partial t = 0$:

$$T = \frac{1}{2} \sum_\alpha \sum_\beta a_{\alpha\beta} \dot{q}_\alpha \, \dot{q}_\beta \tag{6.6}$$

From Eqn. (4.3), $T \geq 0$ and therefore from Eqn. (6.6) $a_{\alpha\beta}$ must be a symmetric positive definite matrix.

Example. Consider a simple pendulum with moving support point (Fig. 6-1). In rectangular components,

$$T = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2)$$

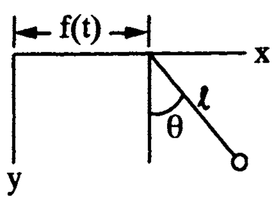


Fig. 6-1

Choose θ as generalized coordinate; the transformation equations are

$$x = f(t) + \ell \sin\theta; \qquad y = \ell \cos\theta$$

Therefore

$$T = \frac{1}{2} m \left[\ell^2 \dot{\theta}^2 + 2\ell\dot{\theta}\cos\theta \dot{f} + \dot{f}^2\right]$$

For one generalized coordinate, Eqn. (6.3) becomes

$$T = \frac{1}{2} a_{11} \dot{\theta}^2 + b_1 \dot{\theta} + c$$

Comparing,

$$a_{11} = m\ell^2 \; ; \qquad b_1 = m\ell \cos\theta \dot{f} \; ; \qquad c = \frac{1}{2} m \dot{f}^2$$

This extra complexity in T due to choosing θ as a coordinate is worth it in the long run. Note that the coefficients $a_{\alpha\beta}$, b_α, and c are in general functions of the generalized coordinates $q_1, \cdot\cdot, q_n$.

Two Equalities. From Eqn. (6.1) we have

$$\frac{\partial \dot{u}_r}{\partial \dot{q}_s} = \frac{\partial u_r}{\partial q_s} \tag{6.7}$$

and

$$\frac{\partial \dot{u}_r}{\partial q_s} = \sum_\alpha \frac{\partial^2 u_r}{\partial q_\alpha \partial q_s} \dot{q}_\alpha + \frac{\partial^2 u_r}{\partial t \, \partial q_s} = \frac{d}{dt} \left(\frac{\partial u_r}{\partial q_s} \right) \tag{6.8}$$

Fundamental Equation. Recall Eqns. (3.7) and (5.10)

$$\overline{\sum_r} \, (m_r \ddot{u}_r - F_r) \delta u_r = 0$$

$$\delta u_r = \sum_s \frac{\partial u_r}{\partial q_s} \delta q_s \; ; \qquad r = 1, \cdot\cdot, N$$

Substituting the second into the first,

$$\sum_s \left(\overline{\sum_r} \, (m_r \ddot{u}_r - F_r) \frac{\partial u_r}{\partial q_s} \right) \delta q_s = 0 \tag{6.9}$$

Now consider

$$\frac{d}{dt} \left(\dot{u}_r \frac{\partial u_r}{\partial q_s} \right) = \ddot{u}_r \, \frac{\partial u_r}{\partial q_s} + \dot{u}_r \, \frac{d}{dt} \left(\frac{\partial u_r}{\partial q_s} \right)$$

$$\ddot{u}_r \, \frac{\partial u_r}{\partial q_s} = \frac{d}{dt} \left(\dot{u}_r \, \frac{\partial \dot{u}_r}{\partial \dot{q}_s} \right) - \dot{u}_r \, \frac{\partial \dot{u}_r}{\partial q_s} \tag{6.10}$$

where Eqns. (6.7) and (6.8) were used.

Substitute Eqn. (6.10) into Eqn. (6.9):

$$\sum_s \left(\overline{\sum_r} \left\{ m_r \left[\frac{d}{dt} \left(\dot{u}_r \frac{\partial \dot{u}_r}{\partial \dot{q}_s} \right) - \dot{u}_r \frac{\partial \dot{u}_r}{\partial q_s} \right] - F_r \frac{\partial u_r}{\partial q_s} \right\} \right) \delta q_s = 0 \quad (6.11)$$

But from Eqn. (4.3),

$$\frac{\partial T}{\partial q_s} = \frac{1}{2} \overline{\sum_r} m_r \, 2\dot{u}_r \frac{\partial \dot{u}_r}{\partial q_s} \quad (6.12)$$

$$\frac{\partial T}{\partial \dot{q}_s} = \frac{1}{2} \overline{\sum_r} m_r \, 2\dot{u}_r \frac{\partial \dot{u}_r}{\partial \dot{q}_s} \quad (6.13)$$

Substituting these in Eqn. (6.11):

$$\sum_s \left[\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_s} \right) - \frac{\partial T}{\partial q_s} - \overline{\sum_r} F_r \frac{\partial u_r}{\partial q_s} \right] \delta q_s = 0 \quad (6.14)$$

Now define the *generalized force* $\underline{Q}$ as that force with components

$$Q_s = \overline{\sum_r} F_r \frac{\partial u_r}{\partial q_s} \quad (6.15)$$

Then we get finally

$$\sum_{s=1}^{n} \left[\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_s} \right) - \frac{\partial T}{\partial q_s} - Q_s \right] \delta q_s = 0 \quad (6.16)$$

This is the fundamental equation in generalized coordinates.

Remarks.

1. $Q_s \delta q_s$ is the work done by generalized force component Q_s in a virtual displacement δq_s. If δq_s is a distance, then Q_s is a force; if δq_s is an angle then Q_s is a moment.
2. There are several advantages to this form of the fundamental equation.
   (i) The coordinates are any convenient set of parameters, provided only they are sufficient to specify the configuration of the system; further they are the minimum number to do so.

(ii) The δq_s must satisfy only the nonholonomic constraints; the holonomic constraints do not appear at all; they have been eliminated by our choice of coordinates.

Conservative Forces and Potential Functions in Generalized Coordinates. Recall from Section 3.5 that if $\underline{F}^c$ is a conservative force, there exists a function $V(u_1, u_2, \cdot\cdot, u_N)$ such that

$$\underline{F}^c = -\text{grad } V^c \tag{6.17}$$

or, in component form,

$$F_s^c = -\frac{\partial V^c}{\partial u_s}; \qquad s = 1, \cdot\cdot, N \tag{6.18}$$

Now make the change of variables to generalized coordinates $u_s(q_1, \cdots, q_n, t)$ in V^c:

$$\frac{\partial V^c}{\partial q_r} = \overline{\sum_s} \frac{\partial V^c}{\partial u_s} \frac{\partial u_s}{\partial q_r} = -\overline{\sum_s} F_s^c \frac{\partial u_s}{\partial q_r} = -Q_r^c$$

Therefore the relations

$$Q_s^c = -\frac{\partial V^c}{\partial q_s}; \qquad s = 1, \cdot\cdot, n \tag{6.19}$$

hold for generalized coordinates.

Now suppose some given forces are conservative and some are not. Each of the conservative forces has a potential function and the sum in terms of components in generalized forces is

$$\sum_p Q_s^c = -\sum_p \frac{\partial V^c}{\partial q_s} = -\frac{\partial V}{\partial q_s}; \qquad s = 1, \cdot\cdot, n \tag{6.20}$$

Recall the definition of the Lagrangian function, Eqn. (4.21), and the fact that V depends on the q_s but not the $\dot{q}_s$; therefore

$$\frac{\partial L}{\partial \dot{q}_s} = \frac{\partial T}{\partial \dot{q}_s} \tag{6.21}$$

Consequently, Eqn. (6.16) can be written

$$\sum_{s=1}^{n} \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) - \frac{\partial L}{\partial q_s} - Q_s^{nc}\right] \delta q_s = 0 \tag{6.22}$$

where the Q_s^{nc} are the components of the nonconservative given forces.

Remark. It is also possible to generalize the concepts of conservative forces and potential functions to include velocity dependence of a certain restricted kind, namely linear in the velocities. This will be done in Section 6.4.

6.2 Multiplier Rule

The Dynamical Problem. Recall the fundamental equation in generalized coordinates, Eqn. (6.16). The q_s are any set of generalized coordinates, that is, any minimal number of independent parameters completely specifying the configuration of the system at any time. The minimal number of coordinates is:

$$n = DSAC = N - L'$$

where, as before, N is three times the number of particles and L' the number of holonomic constraints. The functions T and Q_s are in general functions of $(q_1, \cdot\cdot, q_n, \dot{q}_1, \cdot\cdot, \dot{q}_n, t)$. The virtual displacements are not arbitrary but must satisfy Eqns. (5.15):

$$\sum_{s=1}^{n} B_{rs}\,\delta q_s = 0\,; \qquad r = 1, \cdot\cdot, \ell \tag{5.15}$$

where the constraints in velocity form are given by Eqns. (5.14):

$$\sum_{s=1}^{n} B_{rs}\dot{q}_s + B_r = 0\,; \qquad r = 1, \cdot\cdot, \ell \tag{5.14}$$

As before, any $(q_s, \dot{q}_s)$ satisfying Eqns. (5.14) are a set of possible displacements and velocities; i.e. they are a possible state. Among the possible states is the actual state that also satisfies Eqn. (6.16).

Notation Change. Let

$$\begin{aligned}
R_s &= \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_s}\right) - \frac{\partial T}{\partial q_s} - Q_s\,; \qquad s = 1, \cdot\cdot, n \\
\underline{R} &= (R_1,\ \cdot\cdot,\ R_n) \\
\delta \underline{q} &= (\delta q_1,\ \cdot\cdot,\ \delta q_n) \\
\underset{\sim}{\underline{B}}_r &= (B_{r1}, B_{r2},\ \cdot\cdot,\ B_{rn})\,; \qquad r = 1, \cdot\cdot, \ell
\end{aligned} \tag{6.23}$$

Then Eqns. (6.16) and (5.15) may be written

$$\begin{aligned} \underline{R} \cdot \delta \underline{q} &= 0 \\ \tilde{\underline{B}}_r \cdot \delta \underline{q} &= 0 \,; \qquad r = 1, \cdot\cdot, \ell \end{aligned} \tag{6.24}$$

By assumption, the constraint equations are linearly independent; this means that there do not exist constants c_r; $r = 1, \cdot\cdot, \ell$ not all zero such that

$$\sum_{r=1}^{\ell} c_r \tilde{\underline{B}}_r = \underline{0} \tag{6.25}$$

Multiplier Rule for Single Particle and Single Constraint. For a particle in 3-space with a single nonholonomic constraint, $N = 3$ and $\ell = 1$; thus there is one vector $\tilde{\underline{B}}_1$ and this vector and $\underline{R}$ have three components.

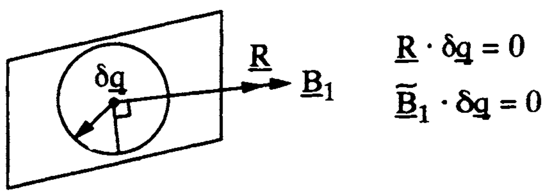


Fig. 6-2

It is clear that there is no c_1 for which $c_1 \tilde{\underline{B}}_1 = \underline{0}$ except $c_1 = 0$. $\tilde{\underline{B}}_1 \cdot \delta \underline{q} = 0$ means that $\delta \underline{q}$ is such that it must be normal to $\tilde{\underline{B}}_1$ but is otherwise arbitrary (Fig. 6-2). $\underline{R} \cdot \delta \underline{q} = 0$ means $\underline{R}$ must be such that it is normal to $\delta \underline{q}$ for all such possible $\delta \underline{q}$. The only possibility is that $\underline{R}$ is in the same direction as $\tilde{\underline{B}}_1$; thus we may write

$$\underline{R} = -\lambda_1 \tilde{\underline{B}}_1$$

Rearranging and dotting with a completely arbitrary $\delta \underline{q}$,

$$(\underline{R} + \lambda_1 \tilde{\underline{B}}_1) \cdot \delta \underline{q} = 0$$

In essence, this combines the fundamental and the constraint equations into a single equation; in this new equation, the $\delta \underline{q}$ are not required to satisfy the constraint.

General Multiplier Rule. We now return to the general problem. Consider vectors[2]

$$\begin{aligned} \underline{a} &= (a_1, \cdot\cdot, a_n) \in I\!E^n \quad \text{given} \\ \underline{x} &= (x_1, \cdot\cdot, x_n) \in I\!E^n \quad \text{arbitrary} \\ \underline{b}_i &= (b_{i_n}, \cdot\cdot, b_{i_n}) \in I\!E^n\,; \qquad i = 1, \cdot\cdot \ell < n, \\ & \quad \text{all } \underline{b}_i \text{ independent, given} \end{aligned}$$

such that

$$\begin{aligned} <\underline{a}, \underline{x}> &= \sum_{i=1}^{n} a_i x_i = a_1 x_1 + a_2 x_2 + \cdot\cdot + a_n x_n = 0 \\ <\underline{b}_i, \underline{x}> &= 0\,\forall\, i \end{aligned}$$

Without the last of these, the first implies $\underline{a} = \underline{0}$; $(a_1, \cdot\cdot, a_n) = (0, \cdot\cdot 0)$. Since the $\underline{b}_i$ are independent, they form a basis in $I\!E^\ell \subset I\!E^n$. Write $I\!E^n = I\!E^\ell \oplus I\!E^\ell_\perp$, where $I\!E^\ell_\perp$ is the orthogonal compliment of $I\!E^\ell$ in $I\!E^n$. Then $\underline{x} \perp \underline{y}\,\forall\, \underline{y} \in I\!E^\ell \iff \underline{x} \in I\!E^\ell_\perp$ and $<\underline{b}_i, \underline{x}>= 0 \implies \underline{x} \in I\!E^\ell_\perp$. Also, $<\underline{a}, \underline{x}>= 0 \implies \underline{a} \notin I\!E^\ell_\perp \implies \underline{a} \in I\!E^\ell \implies \underline{a}$ can be expressed in terms of a basis in $I\!E^\ell$, say $\underline{b}_i$. Therefore,

$$\underline{a} = \sum_{i=1}^{\ell} c_i \underline{b}_i\,, \qquad \text{or,} \qquad a_j = \sum_{i=1}^{\ell} c_i b_{ij}\,; \qquad j = 1, \cdot\cdot, n \tag{6.26}$$

Application to Dynamics. Identify

$$\underline{a} = \underline{R}\,; \qquad \underline{x} = \delta \underline{q}\,; \qquad \underline{b}_i = \underline{\tilde{B}}_r\,; \qquad c_i = -\lambda_i$$

Then Eqn. (6.26) is

$$\underline{R} = -\sum_{r=1}^{\ell} \lambda_r \underline{\tilde{B}}_r \tag{6.27}$$

Dotting with any completely arbitrary vector $\delta \underline{q}$:

$$\left(\underline{R} + \sum_{r=1}^{\ell} \lambda_r \underline{\tilde{B}}_r\right) \cdot \delta \underline{q} = 0$$

Finally, using Eqns. (6.23) to revert to the original notation

$$\sum_{s=1}^{n} \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_s}\right) - \frac{\partial T}{\partial q_s} - Q_s + \sum_{r=1}^{\ell} \lambda_r B_{rs}\right] \delta q_s = 0 \tag{6.28}$$

which is another form of the fundamental equation. This equation is a statement about the system as a whole.

6.3 Lagrange's Equations

Derivation from Fundamental Equation. Since the δq_s are now arbitrary, from Eqn. (6.28) it is necessary and sufficient that

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_s}\right) - \frac{\partial T}{\partial q_s} - Q_s + \sum_{r=1}^{\ell} \lambda_r B_{rs} = 0 \; ; \qquad s = 1, \cdot\cdot, n \tag{6.29}$$

These are *Lagrange's equations of motion.* They are n equations in the $n + \ell$ unknowns $q_1, \cdot\cdot, q_n, \ \lambda_1, \cdot\cdot, \lambda_\ell$; the additional ℓ equations to be satisfied are

$$\sum_{s=1}^{n} B_{rs}\dot{q}_s + B_r = 0 \; ; \qquad r = 1, \cdot\cdot, \ell \tag{5.14}$$

Equations (6.29) are statements about each individual component.

Derivation from Hamilton's Principle. The first form of Hamilton's Principle is Eqn. (4.19):

$$\int_{t_0}^{t_1} (\delta T + \delta W) dt = 0$$

where

$$\delta u_s(t_0) = \delta u_s(t_1) = 0 \; ; \qquad s = 1, \cdot\cdot, N$$

Transforming to generalized coordinates, we now have $T(q_1, \cdot\cdot, q_n, \dot{q}_1, \cdot\cdot, \dot{q}_n, t)$ and

$$\delta W = \sum_{s=1}^{n} Q_s \delta q_s \tag{6.30}$$

The virtual displacements must satisfy Eqns. (5.15),

$$\sum_{s=1}^{n} B_{rs}\delta q_s = 0 \; ; \qquad r = 1, \cdot\cdot, \ell$$

and the possible displacements and velocities must satisfy the nonholonomic constraints given by Eqns. (5.14). By the multiplier rule for integrals, the equivalent problem is

$$\int_{t_0}^{t_1} \left(\delta T + \sum_{s=1}^{n} Q_s \delta q_s - \sum_{s=1}^{n}\sum_{r=1}^{\ell} \lambda_r B_{rs} \delta q_s\right) dt = 0 \tag{6.31}$$

where the δq_s are now arbitrary. The variation of T is

$$\delta T = \sum_{s=1}^{n} \frac{\partial T}{\partial q_s}\delta q_s + \sum_{s=1}^{n} \frac{\partial T}{\partial \dot{q}_s}\delta \dot{q}_s$$

Consider the $\delta \dot{q}_s$ term; integrating by parts:

$$\int_{t_0}^{t_1} \sum_{s=1}^{n} \underbrace{\frac{\partial T}{\partial \dot{q}_s}}_{u} \underbrace{\frac{d}{dt}(\delta q_s)dt}_{dv} = \underbrace{\left[\sum_{s=1}^{n} \frac{\partial T}{\partial \dot{q}_s}\delta q_s\right]_{t_0}^{t_1}}_{=0} - \int_{t_0}^{t_1} \sum_{s=1}^{n} \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_s}\right)\delta q_s dt$$

Consequently, Eqn. (6.31) becomes

$$\int_{t_0}^{t_1} \sum_{s=1}^{n} \left[-\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_s}\right) + \frac{\partial T}{\partial q_s} + Q_s - \sum_{r=1}^{\ell} \lambda_r B_{rs}\right] \delta q_s dt = 0 \qquad (6.32)$$

Since the δq_s are arbitrary, we must have Eqns. (6.29):

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_s}\right) - \frac{\partial T}{\partial q_s} - Q_s + \sum_{r=1}^{\ell} \lambda_r B_{rs} = 0\,; \qquad s = 1, \cdot\cdot, n$$

Historical Remarks. Lagrange was born in 1736 in Turin, France. His abilities were recognized early and he was appointed professor of mathematics at the local technical college at age 16, after having mastered calculus on his own. At this time he began development of the calculus of variations. This subject was originated in correspondence between Euler and Lagrange. Euler and other leading scientists arranged to have him appointed to the Berlin Academy of Sciences. While in Berlin, Lagrange developed a friendship with Fredrick the Great. After Fredrick's death, he returned to France in 1787 where he was appointed to the French Academy of Sciences (before the French Revolution) and then as the first professor of mathematics at the famous Ecole Polytechnique (after). While at the French Academy, the queen, Marie Antoinette, befriended him. It is somewhat remarkable that Lagrange kept his head during the Revolution, a time when many scientists and other intellectuals, and many friends of the royal family, were losing theirs. The last of Lagrange's famous acquaintances was Napoleon, who cultivated socially the leading French intellegencia of the time. Napoleon once said that "Lagrange is the lofty pyramid of the mathematical sciences."

The seminal work of Lagrange's career was the *Mechanique Analytique*. He first conceived of this book when he was 19, but didn't finish it until he was 52 years old. He was somewhat of a perfectionist! For a long period in mid-life, he lost all interest in mathematics and science (we would say he was burned-out). His interest revived in his 70's, and his last effort before dying in 1813 was to revise his great book.

In the *Mechanique Analytique* (1788), Lagrange collected all the known principles of dynamics and presented them in a new, powerful, and unified form, expressed in Eqns. (6.29). In addition, in this book he: (1) Applied his equations to solve a wide variety of problems, (2) Gave a method of approximation for the solution of dynamics problems, (3) Analyzed the stability of equilibrium positions, (4) Asserted that an equilibrium position is stable when the system potential energy is minimum (this was later proved by Dirichlet, see Section 12.5), (5) Studied small motions about equilibrium positions, (6) Recognized that the equations of motion of any dynamical system could be written in first order form (see Section 12.1), (7) Gave the most complete statement of the principle of conservation of energy up to that time, (8) Introduced the use of generalized coordinates (sometimes called Lagrangian coordinates), and (9) Introduced the use of multipliers to account for constraints. He also recognized that there might be constraints which are not holonomic, although he did not pursue the matter.

Lagrange also contributed to many other branches of mathematics (some of which he invented), including the theories of limits, probability, numbers and arithmetic, algebraic equations, functions, sound, vibrations, and orbits in gravitational fields.

One final contribution deserves to be mentioned. Lagrange was president of the committee to reform the system of weights and measures during the Revolution. The result of this committee's deliberations is today's metric system. He was instrumental in the decision to adopt base 10 for the system.

6.4 Special Forms

Conservative Forces. Suppose some of the forces are derivable from a potential energy function $V(q_1, \cdot\cdot, q_n)$ and some are not:

$$Q_s = Q_s^c + Q_s^{nc} = -\frac{\partial V}{\partial q_s} + Q_s^{nc}\,; \qquad s = 1, \cdot\cdot, n \tag{6.33}$$

Also note that, because $V \neq V(\dot{q}_1, \cdot\cdot, \dot{q}_n)$, Eqn. (4.21) gives

$$\frac{\partial L}{\partial \dot{q}_s} = \frac{\partial T}{\partial \dot{q}_s}$$

Consequently, Lagrange's equations may be written as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) - \frac{\partial L}{\partial q_s} - Q_s^{nc} + \sum_{r=1}^{\ell} \lambda_r B_{rs} = 0 \; ; \qquad s = 1, \cdot\cdot, n \tag{6.34}$$

When the system is natural (recall that this means that all given forces are conservative and all constraints are holonomic scleronomic), this reduces to

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) - \frac{\partial L}{\partial q_s} = 0 \; ; \qquad s = 1, \cdot\cdot, n \tag{6.35}$$

This shows that knowledge of the function L is sufficient to derive the equations of motion for a specific system. Such a function is called a ***descriptive function.*** Other descriptive functions will be introduced in later chapters.

Rayleigh's Dissipation Function. A common type of force is the dissipative damping force given by[3]

$$Q_s^d = \sum_{\alpha=1}^{n} d_{s\alpha} \dot{q}_\alpha \, , \qquad d_{s\alpha} \quad \text{negative definite} \tag{6.36}$$

Such a force may be written as

$$Q_s^d = \frac{\partial D}{\partial \dot{q}_s} \tag{6.37}$$

where

$$D = \frac{1}{2} \sum_{\alpha=1}^{n} \sum_{\beta=1}^{n} d_{\alpha\beta} \, \dot{q}_\alpha \, \dot{q}_\beta \tag{6.38}$$

provided d is a symmetric matrix (i.e. $d_{s\alpha} = d_{\alpha s}$).

The proof is as follows:

$$D \;=\; \frac{1}{2}\left[\sum_\alpha d_{\alpha_1} \, \dot{q}_\alpha \, \dot{q}_1 + \sum_\alpha d_{\alpha 2} \, \dot{q}_\alpha \, \dot{q}_2 + \cdots + \sum_\alpha d_{\alpha s} \, \dot{q}_\alpha \, \dot{q}_s + \cdot\cdot\right.$$

$$
\begin{aligned}
&\qquad + \sum_{\alpha} d_{\alpha n}\,\dot{q}_{\alpha}\,\dot{q}_{n}\Bigg] \\
\frac{\partial D}{\partial \dot{q}_s} &= \frac{1}{2}\left[d_{s1}\,\dot{q}_1 + d_{s2}\,\dot{q}_2 + \cdots + \sum_{\alpha} d_{\alpha s}\,\dot{q}_{\alpha} + d_{ss}\,\dot{q}_s + \cdots + d_{sn}\,\dot{q}_n\right] \\
&= \frac{1}{2}\left[\sum_{\alpha} d_{s\alpha}\,\dot{q}_{\alpha} + \sum_{\alpha} d_{\alpha s}\,\dot{q}_{\alpha}\right] \\
&= \frac{1}{2}\sum_{\alpha}(d_{s\alpha} + d_{\alpha s})\,\dot{q}_{\alpha} = \sum_{\alpha} d_{\alpha s}\,\dot{q}_{\alpha}
\end{aligned}
$$

The force form $Q_s^d = \partial D/\partial \dot{q}_s$ is particularly useful in Lagrange's equations just as the conservative force forms are useful. With dissipation forces present, Lagrange's equations become

$$
\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_s}\right) - \frac{\partial T}{\partial q_s} - \frac{\partial D}{\partial \dot{q}_s} - Q_s^{nd} + \sum_{r=1}^{\ell} \lambda_r B_{rs} = 0\,; \qquad s = 1, \cdot\cdot, n \tag{6.39}
$$

where the Q_s^{nd} are the components of the sum of the non-dissipative forces.

6.5 Remarks

Other Derivations. Lagrange's equations are also derivable from the Central Principle and from the Principle of Least Action. Other derivations are possible as well.

General Potential Functions. It is possible to extend the notion of potential and dissipation functions to more general situations. The essential idea is to find a function $\phi(q_1, \cdot\cdot, q_n, \dot{q}_1, \cdot\cdot, \dot{q}_n, t)$ such that a given force may be represented as either $\partial\phi/\partial q_s = Q_s^F$ or $\partial\phi/\partial \dot{q}_s = Q_s^F$. One example is Rayleigh's dissipation function. Another example is Lur'e's dissipation function.[4] A restriction is that ϕ must depend linearly on the velocity components $\dot{q}_s$, otherwise the force would be a function of acceleration components, which is not allowed (Section 2.1).

Non-Minimal Coordinates. Lagrange's equations have been stated in the minimal number of coordinates, $n = N - L'$. There is no requirement, however, to do this; any set of coordinates $(q_1, \cdot\cdot, q_{\overline{n}})$, $\overline{n} \geq n$ that, together with the holonomic constraints, completely define the system configuration (i.e. all components of all the particles) will suffice. On occasion, it's useful to use nonminimal coordinates. One holonomic constraint must be added for every excess coordinate, however.

Dynamic Coupling. Recall that the kinetic energy in generalized coordinates is of the form given by Eqn. (6.3):

$$T = \frac{1}{2}\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n} a_{\alpha\beta}\,\dot{q}_\alpha\,\dot{q}_\beta + \sum_{\alpha=1}^{n} b_\alpha\,\dot{q}_\alpha + c, \qquad a_{\alpha\beta} = a_{\beta\alpha}$$

Now consider the $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_s}\right)$ term of Lagrange's equations:

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_s}\right) &= \frac{d}{dt}\left[\frac{1}{2}\left(\sum_{\alpha=1}^{n} a_{s\alpha}\,\dot{q}_\alpha + \sum_{\alpha=1}^{n} a_{\alpha s}\,\dot{q}_\alpha\right) + b_s\right] \\
&= \frac{d}{dt}\left[\sum_{\alpha=1}^{n} a_{s\alpha}\,\dot{q}_\alpha + b_s\right] \\
&= \sum_{\alpha=1}^{n} a_{s\alpha}\,\ddot{q}_\alpha + \begin{array}{l}\text{terms with the } q_s \text{ and the } \dot{q}_s \\ \text{but not the } \ddot{q}_s \;; \qquad s = 1, \cdot\cdot, n\end{array}
\end{aligned}$$

Thus, in general, every term $\ddot{q}_1, \cdot\cdot, \ddot{q}_n$ appears in every equation; this is called dynamic coupling. Usually, dynamic coupling makes the equations difficult to solve and should be avoided. There is no dynamic coupling if the matrix $a_{s\alpha}$ is diagonal, but this is always possible to achieve by a linear transformation of variables. Because $a_{s\alpha}$ is positive definite, it is nonsingular and therefore there exists a linear transformation of coordinates that diagonalizes $a_{s\alpha}$.

Hamilton's Principle as a Variational Principle. We remarked earlier that Hamilton's Principle for a conservative system, Eqn. (4.22),

$$\int_{t_0}^{t_1} \delta(T - V)dt = 0$$

can be written as a problem in the calculus of variations, Eqn. (4.23),

$$\delta \int_{t_0}^{t_1} (T - V)dt = 0$$

if and only if the system is holonomic.

We now show this directly. For all forces conservative, Eqn. (6.34) with $Q_s^{nc} = 0$ applies and Hamilton's Principle in generalized coordinates becomes

$$\int_{t_0}^{t_1} \sum_{s=1}^{n} \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) - \frac{\partial L}{\partial q_s} + \sum_{r=1}^{\ell} \lambda_r B_{rs} \right] \delta q_s dt = 0 \tag{6.40}$$

On the other hand, suppose we attempt to formulate this problem by applying the multiplier rule to Eqn. (4.23) with constraints given by Eqns. (5.14):

$$\delta \int_{t_0}^{t_1} \left[T - V - \sum_{s=1}^{n} \sum_{r=1}^{\ell} \lambda_r (B_{rs}\dot{q}_s + B_r) \right] dt = 0 \tag{6.41}$$

Carrying out this variation gives a result different than Eqn. (6.40) unless $B_{rs} = B_r = 0$ for all r, s , i.e. unless all constraints are holonomic. Thus Eqn. (6.41) is only valid for the special case $\ell = 0$.

Invariance of Lagrange's Equations. It is often stated that Lagrange's equations are invariant to coordinate transformation. We now show this directly for a natural system. Consider a transformation of generalized coordinates

$$q_r = q_r(q_1', \cdot\cdot, q_n', t); \qquad r = 1, \cdot\cdot, n$$

such that

$$L(q_1, \cdot\cdot, q_n, \dot{q}_1, \cdot\cdot, \dot{q}_n, t) = L'(q_1', \cdot\cdot, q_n', \dot{q}_1', \cdot\cdot, \dot{q}_n', t)$$

Form the derivatives

$$\begin{aligned} \frac{\partial L'}{\partial q_r'} &= \sum_s \frac{\partial L}{\partial q_s} \frac{\partial q_s}{\partial q_r'} + \sum_s \frac{\partial L}{\partial \dot{q}_s} \frac{\partial \dot{q}_s}{\partial q_r'} \\ &= \sum_s \frac{\partial L}{\partial q_s} \frac{\partial q_s}{\partial q_r'} + \sum_s \frac{\partial L}{\partial \dot{q}_s} \frac{d}{dt}\left(\frac{\partial q_s}{\partial q_r'}\right) \end{aligned}$$

$$\begin{aligned} \frac{\partial L'}{\partial \dot{q}_r'} &= \sum_s \frac{\partial L}{\partial q_s} \underbrace{\frac{\partial q_s}{\partial \dot{q}_r'}}_{=0} + \sum_s \frac{\partial L}{\partial \dot{q}_s} \frac{\partial \dot{q}_s}{\partial \dot{q}_r'} \\ &= \sum_s \frac{\partial L}{\partial \dot{q}_s} \frac{\partial q_s}{\partial q_r'} \end{aligned}$$

Now form

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}'_r}\right) - \frac{\partial L'}{\partial q'_r} &= \sum_s \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right)\frac{\partial q_s}{\partial q'_r} + \sum_s \frac{\partial L}{\partial \dot{q}_s}\frac{d}{dt}\left(\frac{\partial q_s}{\partial q'_r}\right) \\
&\quad - \sum_s \frac{\partial L}{\partial q_s}\frac{\partial q_s}{\partial q'_r} - \sum_s \frac{\partial L}{\partial \dot{q}_s}\frac{d}{dt}\left(\frac{\partial q_s}{\partial q'_r}\right) \\
&= \sum_s \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) - \frac{\partial L}{\partial q_s}\right]\frac{\partial q_s}{\partial q'_r}\,; \qquad r = 1, \cdot\cdot, n
\end{aligned}$$

Under our restrictions for transformations, $\partial q_s/\partial q'_r$ is a nonsingular matrix (Jacobian of the transformation is nonzero); consequently

$$\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}'_r}\right) - \frac{\partial L'}{\partial q'_r} = 0 \Longleftrightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) - \frac{\partial L}{\partial q_s} = 0$$

so that Lagrange's equations are the same in *any* set of generalized coordinates.

6.6 Embedding Constraints

Definitions. The class of constraints we are considering (holonomic or nonholonomic) is:

$$\sum_{s=1}^{\bar{n}} B_{rs}\dot{q}_s + B_r = 0\,; \qquad r = 1, \cdot\cdot, L \tag{6.42}$$

where $q_1, q_2, \cdot\cdot, q_{\bar{n}}$ is a set of suitable (not necessarily minimal) coordinates. We have solved the dynamic problem by *adjoining* the constraints to the dynamic equation with multipliers λ_r; $r = 1, \cdot\cdot, L$ to get Lagrange's equations,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) - \frac{\partial L}{\partial q_s} - Q_s^{np} + \sum_{r=1}^{L}\lambda_r B_{rs} = 0\,; \qquad s = 1, \cdot\cdot, \bar{n} \tag{6.43}$$

Alternatively, we may eliminate the constraints directly; this is called *embedding*.

Embedding of Holonomic Constraints. In a holonomic system, and only in a holonomic system, the constraint equations may be solved for L of the velocities $\dot{q}_s$ in terms of the remaining $\bar{n} - L$, the result substituted into the expression for T, and Lagrange's equations applied. This is what was done in the first example of Section 4.3, and will be done many times in what follows.

This is only valid for holonomic systems because in nonholonomic systems virtual displacements from a possible configuration do not lead to another possible displacement.[5] Hence in nonholonomic systems we must embed constraints in terms of virtual displacements and not in terms of possible velocities.

Embedding of Nonholonomic Constraints. Now assume there are ℓ nonholonomic constraints. The virtual displacements satisfy Eqn. (5.14):

$$\sum_{s=1}^{n} B_{rs}\,\delta q_s = 0\,; \qquad r = 1,\cdot\cdot,\ell < n$$

Writing out the i^{th} equation,

$$B_{i1}\,\delta q_1 + \cdot\cdot + B_{i\ell}\,\delta q_\ell + B_{i,\ell+1}\delta q_{\ell+1} + \cdots + B_{in}\,\delta q_n = 0$$

$$B_{i1}\,\delta q_1 + \cdot\cdot + B_{i\ell}\,\delta q_\ell = -\left[B_{i,\ell+1}\,\delta q_{\ell+1} + \cdot\cdot + B_{in}\,\delta q_n\right]$$

All the equations are thus of the form

$$\begin{bmatrix} B_{11} & \cdot\cdot & B_{1\ell} \\ \vdots & & \\ B_{\ell 1} & \cdot\cdot & B_{\ell\ell} \end{bmatrix} \begin{bmatrix} \delta q_1 \\ \vdots \\ \delta q_\ell \end{bmatrix} = - \begin{bmatrix} B_{1,\ell+1} & \cdot\cdot & B_{1n} \\ \vdots & & \\ B_{\ell,\ell+1} & \cdot\cdot & B_{\ell n} \end{bmatrix} \begin{bmatrix} \delta q_{\ell+1} \\ \vdots \\ \delta q_n \end{bmatrix}$$

$$\tilde{\underline{\underline{B}}}\,\delta\tilde{\underline{q}} = -\approx{\underline{\underline{B}}}\,\delta\approx{\underline{q}}$$

Since $\tilde{\underline{\underline{B}}}$ is non singular (the constraints are independent)

$$\begin{aligned} \delta\tilde{\underline{q}} &= -\tilde{\underline{\underline{B}}}^{-1}\;\overset{\approx}{\underline{\underline{B}}}\,\delta\overset{\approx}{\underline{q}} = \underline{\underline{A}}\,\delta\overset{\approx}{\underline{q}} \\ \delta q_i &= \sum_{j=\ell+1}^{n} a_{ij}\,\delta q_j\,; \qquad i = 1,\cdot\cdot,\ell \end{aligned} \tag{6.44}$$

Using the notation of Eqns. (6.23), the fundamental equation may be written as

$$\sum_{s=1}^{n} R_s \delta q_s = 0$$

$$\sum_{s=1}^{\ell} R_s\, \delta q_s + \sum_{s=\ell+1}^{n} R_s\, \delta q_s = 0$$

Substituting Eqn. (6.44),

$$\sum_{s=1}^{\ell} R_s \sum_{j=\ell+1}^{n} a_{sj}\, \delta q_j + \sum_{j=\ell+1}^{n} R_j\, \delta q_j = 0$$

$$\sum_{j=\ell+1}^{n} \left(\sum_{s=1}^{\ell} R_s\, a_{sj} + R_j \right) \delta\, q_j = 0$$

But now there are no side conditions to be satisfied and the δq_j are arbitrary:

$$\sum_{s=1}^{\ell} R_s\, a_{sj} + R_j = 0\,; \qquad j = \ell + 1, \cdot\cdot, n \tag{6.45}$$

These $n - \ell$ equations along with the ℓ equations of constraint give the necessary n equations for the n unknowns q_s.

Notes

1 Again, we use the same symbol, T, for two different functions.

2 In the interest of simplicity, some notation from set theory will be used in this paragraph; also $< . , . >$ will denote the inner product of two vectors.

3 In Section 8.2 we will show that, as a consequence of $d_{s\alpha}$ being negative definite, these dissipative forces cause a loss of system energy. Mathematically, however, there is no reason not to allow $d_{s\alpha}$ to be non-negative definite.

4 See Rosenberg.

5 Rosenberg gives an example that shows explicitly that this procedure for a nonholonomic constraint gives an incorrect result.

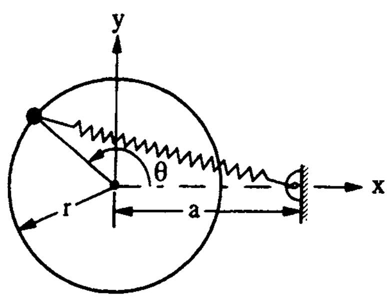
PROBLEMS

6/1. A particle of mass m moves on a smooth surface of revolution about the z axis

(a) What is the kinetic energy in cylindrical coordinates?

(b) Specialize this result for the cases in which the surface is a cone and a sphere.

6/2. A heavy particle of mass m is constrained to move on a circle of radius r which lies in the vertical plane, as shown. It is attached to a linear spring of rate k, which is anchored at a point on the x axis a distance a from the origin of the x, y system, and $a > r$. The free length of the spring is $a - r$. Using the angle θ as generalized coordinate, utilize Eqn. (6.15) to calculate the generalized forces arising from the gravitational and the spring force. Does the answer change if $a < r$ and, if so, how?



Problem 6/2

6/3. A heavy particle of mass m is attached to one extremity of a linear, masslesss spring of rate k, and of free length l. The other extremity of the spring is free to rotate about a fixed point. This system is, therefore, an elastic, spherical pendulum. Using spherical coordinates, calculate the generalized forces acting on the particle.

6/4. The Cartesian components of a force are

$$X = 2ax(y + z),$$
$$Y = 2ay(x + z),$$
$$Z = 2az(x + y).$$

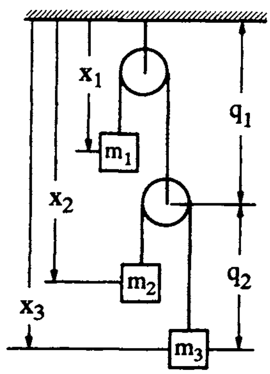
Calculate the generalized forces for cylindrical and spherical coordinates.

6/5. A particle is constrained to move in a plane. What is its kinetic energy in parabolic coordinates ξ, η, where

$$x = l(\xi^2 - \eta^2), \qquad y = 2l\xi\eta \, .$$

Is the transformation one-to-one?

6/6. The three weights of mass m_1, m_2, and m_3, respectively, are the only massive elements of the system of weights, pulleys, and inextensible strings shown.



Problem 6/6

(a) Write down the equation(s) of constraint satisfied by the coordinates x_1, x_2, and x_3 shown.

(b) Calculate the x_i in terms of the q_j and show that the q_j satisfy the equation(s) of constraint identically. Hence, the q_j are generalized coordinates.

(c) Construct Lagrange's equations of motion.

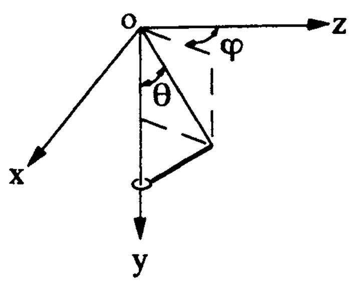
(d) Dynamically uncouple Lagrange's equations when $m_1 = 6$, $m_2 = 1$, $m_3 = 5$.

To solve Problems 6/7 and 6/8 refer to Eqn. (1.59).

6/7. One extremity of a heavy, uniform straight rod of length $2l$ and mass M can slide without friction along a vertical line. The other extremity is connected to one end of a massless, inextensible string of length $2l$ whose other end is tied to a fixed point O on the vertical line. Let θ be the angle subtended by the vertical line and the string, and let φ be the angle which the yz plane makes with the plane formed by the rod and string, as shown.

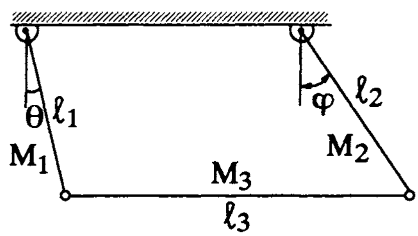
(a) How many and which are the constraints on the Cartesian coordinates of the endpoints of the rod? How many degrees of freedom does the rod have?

(b) Construct Lagrange's equations in terms of the variables θ and φ and their time derivatives.



Problem 6/7

6/8. Three heavy, uniform rods of lengths l_1, l_2, l_3 and of masses M_1, M_2, M_3, respectively, are linked together and can move in a vertical plane as shown.



Problem 6/8

(a) Which are the constraints on the Cartesian coordinates of the end points of the rods?

(b) Is it true that the angles θ and φ, as shown, are generalized coordinates?

(c) Construct Lagrange's equations.

6/9. A smooth thin ring is mounted in the vertical plane on a smooth horizontal table so that it can rotate freely about its vertical diameter. A straight uniform rod of length l and mass m passes through the ring. The rod is set into motion in any way, but so that it remains with all its points on the table. What are Lagrange's equations of motion of the rod so long as it does not slip out of the ring?

6/10. A heavy homogeneous hoop of negligible thickness, of mass m and of radius a, is free to move in a vertical plane. A ring having the same mass as the hoop slides without friction along the hoop. Determine the motion of this system for arbitrary initial conditions.

Chapter 7

Formulation of Equations

7.1 Remarks on Formulating Problems

Procedure for Formulating Problems via Lagrange's Equations. Although there is a wide variety of dynamics problems, the following procedure is generally applicable:

1. Identify and classify all constraints and given forces.
2. Choose suitable coordinates – any set that completely specifies the configuration of the system.
3. Write the kinetic energy, the potential energy, the nonconservative given forces, and the constraints in terms of the choosen coordinates.
4. Substitute the results into Lagrange's equations.

The result is a set of ordinary differential equations, the solution of which gives the motion, that is, the path through the configuration space.

Choice of Coordinates. The goal is to choose the coordinates that make formulating the problem easiest; although this is frequently obvious, especially in simple problems, general guides may be helpful:

1. It is almost always desirable to choose the minimal number of coordinates, i.e. generalized coordinates. These coordinates eliminate the holonomic constraints directly and give the fewest number of differential equations.

2. The holonomic constraints frequently indicate the most desirable coordinates; for example, if the motion is confined to be on the surface of a sphere, spherical coordinates are indicated.

3. Because of the simplicity in T for rectangular coordinates, they should always be considered.

7.2 Unconstrained Particle

Rectangular (Cartesian) Coordinates. Since $n = N = 3$ for a particle moving without constraints in 3-D, the rectangular components (x, y, z) serve as generalized coordinates:

$$q_1 = x\,, \qquad q_2 = y\,, \qquad q_3 = z$$

In these coordinates, from Eqn. (4.2),

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \tag{7.1}$$

Apply Lagrange's equations, Eqns. (6.29),

$$\frac{d}{d}\left(\frac{\partial T}{\partial \dot{q}_s}\right) - \frac{\partial T}{\partial q_s} - Q_s = 0\,; \qquad s = 1, 2, 3 \tag{7.2}$$

The three equations are

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial T}{\partial x} - F_x = 0\,, \\ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{y}}\right) - \frac{\partial T}{\partial y} - F_y = 0\,, \\ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{z}}\right) - \frac{\partial T}{\partial z} - F_z = 0 \end{aligned} \tag{7.3}$$

From Eqn. (7.1) the terms in these equations are:

$$\frac{\partial T}{\partial \dot{x}} = m\dot{x}\,, \qquad \frac{\partial T}{\partial \dot{y}} = m\dot{y}\,, \qquad \frac{\partial T}{\partial \dot{z}} = m\dot{z}$$

$$\frac{\partial T}{\partial x} = 0\,, \qquad \frac{\partial T}{\partial y} = 0\,, \qquad \frac{\partial T}{\partial z} = 0$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) = m\ddot{x}\,, \qquad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{y}}\right) = m\ddot{y}\,, \qquad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{z}}\right) = m\ddot{z}$$

$$Q_1 = F_x\,, \qquad Q_2 = F_y\,, \qquad Q_3 = F_z$$

Thus the equations of motion are

$$m\ddot{x} - F_x = 0\,, \qquad m\ddot{y} - F_y = 0\,, \qquad m\ddot{z} - F_y = 0 \qquad (7.4)$$

Cylindrical Coordinates. Choosing $(q_1, q_2, q_3) = (r, \phi, z)$, the transformation equations giving the rectangular coordinates in terms of the generalized coordinates are Eqns. (1.17) (Fig. 7-1):

$$\begin{aligned} x &= r\cos\phi \\ y &= r\sin\phi \\ z &= z \end{aligned} \qquad (7.5)$$

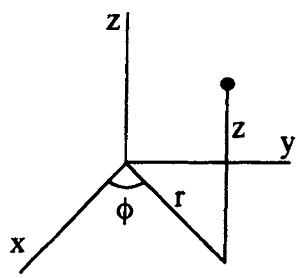


Fig. 7-1

We need to express the kinetic energy in cylindrical coordinates. Differentiating,

$$\dot{x} = \dot{r}\cos\phi - r\dot{\phi}\sin\phi\,; \qquad \dot{y} = \dot{r}\sin\phi + r\dot{\phi}\cos\phi\,; \qquad \dot{z} = \dot{z} \quad (7.6)$$

Substituting into Eqn. (7.1):

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2\right) \qquad (7.7)$$

Also

$$Q_1 = F_r\,, \qquad Q_2 = F_\phi\,, \qquad Q_3 = F_z$$

Lagrange's equations then give the equations of motion; the terms are

$$\frac{\partial T}{\partial \dot r} = m\dot r\,, \qquad \frac{\partial T}{\partial \dot\phi} = mr^2\dot\phi\,, \qquad \frac{\partial T}{\partial \dot z} = m\dot z$$
$$\frac{\partial T}{\partial r} = mr\dot\phi^2\,, \qquad \frac{\partial T}{\partial \phi} = 0\,, \qquad \frac{\partial T}{\partial z} = 0$$

Substituting into Eqns. (6.29):

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot r}\right) - \frac{\partial T}{\partial r} - F_r = 0$$
$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot\phi}\right) - \frac{\partial T}{\partial \phi} - F_\phi = 0$$
$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot z}\right) - \frac{\partial T}{\partial z} - F_z = 0$$

$$\begin{aligned} &m\ddot r - mr\dot\phi^2 - F_r = 0\\ &2mr\dot r\dot\phi + mr^2\ddot\phi - F_\phi = 0\\ &m\ddot z - F_z = 0 \end{aligned} \tag{7.8}$$

If the force is given in rectangular components, Eqns. (6.15) may be used to get the generalized force components:

$$F_s = \sum_{i=1}^{3} F_i \frac{\partial u_i}{\partial q_s}\,; \qquad s = 1, 2, 3$$
$$F_r = F_x\,\frac{\partial x}{\partial r} + F_y\,\frac{\partial y}{\partial r} + F_z\,\frac{\partial z}{\partial r}$$
$$F_\phi = F_x\,\frac{\partial x}{\partial \phi} + F_y\,\frac{\partial y}{\partial \phi} + F_z\,\frac{\partial z}{\partial \phi}$$
$$F_z = F_x\,\frac{\partial x}{\partial z} + F_y\,\frac{\partial y}{\partial z} + F_z\,\frac{\partial z}{\partial z}$$

$$\begin{aligned} F_r &= F_x\cos\phi + F_y\sin\phi\\ F_\phi &= -F_x r\cos\phi + F_y r\cos\phi\\ F_z &= F_z \end{aligned} \tag{7.9}$$

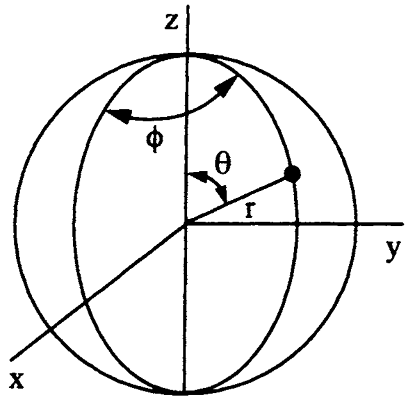


Fig. 7-2

Spherical Coordinates. Choosing $(q_1, q_2, q_3) = (r, \theta, \phi)$, the transformation equations are Eqns. (1.19) (see Fig. 7-2):

$$\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned} \tag{7.10}$$

We now need to express the kinetic energy in spherical coordinates. Differentiating,

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{1}{2}m(\dot{r}^2 + r^2\sin^2\theta\dot{\phi}^2 + r^2\dot{\theta}^2) \tag{7.11}$$

The generalized force components are

$$Q_1 = F_r\ , \qquad Q_2 = F_\theta\ , \qquad Q_3 = F_\phi$$

The partials are

$$\frac{\partial T}{\partial \dot{r}} = m\dot{r}\ ; \qquad \frac{\partial T}{\partial \dot{\theta}} = mr^2\dot{\theta}\ ; \qquad \frac{\partial T}{\partial \dot{\phi}} = mr^2\sin^2\theta\dot{\phi}$$

$$\frac{\partial T}{\partial r} = mr\sin^2\theta\dot{\phi}^2 + mr\dot{\theta}^2$$

$$\frac{\partial T}{\partial \theta} = mr^2\sin\theta\cos\theta\dot{\phi}^2$$

$$\frac{\partial T}{\partial \phi} = 0$$

Thus Lagrange's equations give

$$\begin{aligned}
&\frac{d}{dt}(m\dot{r}) - mr\sin^2\theta\dot{\phi}^2 - mr\dot{\theta}^2 - F_r = 0 \\
&\frac{d}{dt}(mr^2\dot{\theta}) - mr^2\sin\theta\cos\theta\dot{\phi}^2 - F_\theta = 0 \\
&\frac{d}{dt}(mr^2\sin^2\theta\dot{\phi}) - F_\phi = 0
\end{aligned} \tag{7.12}$$

If the force is given in rectangular components,

$$\begin{aligned}
F_s &= \sum_{i=1}^{3} F_i \frac{\partial u_i}{\partial q_s}\,; \qquad s = 1,2,3 \\
F_r &= F_x \sin\theta\cos\phi + F_y \sin\theta\sin\phi + F_z\cos\theta \\
F_\theta &= F_x r\cos\theta\cos\phi + F_y r\cos\theta\sin\phi - F_z r\sin\theta \\
F_\phi &= -F_x r\sin\theta\sin\phi + F_y r\sin\theta\cos\phi
\end{aligned} \tag{7.13}$$

Note that F_r is a force whereas F_θ and F_ϕ are moments. Equations (7.12) will be our starting point for a future topic – central force motion (Chapter 10).

7.3 Constrained Particle

One Holonomic Constraint in Planar Motion. Assume the constraint is rheonomic (scleronomic will be a special case), as shown on Fig. 7-3:

$$y = f(x,t)$$

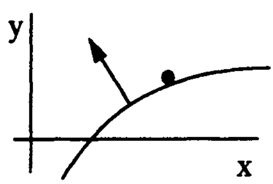


Fig. 7-3

Since the constraint is holonomic, it can be eliminated directly by embedding as follows

$$\dot{y} = f_x \dot{x} + f_t$$
$$T = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2} m(\dot{x}^2 + f_x^2 \dot{x}^2 + 2 f_x \dot{x} f_t + f_t^2)$$

Lagrange's equation gives the differential equation of motion; the terms are

$$\frac{\partial T}{\partial \dot{x}} = \frac{m}{2}(2\dot{x} + 2f_x^2 \dot{x} + 2 f_x f_t)$$

$$\underbrace{\frac{d}{dt}}_{\substack{\text{total}\\ \text{derivative}}} \underbrace{\left(\frac{\partial T}{\partial \dot{x}}\right)}_{\substack{\text{partial}\\ \text{derivative}\\ \text{holding}\\ x \,\&\, t \text{ fixed}}} = m\Big[\ddot{x} + 2 f_x (f_{xx}\dot{x} + f_{xt})\dot{x} + f_x^2 \ddot{x} + (f_{xx}\dot{x} + f_{xt}) f_t + f_x (f_{tx}\dot{x} + f_{tt})\Big]$$

$$\underbrace{\frac{\partial T}{\partial x}}_{\substack{\text{partial derivative}\\ \text{holding } \dot{x} \,\&\, t \text{ fixed}}} = \frac{1}{2} m(2 f_x f_{xx} \dot{x}^2 + 2 f_{xx} \dot{x} f_t + 2 f_x \dot{x} f_{tx} + 2 f_t f_{tx})$$

Substitution into the following equation then gives the equation of motion:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial T}{\partial x} - F_x = 0 \tag{7.14}$$

where F_x is the x-component of the *given* force. The constraint does not appear (either geometrically or as a force) because it has been explicitly removed.

As a special case, suppose the constraint is scleronomic, $y = f(x)$; then Eqn. (7.14) reduces to:

$$m\left[(1 + f_x^2)\ddot{x} + f_x\, f_{xx}\, \dot{x}^2\right] - F_x = 0 \tag{7.15}$$

Nonholonomic Constraints. Here, $\ell > 0$, $DOF = N - L$, $L = L' + \ell$, $n = N - L' > DOF$. This is because a nonholonomic constraint does not affect the accessibility of the configuration space; one particle in 3-space subject to a nonholonomic constraint has 3 generalized coordinates.

Most nonholonomic constraints arise as constraints on velocities, for example in problems involving the rolling of one body on another.

As an example, suppose a particle is subject to a single catastatic nonholonomic constraint

$$a\delta x + b\delta y + c\delta z = 0$$

Comparison with Eqn. (5.15) with $n = 3$ and $\ell = 1$ gives

$$B_{11} = a\ , \qquad B_{12} = b\ , \qquad B_{13} = c$$

Equations (6.29) then give, in rectangular coordinates,

$$\begin{aligned} m\ddot{x} - F_x + \lambda a &= 0 \\ m\ddot{y} - F_y + \lambda b &= 0 \\ m\ddot{z} - F_z + \lambda c &= 0 \end{aligned} \qquad (7.16)$$

These three equations, along with

$$a\dot{x} + b\dot{y} + c\dot{z} = 0$$

provide four equations in the four unknowns x, y, z, and λ.

7.4 Example – Two Link Robot Arm

Problem Definition. Consider two rigid bodies connected together and moving in a plane as shown (Fig. 7-4). This may be considered a typical robot arm. $\underline{M}_1$ and $\underline{M}_2$ are motor torques.

All the constraints are holonomic and there are two generalized coordinates. We choose

$$q_1 = \theta_1\ ; \qquad q_2 = \theta_2$$

We could choose ϕ instead of θ_2 but θ_2 is somewhat easier since we need the velocities and angular velocities relative to the inertial frame (x, y) for use in determining the kinetic energy.

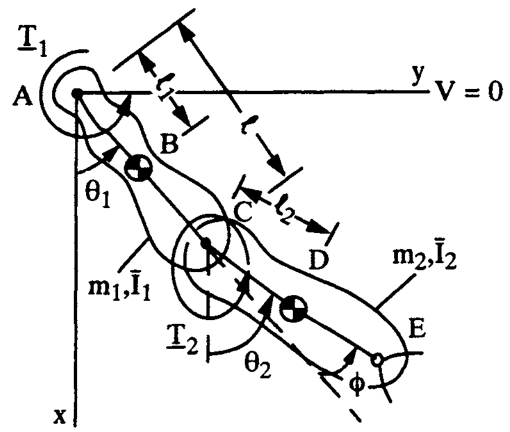


Fig. 7-4

Kinetic Energy. For two rigid bodies, Eqn. (1.56) gives

$$T = \sum_{i=1}^{2} \left(\frac{1}{2} m_i \, \bar{v}_i^2 + \frac{1}{2} \bar{I}_i \, \omega_i^2 \right) \tag{7.17}$$

First consider body # 1 (Fig. 7-5).

$$\begin{aligned} v_B &= \ell_1 \dot{\theta}_1 \\ T_1 &= \frac{1}{2} m_1 v_B^2 + \frac{1}{2} \bar{I}_1 \omega_1^2 = \frac{1}{2} m_1 \ell_1^2 \dot{\theta}_1^2 + \frac{1}{2} \bar{I}_1 \dot{\theta}_1^2 = \frac{1}{2} I_1 \dot{\theta}_1^2 \end{aligned} \tag{7.18}$$

where

$$I_1 = \bar{I}_1 + m_1 \ell_1^2$$

is the moment of inertia about A by the parallel axis theorem. This agrees with the alternative expression given by Eqn. (1.57).

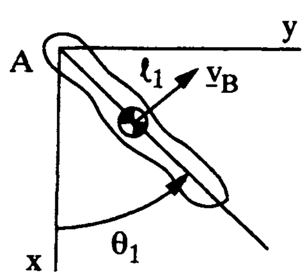


Fig. 7-5

Now consider body # 2 (Fig. 7-6). Let $\{\hat{\imath}_1, \hat{\jmath}_1\}$, $\{\hat{\imath}_2, \hat{\jmath}_2\}$, and $\{\hat{\imath}_3, \hat{\jmath}_3\}$ be reference frames fixed in the ground, in link AC, and in link CD, respectively. Use Eqn. (1.25) to relate the velocity of D relative to $\{\hat{\imath}_3, \hat{\jmath}_3\}$ to its velocity relative to $\{\hat{\imath}_1, \hat{\jmath}_1\}$ and let $\underline{\omega}$ be the angular velocity of $\{\hat{\imath}_3, \hat{\jmath}_3\}$ relative to $\{\hat{\imath}_1, \hat{\jmath}_1\}$:

$$\underline{v}_D = \underline{v}_C + \underline{v}_{\text{rel}} + \underline{\omega} \times \underline{r}$$

where

$$\underline{v}_D = \frac{d\underline{r}_D}{dt}, \qquad \underline{v}_C = \frac{d\underline{r}_C}{dt}$$

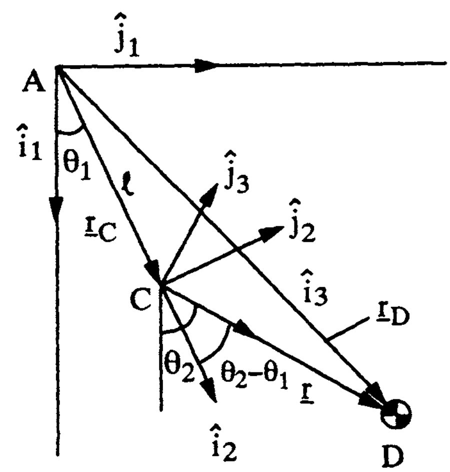


Fig. 7-6

The terms are

$$\underline{v}_C = \ell\dot{\theta}_1\hat{\jmath}_2 , \qquad \underline{v}_{\text{rel}} = \underline{0} , \qquad \underline{\omega} = \dot{\theta}_2\hat{k}_3 , \qquad \underline{r} = \ell_2\hat{\imath}_3$$

The required unit vector transformation is

$$\hat{\jmath}_2 = \sin(\theta_2 - \theta_1)\hat{\imath}_3 + \cos(\theta_2 - \theta_1)\hat{\jmath}_3$$

Substituting,

$$\begin{aligned} \underline{v}_D &= \ell\dot{\theta}_1\hat{\jmath}_2 + \underline{0} + \dot{\theta}_2\hat{k}_3 \times \ell_2\hat{\imath}_3 \\ &= \ell\dot{\theta}_1 \sin(\theta_2 - \theta_1)\hat{\imath}_3 + \left[\ell_2\dot{\theta}_2 + \ell\dot{\theta}_1\cos(\theta_2 - \theta_1)\right]\hat{\jmath}_3 \qquad (7.19) \end{aligned}$$

(This also could be obtained geometrically from the law of cosines.) T_2 is then

$$\begin{aligned} T_2 &= \frac{1}{2}m_2 v_D^2 + \frac{1}{2}\bar{I}_2\omega_2^2 \\ &= \frac{1}{2}m_2\left[\ell^2\dot{\theta}_1^2 + \ell_2^2\dot{\theta}_2^2 + 2\ell\ell_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_2-\theta_1)\right] + \frac{1}{2}\bar{I}_2\dot{\theta}_2^2 \quad (7.20) \end{aligned}$$

Note that is is quadratic in the velocity components with displacement dependent coefficients; the constant and linear terms are missing.

Consequently,

$$T = T_1 + T_2 = \frac{1}{2}I'\dot{\theta}_1^2 + \frac{1}{2}I_2\dot{\theta}_2^2 + m_2\ell\ell_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_2-\theta_1) \quad (7.21)$$

where

$$I' = I_1 + m_2\ell^2\ ; \qquad I_2 = \bar{I}_2 + m_2\ell_2^2$$

Potential Energy and Lagrangian. Since the only given force is gravity,

$$V = -m_1 g\ell_1\cos\theta_1 - m_2 g(\ell\cos\theta_1 + \ell_2\cos\theta_2) \quad (7.22)$$

$$\begin{aligned} L &= T - V = \frac{1}{2}I'\dot{\theta}_1^2 + \frac{1}{2}I_2\dot{\theta}_2^2 + m_2\ell\ell_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_2-\theta_1) \\ &\quad + m_1 g\ell_1\cos\theta_1 + m_2 g(\ell\cos\theta_1 + \ell_2\cos\theta_2) \quad (7.23) \end{aligned}$$

Lagrange's Equations. Equations (6.34) are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}_1}\right) - \frac{\partial L}{\partial\theta_1} = Q_1\ ; \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}_2}\right) - \frac{\partial L}{\partial\theta_2} = Q_2 \quad (7.24)$$

Computing some of the terms

$$\frac{\partial L}{\partial\dot{\theta}_1} = I'\dot{\theta}_1 + m_2\ell\ell_2\dot{\theta}_2\cos(\theta_2-\theta_1)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}_1}\right) = I'\ddot{\theta}_1 + m_2\ell\ell_2\left[\ddot{\theta}_2\cos(\theta_2-\theta_1) - \dot{\theta}_2\sin(\theta_2-\theta_1)(\dot{\theta}_2-\dot{\theta}_1)\right]$$

$$\frac{\partial L}{\partial\theta_1} = m_2\ell\ell_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_2-\theta_1) - m_1 g\ell_1\sin\theta_1 - m_2 g\ell\sin\theta_1$$

Finally, substitution gives the equations of motion

$$\begin{aligned} I'\ddot{\theta}_1 + A\cos(\theta_2-\theta_1)\ddot{\theta}_2 - A\sin(\theta_2-\theta_1)\dot{\theta}_2^2 + B\sin\theta_1 &= M_1 \\ I_2\ddot{\theta}_2 + A\cos(\theta_2-\theta_1)\ddot{\theta}_1 + A\sin(\theta_2-\theta_1)\dot{\theta}_1^2 + C\sin\theta_2 &= M_2 \end{aligned} \tag{7.25}$$

where

$$Q_1 = M_1, \,, \qquad Q_2 = M_2$$

and

$$A = m_2\ell\ell_2\,; \qquad B = m_1g\ell_1 + m_2g\ell\,; \qquad C = m_2g\ell_2$$

Equations (7.25) are dynamically coupled and highly nonlinear, making their solution difficult.

Linearizing for small angles and small angular rates, and setting $M_1 = M_2 = 0$ gives the special case of the double physical pendulum.

$$\begin{aligned} I'\ddot{\theta}_1 + A\ddot{\theta}_2 + B\theta_1 &= 0 \\ I_2\ddot{\theta}_2 + A\ddot{\theta}_1 + C\theta_2 &= 0 \end{aligned} \tag{7.26}$$

These equations are linear but still dynamically coupled. They may be easily dynamically uncoupled by a change of variables. Modal analysis of these equations for typical cases reveals two modes, one rapid and one relatively slow.

7.5 Example – Rolling Disk

Problem Definition. A thin homogeneous disk of radius r rolls without slipping on a horizontal plane (Fig. 7-7). Recall that a rigid body in motion in 3-space without contraints has six degrees of freedom and every point of the body is specified by 6 independent parameters, say (x, y, z), the location of some body-fixed point with respect to an inertial frame, and (ϕ, ψ, θ), three angles of body-fixed lines. If the body rolls on a plane, there is one holonomic constraint ($y = 0$ in our case) and $n = N - L' = 6 - 1 = 5$. For the disk we choose $(x, z, \phi, \psi, \theta)$ as generialized coordinates as shown.

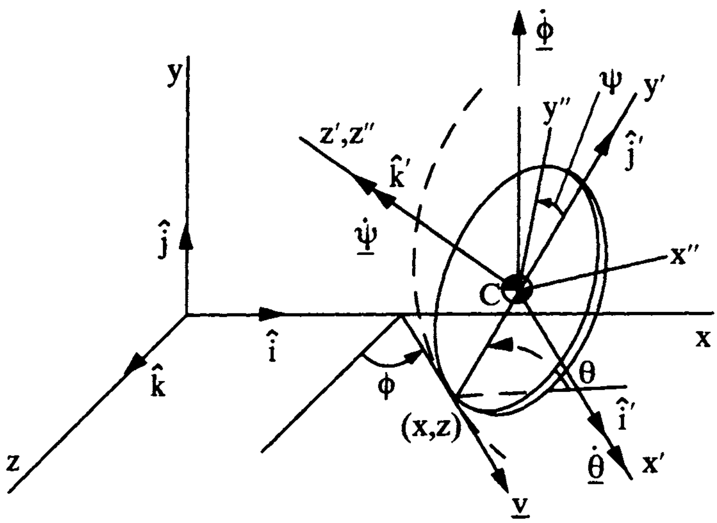


Fig. 7-7

Introduce the following reference frames as shown on Fig. 7-7:

1. $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ ground fixed (inertial).
2. $\{\hat{\imath}', \hat{\jmath}', \hat{k}'\}$ neither ground nor body fixed with $\hat{k}'$ along the axis of symmetry of the disk and $\hat{\jmath}'$ along a diameter of the disk passing through the contact point and the disk center.
3. $\{\hat{\imath}'', \hat{\jmath}'', \hat{k}''\}$ body fixed with $\hat{k}''$ along the axis of symmetry of the disk.

By the definition of pure rolling, the velocity of the disk at the contact point is zero and the contact point itself moves with velocity in the (x, y) plane in the $\hat{\imath}'$ direction; therefore the rolling constraint is

$$\underline{v} = -r\dot{\psi}\hat{\imath}' = -r\dot{\psi}(\cos\phi\hat{k} + \sin\phi\hat{\imath})$$

Since $\underline{v} = \dot{z}\hat{k} + \dot{x}\hat{\imath}$, we have in component form

$$\begin{aligned} &\dot{x} = -r\dot{\psi}\sin\phi \\ &dx + r\sin\phi\, d\,\psi = 0 \\ &\delta x + r\sin\phi\,\delta\,\psi = 0 \end{aligned} \tag{7.27}$$

$$\begin{aligned} &\dot{z} = -r\dot{\psi}\cos\phi \\ &dz + r\cos\phi\, d\,\psi = 0 \\ &\delta z + r\cos\phi\,\delta\,\psi = 0 \end{aligned} \tag{7.28}$$

These are two nonholonomic constraints. (A special case is rolling along a straight line; now, $\theta = \pi/2$, $\phi =$ constant and these constraints are integrable. This shows that the issue of nonholonomic constraints doesn't usually come up in two-dimensional problems.)

Kinetic Energy. For a rigid body,

$$T = \frac{1}{2}m\overline{v}^2 + \frac{1}{2}\overline{I}\omega^2 \tag{1.56}$$

First consider the translational term:

$$\overline{v}^2 = \dot{\overline{x}}^2 + \dot{\overline{y}}^2 + \dot{\overline{z}}^2 \tag{7.29}$$

From Fig. 7-8:

$$\begin{aligned} \overline{x} &= x + r\cos\theta\cos\phi \\ \overline{y} &= r\sin\theta \\ \overline{z} &= z - r\cos\theta\sin\phi \end{aligned}$$

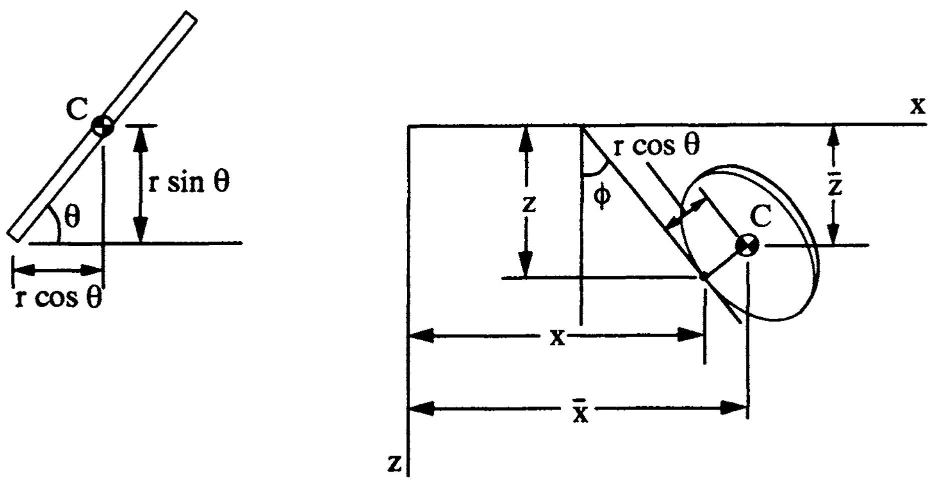


Fig. 7-8

Differentiating

$$\begin{aligned} \dot{\overline{x}} &= \dot{x} - r\dot{\theta}\sin\theta\cos\phi - r\dot{\phi}\cos\theta\sin\phi \\ \dot{\overline{y}} &= r\dot{\theta}\cos\theta \\ \dot{\overline{z}} &= \dot{z} + r\dot{\theta}\sin\theta\sin\phi - r\dot{\phi}\cos\theta\cos\phi \end{aligned}$$

Substitution of these in Eqn. (7.29) gives

$$\begin{aligned}\overline{v}^2 &= \dot{x}^2 + \dot{z}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2\cos^2\theta + 2r(-\dot{x}\dot{\theta}\sin\theta\cos\phi \\ &\quad -\dot{x}\dot{\phi}\cos\theta\sin\phi + \dot{z}\dot{\theta}\sin\theta\sin\phi - \dot{z}\dot{\phi}\cos\theta\cos\phi) \qquad (7.30)\end{aligned}$$

Next consider the rotational term. Figure 7-7 shows the directions of the angular velocity components:

$$\begin{aligned}\underline{\omega} = \dot{\theta}\hat{i}' + \dot{\psi}\hat{k}' + \dot{\phi}\hat{j} &= \dot{\theta}\hat{i}' + \dot{\psi}\hat{k}' + \dot{\phi}(\cos\theta\hat{k}' + \sin\theta\hat{j}') \\ &= \dot{\theta}\hat{i}' + \dot{\phi}\sin\theta\hat{j}' + (\dot{\psi} + \dot{\phi}\cos\theta)\hat{k}' \qquad (7.31)\end{aligned}$$

Because of the rotational mass symmetry, the (x', y', z') axes are principle axes of inertia; let

$$I = I_{x'x'} = I_{y'y'}\ ; \qquad J = I_{z'z'} \qquad (7.32)$$

(The products of inertia are of course all zero.) The angles (θ, ψ, ϕ) are Euler's angles. The rotational term is then

$$\begin{aligned}\frac{1}{2}\overline{I}\omega^2 &= \frac{1}{2}\left(I_{x'x'}\omega_{x'}^2 + I_{y'y'}\omega_{y'}^2 + I_{z'z'}\omega_{z'}^2\right) \\ &= \frac{1}{2}I\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right) + \frac{1}{2}J\left(\dot{\psi} + \dot{\phi}\cos\theta\right)^2 \qquad (7.33)\end{aligned}$$

Also

$$V = mgr\sin\theta \qquad (7.34)$$

and $L = T - V$.

Lagrange's Equations. The nonholonomic constraints are of the form

$$\sum_{s=1}^{5} B_{rs}\,\delta q_s = 0\ ; \qquad r = 1, 2$$

Letting $(q_1, \cdot\cdot, q_5) = (x, z, \theta, \phi, \psi)$ and comparing with the constraint equations, Eqns. (7.27) and (7.28),

$$\begin{array}{lll} B_{11} = 1, & B_{15} = r\sin\phi, & B_{12} = B_{13} = B_{14} = 0 \\ B_{22} = 1, & B_{25} = r\cos\phi, & B_{21} = B_{23} = B_{24} = 0 \end{array} \qquad (7.35)$$

The appropriate form of Lagrange's equation is:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) - \frac{\partial L}{\partial q_s} + \sum_{r=1}^{2}\lambda_r B_{rs} = 0\ ; \qquad s = 1, \cdot\cdot, 5 \qquad (7.36)$$

Combining Eqns. (1.56), (7.30), and (7.33) - (7.35), the resulting equations of motion are:

$$\begin{aligned}
&\frac{d}{dt}[m\dot{x} + mr(-\dot{\theta}\sin\theta\cos\phi - \dot{\phi}\cos\theta\sin\phi] + \lambda_1 = 0\ ,\\
&\frac{d}{dt}[m\dot{z} + mr(\dot{\theta}\sin\theta\sin\phi - \dot{\phi}\cos\theta\cos\phi)] + \lambda_2 = 0\ ,\\
&\frac{d}{dt}[mr^2\dot{\theta} + mr(-\dot{x}\sin\theta\cos\phi + \dot{z}\sin\theta\sin\phi) + I\dot{\theta}]\\
&\quad +mr^2\dot{\phi}^2\cos\theta\sin\theta + mr(\dot{x}\dot{\theta}\cos\theta\cos\phi - \dot{x}\dot{\phi}\sin\theta\sin\phi\\
&\quad -\dot{z}\dot{\theta}\cos\theta\sin\phi - \dot{z}\dot{\phi}\sin\theta\cos\phi) - I\dot{\phi}^2\sin\theta\cos\theta\\
&\quad +J(\dot{\psi} + \dot{\phi}\cos\theta)\dot{\phi}\sin\theta + mgr\cos\theta = 0\ ,\\
&\frac{d}{dt}[mr^2\dot{\phi}\cos^2\theta + mr(-\dot{x}\cos\theta\sin\phi - \dot{z}\cos\theta\cos\phi)\\
&\quad +I\dot{\phi}\sin^2\theta + J(\dot{\psi} + \dot{\phi}\cos\theta)\cos\theta] + mr(-\dot{x}\dot{\theta}\sin\theta\sin\phi\\
&\quad +\dot{x}\dot{\phi}\cos\theta\cos\phi + \dot{z}\dot{\theta}\sin\theta\cos\phi - \dot{z}\dot{\phi}\cos\theta\sin\phi) = 0\ ,\\
&\frac{d}{dt}[J(\dot{\psi} + \dot{\phi}\cos\theta)] + \lambda_1 r\sin\phi + \lambda_2 r\cos\phi = 0\ .
\end{aligned} \tag{7.37}$$

The two nonholonimc constraint equations in velocity form are:

$$\begin{aligned}
\dot{x} + r\dot{\psi}\sin\phi = 0\\
\dot{z} + r\dot{\psi}\cos\phi = 0
\end{aligned} \tag{7.38}$$

Equations (7.37) plus (7.38) are seven equations in the seven unknowns $x, z, \theta, \phi, \psi, \lambda_1$, and λ_2. They are highly coupled and highly nonlinear, and thus difficult to solve.

PROBLEMS

7/1. An unconstrained particle of mass m moves in 3-space under a force

$$F = X_0\hat{\imath} + Y_0\hat{\jmath} + Z_0\hat{k}\ ,$$

where X_0, Y_0, Z_0 are constants. Write the Lagrangian equations of motion in the generalized coordinates ξ, η ζ, which are connected to the Cartesian coordinates by

$$x = l(\xi^2 - \eta^2)\ , \qquad y = 2l\xi\eta\ , \qquad z = \zeta\ .$$

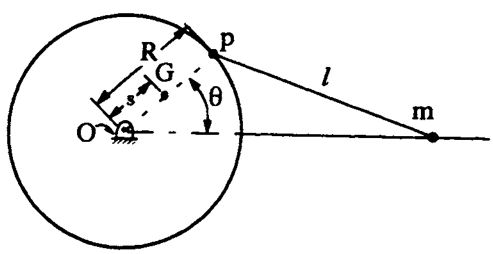
and state why ξ, η, and ζ are called parabolic, cyclindrical coordinates. Calculate an arc length ds in terms of ξ, η, and ζ. Denote the generalized force components by E, H, and Z, respectively.

7/2. Let the Cartesian coordinates of a 4-space be w, x, y, z. A particle of unit mass moves on the surface of a four-dimensional sphere of radius R under a potential force, and the potential energy is constant on the cylindrical surface $w^2 + x^2$ = constant. If θ, φ, and ψ are connected to $w\ x\ y$, and z by

$$w = R\cos\theta\cos\varphi\,, \qquad x = R\cos\theta\sin\varphi\,,$$
$$y = R\sin\theta\cos\psi\,, \qquad z = R\sin\theta\sin\psi\,,$$

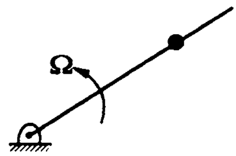
show that θ, φ, and ψ are suitable generalized coordinates, and construct Lagrange's equations in θ, φ, and ψ.

7/3. A heavy eccentric disk can rotate about a fixed, smooth, horizontal axis at O. Let its mass moment of inertia about the axis of rotation be I, and let its mass center G be a distance s from the axis of rotation. A massless connecting rod of length l is smoothly hinged to the disk at a point P a distance R from the axis of rotation, and connected to a particle of mass m, which is constrained to move on a smooth horizontal surface as shown. O, G, and P lie on a straight line. If gravity is the only force acting on the system, define suitable coordinates and construct Lagrange's equations of motion for this system.



Problem 7/3

7/4. A heavy bead of mass m slides on a smooth rod that rotates with constant angular velocity Ω about a fixed point lying on the rod centerline, as shown. What are Lagrange's equations of motion of the bead in suitably chosen generalized coordinates?



Problem 7/4

7/5. Use Lagrange's equations to find the equations of motion of the system described in the first example of Section 4.3.

Chapter 8

Integration of Equations

8.1 Integrals of Motion

Equations in First Order Form. Having formulated Lagrange's equations of motion for a system with n generalized coordinates, we are faced with solving, in general, a set of n second order differential equations in the generalized coordinates; these equations are in general of the form

$$f_1(q_1, \cdot\cdot, q_n, \dot{q}_1, \cdot\cdot, \dot{q}_n, \ddot{q}_1, \cdot\cdot, \ddot{q}_n, t) = 0$$
$$\vdots$$
$$f_n(q_1, \cdot\cdot, q_n, \dot{q}_1, \cdot\cdot, \dot{q}_n, \ddot{q}_1, \cdot\cdot, \ddot{q}_n, t) = 0$$

It is always possible to write a system of ordinary differential equations in first order form. We now do this for Lagrange's equations. Let

$$v_1 = q_1, \cdot\cdot, v_n = q_n, \; v_{n+1} = \dot{q}_1, \cdot\cdot, v_{2n} = \dot{q}_n \tag{8.1}$$

Then the kinetic energy becomes

$$T = T(q_1, \cdot\cdot, q_n, \dot{q}_1, \cdot\cdot, \dot{q}_n, t) = T(v_1, \cdot\cdot, v_{2n}, t)$$

Lagrange's equations in the q_s are given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) - \frac{\partial L}{\partial q_s} - Q_s^{nc} + \sum_{r=1}^{\ell} \lambda_r \, B_{rs} = 0 \, ; \qquad s = 1, \cdot\cdot, n \tag{6.34}$$

In the v_s they become

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v_{n+s}}\right) - \frac{\partial L}{\partial v_s} - Q_s^{nc} + \sum_{r=1}^{\ell} \lambda_r \, B_{rs} = 0 \, ; \qquad s = 1, \cdot\cdot, n \tag{8.2}$$

These are first order equations in the v_s; also to be satisfied are the first order equations

$$\frac{dv_s}{dt} = v_{n+s} \; ; \qquad s = 1, \cdot\cdot, n \tag{8.3}$$

Equations (8.2) and (8.3) give $2n$ first order equations that are entirely equivalent to the n second order equations of Eqns. (6.29).

Also to be satisfied are the nonholonomic constraints

$$\sum_{s=1}^{n} B_{rs} \, \dot{q}_s + B_r = 0 \; ; \qquad r = 1, \cdot\cdot, \ell \tag{5.14}$$

In terms of the v_s these are

$$\sum_{s=1}^{n} B_{rs} \, v_{n+s} + B_r = 0 \; ; \qquad r = 1, \cdot\cdot, \ell \tag{8.4}$$

Definition of an Integral of Motion. If one can find a function $F_\beta(q_1, \cdot\cdot, q_n, \dot{q}_1, \cdot\cdot, \dot{q}_n, t)$ such that

$$\frac{dF_\beta}{dt} = 0 \tag{8.5}$$

whenever $(q_1, \cdot\cdot, q_n, \dot{q}_1, \cdot, \cdot, \dot{q}_n)$ satisfy Lagrange's equations and the nonholonomic constraints, then

$$F_\beta(\cdot) = C_\beta \tag{8.6}$$

is a *first integral* of the motion. There are a total of n independent such first integrals

$$F_\beta(\cdot) = C_\beta \; ; \qquad \beta = 1, \cdot\cdot, n \tag{8.7}$$

The integrals of these n first order equations are of the form

$$G_\gamma(q_1, \cdot\cdot, q_n, t, C_1, \cdot\cdot, C_n) = C'_\gamma \; ; \qquad \gamma = 1, \cdot\cdot, n \tag{8.8}$$

If we can find all $2n$ of the integrals F_β and G_γ, the problem is said to be completely solved.

As an example, for systems in which energy is conserved, the energy is an integral of the motion:

$$T(q_1, \cdot\cdot, q_n, \dot{q}_1, \cdot\cdot, \dot{q}_n, t) + V(q_1, \cdot\cdot, q_n) = h = \text{constant}$$

Solution of the Equations of Motion. A *solution* of a dynamical system consists of a set of functions $u_1 = u_1(t, u_{10}, \cdot\cdot, u_{n0}), \cdot\cdot, u_n = u_n(t, u_{10}, \cdot\cdot, u_{n0})$, where the u_{s0} are the values of the u_s at a specific time t_0, such that when these functions are substituted into the equations of motion identities result. Geometrically, therefore, a solution defines a path, or trajectory, in configuration space.

If all $2n$ integrals of the motion are known and can be expressed in terms of elementary functions, then the solution can be expressed explicitly in terms of elementary functions of time; in this case, we say the solution has been found in *closed form*. If, on the other hand, all integrals are known but some or all cannot be expressed in terms of elementary functions, then we say the solution has been *reduced to quadratures*. In the typical case, part of the solution will be obtained in closed form (the most desirable), part in quadratures, and the remaining part must be obtained by numerical solution of differential equations (the least desirable).

Of course, the equations of motion always can be solved numerically on a digital computer; however, finding closed form or quadrature solutions (in whole or in part) is desirable for reasons of:

- computational efficiency (speed and storage)
- robustness (numerical procedures may become unstable)
- software validation
- insight into solution behavior

For these reasons, obtaining integrals of the motion is important; in fact, much of analytical dynamics is devoted to this goal. We note that numerical evaluation of integrals is much more efficient and robust than is numerical solution of differential equations.

Obtaining a partial closed form solution is sometimes sufficient to obtain a great deal of information about system behavior. The science and art of deducing system behavior without obtaining complete quantitative solutions is sometimes called *qualitative integration*. It is one of the major goals of the theory of nonlinear oscillations. In later chapters, particularly Chapter 11, we shall give examples of qualitative analysis.

8.2 Jacobi's Integral

General Form. We now find a more general form of the energy integral.

Assume:

1. All given forces can be expressed as[1]

$$Q_s = \frac{d}{dt}\frac{\partial V}{\partial \dot{q}_s} - \frac{\partial V}{\partial q_s} \tag{8.9}$$

where $V(q_1, \cdot\cdot, q_n,\ \dot{q}_1, \cdot\cdot, \dot{q}_n)$ is a generalized potential linear in the $\dot{q}_s$. Then Lagrange's equations, Eqns. (6.29), are

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_s}\right) - \frac{\partial T}{\partial q_s} \underbrace{- \frac{d}{dt}\left(\frac{\partial V}{\partial \dot{q}_s}\right) + \frac{\partial V}{\partial q_s}}_{-Q_s} + \sum_{r=1}^{\ell} \lambda_r\, B_{rs} = 0\ ;$$

$$s = 1, \cdot\cdot, n$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) - \frac{\partial L}{\partial q_s} + \sum_{r=1}^{\ell} \lambda_r\, B_{rs} = 0\ ; \qquad s = 1, \cdot\cdot, n \tag{8.10}$$

2. $L \neq L(t)$

3. All constraints catastatic. Then virtual displacements are actual displacements and the Fundamental Equation, Eqn. (6.22), may be written as

$$\sum_{s=1}^{n}\left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) - \frac{\partial L}{\partial q_s}\right]\dot{q}_s = 0 \tag{8.11}$$

To find Jacobi's integral, first consider

$$\frac{d}{dt}\left[\sum_s \dot{q}_s \frac{\partial L}{\partial \dot{q}_s}\right] = \sum_s \dot{q}_s \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) + \sum_s \ddot{q}_s \frac{\partial L}{\partial \dot{q}_s} \tag{8.12}$$

Substituting this into Eqn. (8.11),

$$\frac{d}{dt}\left[\sum_s \dot{q}_s \frac{\partial L}{\partial \dot{q}_s}\right] - \sum_s \ddot{q}_s \frac{\partial L}{\partial \dot{q}_s} - \sum_s \frac{\partial L}{\partial q_s}\dot{q}_s = 0 \tag{8.13}$$

Now consider the total time derivative of L:

$$\frac{dL}{dt} = \sum_s \frac{\partial L}{\partial q_s}\dot{q}_s + \sum_s \frac{\partial L}{\partial \dot{q}_s}\ddot{q}_s \tag{8.14}$$

Combining Eqns. (8.13) and (8.14),

$$\frac{d}{dt}\left[\sum_s \dot{q}_s \frac{\partial L}{\partial \dot{q}_s}\right] - \frac{dL}{dt} = 0$$

This may be integrated to give

$$\sum_{s=1}^{n} \dot{q}_s \frac{\partial L}{\partial \dot{q}_s} - L = h = \text{constant} \tag{8.15}$$

This is Jacobi's integral.

Explicit Form of Jacobi's Integral. Now assume, more strongly, that all forces are conservative, i.e. $V = V(q_i)$. Recall T expressed in generalized coordinates, Eqn. (6.3). Let

$$T = T_2 + T_1 + T_0 \tag{8.16}$$

where

$$\begin{aligned} T_2 &= \frac{1}{2}\sum_\alpha \sum_\beta a_{\alpha\beta}\, \dot{q}_\alpha\, \dot{q}_\beta\,; \qquad \alpha_{\alpha\beta} \text{ positive definite.} \\ T_1 &= \sum_\alpha b_\alpha\, \dot{q}_\alpha \\ T_0 &= c \end{aligned} \tag{8.17}$$

Thus

$$\begin{aligned} \frac{\partial T_2}{\partial \dot{q}_s} &= \sum_\alpha a_{\alpha s}\, \dot{q}_\alpha \Longrightarrow \sum_s \frac{\partial T_2}{\partial \dot{q}_s}\, \dot{q}_s = 2T_2 \\ \frac{\partial T_1}{\partial \dot{q}_s} &= b_s \Longrightarrow \sum_s \frac{\partial T_1}{\partial \dot{q}_s}\, \dot{q}_s = T_1 \end{aligned} \tag{8.18}$$

Substitute Eqn. (8.16) into (8.15) and note that $V \neq V(\dot{q}_i)$:

$$\sum_s \dot{q}_s \left(\frac{\partial T_2}{\partial \dot{q}_s} + \frac{\partial T_1}{\partial \dot{q}_s} + \frac{\partial T_0}{\partial \dot{q}_s}\right) - T_2 - T_1 - T_0 + V = h$$

$$2T_2 + T_1 + 0 - T_2 - T_1 - T_0 + V = h$$

$$T_2 - T_0 + V = h \tag{8.19}$$

where Eqns. (8.18) were used.

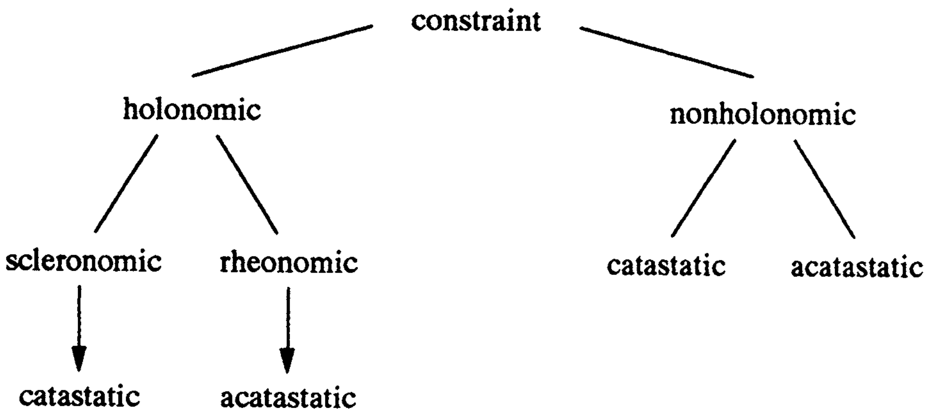


Fig. 8-1

Conservation of Energy. We proved in Section 4.1 that in a closed system energy is conserved, i.e. in the present terms

$$T + V = T_2 + T_1 + T_0 + V = h \tag{8.20}$$

But this seems to be in conflict with Eqn. (8.19). The difference has to do with Eqn. (8.20) being for rectangular coordinates and Eqn. (8.19) being for generalized (minimal) coordinates.

Let's recall the classification of constraints, summarized in Fig. 8-1. When the energy result is stated in rectangular coordinates, all constraints must be considered and "catastatic system" means *all* constraints (holonomic and nonholonomic) are catastatic. When Jacobi's integral is stated in generalized coordinates (minimal implies that all holonomic constraints are eliminated in the coordinate transformation), only the nonholonomic constraints apply and "catastatic system" means all nonholonomic constraints are catastatic. Thus the energy result is more restrictive than Jacobi's result.

Since scleronomic implies catastatic for a holonomic constraint, we would expect that for the case of all holonomic constraints scleronomic and all nonholonomic constraints catastatic, the energy integral and Jacobi's integral would be the same. This is true because for a scleronomic system, the transformation from the u_s to the q_s is time independent:

$$\begin{aligned} q_s &= q_s(u_1, \cdot\cdot, u_N)\ ; \qquad s = 1, \cdot\cdot, n \\ u_r &= u_r(q_1, \cdot\cdot, q_n)\ ; \qquad r = 1, \cdot\cdot, N \end{aligned} \tag{8.21}$$

Therefore, from Eqns. (6.4),

$$\begin{aligned} b_\alpha &= \overline{\sum_r} m_r \frac{\partial u_r}{\partial q_\alpha} \frac{\partial u_r}{\partial t} = 0 \Longrightarrow T_1 = 0 \\ c &= \frac{1}{2}\overline{\sum_r} m_r \left(\frac{\partial u_r}{\partial t}\right)^2 = 0 \Longrightarrow T_0 = 0 \end{aligned} \tag{8.22}$$

and thus Jacobi's integral becomes

$$T_2 + V = T + V = h = \text{constant} \tag{8.23}$$

Example – Rotating Pendulum. A bead slides without friction on a rotating hoop (Fig. 8-2). The constraints are, in spherical coordinates,

1. $r = a$ (holonomic, scleronomic $\Longrightarrow$ catastatic)

2. $\dot{\phi} = \omega$
   $d\phi - \omega dt = 0$ (holonomic, rheonomic, acatastatic)
   $\phi = \int \omega dt +$ constant

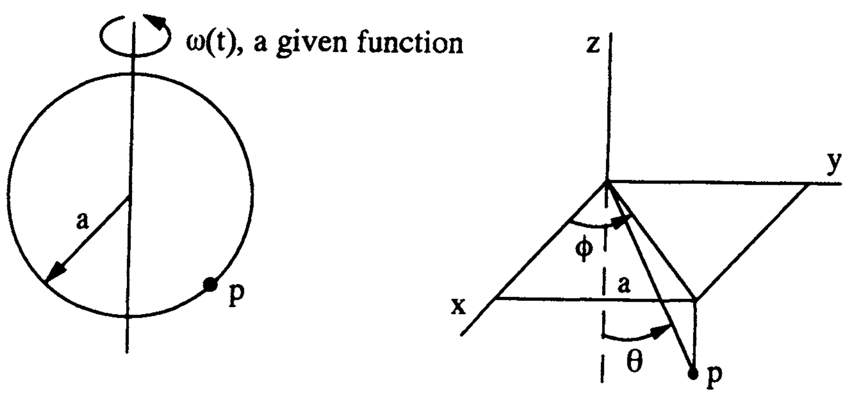


Fig. 8-2

Since *all* constraints are not catastatic, the energy integral in general does not exist. Does Jacobi's integral exist? One requirement, all nonholonomic constraints catastatic, is met. How about the others? Choose θ as generalized coordinate ($n = N - L' = 3 - 2 = 1$). The transformation from generalized to rectangular coordinates is (taking $\phi(0) = 0$):

$$\begin{aligned} x &= a\sin\theta\cos\int\omega dt \\ y &= a\sin\theta\sin\int\omega dt \\ z &= -a\cos\theta \end{aligned}$$

Note the time dependence due to the rheonomic constraint. T, V, and L are

$$\begin{aligned} T &= \frac{1}{2}m(a^2\dot{\theta}^2 + a^2\sin^2\theta\omega^2) \\ V &= -mg\,a\cos\theta \\ L &= T - V \end{aligned}$$

Note that there is no T_1 term in T.

A further requirement for Jacobi's integral is $\partial L/\partial t = 0$; this will be true only if $\omega(t)$ = constant. Under this assumption, Jacobi's integral exists and is

$$T_2 + V - T_0 = \frac{1}{2}ma^2\dot{\theta}^2 - mg\,a\cos\theta - \frac{1}{2}ma^2\sin^2\theta\,\omega^2 = \text{constant}$$

Energy, however, is not generally conserved; as the hoop rotates at constant speed and the bead slides on the hoop, there is a constant interchange of energy in and out of the system, i.e. there is a continually varying torque required to keep ω = constant.

The exception is $\omega = 0$ which implies that $T_0 = 0$, for which

$$T_2 + V = T + V = h = \text{constant}$$

That is, energy is conserved only if $\omega = 0$.

In this problem, there are four integrals of the motion required to completely solve the problem, and we have found only one, Jacobi's integral. This integral gives an equation of the form $\dot{\theta}^2 = f(\theta)$. Study of this equation provides much useful information about the motion of the system;[2] for example, it can be used to establish the stability of equilibrium positions, in the same manner as the energy integral is used in Chapter 11.

Dissipative Forces. We are now in a position to give an interesting physical interpretation of Rayleigh's dissipation function. It will be assumed that all forces except the dissipative forces are conservative. From

Eqn. (6.38),

$$\sum_s \frac{\partial D}{\partial \dot{q}_s}\dot{q}_s = \sum_s \sum_\alpha d_{\alpha s}\dot{q}_\alpha \dot{q}_s = 2D$$

Using Eqn. (6.37), the fundamental equation is now

$$\sum_s \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) - \frac{\partial L}{\partial q_s}\right]\dot{q}_s = \sum_s Q_s^d \dot{q}_s = 2D$$

Proceeding as before, we arrive at

$$\frac{d}{dt}\left[\sum_s \dot{q}_s \frac{\partial L}{\partial \dot{q}_s}\right] - \frac{dL}{dt} = 2D$$

Invoking Eqns. (8.18), this may be written as

$$\frac{d}{dt}(T_2 - T_0 + V) = 2D$$

If the transformation from the u_s to the q_s does not involve time explicitly, then this reduces to

$$\frac{d}{dt}(T + V) = 2D$$

Thus D represents one-half the rate of loss of energy by the dissipation forces (recall that $d_{\alpha s}$ is negative definite and therefore $D < 0$).

Historical Remarks. The concepts of kinetic and potential energy and the principle of conservation of energy pre-date Newton's Laws, and in fact, according to Dugas, can be traced back to the fourteenth century. The first to state these concepts relatively clearly was Huyghens. For a while after Newton, the conservation of energy and Newton's Laws were thought to be independent principles. At this time, kinetic energy was called "living force". D'Alembert, Lagrange, and Carnot recognized that the energy conservation principle was a consequence of Newton's Laws for certain special cases. The principle was first expressed in its modern form by Green and by Helmholtz.

8.3 Ignoration of Coordinates

Example. A massless spring slides without friction on a vertical wire with a particle at it's end (Fig. 8-3). We choose cylindrical coordinates (r, ϕ, z).

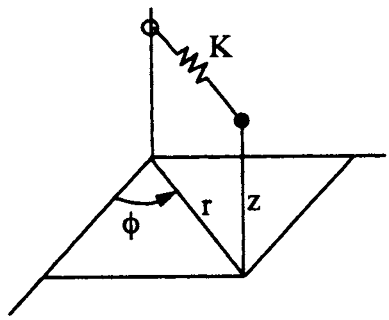


Fig. 8-3

Using Eqn. (7.7), the Lagrangian is

$$\begin{aligned} L &= T - V \\ &= \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2\right) - \frac{1}{2}k(r-\ell)^2 - mgz \end{aligned}$$

We see that $L \neq L(\phi)$ so that Lagrange's equation for ϕ is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = 0$$

Therefore,

$$\frac{d}{dt}(mr^2\dot{\phi}) = 0 \Longrightarrow mr^2\dot{\phi} = \text{constant}$$

Thus we have found one integral of the motion. This motivates the following definitions.

Definitions. A coordinate q_s is *ignorable* if $L \neq L(q_s)$. The *generalized momentum* components are

$$p_s = \frac{\partial T}{\partial \dot{q}_s}\,; \qquad s = 1, \cdot\cdot, n \tag{8.24}$$

If $V = V(q_1, \cdot\cdot, q_n)$, this is equivalent to

$$p_s = \frac{\partial L}{\partial \dot{q}_s}\,; \qquad s = 1, \cdot\cdot, n \tag{8.25}$$

This latter definition is motivated by considering the motion of a particle in rectangular coordinates; the kinetic energy is in this case

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

and thus the x component of the generalized momentum is

$$p_x = \frac{\partial T}{\partial \dot{x}} = m\dot{x}$$

which is the x-component of the linear momentum.

If a system is conservative and holonomic and q_s is ignorable, the Lagrange equation for q_s is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) = \frac{dp_s}{dt} = 0$$

so that

$$p_s = \text{constant}$$

This integral of the motion is termed a *momentum integral.*

Lagrange's Equations with Ignorable Coordinates. Assume that the dynamic system is (1) holonomic, (2) conservative, and that (3) $\partial L/\partial t = 0$. Suppose q_β ; $\beta = 1, \cdot\cdot, b \leq n$ are ignorable and that the other q_s are not. Then

$$L = L(q_{b+1}, \cdot\cdot, q_n, \dot{q}_1, \cdot\cdot, \dot{q}_n) \tag{8.26}$$

and Lagrange's equations are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\beta}\right) = \dot{p}_\beta = 0 ; \qquad \beta = 1, \cdot\cdot, b \tag{8.27}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right) = \dot{p}_\alpha = \frac{\partial L}{\partial q_\alpha} ; \qquad \alpha = b+1, \cdot\cdot, n \tag{8.28}$$

Equations (8.27) give b momentum integrals

$$\frac{\partial L}{\partial \dot{q}_\beta} = p_\beta = C_\beta = \text{constant} ; \qquad \beta = 1, \cdot\cdot, b \tag{8.29}$$

Routhian Function. Continue with the same assumptions. Define the *Routhian* function as:

$$R = \sum_{\beta=1}^{b} \dot{q}_\beta \frac{\partial L}{\partial \dot{q}_\beta} - L \tag{8.30}$$

Take the variation of R, recalling Eqn. (8.26):

$$\delta R = \sum_{\beta=1}^{b} \dot{q}_\beta\, \delta\left(\frac{\partial L}{\partial \dot{q}_\beta}\right) + \sum_{\beta=1}^{b} \frac{\partial L}{\partial \dot{q}_\beta}\, \delta\dot{q}_\beta - \sum_{s=1}^{n} \frac{\partial L}{\partial q_s}\, \delta q_s - \sum_{s=1}^{n} \frac{\partial L}{\partial \dot{q}_s}\, \delta\dot{q}_s$$

$$= \sum_{\beta=1}^{b} \dot{q}_\beta\, \delta C_\beta + \sum_{\beta=1}^{b} \frac{\partial L}{\partial \dot{q}_\beta}\, \delta\dot{q}_\beta - \sum_{\alpha=b+1}^{n} \frac{\partial L}{\partial q_\alpha} \delta q_\alpha$$

$$- \sum_{\beta=1}^{b} \frac{\partial L}{\partial \dot{q}_\beta}\, \delta\dot{q}_\beta - \sum_{\alpha=b+1}^{n} \frac{\partial L}{\partial \dot{q}_\alpha}\, \delta\dot{q}_\alpha \tag{8.31}$$

If we now solve Eqns. (8.29) for the $\dot{q}_\beta$ as functions of the C_r; $\beta, r = 1, \cdot\cdot, b$ and substitute into Eqn. (8.30), the result is

$$R = R(q_{b+1}, \cdot\cdot, q_n,\ \dot{q}_{b+1}, \cdot\cdot, \dot{q}_n,\ C_1, \cdot\cdot, C_b) \tag{8.32}$$

Taking the variation of this function,

$$\delta R = \sum_{\alpha=b+1}^{n} \frac{\partial R}{\partial q_\alpha}\, \delta q_\alpha + \sum_{\alpha=b+1}^{n} \frac{\partial R}{\partial \dot{q}_\alpha}\, \delta\dot{q}_\alpha + \sum_{\beta=1}^{b} \frac{\partial R}{\partial C_\beta}\, \delta C_\beta \tag{8.33}$$

Equating the coefficients of Eqns. (8.31) and (8.33),

$$\dot{q}_\beta = \frac{\partial R}{\partial C_\beta}\,; \qquad \beta = 1, \cdot\cdot, b \tag{8.34}$$

$$-\frac{\partial L}{\partial q_\alpha} = \frac{\partial R}{\partial q_\alpha}\,; \qquad -\frac{\partial L}{\partial \dot{q}_\alpha} = \frac{\partial R}{\partial \dot{q}_\alpha}\,; \qquad \alpha = b+1, \cdot\cdot, n \tag{8.35}$$

Putting Eqns. (8.35) into Lagrange's equations gives

$$\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{q}_\alpha}\right) - \frac{\partial R}{\partial q_\alpha} = 0\,; \qquad \alpha = b+1, \cdot\cdot, n \tag{8.36}$$

Equations (8.34) lead to b integrals of the motion:

$$q_\beta = \int \frac{\partial R}{\partial C_\beta}\, dt + K_\beta\,; \qquad \beta = 1, \cdot\cdot, b \tag{8.37}$$

The Routhian R is in effect the Lagrangian for the nonignorable coordinates.

Example. Two masses connected by a spring slide on a smooth surface (Fig. 8-4). Choose generalized coordinates (x, y); the Lagrangian is

$$L = T - V = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2(\dot{x} + \dot{y})^2 - \frac{1}{2}k(y - \ell)^2$$

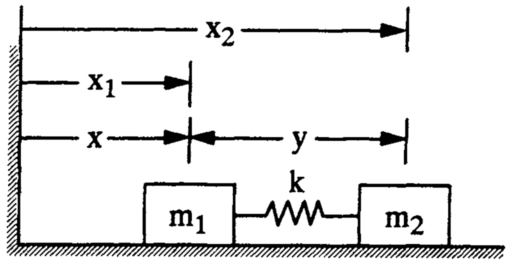


Fig. 8-4

Since $L \neq L(x)$, x is ignorable. (Note that if (x_1, x_2) are selected as coordinates, neither would be ignorable.) From Eqn. (8.30) the Routhian is

$$R = \sum_{\beta=1}^{1} \dot{q}_1 \frac{\partial L}{\partial \dot{q}_1} - L = \dot{x}C - \frac{1}{2}m_1\dot{x}^2 - \frac{1}{2}m_2(\dot{x}+\dot{y})^2 + \frac{1}{2}k(y-\ell)^2$$

where from Eqn. (8.29)

$$C = \frac{\partial L}{\partial \dot{x}} = m_1\dot{x} + m_2(\dot{x}+\dot{y})$$

is one integral of the motion.

Since y is the only non-ignorable coordinate, the one remaining equation of motion is found by substituting $\dot{x}$ for C in R and applying the equation of motion, Eqn. (8.36):

$$\dot{x} = \frac{C - m_2\dot{y}}{m_1 + m_2}$$

$$R = \frac{C - m_2\dot{y}}{m_1+m_2}C - \frac{1}{2}m_1\left(\frac{C-m_2\dot{y}}{m_1+m_2}\right)^2 - \frac{1}{2}m_2\left(\frac{C-m_2\dot{y}}{m_1+m_2}+\dot{y}\right)^2 + \frac{1}{2}k(y-\ell)$$

$$\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{y}}\right) - \frac{\partial R}{\partial y} = 0$$

$$\frac{m_1m_2}{m_1+m_2}\,\ddot{y} + ky = k\ell$$

The Routhian approach removes all traces of the ignorable coordinates from the equations of motion.

8.4 Separation of Variables

Uncoupled Systems. Assume a natural (conservative, holonomic, scleronomic) system with kinetic and potential energies:

$$T = \frac{1}{2}\sum_{s=1}^{n} v_s(q_s)\dot{q}_s^2 \; ; \qquad V = \sum_{s=1}^{n} w_s(q_s) \tag{8.38}$$

(Note that only the quadratic term is present in T in a scleronomic system.)

Substitute Eqns. (8.38) into Lagrange's equations, Eqns. (6.35):

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) - \frac{\partial L}{\partial q_s} = 0 \; ; \qquad s = 1, \cdot\cdot, n$$

$$v_s(q_s)\ddot{q}_s + \frac{1}{2}\frac{dv_s}{dq_s}\dot{q}_s^2 + \frac{dw_s}{dq_s} = 0 \; ; \qquad s = 1, \cdot\cdot, n \tag{8.39}$$

Since each coordinate appears in one and only one equation, and no other coordinate appears in this equation, the system is *uncoupled* and the variables are said to be *separated.* Each equation can be integrated independently in this case. Some, but not all, equations of motion can be uncoupled by a transformation of variables.

Liouville Systems. It can be shown that for systems with energies of the form

$$\begin{aligned} T &= \frac{1}{2}\left[\sum_{\alpha=1}^{n} \tilde{u}_\alpha\,(q_\alpha)\right]\left[\sum_{\beta=1}^{n} v_\beta(q_\beta)\dot{q}_\beta^2\right] \\ V &= \sum_{\alpha=1}^{n} \tilde{w}_\alpha\,(q_\alpha) \Big/ \sum_{\beta=1}^{n} \tilde{u}_\beta\,(q_\beta) \end{aligned} \tag{8.40}$$

that Lagrange's equations are also separated.[3]

Notes

1 For example, if all the forces are either conservative or dissipative damping (see Section 6.4).

2 See Pars.

3 See Rosenberg or Pars.

PROBLEMS

8/1. The simple spherical pendulum with fixed support and constant length possess an energy and a momentum integral. Examine the same question when the point of support is moved in a prescribed fashion $f(t)$ along the horizontal x-axis.

8/2. Examine the same question as in Problem 8/1 when the point of support is moved in a prescribed fashion $f(t)$ along the vertical z-axis.

8/3. Examine the same question as in Problem 8/1 when the point of support is fixed, but the length of the pendulum changes in the prescribed fashion $f(t)$.

8/4. A heavy, uniform rod is constrained to move in the vertical plane, and one of its extremities is constrained to move on a horizontal line so that its distance from a fixed point on that line is a prescribed function $f(t)$. Discuss the existence of integrals.

In the problems that follow, use the constants of the motion to reduce the solution to quadratures to the extent possible.

8/5. The systen of Problem 6/7. Are there conditions such that the system can rotate about the z-axis with $\theta = \text{constant} \neq 0$?

8/6. The system of Problem 6/9.

8/7. The system of Problem 6/10.

8/8. The system of Problem 7/2.

Chapter 9

Examples

9.1 Street Vendor's Cart

Formulation. A two-wheeled cart of dimensions shown on Fig. 9-1 moves in a horizontal plane. This is an example of a multi-body problem with nonholonomic constraints.

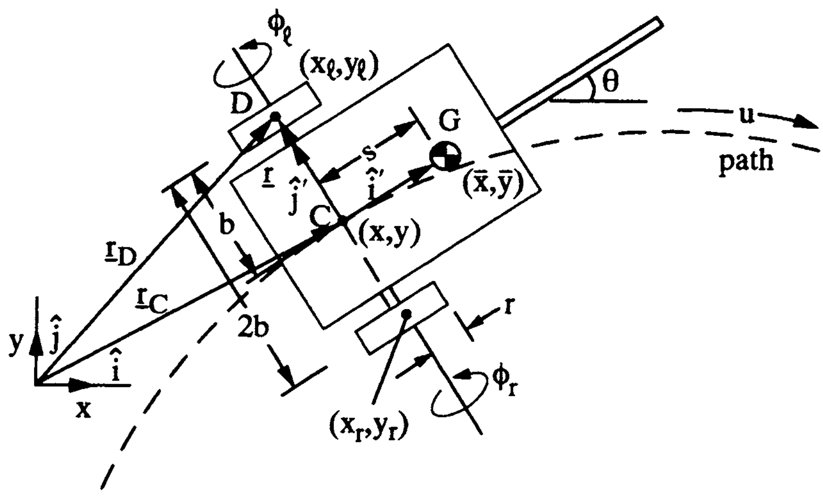


Fig. 9-1

Let:

m_c = mass of the cart without wheels
m_w = mass of each wheel
I_c = mass moment of inertia of the cart without wheels about a vertical axis through point C on the axis of rotation of the wheels
C = mass moment of inertia of each wheel about the axis of rotation
A = mass moment of inertia of each wheel about a diameter
I_c = $\overline{I} + m_c s^2$ (parallel axis theorem)

Constraints. Three rigid bodies in 3-space have $3 \times 6 = 18$ coordinates N. But by inspection, there are 5 generalized coordinates, say $(x, y, \theta, \phi_\ell, \phi_r)$. Therefore, there must be $18 - 5 = 13 = L'$ holonomic constraints, but we need not consider them if we choose generalized coordinates. There are three other (nonholonomic) constraints:

(1) Velocity tangent to path. From the figure,

$$\underline{v} = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath} = v\cos\theta\,\hat{\imath} + v\sin\theta\,\hat{\jmath}$$

$$\dot{x} = v\cos\theta\,; \qquad \dot{y} = v\sin\theta$$

$$\frac{\dot{y}}{\dot{x}} = \frac{v\sin\theta}{v\cos\theta}\,; \qquad \cos\theta\,\dot{y} - \sin\theta\,\dot{x} = 0$$

Thus the constraints on the virtual displacements are

$$\cos\theta\,\delta y - \sin\theta\,\delta x = 0 \tag{9.1}$$

Recall from Section 2.1 that in the dynamics problem of the second kind, the motion is specified and the forces are to be determined. Thus if the path of the cart is specified, say by $y = f(x)$, Eqn. (9.1) contains only two independent variables and the constraint is holonomic. In the problem of the first kind, the forces on the cart are specified and the path is to be determined. In this latter case, which is the case of interest, the constraint Eqn. (9.1) is nonholonomic.

(2) Left wheel rolls without slipping. Using the relative velocity equation, Eqn. (1.25),

$$\underline{v}_D = \underline{v}_C + \underline{v}_{\text{rel}} + \underline{\omega} \times \underline{r}$$

The terms are

$$
\begin{aligned}
\underline{v}_D &= \dot{\phi}_\ell \, r \, \hat{i}' \qquad \text{(no slip condition)} \\
\underline{v}_C &= \dot{x}\hat{i} + \dot{y}\hat{j} \; ; \qquad \underline{v}_{\text{rel}} = \underline{0} \\
\underline{\omega} &= \dot{\theta}\hat{k} \; ; \qquad \underline{r} = b\hat{j}' \\
\hat{i} &= \cos\theta \hat{i}' - \sin\theta \hat{j}' \\
\hat{j} &= \sin\theta \hat{i}' + \cos\theta \hat{j} \\
\hat{k} &= \hat{k}'
\end{aligned}
$$

Therefore

$$\dot{\phi}_\ell \, r \, \hat{i}' = \dot{x}(\cos\theta \, \hat{i}' - \sin\theta \, \hat{j}') + \dot{y}(\sin\theta \, \hat{i}' + \cos\theta \, \hat{j}') + \underline{0} + \dot{\theta}\hat{k}' \times b\hat{j}'$$

Equating components gives

$$
\begin{aligned}
\dot{\phi}_\ell r &= \dot{x}\cos\theta + \dot{y}\sin\theta - \dot{\theta} b \\
0 &= -\dot{x}\sin\theta + \dot{y}\cos\theta
\end{aligned}
$$

The second of these leads to the same constraint as Eqn. (9.1) but the first is an additional nonholonomic constraint:

$$-\cos\theta \, \delta x - \sin\theta \, \delta y + b\delta\theta + r \, \delta\phi_\ell = 0 \tag{9.2}$$

(3) Right wheel rolls without slipping. Proceeding as before,

$$\dot{\phi}_r \, r = \dot{x}\cos\theta + \dot{y}\sin\theta + \dot{\theta} b$$

so that

$$-\cos\theta \, \delta x - \sin\theta \, \delta y - b \, \delta\theta + r \, \delta\phi_r = 0 \tag{9.3}$$

The three nonholonomic constraints are of the form

$$\sum_{s=1}^{5} B_{rs} \, \delta q_s + B_r = 0 \; ; \qquad r = 1, 2, 3 \tag{9.4}$$

Therefore, taking $(q_1, q_2, q_3, q_4, q_5) = (x, y, \theta, \phi_\ell, \phi_r)$, comparison of Eqns. (9.1) - (9.4) gives:

$$
\begin{aligned}
&B_{11} = -\sin\theta \, , \quad B_{12} = \cos\theta \, , \quad B_{13} = B_{14} = B_{15} = 0 \\
&B_{21} = -\cos\theta \, , \quad B_{22} = -\sin\theta \, , \quad B_{23} = b \, , \; B_{24} = r \, , \; B_{25} = 0 \\
&B_{31} = -\cos\theta \, , \quad B_{32} = -\sin\theta \, , \quad B_{33} = -b \, , \; B_{34} = 0 \, , \; B_{35} = r
\end{aligned} \tag{9.5}
$$

Kinetic and Potential Energies. From Fig. (9-1),

$$\bar{x} = x + s\cos\theta\,; \qquad \bar{y} = y + s\sin\theta$$

Forming T of the cart from Eqn. (1.56),

$$\begin{aligned} T_c &= \frac{1}{2}m_c\bar{v}^2 + \frac{1}{2}\bar{I}\omega^2 = \frac{1}{2}m_c(\dot{\bar{x}}^2 + \dot{\bar{y}}^2) + \frac{1}{2}\bar{I}\dot{\theta}^2 \\ &= \frac{1}{2}m_c\left[\dot{x}^2 + \dot{y}^2 + s^2\dot{\theta}^2 - 2s\dot{\theta}\,(\dot{x}\sin\theta - \dot{y}\cos\theta)\right] \\ &\quad + \frac{1}{2}(I_c - m_c s^2)\dot{\theta}^2 \end{aligned} \tag{9.6}$$

T of the left wheel is:

$$\begin{aligned} x_\ell &= x - b\sin\theta\,, \qquad y_\ell = y + b\cos\theta \\ T_\ell &= \frac{1}{2}m_w\left(\dot{x}_\ell^2 + \dot{y}_\ell^2\right) + \frac{1}{2}C\,\dot{\phi}_\ell^2 + \frac{1}{2}A\dot{\theta}^2 \\ &= \frac{1}{2}m_w\left[\dot{x}^2 + \dot{y}^2 - 2b\dot{\theta}\,(\dot{x}\cos\theta + \dot{y}\sin\theta) + b^2\dot{\theta}^2\right] \\ &\quad + \frac{1}{2}A\dot{\theta}^2 + \frac{1}{2}C\,\dot{\phi}_\ell^2 \end{aligned} \tag{9.7}$$

For the right wheel, replace b by $-b$ in this equation. The total system kinetic energy is then

$$\begin{aligned} T &= T_c + T_\ell + T_r \\ T &= \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) + m_c\,s\dot{\theta}\,(\dot{y}\cos\theta - \dot{x}\sin\theta) + \frac{1}{2}I\dot{\theta}^2 \\ &\quad + \frac{1}{2}C\left(\dot{\phi}_r^2 + \dot{\phi}_\ell^2\right) \end{aligned} \tag{9.8}$$

where

$$\begin{aligned} m &= m_c + 2m_w \\ I &= I_c + 2m_w b^2 + 2A \end{aligned}$$

Consider the gravitational forces; are they constraint or given? They are constraint because they do no work (either actual or virtual). Consequently,

$$V = 0 \tag{9.9}$$

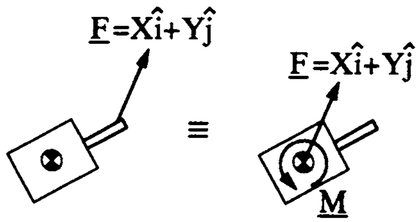


Fig. 9-2

Lagrange's Equations. The applicable form of Eqns. (6.29) is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_s}\right) - \frac{\partial T}{\partial q_s} - Q_s + \sum_{r=1}^{3} \lambda_r B_{rs} = 0 ; \qquad s = 1, \cdot\cdot, 5 \qquad (9.10)$$

Now assume that the cart is pulled by applying a force to the end of the tongue. The equivalent force system consists of the force acting at the center of mass plus a moment (Fig. 9-2). Thus let X, Y, M be the (x, y) components of the given force and the moment acting on the cart. Then from Eqns. (9.5), (9.8), and (9.10) the equations of motion are

$$\begin{aligned} &m\ddot{x} - m_c\, s\left(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta\right) - \lambda_1 \sin\theta - (\lambda_2 + \lambda_3)\cos\theta = X \\ &m\ddot{y} + m_c\, s\left(\ddot{\theta}\sin\theta - \dot{\theta}^2\sin\theta\right) + \lambda_1\cos\theta - (\lambda_2 + \lambda_3)\sin\theta = Y \\ &-m_c\, s\,(\ddot{x}\sin\theta - \ddot{y}\cos\theta) + I\ddot{\theta} + b(\lambda_2 - \lambda_3) = M \\ &C\ddot{\phi}_r + \lambda_2 r = 0 \\ &C\ddot{\phi}_\ell + \lambda_3 r = 0 \end{aligned} \qquad (9.11)$$

These five equations, along with the constraint equations in velocity form, give eight equations in the eight unknowns $(x, y, \theta, \phi_\ell, \phi_r, \lambda_1, \lambda_2, \lambda_3)$.

By a great deal of manipulation, including transformation to pathwise coordinates and eliminating $\phi_r, \phi_\ell, \lambda_1, \lambda_2$, and λ_3, we get[1]

$$\begin{aligned} &D\ddot{u} - m_c\, s\dot{\theta}^2 = F(u, \theta, t) = \text{force component tangent to path.} \\ &J\ddot{\theta} + m_c\, s\dot{\theta}\dot{u} = M(u, \theta, t) \end{aligned} \qquad (9.12)$$

where

$$\begin{aligned} u &= \text{path length (see figure)} \\ D &= m + 2C/r^2 \\ J &= I + 2bC/r^2 \end{aligned}$$

These equations are dynamically uncoupled but still nonlinear. To integrate Eqns. (9.12), we need initial conditions $u(0)$, $\dot{u}(0)$, $\theta(0)$, $\dot{\theta}(0)$, and functions $F(t)$, $M(t)$.

9.2 A Useful Identity

Lemma. Any equation of motion of the form

$$f_1(q)\ddot{q} + f_2(q)f_3(\dot{q}) = 0 \tag{9.13}$$

can always be reduced to quadratures by the identity

$$\ddot{q} = \dot{q}\,\frac{d\dot{q}}{dq} \tag{9.14}$$

Proof of Identity.

$$\frac{d\dot{q}}{dq} = \frac{d\dot{q}}{dt}\,\frac{dt}{dq} = \ddot{q}\,\frac{1}{\dot{q}}$$

Proof of Lemma. Substitute Eqn. (9.14) into (9.13) and integrate:

$$\begin{aligned}
&f_1(q)\dot{q}\,\frac{d\dot{q}}{dq} + f_2(q)f_3(\dot{q}) = 0\\
&\int \frac{\dot{q}}{f_3(\dot{q})}\,d\dot{q} = -\int \frac{f_2(q)}{f_1(q)}\,dq + c_1\\
&F_1(\dot{q}) = F_2(q) + c_1\\
&\dot{q} = F_3(q,\ c_1) = \frac{dq}{dt}\\
&\int dt = \int \frac{dq}{F_3(q,\ c_1)} + c_2\\
&t = F_4(q,\ c_1) + c_2\\
&q = F_5(t,\ c_1,\ c_2)
\end{aligned}$$

Initial conditions $q(0)$, $\dot{q}(0)$ give c_1 and c_2:

$$\begin{aligned}
&F_1\left(\dot{q}(0)\right) = F_2(q(0)) + c_1\\
&0 = F_4(q(0), c_1) + c_2\\
&t = F_6\left(q,\ q(0),\ \dot{q}(0)\right)\\
&q = F_7\left(t, q(0), \dot{q}(0)\right)
\end{aligned}$$

Thus the solution has been reduced to quadratures. This technique is the same as applying the integrating factor $\dot{q}$ to Eqn. (9.13), and the resulting first integral is often equivalent to the energy integral.

Example. Suppose $f_3(\dot{q}) = 1$; that is, Eqn. (9.13) becomes

$$f_1(q)\ddot{q} + f_2(q) = 0 \tag{9.15}$$

Applying Eqn. (9.14) and integrating

$$\int \dot{q}\, d\dot{q} = F_2(q) + c_1 = \frac{1}{2}\dot{q}^2$$

$$\frac{dq}{dt} = \sqrt{2[F_2(q) + c_1]}$$

$$t = \int \frac{dq}{\sqrt{2[F_2(q) + c_1]}} + c_2$$

where

$$F_2(q) = -\int \frac{f_2(q)}{f_1(q)}\, dq$$

9.3 Indian Rope Trick

Problem Formulation. A uniform inextensible rope of length ℓ is thrown into the air bent double vertically. The initial speed of one end is u_0 and that of the other is v_0 (Fig. 9-3). A particle of mass m (a boy or a monkey) climbs along one side of the rope with constant speed v relative to the rope, starting at the end of the rope at $t = 0$. Let ρ be the mass per unit length of the rope.

In terms of x, y, and z, T and V are

$$\begin{aligned} T &= \frac{1}{2}\rho\left[(z-x)\dot{x}^2 + (z-y)\dot{y}^2\right] + \frac{1}{2}m(\dot{x}+v)^2 \\ V &= \rho g\left[(z-x)\frac{1}{2}(z+x) + (z-y)\frac{1}{2}(z+y)\right] + mg(x+vt) \end{aligned} \tag{9.16}$$

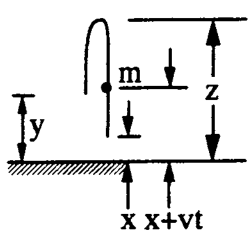


Fig. 9-3

The constraint is

$$(z - x) + (z - y) = \ell \tag{9.17}$$

We pick generalized coordinates x, u where

$$u = y - x \tag{9.18}$$

so that

$$z = \frac{\ell}{2} + \frac{u}{2} + x\,; \qquad y = u + x \tag{9.19}$$

In terms of (x, u),

$$\begin{aligned} T &= \frac{\rho}{4}\left[2\ell\dot{x}^2 + 2\dot{u}\dot{x}(\ell - u) + \dot{u}^2(\ell - u)\right] + \frac{1}{2}m(\dot{x} + v)^2 \\ V &= \rho g\left[x\ell + \frac{1}{2}\ell u - \frac{1}{4}u^2\right] + mg(x + vt) + \frac{1}{4}\rho g\ell^2 \end{aligned} \tag{9.20}$$

Lagrange's Equations. Since all given forces are conservative (gravity) and all constraints are holonomic, the suitable form of Lagrange's equations is Eqn. (6.35):

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) - \frac{\partial L}{\partial q_s} = 0\,; \qquad s = 1, 2 \tag{9.21}$$

where $q_1 = x$, and $q_2 = u$.

Computing the partial derivatives,

$$\begin{aligned} \frac{\partial L}{\partial \dot{x}} &= \frac{\rho}{4}\left[4\ell\dot{x} + 2\dot{u}(\ell - u)\right] + m(\dot{x} + v) \\ \frac{\partial L}{\partial x} &= -\rho g\,\ell - mg \\ \frac{\partial L}{\partial \dot{u}} &= \frac{\rho}{4}[2\dot{x}(\ell - u) + 2\dot{u}(\ell - u)] \\ \frac{\partial L}{\partial u} &= \frac{\rho}{4}[-2\dot{u}\dot{x} - \dot{u}^2 - \rho g\left[\frac{1}{2}\ell - \frac{1}{2}u\right] \end{aligned}$$

so that the equations of motion are

$$\begin{aligned} (\ddot{x} + g)(\rho\ell + m) + \frac{\rho}{2}\left[\ddot{u}(\ell - u) - \dot{u}^2\right] &= 0 \\ (\ddot{x} + g)(\ell - u) + \ddot{u}(\ell - u) - \frac{1}{2}\dot{u}^2 &= 0 \end{aligned} \tag{9.22}$$

Analysis. For convenience, introduce:

$$\mu = \frac{u}{\ell} = \text{dimensionless distance}$$

$$M = \frac{m}{\rho\ell} = \text{dimensionless mass}$$

Noting that variable x appears in Eqns. (9.22) only in the combination $(\ddot{x}+g)$, we eliminate $(\ddot{x}+g)$ to get

$$(1-\mu)\left[\ddot{\mu}(1-\mu)-\dot{\mu}^2\right] = (1+M)\left[2\ddot{\mu}(1-\mu)-\dot{\mu}^2\right]$$

$$\ddot{\mu} = \frac{1}{2}\left(\frac{1}{1-\mu} - \frac{1}{1+2M+\mu}\right)\dot{\mu}^2 \tag{9.23}$$

This is a nonlinear, second order, ordinary differential equation in variable μ.

Let the initial conditions be:

$$\mu(0) = \frac{u(0)}{\ell} = \frac{y(0)-x(0)}{\ell} = 0$$
$$\dot{\mu}(0) = \frac{\dot{\mu}_0}{\sqrt{1+2M}} \tag{9.24}$$

The first of these is the condition that the rope is bent in the middle, initially (Fig. 9-4), and the second is a rescaling of the initial velocity for convenience.

Noting that Eqn. (9.23) is in the form of Eqn. (9.13), we apply Eqn. (9.14):

$$\ddot{\mu} = \dot{\mu}\,\frac{d\dot{\mu}}{d\mu}$$

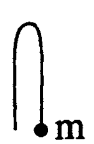


Fig. 9-4

Substituting and integrating once:

$$\begin{aligned}\dot{\mu}\,\frac{d\dot{\mu}}{d\mu} &= \frac{1}{2}\left(\frac{1}{1-\mu}-\frac{1}{1+2M+\mu}\right)\dot{\mu}^2\\ \frac{d\dot{\mu}}{\dot{\mu}} &= \frac{M+\mu}{(1-\mu)(1+2M+\mu)}d\mu\\ \int_{\dot{\mu}(0)}^{\dot{\mu}}\frac{d\dot{\mu}}{\dot{\mu}} &= \int_0^{\mu}\frac{(M+\mu)d\mu}{(1-\mu)(1+2M+\mu)}\end{aligned}$$

$$\dot{\mu} = \frac{\dot{\mu}_0}{\sqrt{(1-\mu)(1+2M+\mu)}} \tag{9.25}$$

Variable μ ranges from 0 at $t = 0$ (rope bent double) to 1 at some time t_1 (rope becomes straight). We see that

$$\lim_{t\to t_1}\dot{\mu} = \lim_{\mu\to 1}\frac{\dot{\mu}_0}{\sqrt{(1-\mu)(1+2M+\mu)}} = \infty$$

Thus the speed of the rope tip becomes infinite as it becomes straight. This explains the crack of a whip. The crack occurs when the tip speed exceeds the speed of sound. It can also be shown that the tension in the rope $\to\infty$ as $\mu\to 1$. In an actual rope, this rising tension brings the elasticity of the rope into play, which keeps the velocity finite.

Integrating Eqn. (9.25),

$$\int_0^{\mu}\sqrt{(1-\mu)(1+2M+\mu)}d\mu = \dot{\mu}_0\int_0^t dt$$

$$\begin{aligned}\dot{\mu}_0 t &= \frac{1}{2}\Bigg\{(\mu-M)\sqrt{(1-\mu)(1+2M+\mu)}+(1+M)^2\sin^{-1}\left(\frac{\mu+M}{M+1}\right)\\ &\quad-\left[M\sqrt{1+2M}+(1+M)^2\sin^{-1}\frac{M}{M+1}\right]\Bigg\}\end{aligned} \tag{9.26}$$

The time t_1 when the rope straightens out is found by putting $\mu = 1$:

$$\dot{\mu}_0 t_1 = \frac{1}{2}\left[(1+M)^2\pi - M\sqrt{1+2M}-(1+M)^2\sin^{-1}\frac{M}{M+1}\right] \tag{9.27}$$

If there is no child or monkey, $M = 0$ and

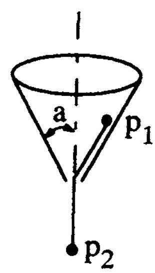
$$t_1 = \frac{\pi}{2\dot{\mu}_0} = \frac{\pi}{2\dot{\mu}(0)} = \frac{\pi}{2\ell[\dot{y}(0)-\dot{x}(0)]} \tag{9.28}$$

Notes

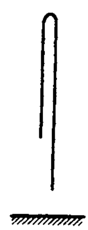
1 See Rosenberg for the details.

PROBLEMS

9/1. Two heavy particles P_1 and P_2 of mass m_1 and m_2, respectively, are interconnected by a massless, inextensible string of length l, as shown. P_1 is constrained to move on the surface of an inverted smooth rigid circular cone. The other particle is constrained to move on a vertical line. (The string passes through a smooth hole in the apex of the cone, as shown.) Examine this system for the exsitence of an energy integral and of momentum integrals and reduce the solution to quadratures to the extent possible.



Problem 9/1



Problem 9/3

9/2. The given forces acting on a particle of unit mass are derivable from the potential energy

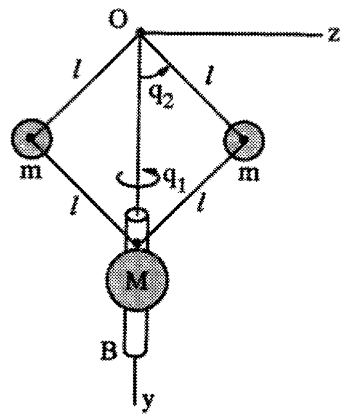
$$V(r,\theta,\varphi) = f(r) + \frac{g(\theta)}{r^2} + \frac{h(\varphi)}{r^2 \sin^2\theta}$$

where r, θ, and φ are spherical coordinates. Show that Lagrange's equations can be reduced to quadratures.

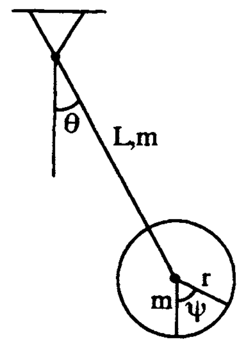
9/3. A uniform heavy rope is thrown in the air bent double, as shown. Let the initial velocity of one extremity be u_0, that of the other v_0, both are vertically up, and $u_0 \neq v_0$. Discuss the motion; in particular, explain what happens when the rope straightens out.

In the problems that follow, use Lagrange's equations to derive the equations of motion, investigate the existence of integrals, and use the integrals and the identity (9.14), as appropriate, to reduce the solution to quadratures to the extent possible.

9/4. The flyball governor shown has four rods that are pivoted at the fixed point O and at each mass in such a way that, as the masses m move outward, the mass M moves smoothly in the vertical direction along rod OB. The moment of inertia of mass M about the OB axis is I and the masses m may be modeled as particles.



Problem 9/4

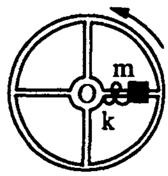


Problem 9/5

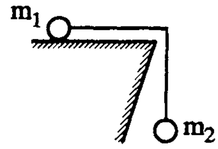
9/5. A uniform rod of length L and mass m is pivoted at one end and can swing in a vertical plane. A homogeneous disk of mass m, radius r, and uniform thickness is attached to the end of the rod and can rotate freely with respect to it. ($\overline{I}_{\text{rod}} = \frac{1}{12}mL^2$, $\overline{I}_{\text{disk}} = \frac{1}{2}mr^2$).

9/6. A flywheel rotating in a verticle plane about its center of mass O carries a particle of mass m that can slide freely along one spoke and is attached to the center of the wheel by a spring with spring constant k. The moment of inertia of the wheel about its axis of rotation is I, and the unrestrained length of the spring is ℓ. Obtain the equations of motion of the system by Lagrange's equations. Use coordinates θ (angle of rotation of the wheel) and x (elongation of the spring).

9/7. An inextensible string of length L lies on a horizontal frictionless table. The mass of the string per unit length is ρ and two particles of mass m_1 and m_2 are attached to the two ends of the string as shown. Initially, a portion of the string of length ℓ hangs over the side of the table and the string is at rest. Obtain a formula for the velocity of the string as m_1 leaves the table.

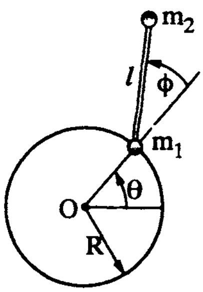


Problem 9/6

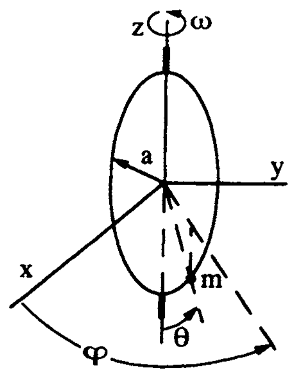


Problem 9/7

9/8. Particles m_1 and m_2, each of mass m, are connected by a massless rod of length ℓ. The particles move on a frictionless horizontal plane such that the motion of m_1, is confined to a fixed frictionless track of radius R.



Problem 9/8



Problem 9/9

9/9. A massless circular hoop of radius a rotates about a vertical axis with given constant angular speed ω. A particle of mass m slides on the hoop without friction (see Section 8.2).

9/10. Same as Problem 9/9 except that the hoop is free to rotate.

Chapter 10

Central Force Motion

10.1 General Properties

Lagrange's Equations. In Section 7.2 we derived Lagrange's equations for an unconstrained particle in spherical coordinates. In central force motion, the only force on the particle is directed towards the origin of an inertial frame, depends only on r, and is conservative (Fig. 10-1). Thus:

$$\begin{aligned} F_r &= F_r(r) = -\frac{\partial V}{\partial r} = -\frac{dV}{dr} \\ F_\theta &= F_\phi = 0 \end{aligned} \tag{10.1}$$

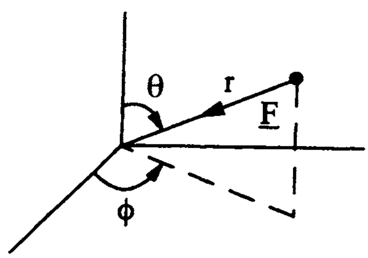


Fig. 10-1

Substituting into Eqns. (7.12):

$$\begin{aligned} & m\ddot{r} - mr\sin^2\theta\dot{\phi}^2 - mr\dot{\theta}^2 + \frac{dV}{dr} = 0 \\ & \frac{d}{dt}(mr^2\dot{\theta}) - mr^2\sin\theta\cos\theta\dot{\phi}^2 = 0 \\ & \frac{d}{dt}(mr^2\sin^2\theta\dot{\phi}) = 0 \end{aligned} \tag{10.2}$$

From these we see that ϕ is an ignorable coordinate with corresponding momentum integral:

$$\begin{aligned} mr^2 \sin^2\theta\dot{\phi} &= k = \text{constant} = p_\phi \\ &= \phi \quad \text{component of generalized momentum} \end{aligned} \tag{10.3}$$

Solving this for $\dot{\phi}$ and substituting into Eqns. (10.2):

$$\begin{aligned} &m\ddot{r} - \frac{k^2}{mr^3\sin^2\theta} - mr\dot{\theta}^2 + \frac{dV}{dr} = 0 \\ &\frac{d}{dt}(mr^2\dot{\theta}) - \frac{k^2\cos\theta}{mr^2\sin^3\theta} = 0 \end{aligned} \tag{10.4}$$

Since these are independent of ϕ, the motion takes place in the (r, θ) plane (this agrees with intuition). Hence, from Eqn. (10.3),

$$\phi = \text{constant} \Longrightarrow \dot{\phi} = 0 \Longrightarrow k = 0 \tag{10.5}$$

and Eqns. (10.4) become

$$\begin{aligned} &m\ddot{r} - mr\dot{\theta}^2 + \frac{dV}{dr} = 0 \\ &\frac{d}{dt}(mr^2\dot{\theta}) = 0 \end{aligned} \tag{10.6}$$

This gives another momentum integral:

$$\begin{aligned} mr^2\dot{\theta} &= K = \text{constant} \\ &= p_\theta = \theta \quad \text{component of generalized momenta} \end{aligned} \tag{10.7}$$

This may be used to prove Kepler's second law; consider the area swept out by the position vector in time dt (Fig. 10-2):

$$\begin{aligned} dA &= \frac{1}{2}r^2\, d\theta \\ \frac{dA}{dt} &= \frac{1}{2}r^2\frac{d\theta}{dt} = \frac{1}{2}\frac{K}{m} = \text{constant} \end{aligned} \tag{10.8}$$

Fig. 10-2

where Eqn. (10.7) was used. Thus "the area is swept out by the position vector at a constant rate."[1]

Next eliminate $\dot{\theta}$ between Eqns. (10.6):

$$m\ddot{r} - \frac{K^2}{mr^3} + \frac{dV}{dr} = 0 \tag{10.9}$$

We see that this is in the form of Eqn. (9.15); using the identity $\ddot{r} = \dot{r}\, d\dot{r}/dr$:

$$m\dot{r}\frac{d\dot{r}}{dr} - \frac{K^2}{mr^3} + \frac{dV}{dr} = 0$$

$$\frac{1}{2}m\dot{r}^2 - \frac{K^2}{m}\int\frac{dr}{r^3} + \int dV = h = \text{const.}$$

$$\frac{1}{2}m\dot{r}^2 + \frac{K^2}{2mr^2} + V(r) = h \tag{10.10}$$

The constant h is in fact the total energy, as can be seen by forming $T + V$.

Reduction to Quadratures and Orbit Equation. Solving Eqn. (10.10) for $\dot{r}$,

$$\dot{r} = \sqrt{\frac{2}{m}\left(h - V - \frac{K^2}{2mr^2}\right)} = f(r) \tag{10.11}$$

Thus $\dot{r}$ depends only on r and not on θ. Integrating,

$$t - t_0 = \int_{r_0}^{r}\frac{dr}{f(r)} \tag{10.12}$$

This gives $r(t)$; $\theta(t)$ is then obtained from Eqn. (10.7):

$$\theta - \theta_0 = \int_{t_0}^{t}\frac{K}{mr^2}dt \tag{10.13}$$

The orbit equation comes from eliminating t between Eqns. (10.7) and (10.11) to get an equation in (r, θ). We have

$$\dot{r} = \frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \frac{K}{mr^2}\frac{dr}{d\theta} = f(r)$$

$$\frac{K}{m}\int_{r_0}^{r}\frac{dr}{r^2 f(r)} = \theta - \theta_0 \tag{10.14}$$

It is convenient to make a change of variable from r to u, where,

$$u = \frac{1}{r}\ ; \qquad du = -\frac{1}{r^2}dr \tag{10.15}$$

Now restrict $F(r)$ to be a power of r

$$F(r) = \alpha r^n \tag{10.16}$$

so that

$$V(r) = -\int F(r)dr = -\frac{\alpha r^{n+1}}{n+1} \tag{10.17}$$

Then Eqn. (10.14) becomes

$$\theta - \theta_0 = \frac{K}{\sqrt{2m}} \int_{u_0}^{u} \frac{-du}{\sqrt{h + \frac{\alpha u^{-n-1}}{n+1} - \frac{K^2u^2}{2m}}} \tag{10.18}$$

It has been determined that this will have a solution in terms of trigonometric functions when, and only when,

$$n = +1, -2, -3$$

10.2 Inverse Square Forces

Gravitation. Equation (1.36) expresses Newton's law of gravitation. If $m_e \gg m$[1], then m_e may be regarded as fixed in an inertial frame and m has central force motion about m_e. Equations (10.16) and (10.17) become

$$F(r) = -\frac{\mu}{r^2}\ ; \qquad V(r) = -\frac{\mu}{r} \tag{10.19}$$

where $\mu = Km_e$ is the gravitational constant of the attracting mass and $F(r)$ is the force per unit mass of the orbiting body (Fig. 10-3).[2] In terms of u, these equations are

$$F(u) = -\mu u^2\ ; \qquad V(u) = -\mu u \tag{10.20}$$

The orbit equation, Eqn. (10.18), becomes

$$\theta - \theta_0 = \frac{K}{\sqrt{2m}} \int_{u_0}^{u} \frac{-du}{\sqrt{h + \mu u - \frac{K^2u^2}{2m}}} \tag{10.21}$$

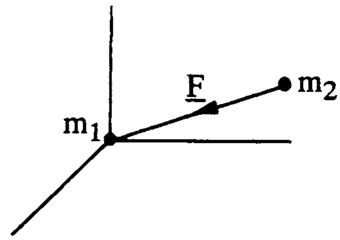


Fig. 10-3

Integrating and substituting $u = \frac{1}{r}$,

$$\frac{1}{r} = \frac{m\mu}{K^2}\left(1 + \sqrt{1 + \frac{2hK^2}{m\mu^2}}\cos(\theta - \theta')\right) \qquad (10.22)$$

where θ' is a constant of integration. The equation of a conic section in polar coordinates with one focus at the origin is

$$\frac{1}{r} = C\left(1 + \epsilon\cos(\theta - \theta')\right) \qquad (10.23)$$

Comparing Eqns. (10.22) and (10.23) we see that the orbit is a conic section, and that

$$C = \frac{m\mu}{K^2}\,; \qquad \epsilon = \sqrt{1 + \frac{2hK^2}{m\mu^2}} \qquad (10.24)$$

where ϵ is called the eccentricity. From the theory of conic sections:

$$\begin{aligned} &h > 0 \Longrightarrow \epsilon > 1\,, && \text{hyperbola} \\ &h = 0 \Longrightarrow \epsilon = 1\,, && \text{parabola} \\ &-\frac{m\mu^2}{2K^2} < h < 0 \Longrightarrow 0 < \epsilon < 1\,, && \text{ellipse} \\ &h = -\frac{m\mu^2}{2K^2} \Longrightarrow \epsilon = 0\,, && \text{circle} \end{aligned} \qquad (10.25)$$

Since the only one of these paths that closes on itself is the ellipse (with the circle as a special case), we have Kepler's First Law: "the planets travel around the sun in ellipses" (Fig. 10-4).

Period of Elliptical Orbits. From Kepler's Second Law, Eqn. (10.8), the area of an ellipse is

$$A = \int_A dA = \int_0^\tau \frac{dA}{dt}dt = \int_0^\tau \frac{1}{2}\frac{K}{m}dt = \frac{K\tau}{2m} \qquad (10.26)$$

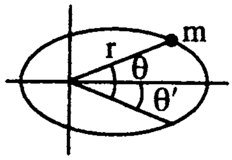


Fig. 10-4

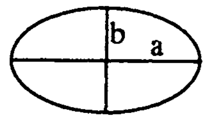


Fig. 10-5

where τ is the orbital period, that is, the time for one complete revolution. But for an ellipse,

$$\begin{aligned} A &= \pi ab \\ b &= a\sqrt{1-\epsilon^2} \\ a &= \frac{1}{C}\,\frac{1}{1-\epsilon^2} \end{aligned} \tag{10.27}$$

where a and b are the semi-major and semi-minor axes, respectively (Fig. 10-5).

Combining these results, the period of the orbit is

$$\tau = \frac{2mA}{K} = \frac{2m\pi ab}{K} = \frac{2m\pi a^2\sqrt{1-\epsilon^2}}{K}$$

$$\tau = \frac{2m\pi}{K\sqrt{C}}a^{3/2} \tag{10.28}$$

Thus "the square of the period is proportional to the cube of the semi-major axis"; this is Kepler's Third Law.

Remarks:

1. Kepler's First and Third Laws are valid for inverse square force only while the Second is true for any central force motion.

2. Newton started with Kepler's Laws and arrived at the inverse square law of attraction, while we have done the reverse.

10.3 The Time Equation

Remarks. Determining the time between two points on an orbit takes one more integration. This is important, for example, for rendenzvous problems. We will do this for an ellipse by a graphical method similar to that first used by Kepler. The derivation for hyperbolic orbits is similar.

Time Equation for Ellipse. Recall Eqns. (10.7) and (10.23):

$$r^2\dot{\theta} = \frac{K}{m} = \text{constant}\,; \qquad \frac{1}{r} = C\left(1 + \epsilon\cos(\theta - \theta')\right)$$

Thus

$$dt = \frac{m}{K}r^2 d\theta = \frac{m}{KC^2(1+\epsilon\cos\theta^*)^2}d\theta \tag{10.29}$$

where $\theta^* = \theta - \theta'$ is the angle from periapsis and is called the true anomaly, and $0 < \epsilon < 1$ for an ellipse.

Also, from Eqn. (10.8):

$$dt = \frac{m}{K}r^2 d\theta = \frac{m}{K}2dA \tag{10.30}$$

At this point we could use substitutions and integral tables to evaluate this integral to get the relation between θ and t. Instead we use a geometrical construction; this has the advantage that it gives a geometric interpretation of the quantities involved.

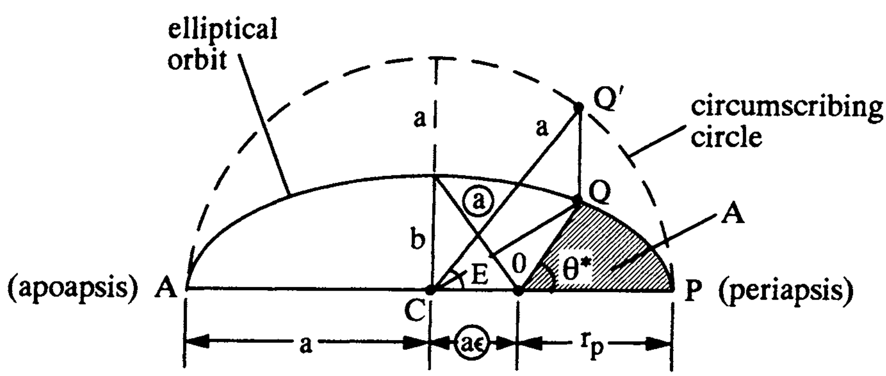


Fig. 10-6

Figure 10-6 shows an elliptical orbit about an attracting center at one of the foci, labeled 0, and the circumscribing circle tangent to the ellipse at apoapsis (maximum distance from 0) and at periapsis (minimum distance from 0). Q is the position of the orbiting object. The circled dimensions are properties of an ellipse not proved here.

From Eqn. (10.23) the radius at periapsis is

$$r_p = r_{\theta^*=0} = \frac{1}{C(1+\epsilon)} \tag{10.31}$$

Thus θ^* is the angle from periapsis. Kepler used the fact that an ellipse is a "squashed" circle in the ratio b/a. Therefore, using Eqn. (10.30),

$$\begin{aligned} t - t_p &= \frac{2m}{K} \ (\text{sector } PCQ - \text{ triangle } OCQ) \\ &= \frac{2m}{K} \ (\text{sector } PCQ' - \text{ triangle } OCQ')\frac{b}{a} \\ &= \frac{2m}{K}\left(\frac{1}{2}a^2 E - \frac{1}{2}a^2 \epsilon \sin E\right)\frac{b}{a} \\ &= \frac{ma^2\sqrt{1-\epsilon^2}}{K}(E - \epsilon \sin E) = \frac{ma^2 M\sqrt{1-\epsilon^2}}{K} \end{aligned} \tag{10.32}$$

where

$$\begin{aligned} E &= \text{eccentric anomaly (see Fig. 10-6)} \\ M &= E - \epsilon \sin E = \text{mean anomaly} \\ t_p &= \text{time at periapsis} \end{aligned}$$

Some Identities. From Fig. 10-7:

$$a\epsilon + r\cos\theta^* = a\cos E \tag{10.33}$$

It can also be shown that

$$\tan\frac{E}{2} = \left(\frac{1-\epsilon}{1+\epsilon}\right)^{\frac{1}{2}} \tan\frac{\theta^*}{2} \tag{10.34}$$

Use of Equations. These equations are of practical use in two ways:

1. Given θ^*, solve Eqn. (10.34) for E and then Eqn. (10.32) for t.

2. Given t, iteratively solve Eqn. (10.32) for E and then Eqn. (10.34) for θ^*.

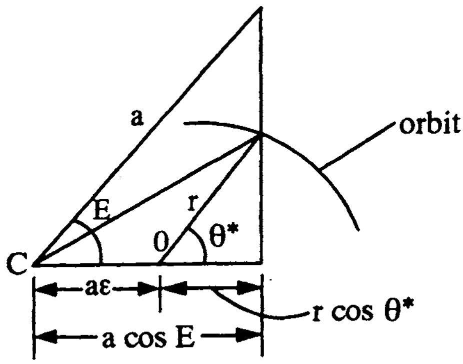


Fig. 10-7

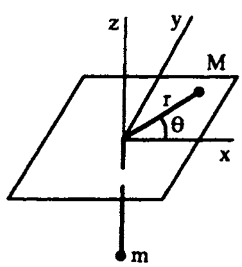
Notes

1 Kepler deduced this from observations of the orbit of Mars.

2 This will be approximately true for the motion of the planets around the sun and for satellites around the earth.

PROBLEMS

10/1. A particle of mass M is connected by an inextensible string of length ℓ and negligible mass to a particle of mass m. The string passes through a hole in a horizontal table. The particle of mass M lies on the table, and the other particle moves on a vertical line beneath the hole. Find the equations of motion if the particle has some nonzero initial angular velocity.



Problem 10/1

10/2. Same as Problem 10/1 except that the string has mass per unit length ρ.

10/3. Two particles of mass m_1 and m_2 move on a smooth horizontal plane in such a way that the line connecting their positions passes for all time through a fixed point in the plane. There are forces equal in magnitude and opposite in direction acting on the particles; these forces act along a line connecting the particles. Use Lagrange's equations to obtain the equations of motion, and find all momentum and energy integrals of the motion.

10/4. Same as Problem 10/3 except that the mutual forces on the particles are due to a linear spring of stiffness k that connects the particles.

10/5. A particle of unit mass is attached to one extremity of a linear spring, the other extremity being hinged at a fixed point in an inertial frame. Use Lagrange's equations to get the equations of motion and use constants of the motion to reduce the solution to quadratures. Show that finite orbits are closed.

10/6. A particle is attracted to a force center by a force which varies inversely as the cube of its distance from the center. Find the equations of motion using Lagrange's equations.

10/7. A particle of mass m, lying on a smooth, horizontal table, is connected to a spring under the table by means of a massless inextensible string which passes through a hole in the table. The spring resists a length change u in the amount $f(u)$, where $f(u)$ is of class C^1. When the spring is unstretched and the string is taut, the distance between the particle and the hole is l. If the particle is set into motion in any way that stretches the spring initially, find the equations of motion and reduce the solution to quadratures.

10/8. For elliptic orbits, verify the formulas

(a) $a(1 - \epsilon \cos E) = r$

(b) $\cos E = \dfrac{\epsilon + \cos\theta^*}{1 + \epsilon\cos\theta^*}$

(c) $\sin E = \dfrac{\sqrt{1-\epsilon^2}\sin\theta^*}{1 + \epsilon\cos\theta^*}$

(d) $\tan\dfrac{E}{2} = \left[\dfrac{1-\epsilon}{1+\epsilon}\right]^{\frac{1}{2}} \tan\dfrac{\theta^*}{2}$

Chapter 11

Gyroscopic Motion

11.1 Rigid Body Motion with One Point Fixed

Kinetic Energy. Consider the motion of a rigid body such that one of its points, say B, is at rest in an inertial frame (Fig. 11-1). Let $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ be body-fixed principal axes of inertia with origin at B.

The velocity of a typical mass particle of the rigid body is given by Eqn. (1.25) as

$$\underline{v}_i = \underline{v}_B + \underline{v}_{\text{rel}} + \underline{\omega} \times \underline{d}_i$$

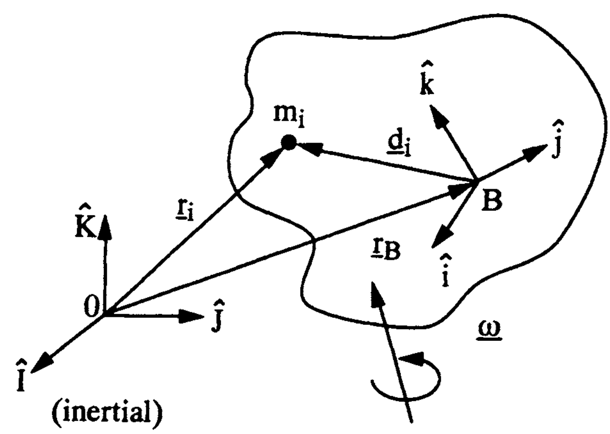


Fig. 11-1

where $\underline{v}_B = \underline{0}$, $\underline{v}_{\text{rel}} = \underline{0}$, and

$$\begin{aligned} \underline{\omega} &= \omega_x \hat{\imath} + \omega_y \hat{\jmath} + \omega_z \hat{k} \\ \underline{d}_i &= x_i \hat{\imath} + y_i \hat{\jmath} + z_i \hat{k} \end{aligned}$$

Thus

$$\begin{aligned} \underline{v}_i &= \left(\omega_x \hat{\imath} + \omega_y \hat{\jmath} + \omega_z \hat{k}\right) \times \left(x_i \hat{\imath} + y_i \hat{\jmath} + z_i \hat{k}\right) \\ &= (\omega_y z_i - \omega_z y_i)\, \hat{\imath} + (\omega_z x_i - \omega_x z_i)\, \hat{\jmath} + (\omega_x y_i - \omega_y x_i)\, \hat{k} \end{aligned} \tag{11.1}$$

Consequently, the kinetic energy of the rigid body is

$$\begin{aligned} T &= \frac{1}{2} \sum_i m_i \underline{v}_i \cdot \underline{v}_i = \frac{1}{2} \sum_i m_i \Big[\left(z_i^2 + y_i^2\right) \omega_x^2 + \left(z_i^2 + x_i^2\right) \omega_y^2 \\ &\quad + \left(y_i^2 + x_i^2\right) \omega_z^2 - 2 z_i\, y_i\, \omega_y\, \omega_z - 2 x_i\, z_i\, \omega_z\, \omega_x - 2 y_i\, x_i\, \omega_x\, \omega_y \Big] \\ &= \frac{1}{2} \left(I_x\, \omega_x^2 + I_y\, \omega_y^2 + I_z\, \omega_z^2\right) \end{aligned} \tag{11.2}$$

where the moments of inertia are

$$\begin{aligned} I_x &= \sum_i m_i (z_i^2 + y_i^2) \\ I_y &= \sum_i m_i (z_i^2 + x_i^2) \\ I_z &= \sum_i m_i (y_i^2 + x_i^2) \end{aligned} \tag{11.3}$$

Note that because $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ are principal axes, all the products of inertia are zero:

$$\begin{aligned} I_{xy} &= \sum_i m_i x_i y_i = 0 \\ I_{yz} &= \sum_i m_i y_i z_i = 0 \\ I_{zx} &= \sum_i m_i z_i x_i = 0 \end{aligned}$$

Euler's Angles. The motion of a rigid body without constraints is described by 6 generalized coordinates. Motion with one point fixed implies three holonomic constraints. Without loss of generality, choose the origin of the inertial frame as point B. Then these holonomic constraints are:

$$x = 0\,, \qquad y = 0\,, \qquad z = 0\,. \tag{11.4}$$

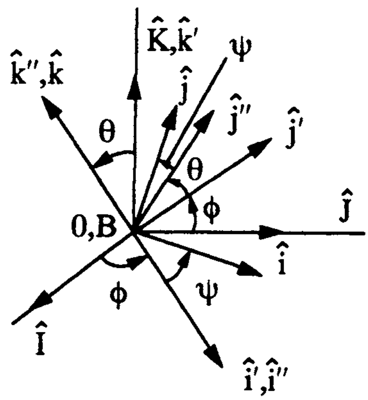


Fig. 11-2

The three generalized coordinates left are orientations or angular measurements. A common choice is the *Euler angles.*[1]

Now consider three reference frames, obtained by rotations relative to the inertial frame $\{\hat{I}, \hat{J}, \hat{K}\}$ (Fig. 11-2):

1. First rotate about $\hat{K} = \hat{k}'$ by ϕ to get $\{\hat{\imath}', \hat{\jmath}', \hat{k}'\}$.
2. Then rotate about $\hat{\imath}' = \hat{\imath}''$ by θ to get $\{\hat{\imath}'', \hat{\jmath}'', \hat{k}''\}$.
3. Finally rotate about $\hat{k}'' = \hat{k}$ by ψ to get $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$.

By a suitable choice of ϕ, θ, ψ, we can get $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ to line up with any arbitrary body-fixed axes. Thus (ϕ, θ, ψ) serve as generalized coordinates.

For the kinetic energy, we need to write ω_x, ω_y, ω_z, the components of $\underline{\omega}$, in body-fixed principal axes, in terms of $\dot{\phi}$, $\dot{\theta}, \dot{\psi}$. From Fig. 11-2, the angular velocity of $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ w.r.t. $\{\hat{I}, \hat{J}, \hat{K}\}$ is

$$\underline{\omega} = \dot{\phi}\hat{k}' + \dot{\theta}\hat{\imath}'' + \dot{\psi}\hat{k} \tag{11.5}$$

where, from Fig. 11-3,

$$\begin{aligned} \hat{\imath}'' &= \cos\psi\hat{\imath} - \sin\psi\hat{\jmath} \\ \hat{\jmath}'' &= \cos\psi\hat{\jmath} + \sin\psi\hat{\imath} \\ \hat{k}' &= \cos\theta\hat{k} + \sin\theta\hat{\jmath}'' \end{aligned} \tag{11.6}$$

Thus

$$\begin{aligned} \underline{\omega} \;=\; & \left(\dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi\right)\hat{\imath} + \left(\dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi\right)\hat{\jmath} \\ & + \left(\dot{\phi}\cos\theta + \dot{\psi}\right)\hat{k} \end{aligned} \tag{11.7}$$

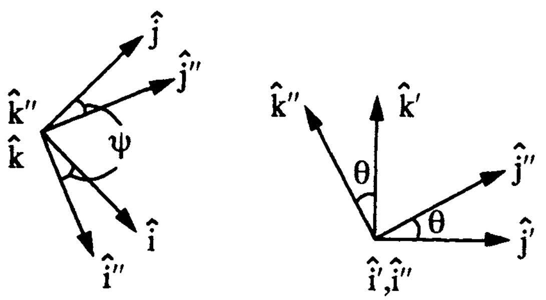


Fig. 11-3

so that

$$
\begin{aligned}
\omega_x &= \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi \\
\omega_y &= \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi \\
\omega_z &= \dot{\phi}\cos\theta + \dot{\psi}
\end{aligned}
\tag{11.8}
$$

Substituting Eqns. (11.8) into Eqn. (11.2) then gives the kinetic energy. Instead of doing this in the general case, we will consider the following special case.

11.2 Heavy Symmetrical Top

Formulation. Now suppose the rigid body is axially symmetric and is spinning about its axis of symmetry, Fig. 11-4.

Let frame $\{\hat{I}, \hat{J}, \hat{K}\}$ be inertial, and $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ be body-fixed. The angular rates in this case have been given names:

$$
\begin{aligned}
\dot{\phi} &= \text{precession} \\
\dot{\theta} &= \text{nutation} \\
\dot{\psi} &= \text{spin}
\end{aligned}
$$

Because of the symmetry, $I_x = I_y$; let

$$
I_x = I_y = I\,; \qquad I_z = J \tag{11.9}
$$

Then, from Eqns. (11.2), (11.8), and (11.9):

$$
\begin{aligned}
T &= \frac{1}{2}\left[I\left(\omega_x^2 + \omega_y^2\right) + J\omega_z^2\right] \\
&= \frac{1}{2}\left[I\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right) + J\left(\dot{\psi} + \dot{\phi}\cos\theta\right)^2\right]
\end{aligned}
\tag{11.10}
$$

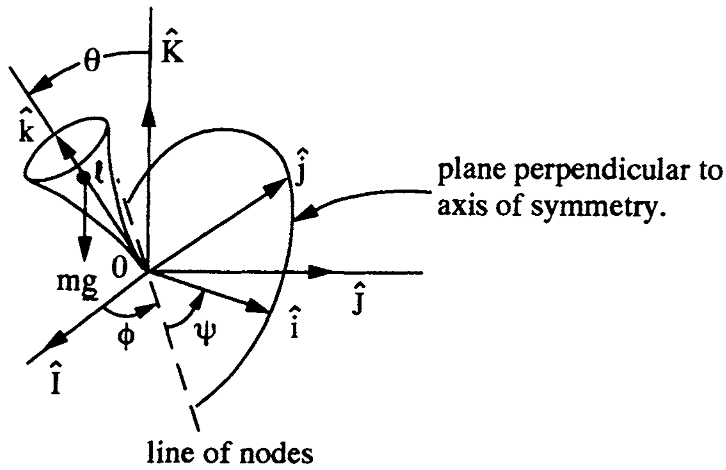


Fig. 11-4

Also,

$$V = mg\ell\cos\theta \tag{11.11}$$

Lagrange's Equations. For a natural system with three generalized coordinates, the proper form of the equations is Eqn. (6.35):

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_s}\right] - \frac{\partial L}{\partial q_s} = 0\,; \qquad s = 1,2,3 \tag{11.12}$$

Computing partials:

$$\frac{\partial L}{\partial \dot{\theta}} = I\dot{\theta}\,; \qquad \frac{\partial L}{\partial \dot{\phi}} = I\dot{\phi}\sin^2\theta + J\left(\dot{\psi} + \dot{\phi}\cos\theta\right)\cos\theta$$

$$\frac{\partial L}{\partial \dot{\psi}} = J\left(\dot{\psi} + \dot{\phi}\cos\theta\right)$$

$$\frac{\partial L}{\partial \theta} = I\dot{\phi}^2\sin\theta\cos\theta - J\left(\dot{\psi} + \dot{\phi}\cos\theta\right)\dot{\phi}\sin\theta + mg\ell\sin\theta$$

$$\frac{\partial L}{\partial \phi} = 0\,; \qquad \frac{\partial L}{\partial \psi} = 0$$

Lagrange's equations are then

$$\frac{d}{dt}\left(I\dot{\theta}\right) - I\dot{\phi}^2 \sin\theta\cos\theta + J\left(\dot{\psi} + \dot{\phi}\cos\theta\right)\dot{\phi}\sin\theta - \underbrace{mg\ell\sin\theta}_{\substack{= Q_\theta = \text{moment} \\ \text{due to gravity}}} = 0$$

$$\frac{d}{dt}\left[I\dot{\phi}\sin^2\theta + J\left(\dot{\psi} + \dot{\phi}\cos\theta\right)\cos\theta\right] = 0 \tag{11.13}$$

$$\frac{d}{dt}\left[J\left(\dot{\psi} + \dot{\phi}\cos\theta\right)\right] = 0$$

We see that ϕ and ψ are ignorable with momentum integrals

$$\begin{aligned} J\left(\dot{\psi} + \dot{\phi}\cos\theta\right) &= p_\psi = P_2 = \text{constant} \\ I\dot{\phi}\sin^2\theta + P_2\cos\theta &= p_\phi = P_1 = \text{constant} \end{aligned} \tag{11.14}$$

Solving these for $\dot{\phi}$ and $\dot{\psi}$ and substituting into the first of Eqns. (11.13) gives a second order differential equation in θ:

$$I\ddot{\theta} - \frac{P_1 - P_2\cos\theta}{I\sin^3\theta}(P_1\cos\theta - P_2) - mg\ell\sin\theta = 0 \tag{11.15}$$

with, from Eqns. (11.14),

$$\dot{\phi} = \frac{P_1 - P_2\cos\theta}{I\sin^2\theta} \qquad \dot{\psi} = \frac{P_2}{J} - \left(\frac{P_1 - P_2\cos\theta}{I\sin^2\theta}\right)\cos\theta \tag{11.16}$$

Energy Integral. Since the system is closed (catastatic and potential),

$$T + V = h = \text{constant}$$

$$\frac{1}{2}\left\{I\left[\dot{\theta}^2 + \left(\frac{P_1 - P_2\cos\theta}{I\sin^2\theta}\right)^2 \sin^2\theta\right] + \frac{P_2^2}{J}\right\} + mg\ell\cos\theta = h \tag{11.17}$$

This is a first integral of Eqn. (11.15); we now have 3 integrals of the motion. The energy integral, Eqn. (11.17), may be written in quadrature as

$$t = \int f(\theta)d\theta + \text{constant}$$

which can be inverted in principle to give $\theta = f_\theta(t)$. Substitution into Eqns. (11.16) then allows integration to get $\phi = f_\phi(t)$ and $\psi = f_\psi(t)$. The solution has been reduced to quadratures.

We now analyze the behavior of the top via "qualitative integration", that is without numerically evaluating the integrals.

Qualitative Analysis. Let

$$\begin{aligned} u &= \cos\theta\,, \qquad a = \frac{2mg\ell}{I}\,, \qquad \alpha = \frac{2h}{I} - \frac{P_2^2}{IJ} \\ \beta &= \frac{P_1}{I}\,, \qquad \gamma = \frac{P_2}{I} \end{aligned} \tag{11.18}$$

Then the energy integral, Eqn. (11.17), becomes

$$\dot{u}^2 = f(u) = (1-u^2)(\alpha - au) - (\beta - \gamma u)^2 \tag{11.19}$$

First we note that for real $\dot{u}$ we must have $f(u) \geq 0$ and that $f(u) = 0$ gives $\dot{u} = 0 \Longrightarrow u = \text{constant} \Longrightarrow \theta = \text{constant}$. The function $f(u)$ is a cubic equation with properties:

(a) $f(u) \to +\infty$ as $u \to +\infty$

$$\left\{ \begin{aligned} \lim_{u \to +\infty} f(u) &= \ell\left[(-u^2)\,(-au) - (\gamma u)^2\right] \\ &= \ell\left[au^3 - \gamma^2 u^2\right] = a\ell\left[u^3\right] \\ &= +\infty \quad \text{because } a > 0 \end{aligned} \right\}$$

(b) $f(u) \to -\infty$ as $u \to -\infty$

(c) $f(\pm 1) = -(\beta \mp \gamma)^2 \leq 0$

(d) $f(u)$ has a zero between $+1$ and $+\infty$.

(e) If $f(u)$ has three zeros, the other two must lie in $-1 \leq u \leq 1$.

Consequently, $f(u)$ looks as shown on Fig. 11-5 ($-1 \leq u \leq +1$ because $u = \cos\theta$), where u_1 and u_2 are the two zeros of $f(u)$, $-1 \leq u \leq 1$, if any exist.

First consider the special case $u_1 = u_2 = u_0$, Fig. 11-6. Now, $\ddot{\theta} = 0$; combining the first of Eqns. (11.13) with Eqns. (11.14) and (11.18) gives

$$\dot{\phi}^2\cos\theta_0 - \gamma\dot{\phi} + \frac{a}{2} = 0$$

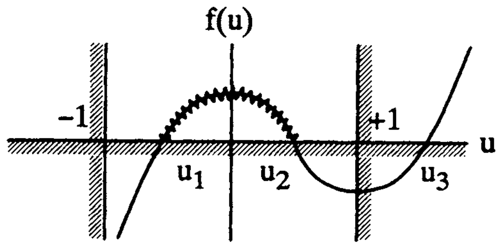


Fig. 11-5

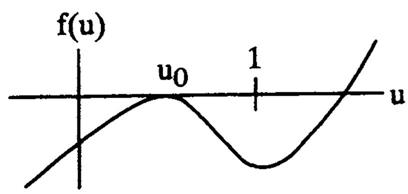


Fig. 11-6

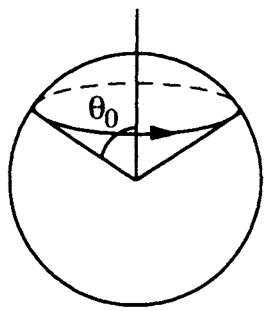


Fig. 11-7

with solution

$$\dot{\phi}_{1,2} = \frac{\gamma}{2\cos\theta_0}\left[1 \pm \left(1 - \frac{2a\cos\theta_0}{\gamma^2}\right)^{1/2}\right] \tag{11.20}$$

Provided

$$\gamma^2 \geq 2a\cos\theta_0$$

that is

$$\left(\dot{\psi} + \dot{\phi}\cos\theta_0\right)^2 \geq \frac{4Img\ell}{J^2} \tag{11.21}$$

there will be two real roots, corresponding to a fast and a slow precession. The inequality will hold when the spin $\dot{\psi}$ is sufficiently high. To visualize the motion, consider the path traced out by body axis z on a sphere (Fig. 11-7). It traces out a circle at cone angle θ_0 at rate $\dot{\phi}_1$ or $\dot{\phi}_2$. This is called steady, or regular, precession.

Another special case is $u_0 = 1(\theta = 0)$ for which the z axis stays vertical. This can be satisfied in either of two ways as shown on Fig. 11-8 (only one is stable). This is called the *sleeping* top. It takes special

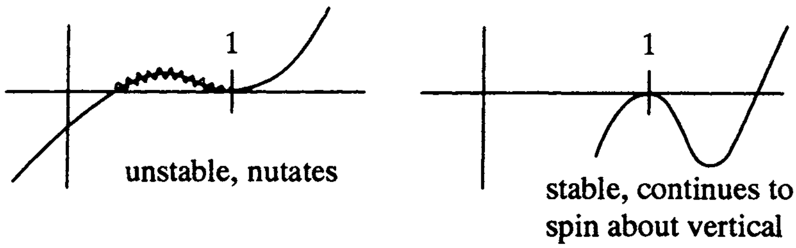


Fig. 11-8

initial conditions $\theta(0)$, $\psi(0)$, $\phi(0)$, $\dot{\theta}(0)$, $\dot{\psi}(0)$, $\dot{\phi}(0)$ to produce these special cases.

Next consider the special case of initial conditions:

$$\theta(0) = \theta_2\,, \qquad \dot{\theta}(0) = 0\,, \qquad \dot{\phi}(0) = 0.$$

That is, the top is released with no precession and no nutation at some angle θ_2. Substituing these initial conditions into Eqns. (11.14) and using Eqns. (11.18):

$$\begin{aligned} J\dot{\psi} &= P_2 = \gamma I \\ P_2 u_2 &= P_1 = \beta I \end{aligned} \tag{11.22}$$

Thus at $t = 0$ Eqn. (11.19) becomes

$$0 = \left(1 - u_2^2\right)(\alpha - a u_2)$$

so that $\alpha = a u_2$, where $u_2 = \cos\theta_2$. Then for any time,

$$\begin{aligned} f(u) &= (1 - u^2)a(u_2 - u) - \gamma^2(u_2 - u)^2 \\ &= (u_2 - u)\left[(1 - u^2)a - \gamma^2(u_2 - u)\right] \end{aligned} \tag{11.23}$$

Clearly, $u = u_2$ is one zero as expected; the other two are given by

$$(1 - u^2)a - \gamma^2(u_2 - u) = 0$$

which implies that

$$u_{1,3} = \frac{\gamma^2}{2a} \pm \left(\frac{\gamma^4}{4a^2} - \frac{\gamma^2}{2}u_2 + 1\right)^{\frac{1}{2}} \tag{11.24}$$

As mentioned previously, only the lower one is physically possible (see Fig. 11-5):

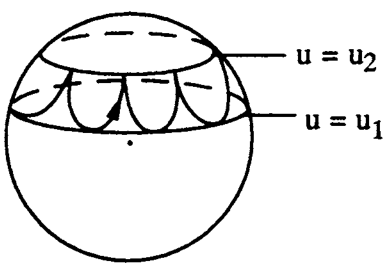
$$u_1 = \cos\theta_1 = \frac{\gamma^2}{2a} - \left(\frac{\gamma^4}{4a^2} - \frac{\gamma^2}{a}\cos\theta_2 + 1\right)^{\frac{1}{2}} \tag{11.25}$$

Therefore the z axis falls from θ_2 to θ_1 and then oscillates between the two values. From the second of Eqns. (11.14), $\dot{\phi}$ is

$$\dot{\phi} = \frac{\gamma(u_2 - u)}{1 - u^2} \tag{11.26}$$

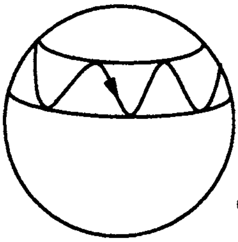
Hence both $\dot{\phi}$ and $\dot{u}$, and thus $\dot{\theta}$ as well, are zero simultaneously at $u = u_2$; geometrically, such a point is cusp. The motion is thus as follows: After release, the top falls under gravity but then begins to precess and nutate (Fig. 11-9). Essentially, the decrease in potential energy is accounted for by an increase in kinetic energy of the same amount. When $u = u_2$ is again reached, the initial state is duplicated.

More generally, if the initial conditions are $\dot{\phi}(0) \neq 0$, we get either of the two cases shown on Fig. 11-10.

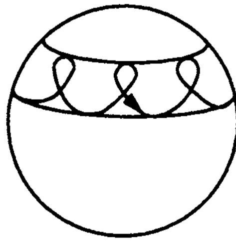


$\dot{\theta}(0) = \dot{\phi}(0) = 0$

Fig. 11-9



$\dot{\phi}(0) > 0$



$\dot{\phi}(0) < 0$

Fig. 11-10

11.3 Some Applications

Precession of the Equinoxes. Because the earth had an initial spin on its polar axis when formed and because it is an oblate spheroid (slightly flattened at the poles), it acts like a top (Fig. 11-11). The torque is due to gravitational attraction, primarily by the sun and moon, and would be zero if the earth was spherical. This torque is extremely weak and gives a processional period of 26,000 years; in 80 years the spin axis precesses 1°.

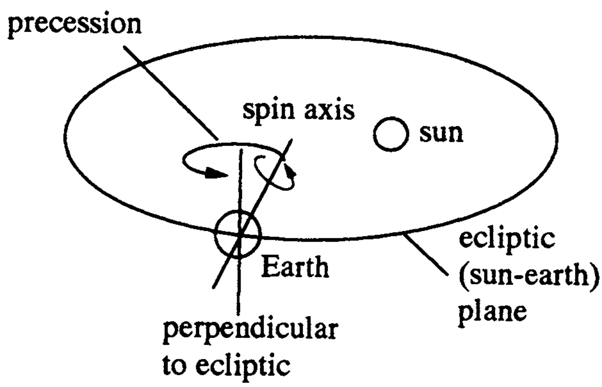


Fig. 11-11

Gyroscope. Consider now a spinning, heavy body with no gravity torque ($\ell = 0$) and constrained such that

$$\theta = \frac{\pi}{2}\,, \qquad \dot\theta = 0\,, \qquad \dot\psi = \text{constant} \tag{11.27}$$

Lagrange's equations are

$$\begin{aligned}
&\frac{d}{dt}\left[I\dot\phi\sin^2\theta + J\left(\dot\psi + \dot\phi\cos\theta\right)\cos\theta\right] - Q_\phi = 0\\
&\frac{d}{dt}\left[J\left(\dot\psi + \dot\phi\cos\theta\right)\right] - Q_\psi = 0\\
&\frac{d}{dt}\left(I\dot\theta\right) - I\dot\phi^2\sin\theta\cos\theta + J\dot\phi\left(\dot\psi - \dot\phi\cos\theta\right)\sin\theta - Q_\theta = 0
\end{aligned} \tag{11.28}$$

where the Q_i's are the components of the torque exerted by the bearings required to keep the motion as specified. Carrying out the differentiation, and using Eqns. (11.27),

$$Q_\phi = 0\,, \qquad Q_\psi = 0\,, \qquad Q_\theta = J\dot\phi\dot\psi \tag{11.29}$$

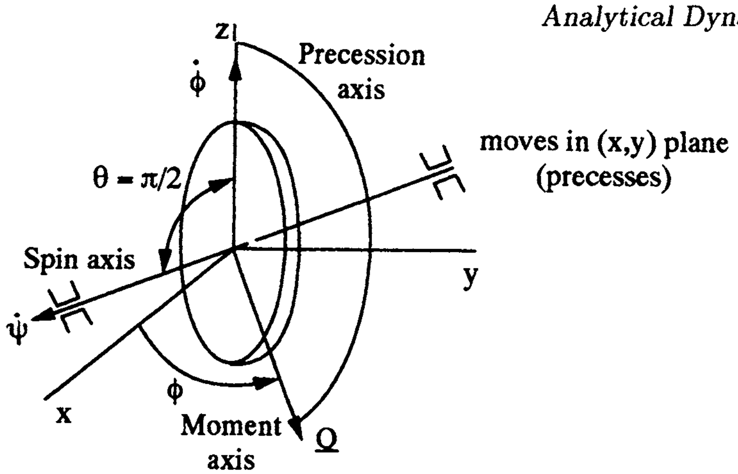


Fig. 11-12

Therefore there is a torque required in the line of nodes (Fig. 11-12), perpendicular to both the $\dot{\phi}$ and $\dot{\psi}$ axes, to keep the motion as specified.

This can be used to *detect* motion. The body is set spinning about its axis of symmetry. Then motion in a perpendicular direction can be detected by measuring moments in the bearings in the third orthogonal direction. This is one application of the gyroscope. Three such devices in perpendicular directions will detect any angular motion.

Another use of the gyroscope is as an angular reference in an *inertial* navigation system. The gyroscope is mounted in gimbals such that it nominally experiences no torque (Fig. 11-13). As the vehicle moves, the gyroscope remains fixed in orientation relative to an inertial reference frame. Therefore, measuring the orientation of the vehicle relative to that of gyroscope gives the orientation of the vehicle relative to an inertial reference frame.

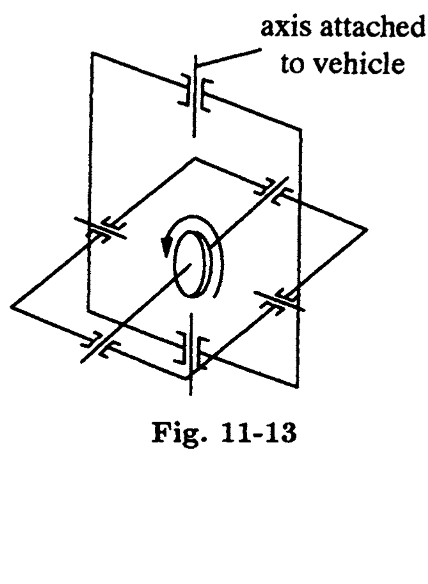


Fig. 11-13

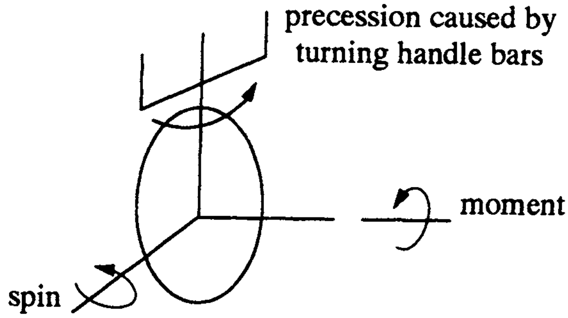


Fig. 11-14

It is very important to minimize friction in the bearings and drag on the spinning disk – these produce moments that make the gyro precess and nutate, known as drift. The latest technology is a "ring-laser" gyro which has very low friction and in which angles are measured by lasers.

Gyrocompass. This is a gyroscope fixed to the earth in such a way that the rotation of the earth causes the gyroscope to precess with a period of one day. This causes bearing torques to act in such a way that the gyroscope axis always lines up with the direction of precession, or the northernly direction.

Other Applications. Gyroscopic motion also partly explains why one can stay up on a bicycle when it's moving (Fig. 11-14) and why a football travels with a constant orientation along its path when spun and thrown[2].

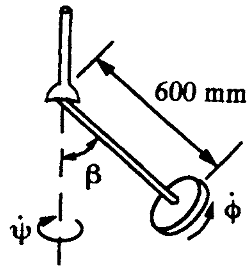
Notes

1 Other choices are the Rodriques parameters or quaternions.

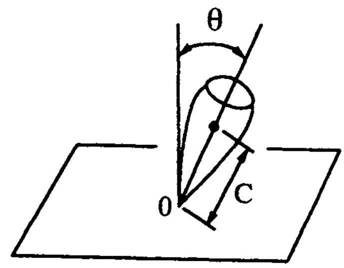
2 See Ardema, *Newton-Euler Dynamics.*

PROBLEMS

11/1. A disk of mass 2 kg and diameter 150 mm is attached to a rod AB of negligible mass to a ball-and-socket joint at A. The disk precesses at a steady rate about the vertical axis of $\dot{\psi} = 36$ rpm and the rod makes an angle of $\beta = 60°$ with the vertical. Determine the spin rate of the disk about rod AB.



Problem 11/1

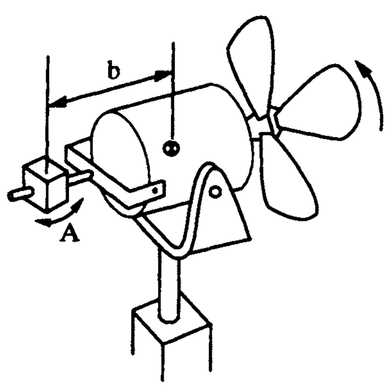


Problem 11/3

11/2. Same as Problem 11/1 except that $\beta = 30°$.

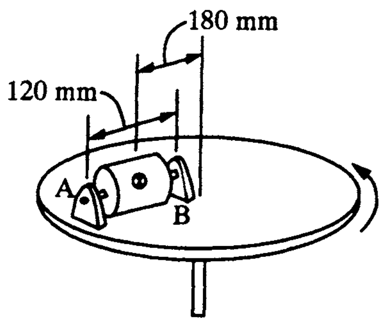
11/3. The figure shows a top weighting 3 oz. The radii of gyration of the top are 0.84 in. and 1.80 in. about the axis of symmetry and about a perpendicular axis passing through the support point 0, respectively. The length $C = 1.5$ in., the steady spin rate of the top about it's axis is 1800 rpm, and $\theta = 30°$. Determine the two possible rates of precession.

11/4. A fan is made to rotate about the vertical axis by using block A to create a moment about a horizontal axis. The parts of the fan that spin when it is turned on have a combined mass of 2.2 kg with a radius of gyration of 60 mm about the spin axis. The block A may be adjusted. With the fan turned off, the unit is balanced when $b = 180$ mm. The fan spins at a rate of 1725 rpm with the fan turned on. Find the value of b that will produce a steady precession about the vertical of 0.2 rad/s.



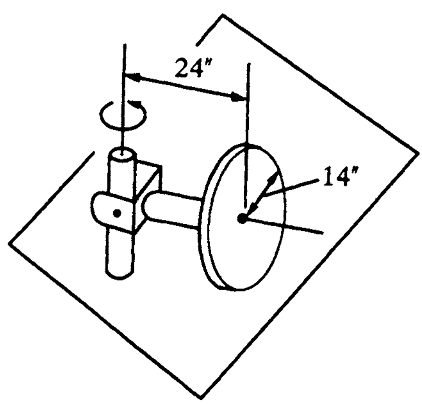
Problem 11/4

11/5. The motor shown has a total mass of 10 Kg and is attached to a rotating disk. The rotating components of the motor have a combined mass of 2.5 Kg and a radius of gyration of 35 mm. The motor rotates with a constant angular speed of 1725 rpm in a counter clockwise direction when viewed from A to B, and the turntable revolves about a vertical axis at a constant rate of 48 rpm in the direction shown. Determine the forces in the bearings A and B.



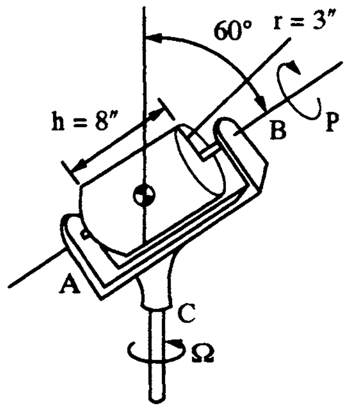
Problem 11/5

11/6. A rigid homogeneous disk of weight 96.6 lb. rolls on a horizontal plane on a circle of radius 2 ft. The steady rate of rotation about the vertical axis is 48 rpm. Determine the normal force between the wheel and the horizontal surface. Neglect the weight of all components except the disk.



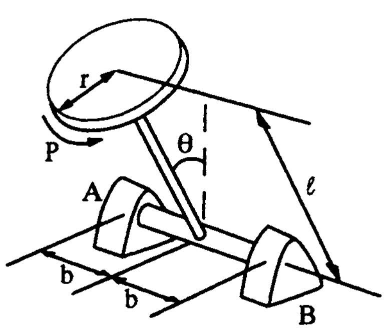
Problem 11/6

11/7. The 64.4 lb homogeneous cylinder is mounted in bearings at A and B to a bracket which rotates about a vertical axis. If the cylinder spins at steady rate $p = 50$ rad/s and the bracket at 30 rad/s, compute the moment that the assembly exerts on the shaft at C. Neglect the mass of everything except the cylinder.



Problem 11/7

11/8. A homogeneous thin disk of mass m and radius r spins on its shaft at a steady rate P. This shaft is rigidly connected to a horizontal shaft that rotates in bearings at A and B. If the assembly is released from rest at the vertical position ($\theta = 0$, $\dot{\theta} = 0$), determine the forces in the bearing at A and B as the horizontal position ($\theta = \pi/2$) is passed. Neglect the masses of all components except that of the disk.



Problem 11/8

Chapter 12

Stability Of Motion

12.1 Introduction

First Order Form of Equations of Motion. As discussed in Section 6.5, application of Lagrange's equations, Eqns. (6.29), to a dynamic system results in a system of differential equations of the form

$$\begin{aligned} &\sum_s a_{1s}\ddot{q}_s + \phi_1(q_1, \cdot\cdot, q_n, \dot{q}_1, \cdot\cdot, \dot{q}_n, t) = 0 \\ &\qquad \vdots \\ &\sum_s a_{ns}\ddot{q}_s + \phi_n(q_1, \cdot\cdot, q_n, \dot{q}_1, \cdot\cdot, \dot{q}_n, t) = 0 \end{aligned} \tag{12.1}$$

where $q_1, \cdot\cdot, q_n$ are suitable generalized coordinates and $\sum\limits_s = \sum\limits_{s=1}^{n}$. Equations (12.1) are called the mathematical model of the system. Note that: (1) These equations are linear in the acceleration components $\ddot{q}_1, \cdot\cdot, \ddot{q}_n$; (2) They are in general dynamically coupled; and (3) The matrix a_{rs} is positive definite.

In Section 8.1, the equations of motion were put into first order, generally coupled, form. We now do this by a different method that results in uncoupled equations. Because a_{rs} is positive definite, there exists a transformation to new generalized coordinates, say, $z_1, \cdot\cdot, z_n$,

$$q_r = \sum_s t_{rs} z_s \tag{12.2}$$

such that in the new variables the equations are dynamically uncoupled:

$$\begin{aligned} \ddot{z}_1 + \phi'_1(z_1, \cdot\cdot, z_n, \dot{z}_1, \cdot\cdot, \dot{z}_n, t) &= 0 \\ &\vdots \\ \ddot{z}_n + \phi'_n(z_1, \cdot\cdot, z_n, \dot{z}_1, \cdot\cdot, \dot{z}_n, t) &= 0 \end{aligned} \tag{12.3}$$

Now let

$$\begin{aligned} y_1 &= z_1 \\ &\vdots \\ y_n &= z_n \\ y_{n+1} &= \dot{z}_1 \\ &\vdots \\ y_{2n} &= \dot{z}_{2n} \end{aligned} \tag{12.4}$$

Then the equations of motion may be written as a system of $2n$ first order equations of the form

$$\begin{aligned} \dot{y}_1 &= Y_1(y_1, \cdot\cdot, y_{2n}, t) \\ &\vdots \\ \dot{y}_{2n} &= Y_{2n}(y_1, \cdot\cdot, y_{2n}, t) \end{aligned} \tag{12.5}$$

where $\underline{y} = (y_1, \cdot\cdot, y_{2n})$ is called the *state vector*.[1] The initial conditions are $\underline{y}(t_0) = (y_1(t_0), \cdot\cdot, y_{2n}(t_0))$. Equations (12.5) are said to be in *state variable form*. This is a convenient form for further analysis and computation.

Intuitive Notion of Stability. Stability has to do with the following question: Does the motion of a system stay close to the motion of some nominal (reference) motion if the conditions are somewhat perturbed? By motion, we mean the solution of Eqns. (12.5). There are generally three types of perturbations of interest:

1. *In initial conditions.* Frequently these are taken as current conditions in control applications. These perturbations may be due to sensor error or to disturbances.

2. *In parameters* (for example, mass, stiffness, or aerodynamic coefficients). This is sometimes called structural uncertainty.

3. *In the dynamic model.* Significant terms may have been neglected in formulating the equations, and there may be significant unmodeled dynamics (for example, the controller dynamics are neglected in many control problems).

Remarks.

1. The reference motion is frequently an equilibrium condition (no motion).

2. Only perturbations in initial conditions are usually considered in dynamics; all three are of importance in control system design.

3. The question of stability is usually of vital importance for a dynamic system because an unstable system is generally not usable.

4. Since any dynamic model is only an approximation of a physical system, there is always unmodeled dynamics.

Example. We investigate the stability of the motion of the harmonic oscillator, whose equation of motion is $\ddot{x} + n^2 x = 0$, when the initial conditions are perturbed. The unperturbed motion is given by

$$x = x_0 \cos nt + \frac{u_0}{n} \sin nt$$

where $x(0) = x_0$ and $\dot{x}(0) = u_0$. Let the perturbed initial conditions be $x(0) = x_0 + \eta_x$ and $\dot{x}(0) = u_0 + \eta_u$ so that the perturbed motion is

$$x' = (x_0 + \eta_x) \cos nt + \frac{(u_0 + \eta_u)}{n} \sin nt$$

The difference between the two is

$$x' - x = \eta_x \cos nt + \frac{\eta_u}{n} \sin nt$$

Since this will stay small if η_x and η_u are small, the motion is stable. Note that the perturbation in the motion does not tend to zero over time, but rather persists at a constant amplitude.

12.2 Definitions of Stability

Geometrical Representations of the Motion. Recall the representations of motion introduced in Section 2.2; in the present terms,

$\underline{z} \in C \subset \mathbb{E}^n$,	where C is the configuration space
$(\underline{z}, t) \in E \subset \mathbb{E}^{n+1}$,	where E is the event space
$\underline{y} \in S \subset \mathbb{E}^{2n}$,	where S is the state space
$(\underline{y}, t) \in T \subset \mathbb{E}^{2n+1}$,	where T is the state-time space

Liapunov Stability (L-Stability). Consider a motion (i.e. a solution of Eqns. (12.5)) $f_1(t), \cdot\cdot, f_{2n}(t)$. This motion is *L-stable* if for each $\epsilon > 0$ there exists a $\eta(\epsilon) > 0$ such that for all disturbed motions $y_1(t), \cdot\cdot, y_{2n}(t)$ with initial disturbances

$$\left| y_s(t_0) - f_s(t_0) \right| \leq \eta(\epsilon) \tag{12.6}$$

we have

$$\left| y_s(t) - f_s(t) \right| < \epsilon \tag{12.7}$$

for all t and $s = 1, \cdot\cdot, 2n$. The situation is depicted in Fig. 12-1 in T space for the case of two states for the general case and in Fig. 12-2 for

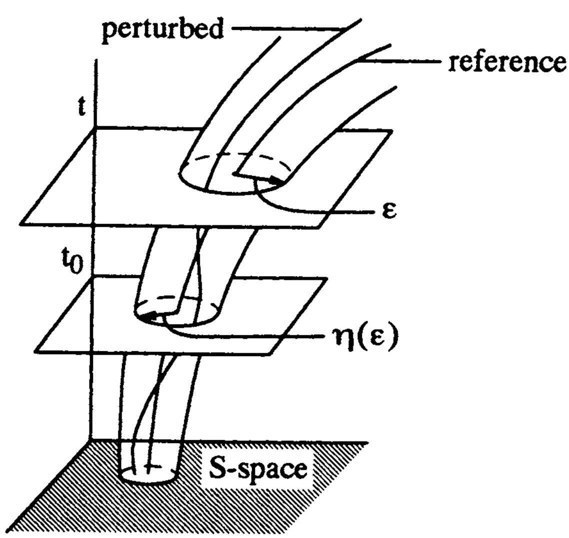


Fig. 12-1

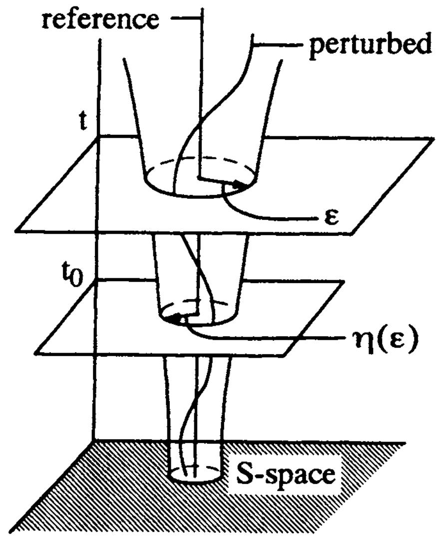


Fig. 12-2

the special case of the reference motion being equilibrium. In words, the motion is L-stable if when the perturbed motions are sufficiently close to the reference motion at some time, then they remain close thereafter.

If, further,

$$\lim_{t \to \infty} \left| y_s(t) - f_s(t) \right| = 0 \tag{12.8}$$

for all s then the motion is *asymptotically stable.*

Poincare Stability (P-Stability). In some cases, a type of stability other than L-Stability is of interest. For example, consider the motion of the harmonic oscillator expressed in the form:

$$x = A \sin(\omega t + B) \tag{12.9}$$

It is clear that this motion is L-Stable with respect to perturbations in both A and B, but is L-Unstable with respect to parameter ω (see Fig. 12-3). If the motion is plotted in the state space, however, we see that the perturbed motion remains close to the reference motion and it may be that this is all that's desired (Fig. 12-4). This motivates another definition of stability.

Consider a motion $f_1(\gamma), \cdot\cdot, f_{2n}(\gamma)$ where γ is an arc length parameter. Then the motion is *P-Stable* if for each $\epsilon > 0$ there exists a $\eta(\epsilon) > 0$ such

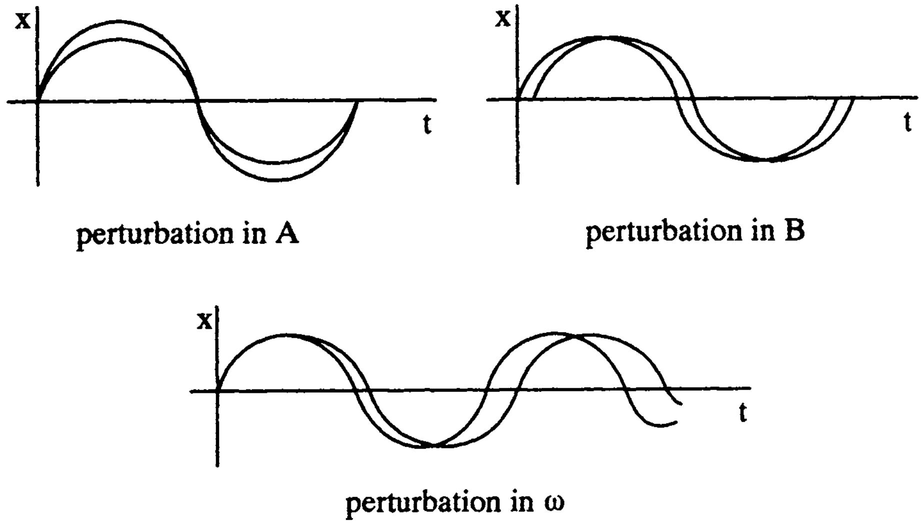


Fig. 12-3

that for all disturbed motions $y_s(\gamma)$ with initial disturbances

$$|y_s(\gamma_0) - f_s(\gamma_0)| \leq \eta(\epsilon) \tag{12.10}$$

we have

$$|y_s(\gamma) - f_s(\gamma)| < \epsilon \tag{12.11}$$

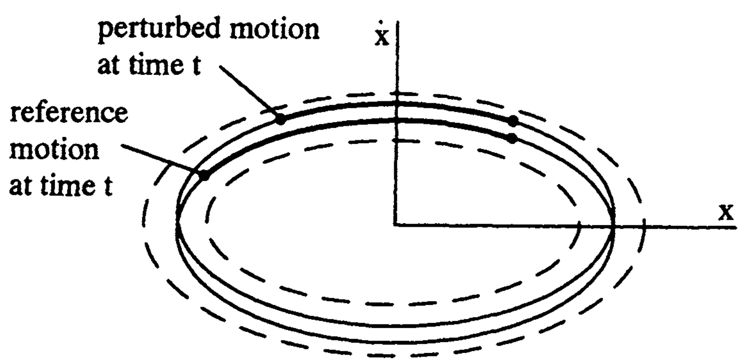


Fig. 12-4

for all γ and s. A motion that is L-Unstable and P-Stable is illustrated on Fig. 12-5 in T space. Note that if a motion is L-Stable, then it is always P-Stable, but not necessarily conversely.

Poincare stability is sometimes called *orbital stability*, for obvious reasons.

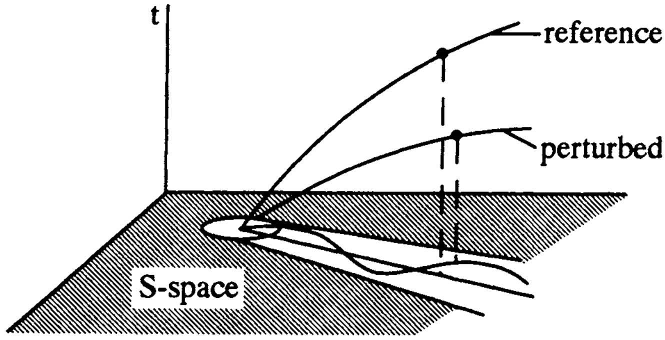


Fig. 12-5

12.3 Indirect Methods

Introductory Remarks. The stability properties of linear systems are well-known as compared with the stability properties of nonlinear systems. This observation suggests the following procedure. The nonlinear system is approximated by *linearizing* about a reference motion, that is, by expanding the disturbed motion in a Taylor series in the perturbations and retaining only the first (linear) terms. The stability of the linear system is then investigated.

There are two potential dangers in this approach:

1. If the disturbances become "large", the first order terms no longer dominate and the approximation is not valid. What constitutes "large", unfortunately, is not usually known.

2. In some exceptional cases, stability of the linear system does not guarantee stability of the nonlinear system, no matter how small the disturbances.

Variational Equations. Let the disturbed motion be equal to the reference motion plus a perturbation:

$$y_s(t) = f_s(t) + \eta_s(t) \; ; \quad s = 1, \cdot\cdot, 2n \tag{12.12}$$

Substitute this into Eqns. (12.5) and expand in a Taylor's series in the perturbations:

$$\dot{f}_s + \dot{\eta}_s \quad = \quad Y_s(f_1 + \eta_1, \cdot\cdot, f_{2n} + \eta_{2n}, t) \; ;$$

$$= \quad Y_s(f_1, \cdot\cdot, f_{2n}) + \sum_{r=1}^{2n} a_{sr}\eta_r$$

$$+\text{nonlinear terms in the } \eta_r; \; s = 1, \cdot\cdot, 2n \qquad (12.13)$$

where $a_{sr} = \dfrac{\partial Y_s}{\partial y_r}$ evaluated at $y_r = f_r$. But $\dot{f}_s = Y_s(f_1, \cdot\cdot, f_{2n}, t)$ because $f_s(t)$ is a motion; therefore, neglecting the nonlinear terms in Eqns. (12.13) gives

$$\dot{\eta}_s = \sum_{r=1}^{2n} a_{sr}\eta_r \; ; \quad s = 1, \cdot\cdot, 2n \qquad (12.14)$$

In general, the a_{sr} will be explicit functions of time. If, however, the reference motion is an equilibrium position ($\dot{f}_s = 0$ for all s, which implies that all velocities and accelerations are zero) and if the functions Y_s do not depend explicitly on time, then the a_{sr} do not depend on time and Eqns. (12.14) are a time-invarient linear system, the stability properties of which are well-known and easily stated.

Stability of Time-Invariant Linear Systems. The key results will be stated without proof. The *characteristic equation* associated with Eqns. (12.14) can be obtained by taking Laplace transforms or by substituting $\eta_1 = A_1 e^{st}, \cdot\cdot, \eta_{2n} = A_{2n}e^{st}$; the result is

$$|a_{sr} - \lambda I_{sr}| = 0 \qquad (12.15)$$

where I_{sr} is the identity matrix. The $2n$ roots of Eqn. (12.15) are called the *eigenvalues* of a_{sr} and their signs determine the stability of Eqns. (12.14) as follows:

(1) If all roots λ_s ; $s = 1, \cdot\cdot, 2n$ have negative real parts, Eqns. (12.14) are asymptotically stable.

(2) If one or more root has a positive real part, the equations are unstable.

(3) If all roots are distinct and some roots have zero and some negative real parts, the equations are stable but not asymptotically stable.

The characteristic equation, Eqn. (12.15) is a polynomial equation of order $2n$. Criteria, called the *Routh-Hurwitz Criteria*, have been developed to determine the stability of a system directly from the coefficients of the system's characteristic equation. This will not be pursued here.

Example – Double Plane Pendulum. Consider a double pendulum with both links of length ℓ and both bobs of mass m (Fig. 12-6). The masses of the links are negligible. In this case, Eqns. (7.25) reduce to

$$\begin{aligned} &2\ddot{\theta}_1 + \cos(\theta_1 - \theta_2)\ddot{\theta}_2 + \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) + 2\frac{g}{\ell}\sin\theta_1 = 0 \\ &\ddot{\theta}_2 + \cos(\theta_1 - \theta_2)\ddot{\theta}_1 - \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) + \frac{g}{\ell}\sin\theta_2 = 0 \end{aligned} \qquad (12.16)$$

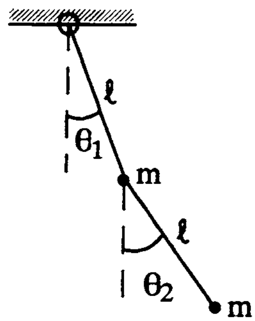


Fig. 12-6

We investigate the stability of the equilibrium positions; these are defined by $\dot{\theta}_1^* = \dot{\theta}_2^* = \ddot{\theta}_1^* = \ddot{\theta}_2^* = 0$. Substitution into Eqns. (12.16) gives $\sin\theta_1^* = 0$ and $\sin\theta_2^* = 0$ so that the four equilibrium positions are given by the combinations of (Fig. 12-7):

$$\theta_1^* = 0°, 180° \; ; \quad \theta_2^* = 0°, 180°$$

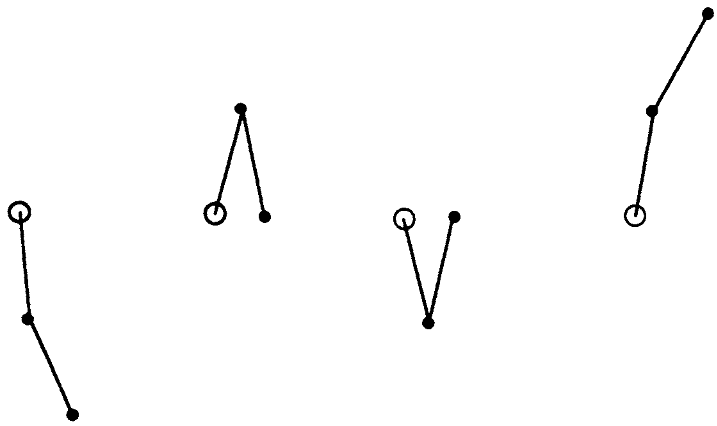
We investigate the stability of these positions by using the rules in the previous section.

First consider equilibrium position (a), $\theta_1^* = \theta_2^* = 0$. Let

$$\theta_1 = \eta_1, \quad \dot{\theta}_1 = \eta_2, \quad \theta_2 = \eta_3, \quad \dot{\theta}_2 = \eta_4$$

where the η_i are small perturbations from equilibrium. Substitution into Eqns. (12.16) and retaining only the first order terms gives:

$$\begin{aligned} &\dot{\eta}_1 = \eta_2 \\ &2\dot{\eta}_2 + \dot{\eta}_4 = -2\frac{g}{\ell}\eta_1 \\ &\dot{\eta}_3 = \eta_4 \\ &\dot{\eta}_4 + \dot{\eta}_2 = -\frac{g}{\ell}\eta_3 \end{aligned}$$



a) $\theta_1^* = 0$, $\theta_2^* = 0$ b) $\theta_1^* = 180°$, $\theta_2^* = 0$ c) $\theta_1^* = 0$, $\theta_2^* = 180°$ d) $\theta_1^* = 180°$, $\theta_2^* = 180°$

Fig. 12-7

(Note that these are not in state variable form.) The eigenvalues of the coefficient matrix of this system are

$$\lambda_{1,2,3,4} = \pm\sqrt{\frac{g}{\ell}(-2 \pm \sqrt{2})}$$

Since $g/\ell > 0$, all four of these eigenvalues have only imaginary parts (Fig. 12-8). Thus this is case (3) above and the equilibrium is stable but not asymptotically stable.

Next consider position (b), $\theta_1^* = \pi$, $\theta_2^* = 0$; proceeding as before,

$$\theta_1 = \pi + \eta_1\ , \quad \dot{\theta}_1 = \eta_2\ , \quad \theta_2 = \eta_3\ , \quad \dot{\theta}_2 = \eta_4$$

$$\begin{aligned} &\dot{\eta}_1 = \eta_2 \\ &2\dot{\eta}_2 + \dot{\eta}_4 = 2\frac{g}{\ell}\eta_1 \\ &\dot{\eta}_3 = \eta_4 \\ &\dot{\eta}_4 + \dot{\eta}_2 = \frac{g}{\ell}\eta_3 \end{aligned}$$

$$\lambda_{1,2,3,4} = \pm\sqrt{\frac{g}{\ell}(2 \pm \sqrt{2})}$$

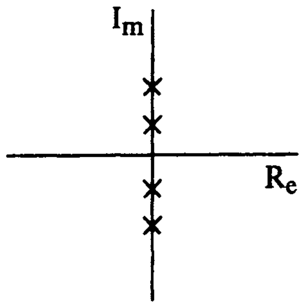


Fig. 12-8

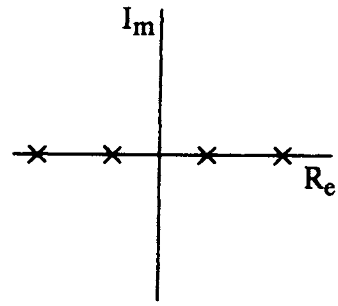


Fig. 12-9

The eigenvalues thus have only real parts and two of them are positive (Fig. 12-9). This is case (2) above and the equilibrium position is unstable.

Equilibrium positions (c) and (d) are also unstable.

Some Simple Examples. Here and in the next section we consider motions in which the reference motion is not an equilibrium position. First, consider the motion of a particle moving vertically near the surface of the earth. The equation of motion is $\ddot{x} = -g$, with x measured upwards from the earth surface. Letting $y = \dot{x}$, the solution with initial conditions $x(0) = x_0$ and $y(0) = y_0$ is

$$x = x_0 + y_0 t - \frac{1}{2}gt^2 , \qquad y = y_0 - gt$$

Suppose the initial conditions are now perturbed so that $x(0) = x_0 + \eta_x$ and $y(0) = y_0 + \eta_y$; then the perturbed motion is given by

$$x' = x_0 + \eta_x + (y_0 + \eta_y)t - \frac{1}{2}gt^2$$

$$y' = y_0 + \eta_y - gt$$

The difference between the two motions is:

$$x' - x = \eta_x + \eta_y t , \qquad y' - y = \eta_y$$

Thus the difference between the reference and the perturbed motion grows with time and the motion is not L-stable.

As a second example, consider the motion defined by the system

$$\dot{x} = -x - y + \frac{ax}{\sqrt{x^2 + y^2}}$$

$$\dot{y} = x - y + \frac{ay}{\sqrt{x^2 + y^2}}$$

Transforming to polar coordinates, the system equations are

$$\dot{r} = a - r , \qquad \dot{\theta} = 1$$

With initial conditions $r(0) = r_0$ and $\theta(0) = \theta_0$, the solution is

$$r = (r_0 - a)e^{-t} + a , \qquad \theta = \theta_0 + t$$

Now suppose the initial conditions are perturbed, $r(0) = r_0 + \eta_r$ and $\theta(0) = \theta_0 + \eta_\theta$. Then the perturbed motion is

$$r' = (r_0 + \eta_r - a)e^{-t} + a$$

$$\theta' = \theta_0 + \eta_\theta + t$$

The difference between the two is

$$r' - r = \eta_r e^{-t} , \qquad \theta' - \theta = \eta_\theta$$

Therefore the perturbation in the motion stays small if η_r and η_θ are small and the system is L-stable. Note that $r' - r \to 0$ as $t \to \infty$ but that $\theta' - \theta$ remains constant. Thus the stability is not asymptotic.

As a third example, consider the motion defined by

$$\dot{x} = -y\sqrt{x^2 + y^2}$$

$$\dot{y} = x\sqrt{x^2 + y^2}$$

Take the initial conditions, without loss of generality, to be

$$x(0) = a \cos \alpha , \qquad y(0) = a \sin \alpha$$

Then the motion is

$$x = a \cos(at + \alpha)$$

$$y = a \sin(at + \alpha)$$

The motion therefore describes a circle in the (x, y) plane with radius a and period $2\pi/a$. Now let the initial conditions be perturbed to $a + \eta_a$ and $\alpha + \eta_\alpha$; then the perturbed motion is

$$x' = (a + \eta_a) \cos[(a + \eta_a)t + \alpha + \eta_\alpha]$$

$$y' = (a + \eta_a) \sin[(a + \eta_a)t + \alpha + \eta_\alpha]$$

Because the period has been changed, the system is not L-stable but is P-stable. The situation is similar to that shown on Fig. 12-4.

12.4 Stability of Orbits in a Gravitational Field

Here we consider the important problem of the stability of closed orbits (i.e. elliptical and circular orbits) of a body moving in a central gravitational field. For specificity, the case of a satellite or space craft in a nominally circular earth orbit will be discussed. The perturbation equations, derived in most books on orbital dynamics, are

$$\begin{aligned} &\ddot{x} - 2n\dot{y} - 3n^2 x = 0 \\ &\ddot{y} + 2n\dot{x} = 0 \\ &\ddot{z} + n^2 z = 0 \end{aligned} \tag{12.17}$$

where $n = \sqrt{\mu/a^3}$ is called the mean motion and μ and a are the earth's gravitational parameter and the radius of the nominal orbit, respectively. Eqns. (12.17) are called in various places the Hill, the Euler-Hill, or the Clohessy-Wiltshire equations. They have found wide application in orbital dynamics; for example they are used in the analysis of rendezvous between two spacecraft in neighboring circular orbits, docking maneuvers between two spacecraft, and orbital station-keeping.

The solution of Eqns. (12.17) is

$$\begin{aligned} x(t) &= 2\left(\frac{\dot{y}_0}{n} + 2x_0\right) - \left(2\frac{\dot{y}_0}{n} + 3x_0\right)\cos nt + \frac{\dot{x}_0}{n}\sin nt \\ y(t) &= y_0 - 2\frac{\dot{x}_0}{n} - 3(\dot{y}_0 + 2nx_0)t + 2\frac{\dot{x}_0}{n}\cos nt \\ &\quad +2\left(2\frac{\dot{y}_0}{n} + 3x_0\right)\sin nt \\ z(t) &= z_0 \cos nt + \frac{\dot{z}_0}{n}\sin nt \end{aligned} \tag{12.18}$$

In these equations, x_0, y_0, z_0, $\dot{x}_0$, $\dot{y}_0$, $\dot{z}_0$ are the perturbations in position and velocity components at time $t = 0$ from a nominal circular orbit, relative to a frame that travels with the orbit (Fig. 12-10), and $x(t)$, $y(t)$, $z(t)$ are the perturbations at some later time t.

Inspection of Eqns. (12.18) shows the following. First, the motion perpendicular to the orbital plane, $z(t)$, is uncoupled from the in-plane motion, and this component of the motion is L-stable but not asymptotically so. Second, because of the term linear in t in the $y(t)$ equation, the in-plane perturbations are not generally bounded and the motion is not L-stable. The radial component, however, is bounded and thus the

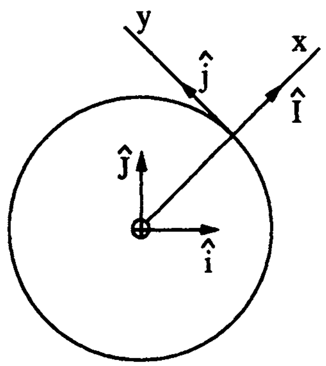


Fig. 12-10

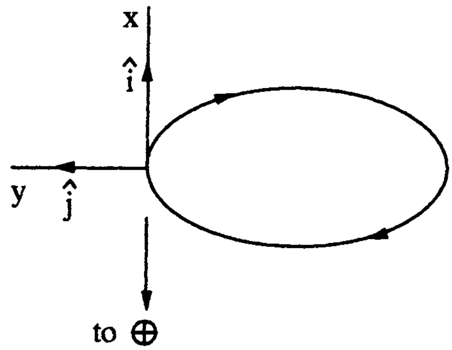


Fig. 12-11

perturbed (elliptical) orbit remains close to the nominal circular one, with an object in the perturbed orbit either pulling ahead or falling back relative to an object in the nominal one. Again the situation is similar to that shown on Fig. 12-4 and the motion is P-stable.

As a simple example, suppose a spacecraft in a circular earth orbit ejects a particle of small mass in the outward radial direction. In this case the initial perturbations are

$$x_0 = y_0 = z_0 = \dot{y}_0 = \dot{z}_0 = 0\,, \quad \dot{x}_0 > 0$$

and Eqns. (12.18) become

$$\begin{aligned} x(t) &= \frac{\dot{x}_0}{n} \sin nt \\ y(t) &= \frac{2\dot{x}_0}{n} \cos nt - \frac{2\dot{x}_0}{n} \\ z(t) &= 0 \end{aligned}$$

These are the equations of an ellipse with semi-major and semi-minor axes of $2\dot{x}_0/n$ and $\dot{x}_0/n$, respectively. Thus, relative to the spacecraft the particle travels in a elliptical orbit (Fig. 12-11) and arrives back at the spacecraft at the time at which the spacecraft has completed one revolution of the earth. In an inertial frame, the particle travels around the earth in an ellipse neighboring the circular orbit with the same period, rendezvousing with the circular orbit after each revolution. This motion is clearly L-stable.

12.5 Liapunov's Direct Method

Autonomous Case. Let $\eta_s(t)$; $s = 1, \cdot\cdot, 2n$ be a perturbation from equilibrium of a dynamical system. Then these functions satisfy equations of the form

$$\dot{\eta}_s = g_s(\eta_1, \cdot\cdot, \eta_{2n}) \; ; \quad s = 1, \cdot\cdot, 2n \tag{12.19}$$

In the autonomous case, these functions do not depend explicitly on t.

We define the following classes of functions:

1. If $V(\eta_1, \cdot\cdot, \eta_{2n})$ is of class C^1 (i.e. continuous with continuous derivatives) in an open region $\Omega \subset \mathbb{E}^{2n}$ containing the origin, if $V(0, \cdot\cdot, 0) = 0$, and if $V(\cdot)$ has the same sign everywhere in Ω except at the origin, then $V(\cdot)$ is called *definite* in Ω.

2. If $V(\cdot)$ is positive everywhere except at the origin, it is *positive definite* (Fig. 12-12); if negative everywhere except at the origin, it is *negative definite* (Fig. 12-13).

3. If $V(\cdot)$ has the same sign everywhere where it is not zero, but it can be zero other than at the origin, it is called *semidefinite.* Figure 12-14 shows a positive semidefinite function.

These definitions are generalizations of the idea of positive definite and negative definite quadratic forms.

Now let the η_s in these definitions be solutions of Eqns. (12.19); then

$$\frac{dV}{dt} = \sum_{s=1}^{2n} \frac{\partial V}{\partial \eta_s} \dot{\eta}_s = \sum_{s=1}^{2n} \frac{\partial V}{\partial \eta_s} g_s \tag{12.20}$$

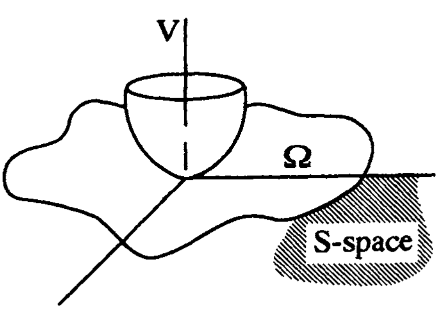


Fig. 12-12

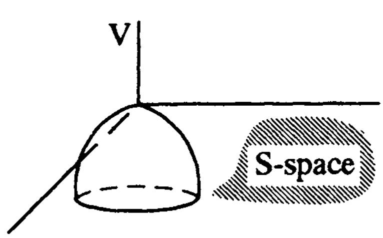


Fig. 12-13

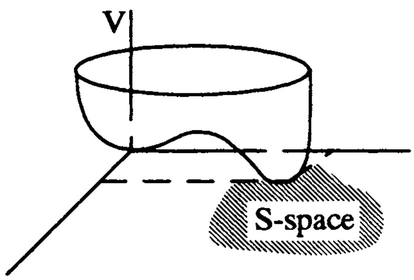


Fig. 12-14

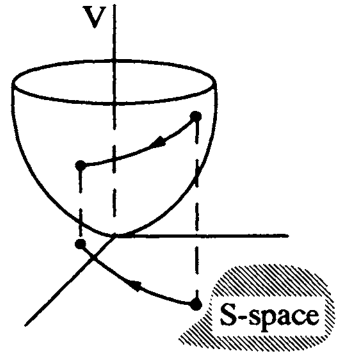


Fig. 12-15

We now define a *Liapunov function* as a function $V(\cdot)$ in Ω definite in sign for which dV/dt is semidefinite and opposite in sign to $V(\cdot)$, or $dV/dt = 0$. The key result is then the following.

Liapunov's Theorem. If a Liapunov function can be constructed for Eqns. (12.19), then the equilibrium position is stable. A geometric proof of the theorem follows from Fig. 12-15. The properties of $V(\cdot)$ ensure that the motions due to small perturbations from equilibrium either tend to zero or remain small.

Application to Dynamics. Consider a natural (holonomic, scleronomic, conservative) system. Lagrange's equations for such a system are[2]

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_s}\right) - \frac{\partial T}{\partial q_s} + \frac{\partial V}{\partial q_s} = 0 \,; \quad s = 1, \cdot\cdot, n \tag{12.21}$$

Because the system is scleronomic, the transformation equations to generalized coordinates are time independent, and therefore, from Eqn. (6.2) the equilibrium condition is $T = 0$. The equations defining the equilibrium condition are thus:

$$\frac{\partial V}{\partial q_s} = 0 \,; \quad s = 1, \cdot\cdot, n \tag{12.22}$$

This equation is a necessary condition for an unbounded extremal point of V; that is, V has a stationary value at an equilibrium point. Also, since the system is closed, energy is conserved:

$$E = T + V = h \tag{12.23}$$

Dirichlet's Stability Theorem. An equilibrium solution of the equations of motion for the class of systems defined above is stable if the stationary value of the potential energy is a minimum relative to neighboring points.

To prove this theorem, it suffices to show that E is a Liapunov function. Let $q_1^*, \cdots, q_n^*$ be the equilibrium values, and let $\eta_1, \cdot\cdot, \eta_n$ be small perturbations from these values; that is $q_r = q_r^* + \eta_r$; $r = 1, \cdot\cdot, n$. Then $V = V(\eta_1, \cdot\cdot, \eta_n)$. Choose the datum for V such that $V(0, \cdot\cdot, 0) = 0$. Then, since V has a minimum at $\eta_1 = 0, \cdot\cdot, \eta_n = 0$, V is positive at neighboring points and V is a positive definite function. From Section 6.1, T is always positive definite so that $E = T + V$ is positive definite. Also, $E = h =$ const. implies that $dE/dt = 0$. Consequently, E is a Liapunov function and by Liapunov's Theorem the equilibrium is stable.

Example. Consider again the double pendulum (Fig. 12-6), for which

$$V = 2mg\ell(1 - \cos\theta_1) + mg\ell(1 - \cos\theta_2)$$

Consider the equilibrium position (a), $\theta_1^* = \theta_2^* = 0$. We see that: (i) $V(0, 0) = 0$ and (ii) $V(\theta_1, \theta_2) > 0$ for all sufficiently small θ_1 and θ_2. Thus V is positive definite. Since the system is closed, energy is conserved and $dE/dt = 0$. Consequently, E is a Liapunov function and the equilibrium is stable.

Remark. In all but the simplest problems, there is no systematic procedure for finding Liapunov functions and they are generally very difficult to find.

Nonantonomous Case. In this case, one or more of the perturbation equations contains time explicitly:

$$\dot{\eta}_s = g_s(\eta_1, \cdot\cdot, \eta_{2n}, t)\ ; \quad s = 1, \cdot\cdot, 2n \tag{12.24}$$

We now need to introduce two functions $V(\eta_1, \cdot\cdot, \eta_{2n}, t)$ and $W(\eta_1, \cdot\cdot, \eta_{2n})$ such that (i) they vanish at $(\eta_1, \cdot\cdot, \eta_{2n}) = (0, \cdot\cdot, 0)$, (ii) they are single-valued and of class C^1 in Ω, and (iii) $W(\cdot)$ is positive definite. Then:

(1) If $V(\cdot) \geq W(\cdot)$ for all $(\eta_1, \cdot\cdot, \eta_{2n}) \in \Omega$, then $V(\cdot)$ is *positive definite.*

(2) If $V(\cdot) \leq W(\cdot)$ for all $(\eta_1, \cdot\cdot, \eta_{2n}) \in \Omega$, then $V(\cdot)$ is *negative definite.* (12.25)

The change in $V(\cdot)$ along a trajectory is now

$$\frac{dV}{dt} = \sum_{s=1}^{2n} \frac{\partial V}{\partial \eta_s} g_s + \frac{\partial V}{\partial t} \tag{12.26}$$

For this case, we call a function $V(\cdot)$ a *Liapunov function* if it is definite in sign in accordance with definitions (12.25) and dV/dt as given in Eqn. (12.26) is semidefinite with opposite sign of $V(\cdot)$.

Liapunov's Theorem (nonantonomous case). If there exists a Liapunov function for Eqns. (12.24), the reference motion is stable.

The proof of this theorem is similar to that for the autonomous case. If the conditions of the theorem are satisfied, the function $V(\cdot)$ will stay "completely inside" the function $W(\cdot)$ (Fig. 12-16), ensuring that motion will tend to zero or remain small. It is clear that this is a much stronger requirement than for the autonomous case, and that Liapunov functions will be even more difficult to find.

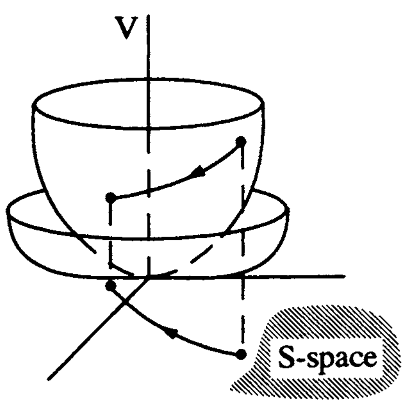


Fig. 12-16

Notes

1 More precisely, $\underline{y} = (y_1, \cdot\cdot, y_{2n})^T$ where T denotes transpose.

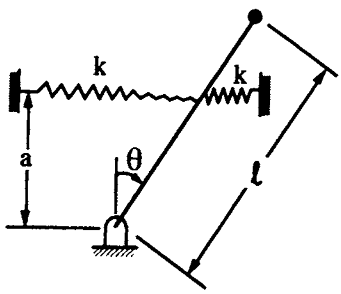
2 Caution: We are using the same symbol, V, for two different functions, Liapunov and potential energy functions.

PROBLEMS

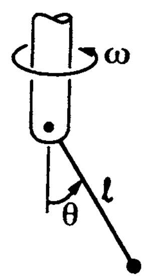
12/1. Investigate the stability of equilibrium position (c) of the double plane pendulum by using linearized equations.

12/2. Investigate the stability of equilibrium position (d) of the double plane pendulum by using linearized equations.

12/3. A heavy, inverted pendulum of mass m is restrained by identical linear springs, as shown. The rigid rod has negligible mass. Examine the stability of small motions about the inverted, vertical position by means of the linearized variational equations.



Problem 12/3



Problem 12/5

12/4. The torque-free motion of a rigid body about a point is given by

$$I_x\dot{\omega}_x - (I_y - I_z)\omega_y\omega_z = 0 \,,$$
$$I_y\dot{\omega}_y - (I_z - I_x)\omega_z\omega_x = 0 \,,$$
$$I_z\dot{\omega}_z - (I_x - I_y)\omega_x\omega_y = 0 \,.$$

Use the linearized variational equations to examine the stability of the steady-state rotation $\omega_z = \Omega = \text{const}$, $\omega_y = \omega_z = 0$. In particular, show that the motion is unstable if I_x is intermediate in magnitude between I_y and I_z.

12/5. A heavy pendulum of mass m rotates with constant angular velocity about the vertical, as shown. The rigid rod has negligible mass. Show that there exist three steady-state motions, for one of which the pendulum angle with the vertical is a non zero constant, and examine the stability of all three steady motions by means of the linearized variational equations.

12/6. Find a Liapunov function for the system of Problem 12/3 thus verifying the results of the linear analysis.

12/7. Investigate the functions

$$I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2$$

and

$$I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2 - I_x\Omega^2$$

as Liapunov functions for the motion of Problem 12/4.

12/8. Investigate the function

$$V = \dot{\theta}^2 + \omega^2\left[\cos^2\theta + 2\alpha(1-\cos\theta)\right]$$

as a Liapunov function for the steady motion with θ not zero of Problem 12/5. If this is not an L-function, can you find one?

12/9. Show that the equilibrium point $\theta = -\frac{\pi}{2}$ of Problem 7/3 is stable by finding a Liapunov function.

12/10. Show, by both the indirect and direct methods, that the equilibrium position of Problem 4/2 is stable.

Chapter 13

Impulsive Motion

13.1 Definitions and Fundamental Equation

Impulsive Force. Until now, we have considered only forces that are everywhere bounded, that is, the strictly Newtonian problem. Now we relax this restriction and consider impulsive forces, that is, the Newtonian problem. An impulsive force is a force that tends to infinity at an isolated instant, say t_j, such that its time integral remains bounded:

$$\lim_{t \to t_j} \int_{t_j}^{t} F^r(\underline{x}^s, \dot{\underline{x}}^s, \tau)d\tau = P^r \tag{13.1}$$

We call P^r the impulse of force F^r at time t_j. In this equation, $\underline{x}^s = (x_1^s, x_2^s, x_3^s)$ denotes the position vector of particle s in an inertial frame, and F^r is the r^{th} component of the impulsive force on particle r. Mathematically, an impulse is an example of what is called a *distribution*; that is, it is a limit of functions that is not itself a function (Fig. 13-1). Distri-

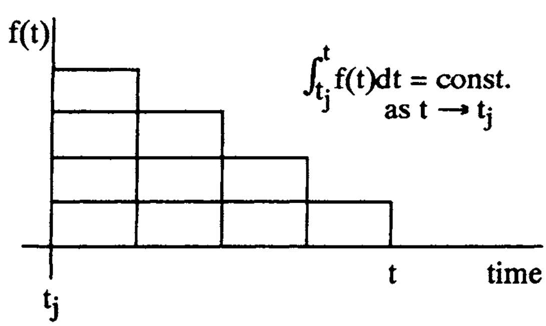


Fig. 13-1

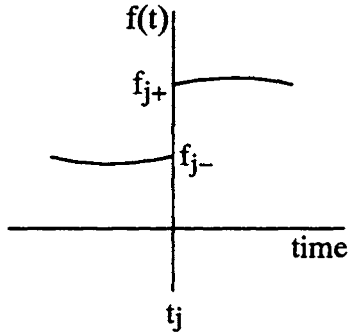


Fig. 13-2

butions have some, but not all, of the properties of functions. Physically, an impulsive force is a very large force that acts over a very short period of time.

If a function f has a discontinuity at $t = t_j$, we write (see Fig. 13-2):

$$\begin{aligned} \lim_{0<\tau_1\to 0} f(t_j - \tau_1) &= f_{j-} \\ \lim_{0<\tau_2\to 0} f(t_j + \tau_2) &= f_{j+} \end{aligned} \tag{13.2}$$

Impulse-Momentum Relationships. Newton's Second Law for particle r is

$$m_r \ddot{\underline{x}}^r = \sum \underline{F}^r \tag{13.3}$$

Since m_r is a constant, this may be integrated once to give

$$m_r \left(\dot{\underline{x}}^r(t) - \dot{\underline{x}}^r(t_0)\right) = \int_{t_0}^{t} \sum \underline{F}^r d\tau \tag{13.4}$$

which states that the change in linear momentum of the particle equals the impulse of the force acting on the particle.

Let an impulsive force act on the particle at time t_j with impulse components

$$\underline{P}^r(t_j) = \begin{pmatrix} P_1^r(t_j) \\ P_2^r(t_j) \\ P_3^r(t_j) \end{pmatrix}$$

Then from Eqn. (13.4),

$$
\begin{aligned}
m_r\left(\dot{x}^r_{1_{j+}} - \dot{x}^r_{1_{j-}}\right) &= P^r_1 \\
m_r\left(\dot{x}^r_{2_{j+}} - \dot{x}^r_{2_{j-}}\right) &= P^r_2 \\
m_r\left(\dot{x}^r_{3_{j+}} - \dot{x}^r_{3_{j-}}\right) &= P^r_3
\end{aligned}
\tag{13.5}
$$

We see that the velocity components are generally *discontinuous* at an impulse, but that the position components are continuous there; this is the fundamental characteristic of an impulse.

The motion proceeds according to Newton's Second Law until time t_j, where there is continuity of displacement and a discontinuity of velocity given by Eqn. (13.5). This equation thus provides the initial conditions for the subsequent motion, which is again governed by Newton's Law. The main goal of the analysis of impulsive forces is to determine the values of the velocity components after an impulse.

Fundamental Equation. Adding up Eqns. (13.4) for all n particles;[1]

$$\sum_{r=1}^{n}\left[m_r\left(\underline{\dot{x}}^r(t) - \underline{\dot{x}}^r(t_0)\right) - \int_{t_0}^{t}\underline{F}^r d\tau\right] = \sum_{r=1}^{n}\int_{t_0}^{t}\underline{F}^{r'} d\tau \tag{13.6}$$

where we have written

$$\sum \underline{F}^r = \underline{F}^r + \underline{F}^{r'} \tag{13.7}$$

where $\underline{F}^r$ is the resultant given force and $\underline{F}^{r'}$ is the resultant constraint force on particle r. Now consider the scalar product of the last term of Eqn. (13.6) with a virtual displacement $\delta\underline{x}^r$; by definition, a constraint force does no virtual work, so that

$$\left(\int_{t_0}^{t}\underline{F}^{r'} d\tau\right)\cdot\delta\underline{x}^r = \int_{t_0}^{t}\left(\underline{F}^{r'}\cdot\delta\underline{x}^r\right)d\tau = 0$$

Thus if we take the scalar product of Eqn. (13.6) with $\delta\underline{x}^r$, we obtain

$$\sum_{r=1}^{n}\left[m_r\left(\underline{\dot{x}}^r(t) - \underline{\dot{x}}^r(t_0)\right) - \underline{P}^r\right]\cdot\delta\underline{x}^r = 0 \tag{13.8}$$

where

$$\underline{P}^r = \int_{t_0}^{t}\underline{F}^r d\tau \tag{13.9}$$

is the impulse of the given force acting on particle r. Now suppose at least some of the forces are impulsive at time t_0; then, at $t = t_0$, Eqn. (13.8) gives

$$\sum_{r=1}^{n} \left[m_r \left(\dot{\underline{x}}^r_+ - \dot{\underline{x}}^r_-\right) - \underline{P}^r\right] \cdot \delta\underline{x}^r = 0 \tag{13.10}$$

This is the *fundamental equation of impulsive motion.* The $\delta\underline{x}^r$ may be subject to *impulsive constraints*, or they may be arbitrary.

In Section 3.4 it was shown that possible velocity changes satisfy the same conditions as virtual displacements; thus another form of Eqn. (13.10) is

$$\sum_{r=1}^{n} \left[m_r \left(\dot{\underline{x}}^r_+ - \dot{\underline{x}}^r_-\right) - \underline{P}^r\right] \cdot \Delta\dot{\underline{x}}^r = 0 \tag{13.11}$$

The displacement components of the particles were introduced in Eqns. (2.4); in terms of displacement components Eqn. (13.11) is

$$\sum_{s=1}^{N} \left[m_s(\dot{u}_{s+} - \dot{u}_{s-}) - P_s\right] \Delta\dot{u}_s = 0 \tag{13.12}$$

where $N = 3n$. This is the impulsive form of Eqn. (3.38). We see that the velocity components after the impulse are governed by *linear* equations and thus this is a relatively simple problem.

13.2 Impulsive Constraints

Definitions. Both the given and the constraint forces may be impulsive. In this Section, we consider the latter case. In an impulse, the constraints must exert very large forces (infinite in the mathematical approximation) over very short times. The assumption is made that the bodies and constraint surfaces do not deform during this time.

The Phaffian form of the constraints on a dynamic system are

$$\sum_{s=1}^{N} A_{rs} du_s + A_r dt = 0 \; ; \quad r = 1, \cdot\cdot, L < N \tag{13.13}$$

The constraint forces associated with these constraints are

$$F'_r = \lambda_r(t) \left[\sum_{s=1}^{N} A_{rs}\dot{u}_s + A_r\right] \; ; \quad r = 1, \cdot\cdot, L \tag{13.14}$$

where the $\lambda_r(t)$ are Lagrange multipliers. Let

$$\begin{aligned} \underline{\lambda} &= (\lambda_1(t), \cdots, \lambda_L(t)) \\ \underline{a} &= \left(\sum_{s=1}^{N} A_{1s}\dot{u}_s + A_1, \ \cdot\cdot, \ \sum_{s=1}^{N} A_{Ls}\dot{u}_s + A_L \right) \\ \underline{F}' &= (F_1', \cdot\cdot, F_L') \end{aligned} \tag{13.15}$$

Then Eqns. (13.14) may be written compactly as

$$\underline{F}' = \underline{\lambda} \cdot \underline{a} \tag{13.16}$$

The impulse of this force at time t_0 is

$$\underline{I} = \lim_{t \to t_0} \int_{t_0}^{t} \underline{F}'(\tau) d\tau \tag{13.17}$$

By definition a constraint is *impulsive* if $\underline{F}'$ is impulsive, that is, if $\underline{F}'$ tends to infinity at t_0 but $\underline{I}$ remains bounded. If $\underline{I} = \underline{0}$, the force is not impulsive. Therefore, we limit our consideration to the case $\underline{0} < \underline{I} < \underline{\infty}$.

Combining Eqns. (13.16) and (13.17) gives

$$\underline{I} = \lim_{t \to t_0} \int_{t_0}^{t} \underline{\lambda}(\tau) \cdot \underline{a}(\tau) d\tau \tag{13.18}$$

Now integrate this equation by parts in two different ways:

$$\begin{aligned} \underline{I} &= \underline{a}(t_0) \cdot (\underline{\Lambda}_{t_0+} - \underline{\Lambda}_{t_0-}) - \lim_{t \to t_0} \int_{t_0}^{t} \underline{\Lambda}(\tau) \cdot \underline{a}'(\tau) d\tau \\ \underline{I} &= \underline{\lambda}(t_0) \cdot (\underline{B}_{t_0+} - \underline{B}_{t_0-}) - \lim_{t \to t_0} \int_{t_0}^{t} \underline{B}(\tau) \cdot \underline{\lambda}'(\tau) d\tau \end{aligned} \tag{13.19}$$

where

$$\begin{aligned} \underline{\Lambda}(t) &= \int \underline{\lambda}(t) dt \\ \underline{B}(t) &= \int \underline{a}(t) dt \end{aligned} \tag{13.20}$$

We see that impulsive constraints may come about in essentially two different ways, as follows.

$\underline{\lambda}(t)$ **Discontinuous.** Assume that $\underline{\lambda}(t)$ is discontinuous at t_0 but that $\underline{a}(t) \in C^1$ there.

Then

$$\lim_{t \to t_0} \int_{t_0}^{t} \underline{\Lambda}(\tau) \cdot \underline{a}'(\tau) d\tau = 0 \tag{13.21}$$

and the first of Eqns. (13.19) becomes

$$\underline{I} = \underline{a}(t_0) \cdot (\underline{\Lambda}_{t_0+} - \underline{\Lambda}_{t_0-}) \tag{13.22}$$

where

$$\underline{\Lambda}_{t_0+} - \underline{\Lambda}_{t_0-} = \lim_{t \to t_0} \int_{t_0}^{t} \underline{\lambda}(\tau) d\tau \tag{13.23}$$

is nonzero by assumption. We see that $\underline{\lambda}(t)$ is discontinuous at t_0.

As an example of this type of constraint, consider a single particle encountering a constraint surface (Fig. 13-3) with velocity not tangent to the surface. For $t < t_0$, the particle is unconstrained. At $t = t_0$ the constraint is encountered, and for $t > t_0$ the particle is constrained to move on the surface. Just before the encounter, $t < t_0$, there is no constraint force and $\lambda = 0$; just after, the constraint force is $\underline{F}' = \lambda \operatorname{grad} \underline{f}$ and $\lambda \neq 0$. Note that the position of the particle is continuous at t_0, but that the velocity is discontinuous. Also note that the DOF changes instantaneously at t_0.

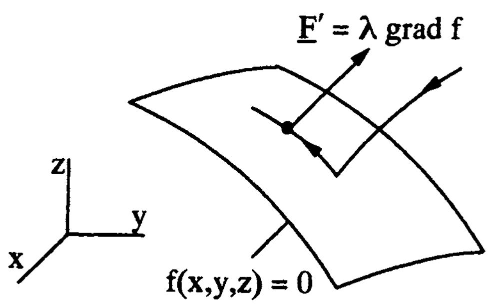


Fig. 13-3

a(t) **Discontinuous.** Now assume $\underline{a}(t)$ is discontinuous at t_0, but that $\underline{\lambda}(t) \in C^1$. Then,

$$\lim_{t \to t_0} \int_{t_0}^{t} \underline{B}(\tau) \cdot \underline{\lambda}'(\tau) d\tau = 0 \tag{13.24}$$

and the second of Eqns. (13.19) becomes

$$\underline{I} = \underline{\lambda}(t_0) \cdot (\underline{B}_{t_0+} - \underline{B}_{t_0-})$$

where

$$\underline{B}_{t_0+} - \underline{B}_{t_0-} = \lim_{t \to t_0} \int_{t_0}^{t} \underline{a}(\tau) d\tau \qquad (13.25)$$

so that $\underline{a}(t)$ is discontinuous at t_0.

Recall from Eqns. (13.15) that

$$a_r = \sum_{s=1}^{N} A_{rs} \dot{u}_s + A_r \qquad (13.26)$$

and let

$$E_{rs} = \int A_{rs}(t) dt \,, \qquad E_r = \int A_r(t) dt \qquad (13.27)$$

Then Eqn. (13.25) becomes, for the $r^{\underline{\text{th}}}$ constraint,

$$\begin{aligned} B^r_{t_0+} - B^r_{t_0-} &= \lim_{t \to t_0} \int_{t_0}^{t} \left[\sum_{s=1}^{N} A_{rs}(\tau) \dot{u}_s(\tau) + A_r(\tau) \right] d\tau \\ &= \lim_{t \to t_0} \left[\sum_{s=1}^{N} (E_{rs}(t) - E_{rs}(t_0)) \dot{u}_s + (E_r(t) - E_r(t_0)) \right] \\ &= \sum_{s=1}^{N} (E_{rs_{t_0+}} - E_{rs_{t_0-}}) \dot{u}_s + (E_{r_{t_0+}} - E_{r_{t_0-}}) \qquad (13.28) \end{aligned}$$

We see that this type of constraint force arises if either some of the E_{rs} or some of the E_r are discontinuous. Thus the impulsive constraints are

$$\sum_{s=1}^{N} E_{rs} \dot{u}_s + E_r = 0 \; ; \quad r = 1, \cdot\cdot, L' \qquad (13.29)$$

where L' is the number of impulsive constraints. Nonimpulsive constraints, of course, may be also present.

If the $r^{\underline{\text{th}}}$ impulsive constraint is holonomic, then it may be expressed as $f_r(u_1, \cdot\cdot, u_N, t) = 0$ and

$$\sum_{s=1}^{N} \frac{\partial f_r}{\partial u_s} \dot{u}_s + \frac{\partial f_r}{\partial t} = 0$$

In this case $A_{rs} = \partial f_r / \partial u_s$ and $A_r = \partial f_r / d\tau$ so that

$$E_{rs} = \int \frac{\partial f_r}{\partial u_s} dt \, , \qquad E_r = \int \frac{\partial f_r}{\partial t} dt \tag{13.30}$$

As an example, consider a single particle and a single holonomic constraint. First suppose that the constraint is scleronomic ($A_r = 0$) and that some of the A_{rs} are unbounded (indeed, if this were not true the constraint would not be impulsive) at some time t_0. Then the function $f(x, y, z)$ is discontinuous there (Fig. 13-4), resulting in discontinuous velocity for the particle (but of course, as usual, continuous displacement). This is called an *inert* impulsive constraint.

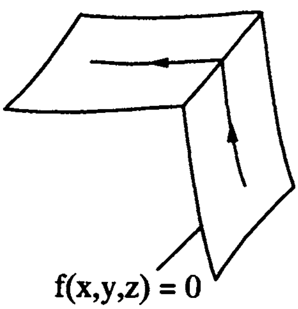


Fig. 13-4

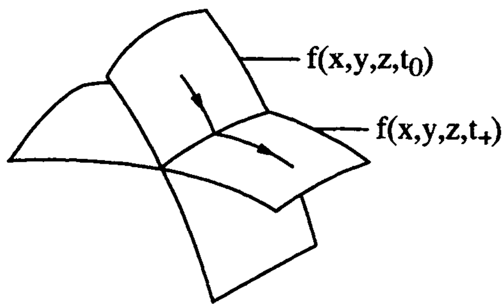


Fig. 13-5

Next suppose that some of the A_r are unbounded but that the A_{rs} are all bounded at some time t_0. Then $f(x, y, z, t)$ is again discontinuous, the surface changing its location instantly (Fig. 13-5). This is called a *live* impulsive constraint. Note that the constraint is not allowed to change in such a way that the position of the particle is discontinuous.

A live impulsive constraint also occurs when there is a discontinuity in the velocity of a constraint surface. For example, suppose an elevator initially at rest is given an instantaneous speed V. Before the impulse, the constraint on a particle on the elevator floor is $\dot{z} = 0$, z being in the direction of the elevator's travel. After the impulse, the constraint is $\dot{z} - V = 0$.

Fundamental Equation. When impulsive given forces and impulsive constraints are present, the fundamental equation is Eqn. (13.12), repeated here:

$$\sum_{s=1}^{N} m_s \left[(\dot{u}_{s+} - \dot{u}_{s-}) - P_s \right] \Delta \dot{u}_s = 0 \tag{13.31}$$

From Eqn. (13.29), the velocity components $\Delta \dot{u}_s$ are subject to

$$\sum_{s=1}^{N} E_{rs} \Delta \dot{u}_s = 0 \; ; \quad r = 1, \cdot\cdot, L' \tag{13.32}$$

There are two important special cases:

(1) If there are no inert constraint forces, i.e. if all constraint forces are live, then all the $E_{rs} = 0$ and the $\Delta \dot{u}_s$ are unconstrained.

(2) If there are no given impulsive forces, then all the $P_s = 0$.

If both (1) and (2) are true, then Eqn. (13.31) is merely a statement that all velocity components are continuous at t_0.

13.3 Impulsive Motion Theorems

Remarks. In this section we consider impulsive given forces only (no impulsive constraints). Because the impulse takes place so quickly, the constraints, even the rheonomic ones, are considered fixed. Since velocities change during impulses, the system kinetic energy generally does also. The theorems which follow in this section relate impulsive forces to energy changes. As before, the following notation is adopted:

$\dot{x}^r_- = \dot{x}^r(t_{0-})$ = velocity of particle r just before impulse at time t_0

$\dot{x}^r_+ = \dot{x}^r(t_{0+})$ = velocity of particle r just after impulse at time t_0

In this section, for convenience, we drop the underbars for vectors. We also let $\sum_r = \sum_{r=1}^{n}$.

Gauss' Principle of Least Constraint.[2] This theorem states that the quantity

$$z = \frac{1}{2} \sum_r m_r \left(\dot{x}^r_+ - \dot{x}^r_- - \frac{P_r}{m_r} \right)^2 \tag{13.33}$$

is a minimum for the actual values of $\dot{x}^r_+$ relative to all other possible velocities $\dot{\overline{x}}^r_+$.

The proof is as follows. Let the difference in z between that for any possible motion and for the actual motion be

$$\Delta z = \overline{z} - z$$

and let the possible motion of particle r after the impulse differ from the actual motion by $\Delta\dot{x}^r$:

$$\dot{\bar{x}}^r_+ = \dot{x}^r_+ + \Delta\dot{x}^r$$

Then

$$\begin{aligned}\Delta z &= \frac{1}{2}\sum_r \left\{ m_r \left[\left(\dot{x}^r_+ + \Delta\dot{x}^r - \Delta\dot{x}^r_- - \frac{P_r}{m_r} \right)^2 \right.\right. \\ &\qquad \left.\left. - \left(\dot{x}^r_+ - \dot{x}^r_- - \frac{P_r}{m_r} \right)^2 \right] \right\} \\ &= \sum_r m_r \left(\dot{x}^r_+ - \dot{x}^r_- - \frac{P_r}{m_r} \right) \Delta\dot{x}^r + \frac{1}{2}\sum_{r=1}^n m_r (\Delta\dot{x}^r)^2\end{aligned}$$

But the first term on the right-hand side is zero by Eqn. (13.11), so that

$$\Delta z = \frac{1}{2}\sum_{r=1}^n m_r (\Delta\dot{x}^r)^2$$

Thus $\Delta z \geq 0$ for all possible velocities and z is a minimum for the actual velocities.

A useful expression may be obtained by applying the first order necessary condition for a minimum of z. Expanding Eqn. (13.33),

$$z = T_+ + T_- + \frac{1}{2}\sum_{r=1}^n \frac{P_r^2}{m_r} - \sum_{r=1}^n m_r \dot{x}^r_+ \dot{x}^r_- - \sum_r (\dot{x}^r_+ - \dot{x}^r_-) P^r$$

where

$$T_+ = \frac{1}{2}\sum_r m_r (\dot{x}^r_+)^2$$

$$T_- = \frac{1}{2}\sum_r m_r (\dot{x}^r_-)^2$$

A necessary condition for $z(\dot{x}^r_+)$ to be a minimum is

$$\frac{dz}{d\dot{x}^r_+} = \frac{\partial T_+}{\partial \dot{x}^r_+} - m_r \dot{x}^r_- - P^r = 0\ ; \quad r = 1, \cdot\cdot, n \tag{13.34}$$

Superposition Theorem. Let $\dot{x}^r_i$ be the velocity of particle r after an impulse P^r_i. Then after k impulses whose sum is

$$P^r = \sum_{i=1}^k P^r_i$$

the velocity is

$$\dot{x}^r = \sum_{i=1}^{k} \dot{x}_i^r$$

The proof follows from repeated use of Eqn. (13.11).

Energy Theorem. This states that the increase in kinetic energy due to an impulse P^r is equal to

$$\Delta T = \frac{1}{2} \sum_r P^r (\dot{x}_+^r + \dot{x}_-^r) \tag{13.35}$$

Since the $\Delta \dot{x}^r$ are unrestricted, we may choose

$$\Delta \dot{x}_r = \dot{x}_+^r + \dot{x}_-^r$$

Substituting in Eqn. (13.11) gives

$$\sum_r m_r (\dot{x}_+^r - \dot{x}_-^r)(\dot{x}_+^r + \dot{x}_-^r) = \sum_r P^r (\dot{x}_+^r + \dot{x}_-^r)$$

$$\sum_r m_r (\dot{x}_+^r)^2 - \sum_r m_r (\dot{x}_-^r)^2 = \sum_r P^r (\dot{x}_+^r + \dot{x}_-^r)$$

But the term on the left-hand side is $2\Delta T$ so that Eqn. (13.35) follows.

In particular, if the system is initially at rest, Eqn. (13.35) reduces to

$$\Delta T = \frac{1}{2} \sum_r P_r \dot{x}_+^r \tag{13.36}$$

Bertrand's Theorem. If an unconstrained system is subjected to an impulse, the kinetic energy of the subsequent motion is greater than if the system had been constrained and subjected to the same impulse.

As an example, Fig. 13-6 shows an unconstrained and two constrained rods subjected to the same impulse P. The theorem says that $T_a > T_b > T_c$. Of course, $T_c = 0$.

To begin the proof of the theorem, let

- T_+ be the kinetic energy after the impulse of the unconstrained system
- T_1 be the kinetic energy after the impulse of the constrained system
- $\dot{x}_+^r$ be the velocity of the unconstrained system after the impulse
- $\dot{x}_1^r$ be the velocity of the constrained system after the impulse

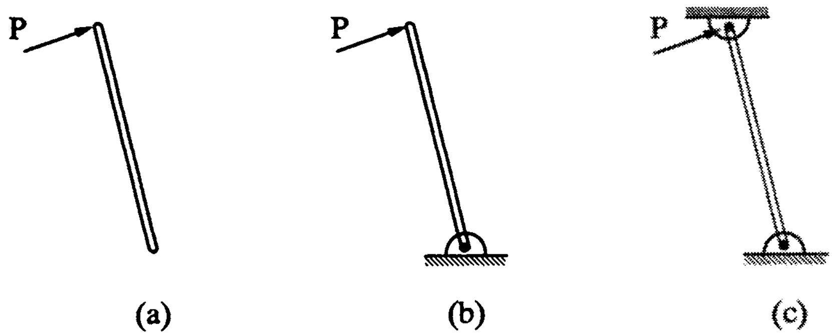


Fig. 13-6

We must show that $T_+ > T_1$. Equation (13.11) applied to the two cases results in

$$\sum_r m_r \dot{x}_+^r \Delta\dot{x}^r = \sum_r P^r \Delta\dot{x}^r$$

$$\sum_r m_r \dot{x}_1^r \Delta\dot{x}^r = \sum_r P^r \Delta\dot{x}^r$$

In the first of these, any $\Delta\dot{x}^r$ is admissible, whereas in the second $\Delta\dot{x}^r = \dot{x}_1^r$ is admissible. Thus we may take $\Delta\dot{x}^r = \dot{x}_1^r$ in both cases. With this choice, subtraction of the two equations gives

$$\sum_r m_r \dot{x}_1^r (\dot{x}_+^r - \dot{x}_1^r) = 0$$

But

$$\dot{x}_1^r(\dot{x}_+^r - \dot{x}_1^r) = \frac{1}{2}\left[(\dot{x}_+^r)^2 - (\dot{x}_1^r)^2 - (\dot{x}_+^r - \dot{x}_1^r)^2\right]$$

Also

$$T_+ = \frac{1}{2}\sum_r m_r (\dot{x}_+^r)^2$$

$$T_1 = \frac{1}{2}\sum_r m_r (\dot{x}_1^r)^2$$

so that

$$R_1 = T_+ - T_1 = \frac{1}{2}\sum_r m_r (\dot{x}_+^r - \dot{x}_1^r)^2 > 0$$

which proves the theorem.

Kelvin's Theorem. If some particles of a connected system are suddenly set into motion with prescribed velocity, the kinetic energy is a minimum relative to other possible motions in which these particles have the same velocity.

The proof proceeds as follows. Let particles $1, \cdot\cdot, m < n$ have prescribed velocities and let

$$\dot{x}_+ = (\dot{x}_+^1, \cdot\cdot, \dot{x}_+^m, \ \dot{x}_+^{m+1}, \cdot\cdot, \dot{x}_+^n)$$
$$\dot{x}_2 = (\dot{x}_+^1, \cdot\cdot, \dot{x}_+^m, \ \dot{x}_2^{m+1}, \cdot\cdot, \dot{x}_2^n)$$

be the actual velocities after the impact and other possible velocities, respectively. Choose $\Delta\dot{x}^r = \dot{x}_+^r - \dot{x}_2^r$ in the fundamental equation:

$$\sum_r m_r \dot{x}_+^r \Delta\dot{x}^r = \sum_r P^r \Delta\dot{x}^r$$

$$\sum_{r=1}^{n} m_r \dot{x}_+^r (\dot{x}_+^r - \dot{x}_2^r) = \sum_{r=1}^{m} P^r (\dot{x}_+^r - \dot{x}_2^r) + \sum_{r=m+1}^{n} P^r (\dot{x}_+^r - \dot{x}_2^r)$$

But $\dot{x}_+^1 - \dot{x}_2^1 = 0, \cdot\cdot, \dot{x}_+^m - \dot{x}_2^m = 0$ and $P^{m+1} = 0, \cdot\cdot, P^n = 0$ so that

$$\sum_r m_r \dot{x}_+^r (\dot{x}_+^r - \dot{x}_2^r) = 0$$

Also,

$$\dot{x}_+^r (\dot{x}_+^r - \dot{x}_2^r) = \frac{1}{2}\left[(\dot{x}_+^r)^2 - (\dot{x}_2^r)^2 + (\dot{x}_+^r - \dot{x}_2^r)^2\right]$$

and

$$T_+ = \frac{1}{2}\sum_r m_r (\dot{x}_+^r)^2$$
$$T_2 = \frac{1}{2}\sum_r m_r (\dot{x}_2^r)^2$$

Therefore,

$$R_2 = T_2 - T_+ = \frac{1}{2}\sum_r m_r (\dot{x}_+^r - \dot{x}_2^r) > 0$$

which proves the theorem.

Taylor's Theorem. The gain R_2 in Kelvin's theorem is greater than the loss R_1 in Bertrand's theorem.

To prove this, begin by recalling from Bertrand's theorem that with $\Delta\dot{x}^r = \dot{x}_1^r$ we obtained

$$\sum_r m_r \dot{x}_1^r(\dot{x}_+^r - \dot{x}_1^r) = 0$$

However, $\Delta\dot{x}^r = \dot{x}_2^r$ is also a possible velocity change in Bertrand's theorem; with this choice, we obtain

$$\sum_r m_r \dot{x}_2^r(\dot{x}_+^r - \dot{x}_1^r) = 0$$

Subtracting these two equations,

$$\sum_r m_r (\dot{x}_+^r - \dot{x}_1^r)(\dot{x}_2^r - \dot{x}_1^r) = 0$$

But

$$(\dot{x}_+^r - \dot{x}_1^r)(\dot{x}_2^r - \dot{x}_1^r) = (\dot{x}_1^r - \dot{x}_+^r)\left[(\dot{x}_1^r - \dot{x}_+^r) - (\dot{x}_2^r - \dot{x}_+^r)\right]$$

Combining results

$$R_{12} = R_2 - R_1 = \frac{1}{2}\sum_r m_r (\dot{x}_2^r - \dot{x}_1^r)^2 > 0$$

which was to be shown.

Example. Three uniform thin rods of equal length and mass, m, are hinged together and are initially at rest in a straight line on a horizontal friction-less surface (Fig. 13-7). An impulse of magnitude P is applied to one end of the assembly. We will analyze the subsequent motion using the previous theorems.

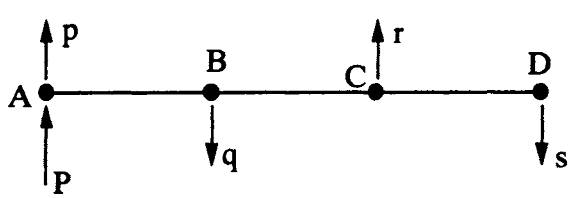


Fig. 13-7

Let the speeds of the ends of rods after the impact be denoted p, q, r and s as shown. Using Eqn. (1.59), the kinetic energy of the system is

$$\begin{aligned} T_+ &= T_{AB} + T_{BC} + T_{CD} = \frac{1}{6}M(p^2 - pq + q^2) + \frac{1}{6}M(q^2 - qr + r^2) \\ &\quad + \frac{1}{6}M(r^2 - rs + s^2) \\ &= \frac{1}{6}M(p^2 + 2q^2 + 2r^2 + s^2 - pq - qr - rs) \end{aligned}$$

First we apply Gauss' principle, Eqn. (13.34), which is in this case

$$P^r = \frac{\partial T_+}{\partial \dot{x}_+^r}\,; \qquad \dot{x}_+^r = p, q, r, s$$

This gives

$$\begin{aligned} P &= \frac{\partial T_+}{\partial p} = \frac{1}{6}M(2p - q) \\ 0 &= \frac{\partial T_+}{\partial q} = \frac{1}{6}M(4q - p - r) \\ 0 &= \frac{\partial T_+}{\partial r} = \frac{1}{6}M(4r - q - s) \\ 0 &= \frac{\partial T_+}{\partial s} = \frac{1}{6}M(2s - r) \end{aligned}$$

Solving these equations simultaneously gives

$$p = \frac{52}{15}\frac{P}{M}, \quad q = \frac{14}{15}\frac{P}{M}, \quad r = \frac{4}{15}\frac{P}{M}, \quad s = \frac{2}{15}\frac{P}{M}$$

Next, we apply the Energy theorem to compute the kinetic energy. Applying Eqn. (13.36),

$$T_+ = \Delta T = \frac{1}{2}\sum_{r=1}^{4} p_r \dot{x}_+^r = \frac{1}{2}Pp = \frac{26}{15}\frac{P^2}{M}$$

Now suppose point D is held fixed. Then $s = 0$ and in this case Gauss' principle gives

$$p = \frac{45}{13}\frac{P}{M}, \quad q = \frac{12}{13}\frac{P}{M}, \quad r = \frac{3}{13}\frac{P}{M}$$

and the energy after the impact is

$$T_1 = \frac{1}{2}Pp = \frac{45}{26}\frac{P^2}{M}$$

Thus $T_+ > T_1$, verifying Bertrand's theorem.

Next suppose that P is changed such that end A has the same speed p after the impact in the two cases above. Then,

$$p = \frac{52}{15}\frac{P}{M}, \quad T_+ = \frac{1}{2}\frac{15}{52}Mp^2, \quad T_2 = \frac{1}{2}\frac{13}{45}Mp^2$$

where T_+ and T_2 are the energies with end A free and fixed, respectively. Thus $T_+ > T_2$, verifying Kelvin's theorem.

Finally,

$$R_{12} = R_2 - R_1 = (T_2 - T_+) - (T_+ - T_1) = T_2 + T_1 - 2T_+ > 0$$

verifying Taylor's theorem.

Carnot's Theorem. Finally, we give one result for impulsive constraint forces, without proof. Carnot's theorem states that the change in energy due to an impulsive inert constraint is always negative.

13.4 Lagrange's Equations for Impulsive Motion

Lagrange's Equations. So far in this chapter we have formulated problems in terms of position vectors and rectangular components. Now we consider systems defined in terms of generalized coordinates. If there are no constraints, Eqns. (6.29) are

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_s}\right) - \frac{\partial T}{\partial q_s} - Q_s = 0\,; \quad s = 1, \cdot\cdot, n \tag{13.37}$$

where n is now the number of generalized coordinates. Let Q_s be an impulsive force component at t_0; then the impulse component is

$$R_s = \lim_{t \to t_0} \int_{t_0}^{t} Q_s dt \tag{13.38}$$

Now integrate Eqns. (13.37) and take the limit $t \to t_0$ to obtain

$$\lim_{t \to t_0} \int_{t_0}^{t} \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_s}\right) - \frac{\partial T}{\partial q_s}\right] dt = \lim_{t \to t_0} \int_{t_0}^{t} Q_s dt$$

$$\Delta\left(\frac{\partial T}{\partial \dot{q}_s}\right) = R_s\,; \quad s = 1, \cdot\cdot, n \tag{13.39}$$

This is in fact just another form of Eqn. (13.34).

All of the theorems of the previous section also apply when the system is defined in terms of generalized coordinates.

Notes

1 In this chapter, n refers to the number of particles, not the number of generalized coordinates, unless stated otherwise.

2 This is an adaptation of a similar principle for the strictly Newtonian problem. This principle can be applied also to inert constraints. See Pars.

PROBLEMS

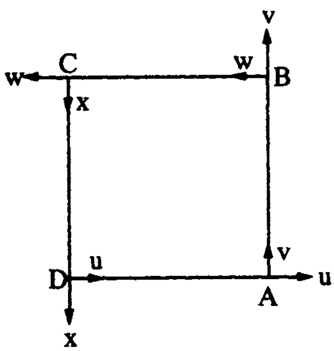
13/1. A homogeneous rod of length l and mass m lies on a smooth horizontal table. It is struck a blow in the plane of the table a distance nl from one end $(0 < n < 1)$, and in a direction normal to the rod. The blow is such that it would give a velocity v to a particle at rest of mass m. Find the velocities of the ends of the rod immediately after the blow is struck.

13/2. A heavy uniform rod of mass m and length l hangs vertically from a smooth pin a distance nl from one end $(0 < n < 0.5)$. What is the smallest blow that the rod can be struck at the bottom at right angles to the bar which will just make the bar reach the inverted position?

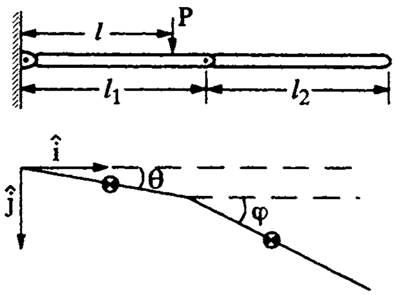
13/3. Repeat the problem of the four rods of the example of Section 3.3 except that the impulse P is applied at point B.

13/4. Four uniform thin rods, each of mass M and length l, are freely hinged together at their corners as shown. Two opposite corners are connected by a light inelastic string of length $l/\sqrt{2}$ so that the string is taunt when the framework is square. The framework moves on a smooth horizontal surface. The framework moves such that at time t_1 the string becomes taunt. Find the motion immediately after the impulse at t_1. Label the components of velocity of the ends of the rods u, v, w, x as shown.

13/5. Two homogeneous rods OA and AB have equal mass m, but OA has length l_1 and AB has length l_2. They are smoothly hinged at A and lie on a smooth horizontal table. The rod OA is smoothly hinged to a vertical pin at O. Initially, the rods form a straight line. An impulse P is applied normal to OA and in the plane of the table to a point C lying between O and A. Calculate the reactions at the hinge pins at O and A and the angular velocities of the two rods immediately after the impulse.

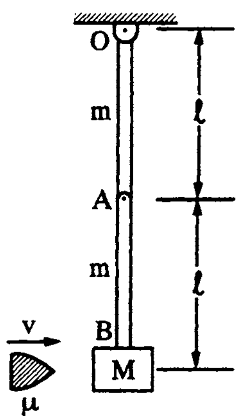


Problem 13/4

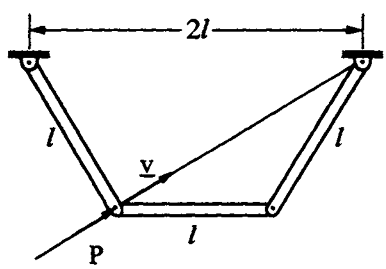


Problem 13/5

13/6. Two equal heavy, homogeneous rod OA and AB of mass m and length l are smoothly hinged at A, and OB is smoothly hinged at O to a fixed point. The rod AB has fixed to it at B a block of mass M and of negligible dimension. This mechanism hangs at rest when a bullet of mass μ is shot with horizontal velocity v into the block as shown. Find the angular velocities of the two rods immediately after the impact.



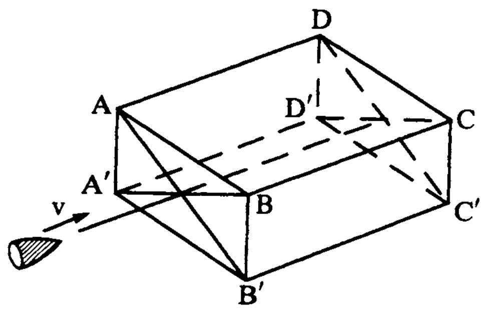
Problem 13/6



Problem 13/7

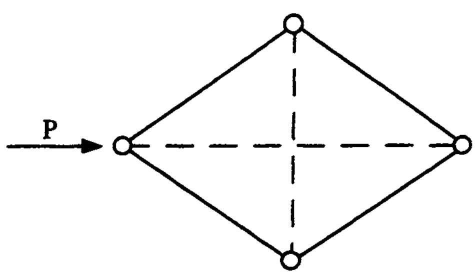
13/7. Three heavy equal homogeneous rods AB, BC, CD are smoothly hinged at B and C. This assembly is suspended at A and D from smooth pins so that the distance between A and D is twice the rod length, and so that all rods are in a vertical plane. An impulse is applied at B in such a way as to give B a velocity v toward D. Find the initial motion.

13/8. A box of dimensions $AB = 2a$, $AD = b$, $AA' = 2c$ rests on a smooth horizontal table. It is filled with sand, and the mass of box and sand is M. The density of the material of the box is the same as that of the sand. A bullet of mass m is fired with velocity v into the center of the face $AA'BB'$, as shown. Calculate the kinetic energy imparted to the box by the bullet when the edge DD' is fixed, and compare it to the kinetic energy when DD' is not fixed.



Problem 13/8

13/9. Four equal, homogeneous rods are smoothly hinged together to form a rhombus which is initially at rest on a smooth horizontal table. Let an impulsive force acting on a hinge point along one of the diagonals give the hinge point a velocity v as shown. Find the angular velocity of the rods immediately after the impact.



Problem 13/9

Chapter 14

Gibbs-Appell Equations

14.1 Quasi-Coordinates

Introduction. Nonholonomic constraints are accounted for in Lagrange's equations by the use of Lagrange multipliers. We now develop an approach to systems with nonholonomic constraints that does not depend on multipliers – the use of quasi-coordinates and the Gibbs-Appell equations.

Quasi-coordinates are analogous to nonholonomic constraints in that they are defined by differential relations that are not integrable. Thus the requirement that the displacement components of the particles are explicit functions of the generalized coordinates is relaxed (see Eqns. (5.7)), and we consider coordinates such that the velocity components are explicit, linear, nonintegrable functions of the time derivatives of the generalized coordinates.

In this and the next four chapters, we will loosely follow Pars. Therefore, we make a notation change to bring our notation into line with that of Pars. The displacement components will now be denoted by x_r, that is

$$\begin{aligned} x_1 &= u_1 = x_1^1 \\ x_2 &= u_2 = x_2^1 \\ x_3 &= u_3 = x_3^1 \\ x_4 &= u_4 = x_1^2 \\ &\vdots \\ x_N &= u_N = x_3^{N/3} \end{aligned}$$

where $\underline{x}^r = (x_1^r, x_2^r, x_3^r)$; $r = 1, \cdot\cdot, \nu$ are the position vectors of the particles. Now, ν is the number of particles and $N = 3\nu$ is the number of displacement components; as before, n will be the number of generalized coordinates, L will be the total number of constraints, ℓ will be the number of nonholonomic constraints, and $k = N - L$ will be the degrees of freedom of the dynamic system. As usual, $\sum_s$ will denote $\sum_{s=1}^{n}$.

Quasi-Coordinates. Consider a dynamic system with ν particles, ℓ nonholonomic constraints, and no holonomic ones. Let a set of generalized coordinates be

$$q_r \; ; \; r = 1, \cdot\cdot, k + \ell = n = 3\nu = N$$

The ℓ constraints are

$$\sum_s B_{rs} dq_s + B_r dt = 0 \; ; \quad r = 1, \cdot\cdot, \ell \tag{14.1}$$

Introduce p new coordinates θ_r, called *quasi coordinates*, such that

$$d\theta_r = \sum_s C_{rs} dq_s + C_r dt \; ; \quad r = 1, \cdot\cdot, p \tag{14.2}$$

where $C_{rs}, \; C_r \in C^1(q,t)$. The total number of coordinates is now $k+\ell+p$ where

$$q_{k+\ell+r} = \theta_r \; ; \quad r = 1, \cdot\cdot, p$$

Relabel this new set of coordinates $q_1, \cdot\cdot, q_{k+\ell+p}$ so that Eqn. (14.2) may be rewritten as

$$dq_{k+\ell+r} = \sum_s C_{rs} dq_s + C_r dt \; ; \quad r = 1, \cdot\cdot, p \tag{14.3}$$

We now require that the matrix $\begin{bmatrix} B_{rs} \\ C_{rs} \end{bmatrix}$ have maximum rank. (Since B_{rs} is $(k+\ell) \times \ell$ and C_{rs} is $(k+\ell) \times p$, the matrix is $(k+\ell) \times (\ell + p)$.) Under this condition the implicit function theorem guarantees that we may solve for $\ell + p$ of the dq_r as functions of the remaining k dq_r. Call the remaining ones ρ_s. Then

$$dq_r = \sum_{s=1}^{k} D_{rs} d\rho_s + D_r dt \; ; \quad r = 1, \cdot\cdot, \ell + p \tag{14.4}$$

Equation (14.4) is equivalent to Eqns. (14.1) and (14.3). In general, the coefficients D_{rs} and D_r will be functions of all $k + \ell$ generalized coordinates q_r.

The displacement components in terms of the generalized coordinates are, as usual, given by Eqns. (5.9):

$$dx_r = \sum_{s=1}^{k+\ell} \frac{\partial x_r}{\partial q_s} dq_s + \frac{\partial x_r}{\partial t} dt \; ; \quad r = 1, \cdot\cdot, N \tag{14.5}$$

Next we relabel $\rho_s = q_s$; $r = 1, \cdot\cdot, k$ and use Eqn. (14.4) to eliminate the superfluous coordinates in Eqn. (14.5):

$$dx_r = \sum_{s=1}^{k} \alpha_{rs} dq_s + \alpha_r dt \; ; \quad r = 1, \cdot\cdot, N \tag{14.6}$$

Equation (14.4) becomes

$$dq_r = \sum_{s=1}^{k} \beta_{rs} dq_s + \beta_r dt \; ; \quad r = 1, \cdot\cdot, n = k + \ell \tag{14.7}$$

Comparing Eqns. (14.5) and (14.6), we see that we have reduced the number of dq_r upon which the dx_r depend to k, the degrees of freedom. Thus the system now *appears to be holonomic* because it takes k coordinates to specify the system, and no multipliers will be needed.

From Eqns. (14.6) and (14.7), virtual displacements satisfy

$$\delta x_r = \sum_{s=1}^{k} \alpha_{rs} \delta q_s \; ; \quad r = 1, \cdot\cdot, N \tag{14.8}$$

$$\delta q_r = \sum_{s=1}^{k} \beta_{rs} \delta q_s \; ; \quad r = 1, \cdot\cdot, n \tag{14.9}$$

Example. First consider a particle moving in a plane (Fig. 14-1). Two possible choices of generalized coordinates are rectangular, (x, y), and polar, (r, θ). A possible quasi-coordinate is q, defined by

$$dq = x dy - y dx$$

It is easy to show that this is nonintegrable (see Section 2.6). We have

$$\dot{q} = x\dot{y} - y\dot{x}$$

$$q = \int_{t_0}^{t} (x\dot{y} - y\dot{x}) dt$$

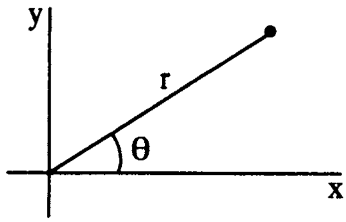


Fig. 14-1

so that q is twice the area swept out in time $(t - t_0)$ by the position vector. We will come back to this problem later.

As a second example, the total rotation about a given line of a rigid body is often a convenient quasi coordinate. For example, from Eqn. (11.8) the total rotation about the axis of a spinning top is q, where

$$dq = d\psi + \cos\theta d\phi$$

where ψ, θ, and ϕ are the spin, nutation, and precession angles, respectively.

14.2 Fundamental Equation

Fundamental Equation with Quasi-Coordinates. Recall the three forms of the fundamental equation established in Chapter 3, namely Eqns. (3.7), (3.38) and (3.39), repeated here in the new notation:

$$\sum_{r=1}^{N}(m_r\ddot{x}_r - F_r)\delta x_r = 0 \tag{14.10}$$

$$\sum_{r=1}^{N}(m_r\ddot{x}_r - F_r)\Delta\dot{x}_r = 0 \tag{14.11}$$

$$\sum_{r=1}^{N}(m_r\ddot{x}_r - F_r)\Delta\ddot{x}_r = 0 \tag{14.12}$$

We want to obtain a fundamental equation similar to Eqn. (14.12) in our new coordinates.

First, the virtual work is obtained using Eqn. (14.8):

$$\sum_{r=1}^{N} F_r\delta x_r = \sum_{r=1}^{N} F_r \sum_{s=1}^{k} \alpha_{rs}\delta q_s = \sum_{s=1}^{k} Q_s\delta q_s$$

so that

$$Q_s = \sum_{r=1}^{N} F_r \alpha_{rs} \tag{14.13}$$

From Eqn. (14.6),

$$\dot{x}_r = \sum_{s=1}^{k} \alpha_{rs}\dot{q}_s + \alpha_r \ ; \quad r = 1, \cdot\cdot, N$$

$$\ddot{x}_r = \sum_{s=1}^{k} \alpha_{rs}\ddot{q}_s + \text{terms without the } \ddot{q}_s \ ; \quad r = 1, \cdot\cdot, N \tag{14.14}$$

Consider another possible acceleration $\ddot{x}_r + \Delta\ddot{x}_r$; then

$$\ddot{x}_r + \Delta\ddot{x}_r = \sum_{s=1}^{k} \alpha_{rs}(\ddot{q}_s + \Delta\ddot{q}_s) + \text{terms without the } \ddot{q}_s \ ; \quad r = 1, \cdot\cdot, N$$

Thus

$$\Delta\ddot{x}_r = \sum_{s=1}^{k} \alpha_{rs}\Delta\ddot{q}_s \ ; \quad r = 1, \cdot\cdot, N \tag{14.15}$$

Substitute Eqn. (14.15) into (14.12) and use (14.13):

$$\sum_{r=1}^{N} m_r \ddot{x}_r \Delta\ddot{x}_r - \sum_{r=1}^{N} F_r \sum_{s=1}^{k} \alpha_{rs}\Delta\ddot{q}_s = 0$$

$$\sum_{r=1}^{N} m_r \ddot{x}_r \Delta\ddot{x}_r - \sum_{s=1}^{k} Q_s \Delta\ddot{q}_s = 0 \tag{14.16}$$

which is what we wanted to derive. Note that this involves a mixture of rectangular and generalized coordinates.

14.3 Gibbs' Theorem and the Gibbs-Appell Equations

Gibbs' Function. Define the acceleration, or Gibbs, function by

$$G = \frac{1}{2}\sum_{r=1}^{N} m_r \ddot{x}_r^2 \tag{14.17}$$

Note that this is similar to the definition of kinetic energy, except that accelerations are used instead of velocities. Substituting Eqn. (14.14) into (14.17) gives a function of the form

$$G = G_2 + G_1 + G_0$$

where G_2 is quadratic in the $\ddot{q}_r$, G_1 is linear in the $\ddot{q}_r$, and G_0 does not contain the $\ddot{q}_r$. We now state the following.

Gibb's Theorem. Given the displacements and velocities at some time t, the accelerations at that time are such that

$$G - \sum_{s=1}^{k} Q_s \ddot{q}_s$$

is a minimum with respect to the $\ddot{q}_r$.

The proof is as follows. Let $\ddot{q}_s$ be the actual accelerations and let $\ddot{q}_s + \Delta\ddot{q}_s$ be possible ones. Form the change in the function just above and use Eqn. (14.16):

$$\begin{aligned}\Delta\left(G - \sum_{s=1}^{k} Q_s\ddot{q}_s\right) &= \frac{1}{2}\sum_{r=1}^{N} m_r(\ddot{x}_r + \Delta\ddot{x}_r)^2 - \sum_{s=1}^{k} Q_s(\ddot{q}_s + \Delta\ddot{q}_s) \\ &\quad -\frac{1}{2}\sum_{r=1}^{N} m_r\ddot{x}_r^2 + \sum_{s=1}^{k} Q_s\ddot{q}_s \\ &= \frac{1}{2}\sum_{r=1}^{N} m_r(\Delta\ddot{x}_r)^2 + \left(\sum_{r=1}^{N} m_r\ddot{x}_r\Delta\ddot{x}_r - \sum_{s=1}^{k} Q_s\Delta\ddot{q}_s\right) \\ &= \frac{1}{2}\sum_{r=1}^{N} m_r(\Delta\ddot{x}_r)^2 > 0 \qquad (14.18)\end{aligned}$$

which proves the theorem.

Gibbs-Appell Equations. These equations are the first order necessary conditions associated with Gibbs' Theorem, namely,

$$Q_s = \frac{\partial G}{\partial \ddot{q}_s}\,; \quad s = 1, \cdot\cdot, k \qquad (14.19)$$

Also to be satisfied are the constraint equations, obtained from Eqns. (14.7):

$$\dot{q}_r = \sum_{s=1}^{k} \beta_{rs}\dot{q}_s + \beta_r\,; \quad r = k+1, \cdot\cdot, n \qquad (14.20)$$

Equations (14.19) and (14.20) serve to determine the equations of motion of a dynamic system.

Remarks.

1. The q_r are in general a mixture of generalized coordinates and quasi-coordinates.

2. Equations (14.19) were first discovered by Gibbs but attracted little attention. They were later discovered independently by Appell who first realized their full importance.

3. The Gibbs-Appell equations are equivalent to Kane's equations (see Baruh and Kane and Levinson)

Solution Procedure. To solve problems using Eqns. (14.19), the following steps are required.

1. Determine $k = N - \ell$, the degrees of freedom of the system.

2. Obtain G by expressing the $\ddot{x}_r^2$ in terms of k of the $\ddot{q}_r$ (see Eqn. (14.17)). Note that generally all the q_r and $\dot{q}_r$ will appear in G, but only k of the $\ddot{q}_r$. The k preferred q_r may be either generalized or quasi-coordinates.

3. Consider the work done in a virtual displacement to get $\sum_{s=1}^{k} Q_s \delta q_s$ and hence the Q_s.

4. Form the equations of motion from Eqns. (14.19) and (14.20).

14.4 Applications

Particle in a Plane. Let a particle in a plane be subjected to a force with radial and transverse components R and S, respectively, as shown on Fig. 14-2. As mentioned previously, either (x, y) or (r, θ) serve as generalized coordinates. We pick coordinates (r, q) defined by

$$r^2 = x^2 + y^2$$

$$dq = xdy - ydx$$

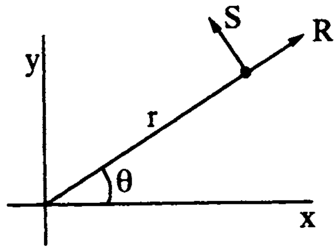


Fig. 14-2

Coordinates r and q are generalized and quasi-coordinates, respectively. To form the Gibbs function G from Eqn. (14.17), $\ddot{x}$ and $\ddot{y}$ are needed:

$$r\dot{r} = x\dot{x} + y\dot{y}$$
$$\dot{r}^2 + r\ddot{r} = \dot{x}^2 + x\ddot{x} + \dot{y}^2 + y\ddot{y}$$

$$\dot{q} = x\dot{y} - y\dot{x}$$
$$\ddot{q} = \dot{x}\dot{y} + x\ddot{y} - \dot{y}\dot{x} - y\ddot{x} = x\ddot{y} - y\ddot{x}$$

These expressions are to be solved for $\ddot{x} = \ddot{x}(\ddot{r}, \dot{r}, r, \ddot{q}, \dot{q}, q)$ and $\ddot{y} = \ddot{y}(\ddot{r}, \dot{r}, r, \ddot{q}, \dot{q}, q)$ and substituted into

$$G = \frac{1}{2}m(\ddot{x}^2 + \ddot{y}^2)$$

The result is

$$G = \frac{1}{2}m\left(\ddot{r}^2 - \frac{2}{r^3}\dot{q}^2\ddot{r} + \frac{1}{r^2}\ddot{q}^2\right)$$

where all terms not having the factors $\ddot{r}$ or $\ddot{q}$ have been omitted because in view of Eqn. (14.19) they do not enter into the equations of motion.

The generalized forces are obtained by considering the virtual work done by R and S. We have

$$\begin{aligned} \dot{q} &= x\dot{y} - y\dot{x} \\ &= (r\cos\theta)(\dot{r}\sin\theta + r\dot{\theta}\cos\theta) \\ -(r\sin\theta)(\dot{r}\cos\theta - r\dot{\theta}\sin\theta) &= r^2\dot{\theta} \end{aligned}$$

Thus

$$dq = r^2 d\theta$$

and

$$\delta q = r^2 \delta\theta$$

and therefore

$$\delta W = R\delta r + Sr\delta\theta = R\delta r + Sr\frac{\delta q}{r^2}$$

Consequently,

$$Q_r = R\,, \qquad Q_q = \frac{S}{r}$$

Now we apply the Gibbs-Appell equations, Eqns. (14.19),

$$\frac{\partial G}{\partial \ddot{r}} = Q_r\,, \qquad \frac{\partial G}{\partial \ddot{q}} = Q_q$$

to obtain

$$m\left(\ddot{r} - \frac{\dot{q}^2}{r^3}\right) = R\,, \qquad \frac{m\ddot{q}}{r^2} = \frac{S}{r}$$

Consider the special case of central force motion with conservative force; in this case

$$S = 0, \quad R = -m\frac{dV}{dr}$$

and

$$\dot{q} = \alpha = \text{constant}, \qquad m\left(\ddot{r} - \frac{\alpha^2}{r^3}\right) = -m\frac{dV}{dr}$$

Using the identity of Eqn. (9.14), the second of these integrates to

$$\dot{r}^2 + 2V + \frac{\alpha^2}{r^2} = 2h = \text{constant}$$

which is the energy integral.

Analogue of Koenig's Theorem. Let G be the center of mass of a rigid body and fix an axis system at G that does not rotate relative to an inertial frame (but the body may rotate) as shown on Fig. 14-3. Recall that Koenig's theorem states that the kinetic energy of the body is given by Eqn. (1.58). Since the Gibbs function is analogous to the kinetic energy, with accelerations replacing velocities, we have immediately, for the same situation,

$$G = \frac{1}{2}M(\ddot{\bar{x}}^2 + \ddot{\bar{y}}^2 + \ddot{\bar{z}}^2) + \frac{1}{2}\sum m_r(\ddot{\zeta}_r^2 + \ddot{\eta}_r^2 + \ddot{\nu}_r^2) \tag{14.21}$$

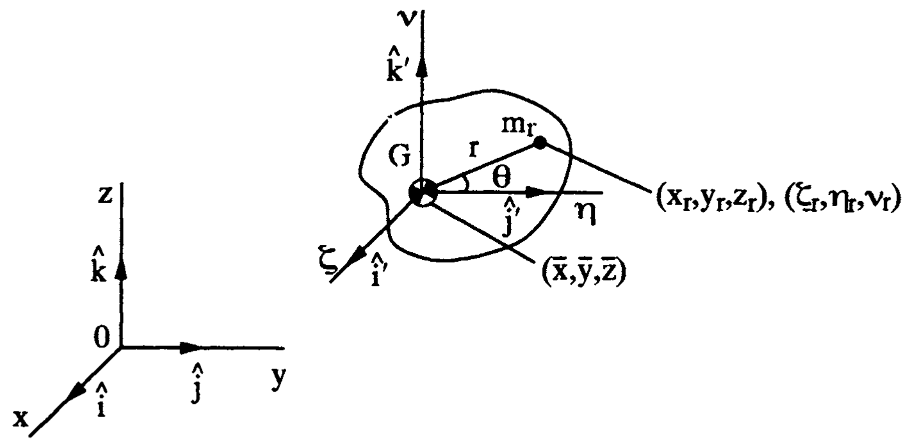


Fig. 14-3

where $M = \sum m_r$ is the mass of the rigid body.

Two-Dimensional Problems. Let a rigid body move in a plane (Fig. 14-4). We note that r is a constant for each particle, but θ varies with time. Consequently,

$$\begin{aligned}\zeta &= r\cos\theta \\ \dot{\zeta} &= -r\sin\theta\dot{\theta} \\ \ddot{\zeta} &= -r\cos\theta\dot{\theta}^2 - r\sin\theta\ddot{\theta}\end{aligned}$$

$$\begin{aligned}\eta &= r\sin\theta \\ \dot{\eta} &= r\cos\theta\dot{\theta} \\ \ddot{\eta} &= -r\sin\theta\dot{\theta}^2 + r\cos\theta\ddot{\theta}\end{aligned}$$

so that

$$\ddot{\zeta}^2 + \ddot{\eta}^2 = r^2\ddot{\theta}^2 + r^2\dot{\theta}^4$$

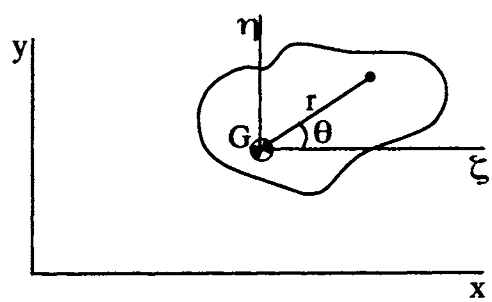


Fig. 14-4

Substituting into Eqn. (14.21),

$$G = \frac{1}{2}Mf^2 + \frac{1}{2}\bar{I}\ddot{\theta}^2 \tag{14.22}$$

where $f^2 = \ddot{\bar{x}}^2 + \ddot{\bar{y}}^2$ is the acceleration of G squared, $\bar{I} = \sum m_r r^2$ is the moment of inertia relative to an axis passing through G and perpendicular to the plane of the motion, and the $r^2\dot{\theta}^4$ term has been omitted because it does not contain any acceleration factors.

Cylinder Rolling in a Cylinder. We first get the rolling without slipping condition (Fig. 14-5) by noting that A' is at A when $\theta = 0$. Letting $c = b - a$,

$$AB = A'B$$
$$b\theta = a(\theta + \phi)$$
$$a\phi = c\theta$$
$$a\dot{\phi} = c\dot{\theta}$$
$$a\ddot{\phi} = c\ddot{\theta}$$

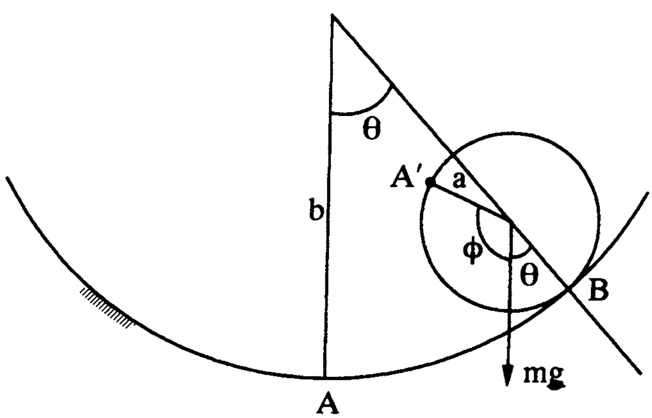


Fig. 14-5

From Eqn. (14.22),

$$G = \frac{1}{2}M(c^2\ddot{\theta}^2 + c^2\dot{\theta}^4) + \frac{1}{2}\left(\frac{1}{2}Ma^2\right)\ddot{\phi}^2$$

Because $k = 1$, we must write this in terms of only one acceleration component; the rolling constraint is used to do this:

$$G = \frac{1}{2}M(c^2\ddot{\theta}^2 + c^2\dot{\theta}^4) + \frac{1}{4}Ma^2\left(\frac{c}{a}\ddot{\theta}\right)^2$$

$$G = \frac{3}{4}Mc^2\ddot{\theta}^2 + \text{terms without } \ddot{\theta}$$

Since the contact force does no virtual work, the only given force doing virtual work is gravity:

$$\delta W = Mg\delta(c\cos\theta) = -Mgc\sin\theta\delta\theta$$

so that $Q_\theta = -Mgc\sin\theta$. Equation (14.19) then gives the equation of motion:

$$-Mgc\sin\theta = \frac{3}{2}Mc^2\ddot{\theta}$$

$$\ddot{\theta} + \frac{2}{3}\frac{g}{c}\sin\theta = 0$$

which is a form of the equation of a simple pendulum.

Sphere Rolling on a Rotating Plane. In the preceeding two examples, the systems were holonomic and the equations of motion could have been obtained by more elementary means. Now we consider a nonholonomic system, the situation in which the use of quasi-coordinates and the Gibbs-Appell equations is particularly advantageous.

Consider a spherical rigid body with radius a and radial mass symmetry (i.e. the mass density depends only on the distance from the center) rolling without slipping on a rotating plane (Fig. 14-6). The plane rotates with variable rate $\Omega(t) \in C^1$ about the z-axis. The $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ frame is fixed (inertial) with origin at the center of rotation and the $\{\hat{\imath}', \hat{\jmath}', \hat{k}'\}$ frame is parallel to the fixed frame with origin at G, the center of mass of the sphere. The rectangular coordinates of the center of mass relative to the fixed frame are (x, y, a). Let the angular velocity of the body be $\underline{\omega} = \omega_x\hat{\imath} + \omega_y\hat{\jmath} + \omega_z\hat{k}$.

If the plane were at rest, the rolling-without-slipping conditions would be $\dot{x} = a\omega_y$ and $\dot{y} = -a\omega_x$. If the sphere were at rest on the rotating

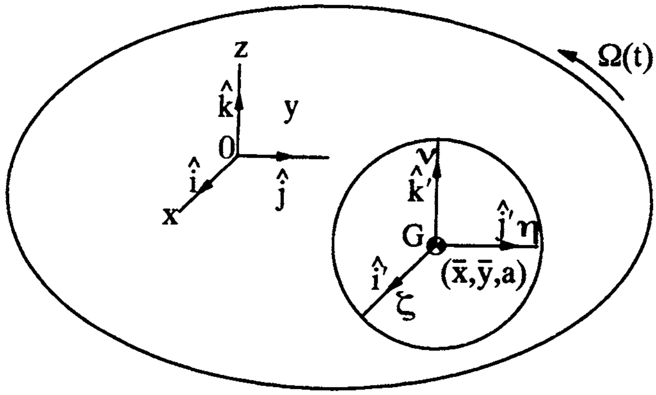


Fig. 14-6

plane, $\dot{x} = -\Omega y$ and $\dot{y} = \Omega x$. Combining the rotating and rolling gives the nonholonomic constraints:

$$\begin{aligned} \dot{x} - a\omega_y &= -\Omega y \\ \dot{y} + a\omega_x &= \Omega x \end{aligned} \tag{14.23}$$

Thus there are one holonomic ($z = a$) and two nonholonomic constraints on the motion so that $L' = 1$, $\ell = 2$ and $k = 3$. We choose the five coordinates x, y, q_x, q_y, q_z where x and y are generalized coordinates and the three quasi-coordinates are defined by

$$\dot{q}_x = \omega_x \,, \qquad \dot{q}_y = \omega_y \,, \qquad \dot{q}_z = \omega_z \tag{14.24}$$

Using the analoque of Koenig's theorem, the Gibbs function is[1]

$$G = \frac{1}{2}M\left(\ddot{x}^2 + \ddot{y}^2\right) + \frac{1}{2}I\left(\ddot{q}_x^2 + \ddot{q}_y^2 + \ddot{q}_z^2\right) \tag{14.25}$$

where all non-essential terms have been omitted, and where I is the moment of inertia of the body about any axis passing through G. (For a body with radial mass symmetry any axis passing through G is a principal axis of inertia and the moment of inertia about all such axes is the same.)

The Gibbs function must now be expressed in terms of the acceleration components of three of the coordinates; we choose x, y, and q_z. Differentiating Eqns. (14.23),

$$\begin{aligned} a\ddot{q}_y &= \ddot{x} + \Omega\dot{y} + \dot{\Omega}y \\ a\ddot{q}_x &= -\ddot{y} + \Omega\dot{x} + \dot{\Omega}x \end{aligned} \tag{14.26}$$

Substituting these relations into Eqn. (14.25) gives

$$\begin{aligned} G &= \frac{1}{2}M\left(\ddot{x}^2+\ddot{y}^2\right)+\frac{I}{2a^2}\left(\ddot{x}+\Omega\dot{y}+\dot{\Omega}y\right)^2 \\ &\quad +\frac{I}{2a^2}\left(-\ddot{y}+\Omega\dot{x}+\dot{\Omega}x\right)^2+\frac{1}{2}I\ddot{q}_z^2 \end{aligned} \tag{14.27}$$

Now suppose that the external force system acting on the body has been resolved into a force $\underline{F} = F_x\hat{\imath}+F_y\hat{\jmath}+F_z\hat{k}$ acting at the center of the sphere and a moment $\underline{M} = M_x\hat{\imath}+M_y\hat{\jmath}+M_z\hat{k}$ about the center. From Eqns. (14.23) and (14.24)

$$dx - a\,dq_y = -\Omega y\,dt$$

$$dy + a\,dq_x = \Omega x\,dt$$

so that virtual displacements satisfy

$$\delta x - a\,\delta q_y = 0$$

$$\delta y + a\,\delta q_x = 0$$

Consequently, the work done in a virtual displacement is

$$\begin{aligned} &F_x\delta x + F_y\delta y + F_z\delta z + M_x\delta q_x + M_y\delta q_y + M_z\delta q_z \\ &\quad = \left(F_x+\frac{M_y}{a}\right)\delta x + \left(F_y-\frac{M_x}{a}\right)\delta y + M_z\delta q_z \end{aligned} \tag{14.28}$$

where, of course, $\delta z = 0$.

We are now in a position to apply the Gibbs-Appell equations, Eqns. (14.19); the result is

$$\begin{aligned} M\ddot{x}+\frac{I}{a^2}\left(\ddot{x}+\Omega\dot{y}+\dot{\Omega}y\right) &= F_x+\frac{M_y}{a} \\ M\ddot{y}+\frac{I}{a^2}\left(\ddot{y}-\Omega\dot{x}-\dot{\Omega}x\right) &= F_y-\frac{M_x}{a} \end{aligned} \tag{14.29}$$

$$I\ddot{q}_z = M_z$$

Consider the following special case: (i) the rotation Ω = const., (ii) the body is a homogeneous sphere, so that $I = \frac{2}{5}Ma^2$, and (iii) there is no external moment acting on the sphere. Then the equations of motion of the mass center reduce to:

$$
\begin{aligned}
7\ddot{x} + 2\Omega\dot{y} &= \frac{5F_x}{M} \\
7\ddot{y} - 2\Omega\dot{x} &= \frac{5F_y}{M}
\end{aligned}
\tag{14.30}
$$

These linear equations are easily solved in terms of convolution integrals.

Notes

1 See Section 11.1

PROBLEMS

14/1. Consider a rigid body moving in space under the action of any given system of forces. Let $\{\hat{i}', \hat{j}', \hat{k}'\}$ be a body-fixed frame aligned with the principal axes of inertia, and $\{\hat{I}, \hat{J}, \hat{K}\}$ be non-moving (inertial) axes. Let the moments of inertia be I_x, I_y, and I_z and let the mass be m. Suppose the resultant force has components F_x, F_y, and F_z along the inertial axes and the resultant moment has components M_x, M_y, and M_y about the center of mass along the body-fixed axes. Then the Gibbs function is (Pars, pp. 216):

$$
G = \frac{1}{2}M\left(\ddot{x}^2 + \ddot{y}^2 + \ddot{z}^2\right) + \frac{1}{2}\Big[I_x\dot{\omega}_x^2 - 2(I_y - I_z)\omega_y\omega_z\dot{\omega}_x + I_y\dot{\omega}_y^2 \\
-2(I_z - I_x)\omega_z\omega_x\dot{\omega}_y + I_z\dot{\omega}_z^2 - 2(I_x - I_y)\omega_x\omega_y\dot{\omega}_z\Big]
$$

where (x, y, z) are the coordinates of the center of mass relative to $\{\hat{I}, \hat{J}, \hat{K}\}$, $\omega_x = \dot{q}_x$, $\omega_y = \dot{q}_y$, and $\omega_z = \dot{q}_z$, and where $\dot{q}_x$, $\dot{q}_y$, and $\dot{q}_z$ are the components along $\{\hat{i}', \hat{j}', \hat{k}'\}$ of the angular velocity of the body. The system is holonomic with 6 DOF. Choose as coordinates x, y, z, which are generalized coordinates, and q_1, q_2, q_3, which are quasi-coordinates.

Use the Gibbs-Appell Eqns. to generate the equations of motion, three of which are called in this case Euler's equations.

14/2. Fill in the details of the particle in a plane problem.

14/3. Fill in the details of the cylinder rolling in a cylinder problem.

14/4. Fill in the details of the sphere rolling on a turntable problem.

Chapter 15

Hamilton's Equations

15.1 Derivation of Hamilton's Equations

Another Fundamental Equation. Consider a holonomic system with all forces embodied in a function $V = V(q_r, t)$. (Strictly speaking, the system is not conservative because in that case $V = V(q_r)$ is required.) From Eqn. (6.28), the fundamental equation in this case is

$$\sum_{r=1}^{n} \left[\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_r} \right) - \frac{\partial L}{\partial q_r} \right] \delta q_r = 0 \tag{15.1}$$

where n (the number of generalized coordinates)=k (the degrees of freedom). Because the system is holonomic, the δq_r are independent, leading directly to Lagrange's equations, Eqns. (6.35).

Since $L = L(q_r, \dot{q}_r, t)$, the variation of L is[1]

$$\delta L = \sum_r \left(\frac{\partial L}{\partial q_r} \delta q_r + \frac{\partial L}{\partial \dot{q}_r} \delta \dot{q}_r \right) \tag{15.2}$$

with, as usual, $\sum\limits_r = \sum\limits_{r=1}^{n}$. Recall that the generalized momenta are defined by Eqn. (8.25), repeated here:

$$p_r = \frac{\partial L}{\partial \dot{q}_r} \, ; \quad r = 1, \cdot\cdot, n \tag{15.3}$$

Thus Eqn. (15.1) implies

$$\dot{p}_r - \frac{\partial L}{\partial q_r} = 0 \, ; \quad r = 1, \cdot\cdot, n \tag{15.4}$$

Substituting Eqns. (15.3) and (15.4) into (15.2) gives

$$\delta L = \sum_r (\dot{p}_r \delta q_r + p_r \delta \dot{q}_r) \tag{15.5}$$

Pars calls this the sixth form of the fundamental equation. Note that Eqns. (15.4) are just Lagrange's equations.

Hamilton's Equations. Define the Hamiltonian function by

$$H = \sum_r p_r \dot{q}_r - L \tag{15.6}$$

Form the variation of this function and use Eqn. (15.5) to get

$$\begin{aligned} \delta H &= \sum_r \dot{q}_r \delta p_r + \sum_r p_r \delta \dot{q}_r - \delta L \\ \delta H &= \sum_r (\dot{q}_r \delta p_r - \dot{p}_r \delta q_r) \end{aligned} \tag{15.7}$$

Next recall that the kinetic energy in generalized coordinates is given by Eqn. (6.3) and therefore

$$p_r = \frac{\partial L}{\partial \dot{q}_r} = \frac{\partial T}{\partial \dot{q}_r} = \sum_s a_{rs} \dot{q}_s + b_r \; ; \quad r = 1, \cdot\cdot, n \tag{15.8}$$

Since a_{rs} is nonsingular, we may solve these equations for the $\dot{q}_s$ in terms of the q_s and p_s. If the result is substituted into Eqn. (15.6), H will be of the form $H = H(q_r, p_r, t)$. Now take the variation of this function:

$$\delta H = \sum_r \frac{\partial H}{\partial q_r} \delta q_r + \sum_r \frac{\partial H}{\partial p_r} \delta p_r \tag{15.9}$$

Combining Eqns. (15.7) and (15.9) gives

$$\sum_r \left[\left(\frac{\partial H}{\partial q_r} + \dot{p}_r \right) \delta q_r + \left(\frac{\partial H}{\partial p_r} - \dot{q}_r \right) \delta p_r \right] = 0 \tag{15.10}$$

But because the δq_r and δp_r are independent, this gives

$$\dot{q}_r = \frac{\partial H}{\partial p_r} \; ; \quad \dot{p}_r = -\frac{\partial H}{\partial q_r} \; ; \quad r = 1, \cdot\cdot, n \tag{15.11}$$

These are *Hamilton's equations.* Their special form is called *canonical.* They give the motion in the *phase space* P, defined by:

$$(\underline{q}, \underline{p}) = \begin{pmatrix} q_1 \\ \vdots \\ q_n \\ p_1 \\ \vdots \\ p_n \end{pmatrix} \in P \subset I\!E^{2n} \qquad (15.12)$$

Note that the first set of Eqns. (15.11) follows directly from the definition of H and is equivalent to the set Eqns. (15.3), which is the definition of the p_r. The second set expresses the dynamics. Although these two sets of equations have different meanings, they are to be treated mathematically as of equal stature.

Next we compute the total time derivative of H; using Eqns. (15.11) this is

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} + \sum_r \frac{\partial H}{\partial q_r}\dot{q}_r + \sum_r \frac{\partial H}{\partial p_r}\dot{p}_r = \frac{\partial H}{\partial t} \qquad (15.13)$$

Natural Systems. Recall that (Section 3.5) a natural system is one that is holonomic, scleronomic, and conservative. In such a system, $T = T(q_r, \dot{q}_r)$ and $V = V(q_r)$. Thus, $H \neq H(t)$ and from Eqn. (15.13) $H =$ constant. Also, for such a system Eqns. (6.4) give $b_r = 0$ so that Eqns. (15.8) become

$$p_r = \sum_s a_{rs}\dot{q}_s \; ; \quad r = 1, \cdot\cdot, n \qquad (15.14)$$

Consequently, using this and Eqns. (15.6) and (6.6), the H function becomes

$$\begin{aligned} H &= \sum_r \sum_s a_{rs}\dot{q}_s\dot{q}_r - (T - V) = 2T - T + V \\ &= T + V = E = h = \text{constant}. \end{aligned} \qquad (15.15)$$

Thus for a natural system, the Hamiltonian is equal to the system mechanical energy and is an integral of the motion.

An explicit form of H for a natural system[2] may be obtained as follows. If Eqns. (15.14) are inverted, there results

$$\dot{q}_s = \sum_r c_{sr} p_r \qquad (15.16)$$

Substitute this into Eqn. (15.6) to obtain

$$\begin{aligned} H &= \sum_r \sum_s p_r c_{rs} p_s - \left(\frac{1}{2} \sum_r \sum_s a_{rs} \dot{q}_r \dot{q}_s - V \right) \\ &= \frac{1}{2} \sum_r \sum_s p_r c_{rs} p_s + V \end{aligned} \tag{15.17}$$

General Systems. If there are nonholonomic constraints and forces not derivable from a function $V(q_r, t)$, Eqns. (6.34), (8.25), and (15.6) lead to

$$\dot{q}_r = \frac{\partial H}{\partial p_r}\,; \qquad \dot{p}_r = -\frac{\partial H}{\partial q_r} + Q_r^{nc} - \sum_{s=1}^{\ell} \lambda_s B_{sr}\,; \quad r = 1, \cdot\cdot, n \tag{15.18}$$

which are no longer of canonical form. In what follows, we will consider only holonomic systems with forces derivable from $V(q_r, t)$, that is the canonical Eqns. (15.11).

Remarks.

1. The functions L and H are sometimes termed *descriptive functions*, because once they are known for a given system the equations of motion of the system can be produced. In subsequent chapters we will identify other such functions.

2. The important difference between L and H is that we regard $L = L(q_r, \dot{q}_r, t)$ and $H = H(q_r, p_r, t)$.

3. As remarked in Section 1.6, one of the main aims of analytical dynamics is to find integrals of the motion (i.e. to "solve" the dynamics problem in whole or in part). This will be the primary motivation for the developments in the remaining chapters.

15.2 Hamilton's Equations as a First Order System

Remarks. Recall from Section 12.1 that Lagrange's equations always may be written in state variable form, that is as a system of uncoupled first order differential equations. It is clear that Hamilton's equations, Eqns. (15.11), come naturally in this form. This will now be made explicit.

Hamilton's Equations in First Order Form. Set

$$\begin{array}{ll} x_1 = q_1 & x_{n+1} = p_1 \\ \vdots & \vdots \\ x_n = q_n & x_{2n} = p_n \end{array} \tag{15.19}$$

$$\begin{array}{ll} X_1 = \dfrac{\partial H}{\partial p_1} & X_{n+1} = -\dfrac{\partial H}{\partial q_1} \\ \vdots & \vdots \\ X_n = \dfrac{\partial H}{\partial p_n} & X_{2n} = -\dfrac{\partial H}{\partial q_n} \end{array} \tag{15.20}$$

Then Hamilton's equations are in state variable form.

We can also write these equations compactly in matrix form. Let

$$x = \begin{pmatrix} q_1 \\ \vdots \\ q_n \\ p_1 \\ \vdots \\ p_n \end{pmatrix}; \qquad H_Z = \begin{pmatrix} \partial H/\partial q_1 \\ \vdots \\ \partial H/\partial q_n \\ \partial H/\partial p_1 \\ \vdots \\ \partial H/\partial p_n \end{pmatrix}$$

and define the $2n \times 2n$ matrix Z by

$$Z = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

where I and 0 are the $n \times n$ identity and zero matrices, respectively. Then Eqns. (15.11) are

$$\dot{x} = ZH_Z \tag{15.21}$$

15.3 Examples

Example. Use of Hamilton's equations is often a convenient method for solving specific problems, and we first illustrate this use by obtaining the

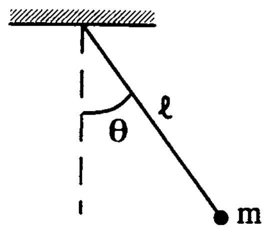


Fig. 15-1

equation of motion of the simple pendulum (Fig. 15-1). For this system,

$$\begin{aligned} T &= \frac{1}{2}m\ell^2\dot{\theta}^2\,, \quad V = mg\ell(1-\cos\theta) \\ L &= \frac{1}{2}m\ell^2\dot{\theta}^2 - mg\ell(1-\cos\theta) \\ p_\theta &= \frac{\partial L}{\partial \dot{\theta}} = m\ell^2\dot{\theta} \\ H &= p_\theta\dot{\theta} - L = \frac{1}{2}m\ell^2\dot{\theta}^2 + mg\ell(1-\cos\theta) \\ &= \frac{1}{2}\frac{p_\theta^2}{m\ell^2} + mg\ell(1-\cos\theta) \end{aligned}$$

Applying Eqns. (15.11),

$$\begin{aligned} \dot{\theta} &= \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{m\ell^2} \Longrightarrow \dot{p}_\theta = m\ell^2\ddot{\theta} \\ \dot{p}_\theta &= -\frac{\partial H}{\partial \theta} = -mg\ell\sin\theta \end{aligned}$$

Thus,

$$\ddot{\theta} + \frac{g}{\ell}\sin\theta = 0$$

Example.[3] We next consider a more substantial example. Figure 15-2 shows two bodies of equal mass connected by a rigid, massless tether. The system is traveling in a planar earth orbit. Only the motion of the mass centers of the bodies is of interest. It is desired to obtain the equations of motion of the system.

The system is natural with three degrees of freedom; we choose $(r,\ \rho,\ \theta)$ as generalized coordinates. The length z is a known function

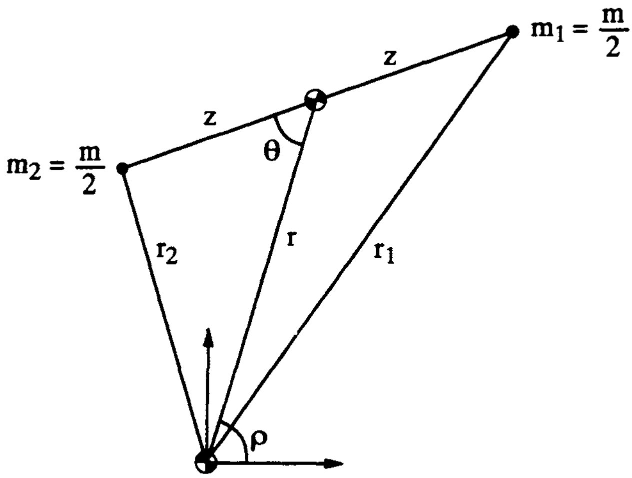


Fig. 15-2

of time. The potential and kinetic energies are (using Koenig's theorem for the latter),

$$V = -\frac{m}{2}\frac{\mu}{r_1} - \frac{m}{2}\frac{\mu}{r_2}$$

$$T = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\rho}^2\right) + \frac{m}{2}\left(\dot{z} + z^2(\dot{\rho} - \dot{\theta})^2\right)$$

where

$$r_1^2 = r^2 + z^2 + 2zr\cos\theta$$
$$r_2^2 = r^2 + z^2 - 2zr\cos\theta$$

The L function and the generalized momenta are

$$L = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\rho}^2 + \dot{z}^2 + z^2(\dot{\rho} - \dot{\theta})^2\right) + \frac{m\mu}{2}\left(\frac{1}{r_1} + \frac{1}{r_2}\right)$$

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = -mz^2(\dot{\rho} - \dot{\theta})$$

$$p_\rho = \frac{\partial L}{\partial \dot{\rho}} = m\left(r^2\dot{\rho} + z^2(\dot{\rho} - \dot{\theta})\right)$$

Consequently, the Hamiltonian function is

$$
\begin{aligned}
H &= \sum_{r=1}^{3} p_r \dot{q}_r - L = p_r \dot{r} + p_\theta \dot{\theta} + p_\rho \dot{\rho} - L \\
&= \frac{1}{2m}\left(p_r^2 + \frac{1}{r^2}(p_\rho + p_\theta)^2 + \frac{p_\theta^2}{z^2}\right) - \frac{m}{2}\dot{z}^2 - \frac{m\mu}{2}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)
\end{aligned}
$$

Hamilton's equations then give the equations of motion:

$$
\begin{aligned}
\dot{r} &= \frac{\partial H}{\partial p_r} = \frac{1}{m} p_r \\
\dot{\rho} &= \frac{\partial H}{\partial p_\rho} = \frac{1}{mr^2}(p_\rho + p_\theta) \\
\dot{\theta} &= \frac{\partial H}{\partial p_\theta} = \frac{1}{mr^2}(p_\rho + p_\theta) + \frac{1}{mz^2} p_\theta \\
\dot{p}_r &= -\frac{\partial H}{\partial r} = \frac{1}{mr^3}(p_\rho + p_\theta)^2 - \frac{m\mu}{2}\left[\frac{1}{r_1^3}(r + z\cos\theta)\right. \\
&\quad \left. + \frac{1}{r_2^3}(r - z\cos\theta)\right] \\
\dot{p}_\rho &= -\frac{\partial H}{\partial \rho} = 0 \qquad (\rho \text{ is ignorable}) \\
\dot{p}_\theta &= -\frac{\partial H}{\partial \theta} = \frac{m\mu z r \sin\theta}{2}\left(\frac{1}{r_1^3} - \frac{1}{r_2^3}\right)
\end{aligned}
$$

Of particular interest in applications is the "spoke equilibrium", defined by

$$
r = r_0 \,, \quad \rho = \omega t \,, \quad \theta = 0 \,, \quad z = z_0
$$

where r_0, ω and z_0 are constants. It may be shown (see Problems) that this equilibrium is possible only for a certain specific value of ω.

15.4 Stability of Hamiltonian Systems

Variational Equations. Here we write Eqns. (15.11) in vector form as

$$
\dot{q} = H_p \,; \qquad \dot{p} = -H_q \tag{15.22}
$$

where

$$q = \begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix} ; p = \begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix} ; \quad H_p = \begin{pmatrix} \frac{\partial H}{\partial p_1} \\ \vdots \\ \frac{\partial H}{\partial p_n} \end{pmatrix} ; H_q = \begin{pmatrix} \frac{\partial H}{\partial q_1} \\ \vdots \\ \frac{\partial H}{\partial q_n} \end{pmatrix}$$

Only the case of H not an explicit function of t will be considered.

Now suppose $q^*(t)$, $p^*(t)$ is a reference motion satisfying Hamilton's equations, and that a perturbed motion is

$$q(t) = q^*(t) + \alpha(t) ; \qquad p(t) = p^*(t) + \beta(t)$$

where $\alpha(t)$ and $\beta(t)$ are small perturbations. Substituting these into Eqns. (15.22), expanding, and retaining only first order terms,[4]

$$\begin{aligned} \dot{q}^* + \dot{\alpha} &= H_p(q^* + \alpha, p^* + \beta) = H_p(q^*, p^*) + H_{pq}(q^*, p^*)\alpha \\ &\quad + H_{pp}(q^*, p^*)\beta \end{aligned}$$

$$\begin{aligned} \dot{p}^* + \dot{\beta} &= -H_q(q^* + \alpha, p^* + \beta) = -H_q(q^*, p^*) - H_{qq}(q^*, p^*)\alpha \\ &\quad - H_{qp}(q^*, p^*)\beta \end{aligned}$$

But since q^*, p^* satisfy Eqns. (15.22),

$$\begin{aligned} \dot{\alpha} &= H^*_{pq}\alpha + H^*_{pp}\beta \\ \dot{\beta} &= -H^*_{qq}\alpha - H^*_{qp}\beta \end{aligned} \tag{15.23}$$

These equations are of Hamiltonian form with Hamiltonian

$$H' = \frac{1}{2}\alpha^T H^*_{qq}\alpha + \beta^T H^*_{pq}\alpha + \frac{1}{2}\beta^T H^*_{pp}\beta$$

Stability of Motion. Recognizing that $H^*_{qp} = H^{*T}_{pq}$, where T denotes transpose, Eqns. (15.23) may be written

$$\begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \end{pmatrix} = \begin{pmatrix} H^*_{pq} & H_{pp} \\ -H^*_{qq} & -H^{*T}_{pq} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \tag{15.24}$$

It may be shown that the eigenvalues of such a matrix occur in positive and negative pairs. Further, since the coefficients of the matrix in Eqn.

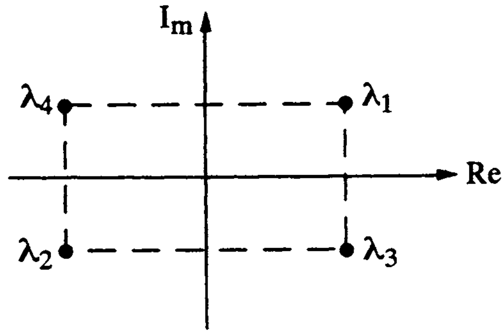


Fig. 15-3

(15.24) are real, the eigenvalues occur in complex conjugate pairs. Thus if $\lambda_1 = a + ib$ is one eigenvalue then so are $\lambda_2 = -a - ib$, $\lambda_3 = a - ib$, and $\lambda_4 = -a + ib$ (some of these may not be distinct; for example if $b = 0$ then $\lambda_1 = \lambda_3$ and $\lambda_2 = \lambda_4$). Plotted in the complex plane, these eigenvalues exhibit a "box form" (Fig. 15-3). Since eigenvalues with positive real parts and those with negative real parts denote unstable and stable modes, respectively, speaking loosely we may say that the system is "half stable and half unstable", unless, of course, some eigenvalues have zero real parts. This stability property may cause numerical problems when integrating Hamilton's equations.

15.5 Poisson Brackets

Definitions. Recall from Section 8.1 that an *integral of the motion* is a function that remains constant along a solution in state-time space (x, t), where we now take the state x as $x = (q_r, p_r)$:

$$F(x, t) = \text{constant}. \tag{15.25}$$

Taking the total derivative of this:

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \sum_{r=1}^{2n} \frac{\partial F}{\partial x_r} \dot{x}_r = 0 \tag{15.26}$$

$$\frac{\partial F}{\partial t} + \sum_{r=1}^{n} \left(\frac{\partial F}{\partial q_r} \frac{\partial H}{\partial p_r} - \frac{\partial F}{\partial p_r} \frac{\partial H}{\partial q_r} \right) = 0 \tag{15.27}$$

where Eqns. (15.11) were used.

Define the *Poisson bracket* of F and H by

$$(F,H) = \sum_{r=1}^{n} \left(\frac{\partial F}{\partial q_r}\frac{\partial H}{\partial p_r} - \frac{\partial F}{\partial p_r}\frac{\partial H}{\partial q_r} \right) = \sum_{r=1}^{n} \frac{\partial(F,H)}{\partial(q_r,p_r)} \qquad (15.28)$$

Thus Eqn. (15.27) may be written as

$$\frac{\partial F}{\partial t} + (F,H) = 0 \qquad (15.29)$$

This equation is satisfied by any function F that is an integral of the motion. Consider (x_r, H); by definition:

$$(x_r, H) = \sum_{s} \left(\frac{\partial x_r}{\partial q_s}\frac{\partial H}{\partial p_s} - \frac{\partial x_r}{\partial p_s}\frac{\partial H}{\partial q_s} \right)$$

Since $x_r = q_r$ for $r = 1, \cdot\cdot, n$ and $x_r = p_r$ for $r = n+1, \cdot\cdot, 2n$, and the q_r and p_r are independent, this reduces to

$$(x_r, H) = \frac{\partial H}{\partial p_r} = \dot{q}_r \; ; \quad r = 1, \cdot\cdot, n$$

$$(x_r, H) = -\frac{\partial H}{\partial q_{r-n}} = \dot{p}_{r-n} \; ; \quad r = n+1, \cdot\cdot, 2n$$

where Eqns. (15.11) were used. Consequently Hamilton's equations in terms of Poisson's brackets are

$$\dot{x}_r = (x_r, H) \; ; \quad r = 1, \cdot\cdot, 2n \qquad (15.30)$$

Properties. Let u, v, and w be class C^2 functions of (q_r, p_r, t) and let c be a constant. Then the following properties of the Poisson brackets follow directly from Eqn. (15.28):

(i) $(u,u) = (u,c) = (c,u) = 0$

(ii) $(v,u) = (-u,v) = (u,-v) = -(u,v)$

(iii) $\dfrac{\partial}{\partial t}(u,v) = \left(\dfrac{\partial u}{\partial t}, v\right) + \left(u, \dfrac{\partial v}{\partial t}\right)$

(iv) $(u,(v,w)) + (v,(w,u)) + (w,(u,v)) = 0$

Poisson's Theorem. If ϕ and ψ are functions of class C^2 and are integrals of Hamilton's equations then (ϕ, ψ) is also an integral.

This result provides a means of constructing a new integral of the motion if at least two are already known. This new integral, however, may or may not be independent of the two used to generate it. It is obvious that independent integrals cannot be constructed indefinitely by this method, because only $2n$ such integrals exist. Furthermore, sometimes the new integral produced is identically zero.

We now prove the theorem. Since ϕ and ψ are integrals, Eqn. (15.29) gives

$$\frac{\partial \phi}{\partial t} + (\phi, H) = 0\,, \qquad \frac{\partial \psi}{\partial t} + (\psi, H) = 0$$

We need to show that

$$\frac{\partial}{\partial t}(\phi, \psi) + ((\phi, \psi), H) = 0$$

Using the properties of the Poisson bracket stated above and Eqn. (15.29),

$$\begin{aligned}\frac{\partial}{\partial t}(\phi, \psi) + ((\phi, \psi), H) &= \left(\frac{\partial \phi}{\partial t}, \psi\right) + \left(\phi, \frac{\partial \psi}{\partial t}\right) + ((\phi, \psi), H) \\ &= -((\phi, H), \psi) - (\phi, (\psi, H)) + ((\phi, \psi), H) \\ &= (\psi, (\phi, H)) + (\phi, (H, \psi)) + (H, (\psi, \phi)) = 0\end{aligned}$$

An important special case is when the system is natural. Then $H \neq H(t)$ and, in view of property (i), $F = H$ satisfies Eqn. (15.29) and consequently H is a constant of the motion, which of course we already knew.

Example – Central Force Motion. Consider a particle of unit mass subject to a conservative central force (Fig. 15-4). For this system,

$$r^2 = x^2 + y^2 + z^2$$

$$T = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

$$H = \sum_{i=1}^{3} p_i \dot{q}_i - L = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) + V(r)$$

It is known that in this system angular momentum is conserved but linear momentum is not. We show this by using Poisson's brackets. The

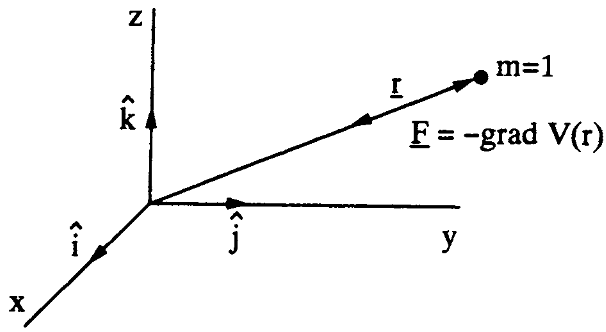


Fig. 15-4

angular momentum and its z component are

$$\underline{\ell} = \underline{r} \times \underline{v} = (x\hat{i} + y\hat{j} + z\hat{k}) \times (\dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k})$$

$$\ell_z = \underline{\ell} \cdot \hat{k} = x\dot{y} - y\dot{x} = xp_y - yp_x$$

This is an integral of the motion because

$$\begin{aligned}\frac{\partial \ell_z}{\partial t} + (\ell_z, H) &= 0 + \sum_{r=1}^{3} \left(\frac{\partial \ell_z}{\partial q_r}\frac{\partial H}{\partial p_r} - \frac{\partial \ell_z}{\partial p_r}\frac{\partial H}{\partial q_r} \right) \\ &= p_y p_x + y\frac{\partial V}{\partial x} - p_x p_y - x\frac{\partial V}{\partial y} = y\frac{\partial V}{\partial x} - x\frac{\partial V}{\partial y} \\ &= y\frac{x}{r}\frac{dV}{dr} - x\frac{y}{r}\frac{dV}{dr} = 0\end{aligned}$$

where the properties of the Poisson brackets were used and where

$$\frac{\partial V}{\partial x} = \frac{\partial V}{\partial r}\frac{\partial r}{\partial x} = \frac{x}{r}\frac{dV}{dr}, \qquad \frac{\partial V}{\partial y} = \frac{\partial V}{\partial r}\frac{\partial r}{\partial y} = \frac{y}{r}\frac{dV}{dr}$$

We can show that, similarly, ℓ_y, the y component of $\underline{\ell}$, is an integral of the motion. Next Poisson's theorem is used to get a third constant of the motion:

$$(\ell_z, \ell_y) = \sum_{r=1}^{3} \left(\frac{\partial \ell_z}{\partial q_r}\frac{\partial \ell_y}{\partial p_r} - \frac{\partial \ell_z}{\partial p_r}\frac{\partial \ell_y}{\partial q_r} \right) = p_y z - y p_z = \ell_x$$

In fact, taking the Poisson bracket of any two components of $\underline{\ell}$ gives the third.

Finally we show that the components of the linear momentum are not integrals of the motion. The linear momentum is $\underline{h} = \underline{v}$ and its z component is

$$h_z = \underline{v} \cdot \hat{k} = \dot{z} = p_z$$

Thus

$$\frac{\partial h_z}{\partial t} + (h_z, H) = \sum_{r=1}^{3} \left(\frac{\partial h_z}{\partial q_r} \frac{\partial H}{\partial p_r} - \frac{\partial h_z}{\partial p_r} \frac{\partial H}{\partial q_r} \right) = -\frac{z}{r} \frac{dV}{dr} \neq 0$$

and similarly for the other two components.

15.6 Reduction of System Order

Use of the Energy Integral. For a natural system, the Hamiltonian does not depend on time explicitly and $H = T + V$ is an integral of the motion:

$$H(q_1, \cdot\cdot, q_n, p_1, \cdot\cdot, p_n) = h = \text{constant.} \tag{15.31}$$

Suppose that we solve this equation for one of the q_r, say q_1:

$$q_1 = \phi(q_2, \cdot\cdot, q_n, p_1, \cdot\cdot, p_n, h) \tag{15.32}$$

Now substitute this into Eqn. (15.31) and take the derivatives with respect to the last $n - 1$ of the p_r:

$$\frac{\partial H}{\partial p_r} + \frac{\partial H}{\partial q_1} \frac{\partial \phi}{\partial p_r} = 0 \, ; \qquad r = 2, \cdot\cdot, n \tag{15.33}$$

Therefore, using Eqns. (15.11),

$$\frac{dq_r}{dp_1} = \frac{\dot{q}_r}{\dot{p}_1} = -\frac{\frac{\partial H}{\partial p_r}}{\frac{\partial H}{\partial q_1}} = \frac{\partial \phi}{\partial p_r} \, ; \qquad r = 2, \cdot\cdot, n \tag{15.34}$$

Similarly, substituting Eqn. (15.32) into Eqn. (15.31) and differentiating with respect to the last $n - 1$ of the q_r:

$$\frac{\partial H}{\partial q_r} + \frac{\partial H}{\partial q_1} \frac{\partial \phi}{\partial q_r} = 0 \tag{15.35}$$

which gives

$$\frac{dp_r}{dp_1} = \frac{\dot{p}_r}{\dot{p}_1} = \frac{\frac{\partial H}{\partial q_r}}{\frac{\partial H}{\partial q_1}} = -\frac{\partial \phi}{\partial q_r}; \qquad r = 2, \cdot\cdot, n \tag{15.36}$$

Equations (15.34) and (15.36) are a new Hamiltonian system with p_1 the independent variable (taking the role formerly played by t) and with ϕ the Hamiltonian function (taking the role formerly played by H). Note that the new Hamiltonian system is of order $2(n-1)$ and that the new Hamiltonian system is nonautonomous because ϕ is a function of independent variable p_1.

Example. Consider a system for which the Hamiltonian is

$$H = \frac{1}{2}(p_x^2 + p_y^2) - kyp_x + \left(\frac{1}{2}k^2y^2 - gx\right)$$

where x and y are generalized coordinates and k and g are positive constants. Solving $H = h$ for x we have

$$x = \phi(y, p_x, p_y, h) = \frac{1}{2g}\left[(p_x - ky)^2 + p_y^2\right] - \frac{h}{g}$$

Using ϕ as the new Hamiltonian, p_x as the independent variable, and y and p_y as the remaining dependent variables, Eqns. (15.34) and (15.36) give

$$\frac{dy}{dp_x} = \frac{\partial \phi}{\partial p_y} = \frac{p_y}{g}$$

$$\frac{dp_y}{dp_x} = -\frac{\partial \phi}{\partial y} = \frac{k}{g}(p_x - ky)$$

The solution of these equations is obtained as follows:

$$\frac{d^2p_y}{dp_x^2} = \frac{k}{g} - \frac{k^2}{g}\frac{dy}{dp_x} = \frac{k}{g} - \frac{k^2}{g^2}p_y$$

$$p_y = \frac{g}{k} + A\cos\frac{k}{g}p_x + B\sin\frac{k}{g}p_x$$

$$y = \frac{1}{g}\int\left(\frac{g}{k} + A\cos\frac{k}{g}p_x + B\sin\frac{k}{g}p_x\right)dp_x + C$$

$$y = \frac{p_x}{k} + \frac{A}{k}\sin\frac{k}{g}p_x - \frac{B}{k}\cos\frac{k}{g}p_x + C$$

The variable p_x is obtained from Eqns. (15.11) as

$$\dot{p}_x = -\frac{\partial H}{\partial x} = g$$

$$p_x = gt + D$$

We now have all the equations necessary to express the solution of the problem as

$$x = x(t; A, B, C, D) \; ; \qquad y = y(t; A, B, C, D)$$

Use of a Momentum Integral. When there is an ignorable coordinate, the corresponding momentum integral may be used in the same way as the energy integral to reduce the order of a Hamiltonian system.[5]

Theorem of the Last Multiplier. If we have found $2n-2$ integrals, this theorem tells us how to find the last two. Consider a system of first order differential equations:

$$\frac{dx_r}{dt} = X_r(x_r) \; ; \qquad r = 1, \cdot\cdot, m \tag{15.37}$$

or equivalently

$$\frac{dx_1}{X_1} = \frac{dx_2}{X_2} = \cdot\cdot = \frac{dx_m}{X_m} \tag{15.38}$$

Suppose we have found $(m-2)$ independent integrals of the motion given by

$$f_r(x_r, t) = c_r \; ; \qquad r = 1, \cdot\cdot, m-2 \tag{15.39}$$

Then to complete the solution for the trajectories we need only to integrate

$$\frac{dx_{m-1}}{X'_{m-1}} = \frac{dx_m}{X'_m}$$

which is equivalent to

$$X'_m dx_{m-1} - X'_{m-1} dx_m = 0 \tag{15.40}$$

where $X'_{m-1}(x_{m-1}, x_m) = X_{m-1}(c_1, \cdot\cdot, c_{m-2}, x_{m-1}, x_m)$ and $X'_m(x_{m-1}, x_m) = X_m(c_1, \cdot\cdot, c_{m-2}, x_{m-1}, x_m)$. *Jacobi's theorem of the last multiplier* (TLM)[6] then states that an additional integral of motion is given by

$$f_{m-1} = \int \frac{M'}{K'}(X'_m dx_{m-1} - X'_{m-1} dx_m) = c_{m-1} \tag{15.41}$$

where

$$K = \frac{\partial(f_1, \cdot\cdot, f_{m-2})}{\partial(x_1, \cdot\cdot, x_{m-2})}$$

and M is any solution of the partial differential equation

$$\frac{\partial}{\partial x_1}(MX_1) + \frac{\partial}{\partial x_2}(MX_2) + \cdot\cdot + \frac{\partial}{\partial x_m}(MX_m) = 0 \tag{15.42}$$

The functions $M(x_1, \cdot\cdot, x_m)$ satisfying this equation are called the *multipliers*[7] for the system of Eqns. (15.37).

The final integral, to obtain the time, is then obtained by one more application of the theorem. The final system equation is

$$\frac{dx_m}{X_m} = \frac{dt}{1} \tag{15.43}$$

The TLM then provides an integrating factor for

$$dx_m - X'_m dt = 0 \tag{15.44}$$

where $X'_m(x_m) = X_m(c_1, \cdot\cdot, c_{m-1}, x_m)$. Although we have considered an autonomous system, the TLM works for nonautonomous systems as well.

Application of the TLM to Hamiltonian Systems. Now consider an autonomous Hamiltonian system with two degrees of freedom. Then $m = 2n = 4$ and Eqns. (15.38) are

$$\frac{dq_1}{\partial H/\partial p_1} = \frac{dq_2}{\partial H/\partial p_2} = \frac{dp_1}{-\partial H/\partial q_1} = \frac{dp_2}{-\partial H/\partial q_2} \tag{15.45}$$

where $H = H(q_1, q_2, p_1, p_2)$. Assume that two integrals of the motion are known, the energy integral plus one other; thus

$$\begin{aligned} H(q_1, q_2, p_1, p_2) &= h \\ F(q_1, q_2, p_1, p_2) &= \alpha \end{aligned} \tag{15.46}$$

It is assumed that the Jacobian

$$J = \frac{\partial(H,F)}{\partial(p_1,p_2)} = \frac{\partial H}{\partial p_1}\frac{\partial F}{\partial p_2} - \frac{\partial H}{\partial p_2}\frac{\partial F}{\partial p_1} \tag{15.47}$$

is not zero. For this system, $M = 1$ is a multiplier and we use it. Then Eqn. (15.41) gives a third integral as

$$\int \frac{1}{J}\left(\frac{\partial H}{\partial p_2}dq_1 - \frac{\partial H}{\partial p_1}dq_2\right) = \text{constant} \tag{15.48}$$

where the coefficients are expressed in terms of $q_1, q_2, h,$ and α.

The integral may be put into a different form. Suppose Eqns. (15.46) are solved for p_1 and p_2:

$$p_1 = f_1(q_1,q_2,h,\alpha)\ ; \qquad p_2 = f_2(q_1,q_2,h,\alpha)$$

Now differentiate Eqns. (15.46) with respect to α:

$$\begin{aligned} \frac{\partial H}{\partial p_1}\frac{\partial f_1}{\partial \alpha} + \frac{\partial H}{\partial p_2}\frac{\partial f_2}{\partial \alpha} &= 0 \\ \frac{\partial F}{\partial p_1}\frac{\partial f_1}{\partial \alpha} + \frac{\partial F}{\partial p_2}\frac{\partial f_2}{\partial \alpha} &= 1 \end{aligned} \tag{15.49}$$

Using Eqns. (15.47) and (15.49) in (15.48), we arrive at

$$\int \left(\frac{\partial f_1}{\partial \alpha}dq_1 + \frac{\partial f_2}{\partial \alpha}dq_2\right) = \text{constant} \tag{15.50}$$

Thus $f_1\, dq_1 + f_2 dq_2$ is a perfect differential and there is a function $K(q_1,q_2,h,\alpha)$ such that $dK = f_1 dq_1 + f_2 dq_2$ and the third integral, Eqn. (15.50), may be written as

$$\int \frac{\partial K}{\partial \alpha} = \text{constant} \tag{15.51}$$

Equating the expressions in Eqn. (15.45) to dt and using the same procedure, we arrive at the fourth integral,

$$\int \frac{\partial K}{\partial h} = t + \text{constant} \tag{15.52}$$

In summary, the four integrals of the motion are

$$\begin{aligned} &H = h \\ &F = \alpha \\ &\int \frac{\partial K}{\partial \alpha} = -\beta \\ &\int \frac{\partial K}{\partial h} = t + t_0 \end{aligned} \tag{15.53}$$

In many specific problems, the second integral will be a momentum integral corresponding to an ignorable coordinate. Suppose q_2 is ignorable; then two known integrals are

$$H(q_1, p_1, p_2) = h\,; \qquad p_2 = \alpha \tag{15.54}$$

In this case, $f_2 = \alpha$ and $f_1(q_1, h, \alpha)$ is obtained by solving $H(q_1, p_1, \alpha) = h$ for p_1. Now $dK = f_1 dq_1 + \alpha dq_2$ and the last two of Eqns. (15.53) give the third and fourth integrals as

$$\begin{aligned} \int \frac{\partial f_1}{\partial \alpha} dq_1 + q_2 &= -\beta \\ \int \frac{\partial f_1}{\partial h} dq_1 &= t - t_0 \end{aligned} \tag{15.55}$$

Example. Consider a particle of unit mass in a central force field with potential energy function $V(r)$. From Eqns. (7.11) and (8.24), $p_r = \dot{r}$ and $p_\theta = r^2\dot{\theta}$ so that Eqn. (15.15) gives

$$H = \frac{1}{2}\left(p_r^2 + \frac{1}{r^2}p_\theta^2\right) + V$$

Since energy is conserved and θ is ignorable, two integrals of the motion are

$$H(r, p_r, p_\theta) = h$$
$$p_\theta = \alpha$$

Solving the first of these for p_r gives

$$p_r = f_r(r, h, \alpha) = \sqrt{2h - 2V - \frac{\alpha^2}{r^2}}$$

Equations (15.55) then give the other two integrals as

$$\begin{aligned} -\int_{r_0}^{r} \frac{(\alpha/\zeta^2)}{f_r(\zeta)} d\zeta + \theta &= -\beta \\ \int_{r_0}^{r} \frac{d\zeta}{f_r(\zeta)} &= t - t_0 \end{aligned}$$

which are the same equations as were obtained in Chapter 10.

Notes

1 Recall from Section 3.3 that in the δ operation t is not varied.

2 Pars gives explicit forms of H for types of systems other than natural.

3 Anderson, K.S., and Hagedorn, P., "Control of Orbital Drift of Geostationary Tethered Satellites", *Journal of Guidance, Control, and Dynamics*, Vol. 17, No. 1, Jan–Feb 1994.

4 Subscripts here will denote partial derivatives; for example, H_{pq} is the matrix $\|\partial^2 H/\partial p_r \partial q_r\|$.

5 See Pars for the details.

6 See Pars or Whittaker for the proof.

7 Not to be confused with Lagrange multipliers.

PROBLEMS

Obtain the equations of motion using Hamilton's equations for the systems described in the following six problems:

15/1. Problem 4/2.

15/2. Problem 6/7.

15/3. Problem 6/10.

15/4. Problem 7/1, with $X_0 = Y_0 = Z_0 = 0$.

15/5. Problem 7/3.

15/6. Problem 10/1.

15/7. Prove properties (i) – (iii) of Poisson's brackets.

15/8. Prove property (iv) of Poisson's brackets.

Chapter 16

Contact Transformations

16.1 Introduction

The Nature of Hamiltonian Dynamics. In the last chapter we have seen that for solving specific problems, use of Hamilton's equations offers no particular advantage over Lagrange's equations; the procedures and the amount of work required are essentially the same. Rather, the Hamiltonian formulation offers a new point of view, one that will be exploited in this and the following chapters.

In the Lagrangean formulation, q_i and $\dot{q}_i$ are regarded as the independent variables. There is an obvious connection between the two sets q_i and $\dot{q}_i$, however, the latter being the derivatives of the former. In the Hamiltonian formulation, the variables q_i and p_i must be regarded as truly independent. The motion is now envisioned as the motion of a point in phase space (see Eqn. (15.12)). In modern mathematical terms, the motion of a dynamical system defines a *continuous group* of transformations in phase space that carry the point

$$(q_1^0, \cdot\cdot, q_n^0,\ p_1^0, \cdot\cdot, p_n^0)$$

at $t = 0$ to the point

$$(q_1, \cdot\cdot, q_n,\ p_1, \cdot\cdot, p_n)$$

at time t. The equations defining this transformation are the solutions of Hamilton's equations, say

$$q_r = \phi_r(q_r^0, p_r^0, t)\ ; \qquad p_r = \phi_{n+r}(q_r^0, p_r^0, t)\ ; \qquad r = 1, \cdot\cdot, n \qquad (16.1)$$

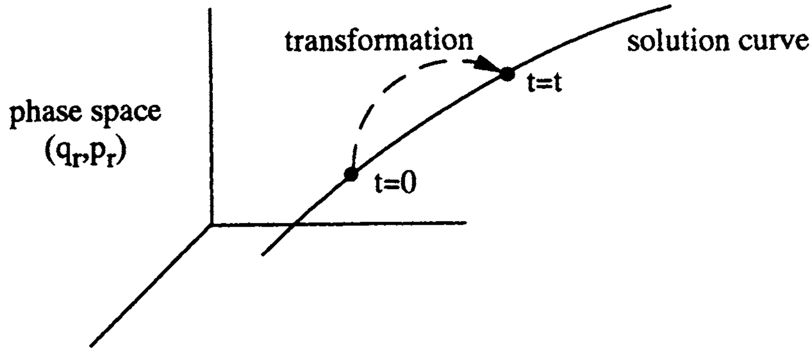


Fig. 16-1

Thus the dynamics problem becomes the study of transformations (Fig. 16-1).

Ignorable Coordinates. Other types of transformations are also of interest, for example a transformation of coordinates. Consider the case of a natural system, for which H = constant, and suppose all of the coordinates are ignorable. Then $L \neq L(q_k)$ and from Eqn. (15.6) $H = H(p_k)$. Equations (15.11) then give

$$\dot{q}_r = \frac{\partial H}{\partial p_r}\,; \qquad \dot{p}_r = 0\,; \qquad r = 1,\cdot\cdot,n \tag{16.2}$$

These have solutions

$$p_r = p_r^0 = \text{constant}\,; \qquad q_r = \omega_r t + q_r^0\,; \qquad r = 1,\cdot\cdot,n \tag{16.3}$$

where $\omega_r = \omega_r\ (p_s^0)$ are constants. Thus in this special case the dynamics problem is easily completely solved.

This case is not quite as restricted as it first appears. A dynamic system may be described by any set of suitable generalized coordinates, and for some choices more of the coordinates may be ignorable than for others. For example, for the problem of Fig. 8-3 none of the rectangular coordinates, x, y, z, are ignorable but one of the cylindrical coordinates, ϕ, is. As another example, for the problem of Fig. 8-4 neither x_1, x_2 are ignorable but for the choice x, y one is, namely y.

Idea of Contact Transformations. Based on the preceeding observations, it would be of great value if we could find transformations such that either the new variables were constants, say the initial conditions q_k^0, p_k^0, or the new coordinates were ignorable. In either case, the problem would be completely solved. Such transformations must have the property that if the original variables are Hamiltonian (i.e. satisfy Hamilton's equations) then the new ones must be as well. It must be remembered that not only the generalized coordinates but also their corresponding generalized momenta must be transformed. Such transformations are called *contact transformations.*[1] We will first define and study contact transformations in general, and then prove that such transformations have the desired property of preserving the Hamiltonian structure. In the following chapter, the most important application of contact transformations will be covered.

16.2 General Contact Transformations

Definition. Now consider a general transformation of $2n$ variables, one not necessarily giving the motion of a dynamic system:

$$(q_1, \cdot\cdot, q_n,\ p_1, \cdot\cdot, p_n) \longrightarrow (Q_1, \cdot\cdot, Q_n,\ P_1, \cdot\cdot, P_n)$$

such that the differential relation is true:

$$\sum_r P_r dQ_r = \sum_r p_r dq_r + Rdt - dW \tag{16.4}$$

This equation defines a *contact transformation* (CT). The function W is called the *generating function* of the CT. The CT generates the transformation

$$Q_r = \phi_r(q_s, p_s, t)\ ; \qquad P_r = \phi_{r+n}(q_s, p_s, t)\ ; \quad r = 1, \cdot\cdot, n \tag{16.5}$$

These are $2n$ equations in $4n$ variables. Because any set of the q_k, p_k combined with any set of the Q_k, P_k may be regarded as the $2n$ independent variables of the transformation, there are four possibilities: $W = W_1(q_r, Q_r, t)$, $W = W_2(q_r, P_r, t)$, $W = W_3(p_r, Q_r, t)$, and $W = W_4(p_r, P_r, t)$.

Case $W = W_1(q_r, Q_r, t)$. First suppose that the Jacobian

$$\frac{\partial(\phi_1, \cdot\cdot, \phi_n)}{\partial(p_1, \cdot\cdot, p_n)} \neq 0 \tag{16.6}$$

In this case, by the Implicit Function Theorem, we can solve the first set of Eqns. (16.5) for the p_r in terms of the q_r, Q_r, and t. The functions R

and W can then be expressed in terms of the q_r, Q_r, and t and there is no relation connecting these variables. Taking the differential of W_1:

$$dW_1 = \sum_r \frac{\partial W_1}{\partial q_r} dq_r + \sum_r \frac{\partial W_1}{\partial Q_r} dQ_r + \frac{\partial W_1}{\partial t} dt \tag{16.7}$$

Since the q_r, Q_r, t are independent, comparison of Eqns. (16.4) and (16.7) gives

$$\begin{aligned} p_r &= \frac{\partial W_1}{\partial q_r}\,; \qquad P_r = -\frac{\partial W_1}{\partial Q_r}\,; \quad r = 1, \cdot\cdot, n \\ R &= \frac{\partial W_1}{\partial t} \end{aligned} \tag{16.8}$$

These are the explicit equations of the CT.

If the Jacobian defined in Eqn. (16.6) is zero, we proceed as follows. Suppose, for definiteness, that the rank is $n-1$; then there is one relation

$$\theta(q_r, Q_r, t) = 0 \tag{16.9}$$

Take the differential of this:

$$d\theta = \sum_r \frac{\partial \theta}{\partial q_r} dq_r + \sum_r \frac{\partial \theta}{\partial Q_r} dQ_r + \frac{\partial \theta}{\partial t} dt = 0 \tag{16.10}$$

Now, the q_r, Q_r, t are not independent but are constrained by Eqn. (16.10). Using a Lagrange multiplier λ to account for the constraint,

$$\begin{aligned} p_r &= \frac{\partial W_1}{\partial q_r} + \lambda \frac{\partial \theta}{\partial q_r}\,; \quad r = 1, \cdot\cdot, n \\ P_r &= -\frac{\partial W_1}{\partial Q_r} - \lambda \frac{\partial \theta}{\partial Q_r}\,; \quad r = 1, \cdot\cdot, n \\ R &= \frac{\partial W_1}{\partial t} + \lambda \frac{\partial \theta}{\partial t} \end{aligned} \tag{16.11}$$

There will be a Lagrange multiplier for each such relation θ.

The Other Cases. Next consider the case $W = W_2(q_r, P_r, t)$. Now take as the generating function

$$W_2 = W_1 + \sum_r P_r Q_r \tag{16.12}$$

Forming dW_2 and using Eqns. (16.7) and (16.8) gives:

$$\begin{aligned} dW_2 &= dW_1 + \sum_r P_r dQ_r + \sum_r Q_r dP_r \\ &= \sum_r p_r dq_r + \sum_r Q_r P_r + R dt \end{aligned} \tag{16.13}$$

Since in this case the q_r, P_r, t are independent, this implies

$$\begin{aligned} p_r &= \frac{\partial W_2}{\partial q_r}; \qquad Q_r = \frac{\partial W_2}{\partial P_r}; \quad r = 1, \cdot\cdot, n \\ R &= \frac{\partial W_2}{\partial t} \end{aligned} \tag{16.14}$$

The next case[2] is $W = W_3(p_r, Q_r, t)$. The appropriate generating function is

$$W_3 = W_1 - \sum_r q_r p_r \tag{16.15}$$

and this gives

$$\begin{aligned} q_r &= -\frac{\partial W_3}{\partial p_r}; \qquad P_r = -\frac{\partial W_3}{\partial Q_r}; \qquad r = 1, \cdot\cdot, n \\ R &= \frac{\partial W_3}{\partial t} \end{aligned} \tag{16.16}$$

The final case is $W = W_4(p_r, P_r, t)$. The generating function is

$$W_4 = W_1 + \sum_r P_r Q_r - \sum_r p_r q_r \tag{16.17}$$

which gives

$$\begin{aligned} q_r &= -\frac{\partial W_4}{\partial p_r}; \qquad Q_r = \frac{\partial W_4}{\partial P_r}; \qquad r = 1, \cdot\cdot, n \\ R &= \frac{\partial W_4}{\partial t} \end{aligned} \tag{16.18}$$

Of course, in all these cases if there are relations among the variables one Lagrange multiplier will have to be introduced for each such relation.

Example. Consider the transformation described by the generating function of the first kind:

$$W_1 = \sum_r q_r Q_r$$

From Eqns. (16.8),

$$p_r = \frac{\partial W_1}{\partial q_r} = Q_r \; ; \; r = 1, \cdot\cdot, n$$

$$P_r = -\frac{\partial W_1}{\partial Q_r} = -q_r \; ; \; r = 1, \cdot\cdot, n$$

Thus this transformation interchanges the q_r and the p_r (except for a change of sign).

Example. Next consider an example of a CT with a generating function of the second type given by

$$W_2 = \sum_r q_r P_r$$

Applying Eqns. (16.14),

$$p_r = \frac{\partial W_2}{\partial q_r} = P_r$$

$$Q_r = \frac{\partial W_2}{\partial P_r} = q_r$$

Thus the old and new variables are the same; W_2 generates the identity transformation.

16.3 Homogeneous Contact Transformations

Definition. The special case of a CT with $Rdt - dW = 0$ is called a *homogeneous contact transformation* (HCT). From Eqn. (16.4) this is defined by

$$\sum_r P_r dQ_r = \sum_r p_r dq_r \tag{16.19}$$

It is assumed that the Q_r, P_r are independent functions of the $q_r, p_r \in C^1$; that is, the Jacobian of the transformation is not zero:

$$\left| \frac{\partial(Q_r, P_r)}{\partial(q_r, p_r)} \right| \neq 0$$

Transformation of Coordinates. One application of HCT's is the transformation of one set of generalized coordinates, q_r, to another, Q_r. Let the inverse transformation be

$$q_r = F_r(Q_s) \; ; \qquad r = 1, \cdot\cdot, n \tag{16.20}$$

where $F_r \in C^2$. Such a transformation is sometimes referred to as a *continuous point transformation.* The corresponding generalized momenta are denoted p_r and P_r. We assume a natural system so that the transformation does not explicitly contain time. For such a system (see Sections 6.1 and 15.1),

$$T = \frac{1}{2}\sum_r p_r \dot{q}_r = \frac{1}{2}\sum_r P_r \dot{Q}_r \tag{16.21}$$

Thus

$$\sum_r P_r dQ_r = \sum_r p_r dq_r$$

so that the transformation is a HCT.

We now obtain the explicit relations for the transformation of the corresponding generalized momenta. From Eqn. (6.3), for a natural system,

$$2T = \sum_r \sum_s a_{rs}\dot{q}_r \dot{q}_s = \sum_r \sum_s A_{rs}\dot{Q}_r \dot{Q}_s$$

so that, using Eqn. (16.20),

$$P_r = \frac{\partial T}{\partial \dot{Q}_r} = \sum_i \sum_j \sum_s a_{ij} \frac{\partial F_i}{\partial Q_r}\frac{\partial F_j}{\partial Q_s}\dot{Q}_s = \sum_i \sum_j a_{ij}\frac{\partial F_i}{\partial Q_r}\dot{q}_j$$

$$P_r = \sum_i p_i \frac{\partial F_i}{\partial Q_r}\,; \qquad r = 1, \cdot\cdot, n \tag{16.22}$$

Equations (16.20) and (16.22) define the CT. Note that the P_r are homogeneous linear functions of the p_r.

Example. Consider the transformation from rectangular coordinates to polar coordinates for a particle moving in the (x, y) plane. The transformation is given by

$$x = r\cos\theta\,, \qquad y = r\sin\theta$$

Letting $q_1 = x$, $q_2 = y$, $Q_1 = r$, and $Q_2 = \theta$, this is

$$q_1 = Q_1\cos Q_2 = F_1(Q_1, Q_2)\,, \qquad q_2 = Q_1 \sin Q_2 = F_2(Q_1, Q_2)$$

These correspond to Eqns. (16.20). Equations (16.22) give expressions for the new generalized momenta in terms of the old:

$$P_1 = p_1\frac{\partial F_1}{\partial Q_1} + p_2\frac{\partial F_2}{\partial Q_1} = p_1 \cos Q_2 + p_2 \sin Q_2$$

$$P_2 = p_1\frac{\partial F_1}{\partial Q_2} + p_2\frac{\partial F_2}{\partial Q_2} = Q_1(-p_1 \sin Q_2 + p_2 \cos Q_2)$$

where $p_1 = p_x$, $p_2 = p_y$, $P_1 = p_r$, and $P_2 = p_\theta$.

16.4 Conditions for a Contact Transformation

Remarks. It may be required to determine whether or not a given transformation is a CT. If we can find a function W such that Eqn. (16.4) is satisfied, then the transformation is a CT, but this is often difficult to do. In this section, we derive tests that are usually easier to apply.

Liouville's Theorem. Suppose the transformation $(q_r, p_r) \to (Q_r, P_r)$ is a CT with no relations between the Q_r and q_r. Consider Case 1 of Section 16.2 for which Eqns. (16.8) define the transformation. Forming the Jacobian $\partial(Q_r, P_r)/\partial(q_r, p_r)$,

$$\frac{\partial(Q_r, P_r)}{\partial(q_r, p_r)} = \frac{\frac{\partial(Q_r,P_r)}{\partial(q_r,Q_r)}}{\frac{\partial(q_r,p_r)}{\partial(q_r,Q_r)}} = (-1)^n \frac{\frac{\partial(P_r)}{\partial(q_r)}}{\frac{\partial(p_r)}{\partial(Q_r)}} = \frac{\frac{\partial\left(\frac{\partial W_1}{\partial Q_r}\right)}{\partial(q_r)}}{\frac{\partial\left(\frac{\partial W_1}{\partial q_r}\right)}{\partial(Q_r)}} = 1 \qquad (16.23)$$

The same result holds for the other cases. This is sometimes called Liouville's theorem. Equation (16.23) expresses the fact that the transformation is *measure preserving.*

Lagrange Brackets. Let $q_1, \cdot\cdot, q_n$, $p_1, \cdot\cdot, p_n$ be C^2 functions of variables u and v. Then the *Lagrange bracket* of u and v is

$$[u, v] = \sum_r \left(\frac{\partial q_r}{\partial u}\frac{\partial p_r}{\partial v} - \frac{\partial p_r}{\partial u}\frac{\partial q_r}{\partial v}\right) = \sum_r \frac{\partial(q_r, p_r)}{\partial(u, v)} \qquad (16.24)$$

Theorem. The transformation from (q_r, p_r) to (Q_r, P_r) is a CT if and only if,

$$\begin{aligned}&(1)\ \ [Q_r, Q_s] = 0\\ &(2)\ \ [P_r, P_s] = 0\\ &(3)\ \ [Q_r, P_s] = \delta_{rs} = \left\{\begin{array}{l} 0 \text{ if } s \neq r \\ 1 \text{ if } s = r \end{array}\right\}\end{aligned} \tag{16.25}$$

for all $r,\ s = 1, \cdot\cdot, n$ and fixed t.

The proof is as follows. Treating the Q_r and P_r as independent variables, with t fixed, in Eqn. (16.4), one obtains

$$\begin{aligned} dW &= \sum_r p_r dq_r - \sum_r P_r dQ_r \\ &= \sum_r p_r \left(\sum_s \frac{\partial q_r}{\partial Q_s} dQ_s + \sum_s \frac{\partial q_r}{\partial P_s} dP_s \right) - \sum_r P_r dQ_r \\ &= \sum_s \left(\sum_r p_r \frac{\partial q_r}{\partial Q_s} - P_s \right) dQ_s + \sum_s \left(\sum_r p_r \frac{\partial q_r}{\partial P_s} \right) dP_s \end{aligned} \tag{16.26}$$

For such a generating function to exist, this must be a perfect differential. The necessary and sufficient conditions for this to be a perfect differential is that Eqn. (2.24) be satisfied. But Eqns. (16.25) are just the conditions for this to be true, and the theorem is proved.

Example. Consider the rectilinear motion of a particle in a uniform gravitational field. The familiar solution of the equation of motion is:

$$x = x_0 + \dot{x}_0 t + \frac{1}{2} g t^2 \,, \qquad \dot{x} = \dot{x}_0 + gt$$

In our current viewpoint, this is regarded as the transformation of the generalized coordinates and momenta at time zero to those at time t, $(q, p) \to (Q, P)$, given by

$$Q = q + pt + \frac{1}{2} g t^2 \,, \qquad P = p + gt$$

We check to see if this is a CT by the previous two methods.

First,

$$\frac{\partial(Q, P)}{\partial(q, p)} = \begin{vmatrix} 1 & t \\ 0 & 1 \end{vmatrix} = 1$$

so that Eqn. (16.23) is satisfied and the transformation is a CT.

Second, we check Eqns. (16.25). Inverting the transformation,

$$q = Q - Pt + \frac{1}{2}gt^2 , \qquad p = P - gt$$

Thus conditions (1) and (2) are satisfied trivially and condition (3) is

$$[Q, P] = \frac{\partial q}{\partial Q}\frac{\partial p}{\partial P} - \frac{\partial p}{\partial Q}\frac{\partial q}{\partial P} = 1$$

again verifying that the transformation is a CT.

Relation Between Poisson and Lagrange Brackets. Comparison of Eqns. (15.28) and (16.24) seems to indicate that there is some sort of inverse relationship between Poisson and Lagrange brackets. This is indeed the case. Consider independent functions $u_1(q_r, p_r), \cdot\cdot, u_{2n}(q_r, p_r)$. Then it can be proved that[3]

$$\sum_{s=1}^{2n}[u_s, u_i](u_s, u_j) = \delta_{ij} \tag{16.27}$$

The necessary and sufficient conditions for the transformation $q_r, p_r \rightarrow Q_r P_r$ to be a CT in terms of Poisson brackets are

$$\begin{aligned} (Q_r, Q_s) &= 0 \\ (P_r, P_s) &= 0 \\ (Q_r, P_s) &= \delta_{rs} \end{aligned} \tag{16.28}$$

for all $r, s = 1, \cdot\cdot, n$ and fixed t.

16.5 Jacobi's Theorem

Remarks. Up to now, we have been considering general contact transformations. Now we will consider the transformation of the generalized coordinates and momenta of a dynamics problem. One of the reasons CT's are important is because if the original variables satisfy Hamilton's equations, then the transformed ones do also, as will be proved shortly.

In the original variables, the q_r and the p_r, $H = H(q_r, p_r, t)$ and Hamilton's equations are Eqns. (15.11), repeated here:

$$\dot{q}_r = \frac{\partial H}{\partial p_r} ; \qquad \dot{p}_r = -\frac{\partial H}{\partial q_r} ; \qquad r = 1, \cdot\cdot, n \tag{16.29}$$

Consider a CT defining new variables P_r, Q_r:

$$\sum_r P_r dQ_r = \sum_r p_r dq_r + Rdt - dW \tag{16.30}$$

Jacobi's Theorem. If the equations of motion for q_r, p_r are Hamiltonian (that is, if they satisfy Eqns. (16.29)) then they are also Hamiltonian for P_r, Q_r as given by Eqn. (16.30). To prove this, two lemmas are needed.

Lemma 1. If

$$\begin{aligned} q_r &= \rho_r(\gamma_1, \cdot\cdot, \gamma_{2n}, t) \; ; \qquad r = 1, \cdot\cdot, n \\ p_r &= \rho_{r+n}(\gamma_1, \cdot\cdot, \gamma_{2n}, t) \; ; \qquad r = 1, \cdot\cdot, n \end{aligned} \tag{16.31}$$

are the general solutions of Eqns. (16.29) and if we substitute Eqns. (16.31) in H to get $H = G(\gamma_1, \cdot\cdot, \gamma_{2n}, t)$ then $G \in C^2$ and

$$\frac{\partial G}{\partial \gamma_i} = [t, \gamma_i] \; ; \qquad i = 1, \cdot\cdot, 2n \tag{16.32}$$

To prove this, we use Eqns. (16.29) and (16.24):

$$\begin{aligned} \frac{\partial G}{\partial \gamma_i} &= \sum_r \frac{\partial H}{\partial q_r}\frac{\partial q_r}{\partial \gamma_i} + \sum_r \frac{\partial H}{\partial p_r}\frac{\partial p_r}{\partial \gamma_i} \\ &= \sum_r \left(\dot{q}_r \frac{\partial p_r}{\partial \gamma_i} - \dot{p}_r \frac{\partial q_r}{\partial \gamma_i} \right) = [t, \gamma_i] \; ; \qquad i = 1, \cdot\cdot, 2n \end{aligned}$$

Lemma 2. This is the converse of Lemma 1. If q_r, p_r are $2n$ independent functions of $\gamma_1, \cdot\cdot, \gamma_{2n}, t$ and if there exists a function $G(\gamma_1, \cdot\cdot, \gamma_{2n}, t)$ such that

$$\frac{\partial G}{\partial \gamma_i} = [t, \gamma_i] \; ; \qquad i = 1, \cdot\cdot, 2n$$

then q_r, p_r satisfy Eqn. (16.29).

To prove this, think of the functions of Eqns. (16.31) as the functions just described and solve them for the $\gamma_1, \cdot\cdot, \gamma_{2n}$. Substitute the result into G to get a function H such that

$$H(q_r, p_r, t) = G(\gamma_r, t)$$

Now differentiate

$$\frac{\partial G}{\partial \gamma_i} = \sum_r \left(\frac{\partial H}{\partial q_r}\frac{\partial q_r}{\partial \gamma_i} + \frac{\partial H}{\partial p_r}\frac{\partial p_r}{\partial \gamma_i} \right) = [t, \gamma_i]$$

$$= \sum_r \left(\frac{\partial q_r}{\partial t}\frac{\partial p_r}{\partial \gamma_i} - \frac{\partial p_r}{\partial t}\frac{\partial q_r}{\partial \gamma_i} \right) ; \qquad i = 1, \cdot\cdot, 2n$$

where Eqns. (16.32) and (16.24) were used. Since the $\partial p_r / \partial \gamma_i$ and $\partial q_r / \partial \gamma_i$ are all independent, this equation implies Eqns. (16.29).

Proof of Jacobi's Theorem. Choose generating function

$$W = W_1(q_r, Q_r, t) = F(\gamma_i, t) \tag{16.33}$$

Using Eqns. (16.8),

$$\frac{\partial F}{\partial \gamma_i} = \sum_r \left(\frac{\partial W_1}{\partial q_r}\frac{\partial q_r}{\partial \gamma_i} + \frac{\partial W_1}{\partial Q_r}\frac{\partial Q_r}{\partial \gamma_i} \right)$$

$$= \sum_r \left(p_r \frac{\partial q_r}{\partial \gamma_i} - P_r \frac{\partial Q_r}{\partial \gamma_i} \right) ; \qquad i = 1, \cdot\cdot, 2n \tag{16.34}$$

$$\frac{\partial F}{\partial t} = \sum_r \left(\frac{\partial W_1}{\partial q_r}\frac{\partial q_r}{\partial t} + \frac{\partial W_1}{\partial Q_r}\frac{\partial Q_r}{\partial t} \right) + \frac{\partial W_1}{\partial t}$$

$$= \sum_r \left(p_r \frac{\partial q_r}{\partial t} - P_r \frac{\partial Q_r}{\partial t} \right) + \frac{\partial W_1}{\partial t} \tag{16.35}$$

The function $F(\gamma_i, t)$ is of class C^2 so that

$$\frac{\partial^2 F}{\partial \gamma_i \partial t} = \frac{\partial^2 F}{\partial t \partial \gamma_i} ; \qquad i = 1, \cdot\cdot, 2n \tag{16.36}$$

From Eqns. (16.34) - (16.36):

$$\frac{\partial}{\partial t} \sum_r \left(p_r \frac{\partial q_r}{\partial \gamma_i} - P_r \frac{\partial Q_r}{\partial \gamma_i} \right) = \frac{\partial}{\partial \gamma_i} \left[\sum_r \left(p_r \frac{\partial q_r}{\partial t} - P_r \frac{\partial Q_r}{\partial t} \right) + \frac{\partial W}{\partial t} \right] ;$$

$$\sum_r \left(\frac{\partial Q_r}{\partial t}\frac{\partial P_r}{\partial \gamma_i} - \frac{\partial P_r}{\partial t}\frac{\partial Q_r}{\partial \gamma_i} \right) = \sum_r \left(\frac{\partial q_r}{\partial t}\frac{\partial p_r}{\partial \gamma_i} - \frac{\partial p_r}{\partial t}\frac{\partial q_r}{\partial \gamma_i} \right) + \frac{\partial^2 W}{\partial \gamma_i \partial t} ;$$

$$[t, \gamma_i] = \frac{\partial G}{\partial \gamma_i} \qquad i = 1, \cdot\cdot, 2n \tag{16.37}$$

where, using Eqns. (16.29),

$$\begin{aligned} \frac{\partial H}{\partial \gamma_i} &= \sum_r \left(\frac{\partial H}{\partial q_r} \frac{\partial q_r}{\partial \gamma_i} + \frac{\partial H}{\partial p_r} \frac{\partial p_r}{\partial \gamma_i} \right) \\ &= \sum_r \left(\frac{\partial q_r}{\partial t} \frac{\partial p_r}{\partial \gamma_i} - \frac{\partial p_r}{\partial t} \frac{\partial q_r}{\partial \gamma_i} \right) ; \qquad i = 1, \cdot\cdot, 2n \end{aligned}$$

was used, and where $G = H + \partial W/\partial t$ and $[t, \gamma_i]$ is the Lagrange bracket of t and γ_i in terms of the Q_r, P_r.

By Lemma 2, P_r, Q_r satisfy Hamilton's equations with Hamiltonian function G, that is,

$$\frac{\partial Q_r}{\partial t} = \frac{\partial H^*}{\partial P_r} ; \qquad \frac{\partial P_r}{\partial t} = -\frac{\partial H^*}{\partial Q_r} ; \qquad r = 1, \cdot\cdot, n \tag{16.38}$$

where the new Hamiltonian is

$$H^* = H + \frac{\partial W}{\partial t} \tag{16.39}$$

Example.[4] Consider a linear harmonic oscillator with linear restoring force constant k. Then

$$\begin{aligned} &T = \frac{1}{2} m \dot{q}^2 , \qquad V = \frac{kq^2}{2} , \qquad p = \frac{\partial L}{\partial \dot{q}} = m\dot{q} \\ &L = \frac{1}{2} m \dot{q}^2 - \frac{kq^2}{2} \\ &H = \frac{p^2}{2m} + \frac{m\omega^2}{2} q^2 \end{aligned}$$

where $\omega^2 = k/m$. Now consider a CT from q, p to new variables Q, P as defined by the generating function

$$W = W_1(q, Q) = \frac{m}{2} \omega q^2 \cot Q$$

Equations (16.8) give

$$\begin{aligned} p &= \frac{\partial W_1}{\partial q} = m\omega q \cot Q \\ P &= -\frac{\partial W_1}{\partial Q} = \frac{m\omega q^2}{2 \sin^2 Q} \end{aligned}$$

Solving for q, p in terms of Q, P,

$$q = \sqrt{\frac{2P}{m\omega}} \sin Q$$

$$p = \sqrt{2m\omega P} \cos Q$$

Since the transformation does not contain t explicitly, Eqn. (16.39) gives $H^*(Q, P) = H(p, q)$; thus

$$H = \omega P \cos^2 Q + \omega P \sin^2 Q = \omega P$$

Therefore, Q is an ignorable coordinate with momentum integral

$$P = \frac{H}{\omega} = \frac{E}{\omega} = \text{constant}$$

The equation of motion for Q is now easy to solve:

$$\dot{Q} = \frac{\partial H}{\partial P} = \omega$$

$$Q = \omega t + \alpha$$

In terms of the original generalized coordinate, the well-known solution to this problem is obtained as

$$q = \sqrt{\frac{2E}{m\omega^2}} \sin(\omega t + \alpha)$$

This is an example of how a CT can be used to obtain a Hamiltonian for which all coordinates are ignorable.

Notes

1 The term originated in optics where it has to do with preserving the contact point between wave fronts; they are called *canonical transformations* in some texts.

2 The details of the analysis of the last two cases may be found in Goldstein.

3 See Pars or Goldstein.

4 Goldstein.

PROBLEMS

In the two following problems, show that the indicated transformation is a contact transformation by three methods: (i) Directly using the definition of a CT, (ii) Using Liouville's theorem, and (iii) Using the Lagrange brackets or Poisson brackets tests.

16/1. $Q = e^{kp}\sqrt{q+a}\,, \qquad P = -\left(\frac{1}{k}\right) e^{-kp}\sqrt{q+a}$

16/2. $Q = \ln\left(\frac{1}{q}\sin p\right)\,, \qquad P = q\cot p$

16/3. For what values of α and β is

$$Q = q^{\alpha}\cos\beta p\,, \qquad P = q^{\alpha}\sin\beta p$$

a CT?

16/4. Prove Liouville's theorem for the case $n = 1$.

16/5. Consider the transformation described by the generating function $W_2 = \sum_i f_i(q_1, \cdot\cdot, q_n, t)P_i$ where the f_i are any smooth functions. Show that the new coordinates depend only on the old coordinates and time and thus W_2 generates a continuous point transformation.

16/6. Show that the generating function $W_2 = \sum_{i,k} a_{ik}q_k p_i$ generates a linear transformation of coordinates, $Q_i = \sum_k a_{ik}q_k$ and that the generalized momenta also transform linearly.

Chapter 17

Hamilton-Jacobi Equation

17.1 The Principal Function

Hamilton's Principle Again. Consider the motion of a holonomic conservative system in configuration space and consider a varied path such that the δq_r occur at a fixed time. Figure 17-1 shows the situation for two generalized coordinates. In this case,

$$\frac{d}{dt}(\delta q) = \delta \dot{q}$$

and Eqn. (15.5) may be written:

$$\delta L = \frac{d}{dt}\left(\sum_r p_r \delta q_r\right) \tag{17.1}$$

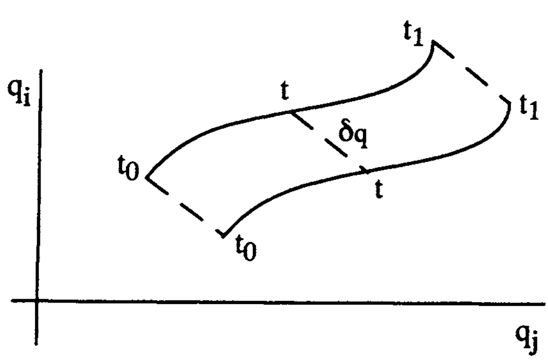


Fig. 17-1

This leads directly to Hamilton's principle, as follows. Integrating Eqn. (17.1) from t_0 to t_1 with the variations δq_r zero at the endpoints gives

$$\int_{t_0}^{t_1} \delta L dt = \sum_r p_r \delta q_r \Big|_{t_0}^{t_1}$$

$$\delta \int_{t_0}^{t_1} L dt = 0$$

which is the third form of Hamilton's principle (see Section 4.3).

Principal Function. Define *Hamilton's principal function* by

$$S = \int_{t_0}^{t_1} L dt \tag{17.2}$$

Now suppose that all n integrals of Lagrange's equations are known; they will be functions of the form

$$q_s(t) = \rho_s(q_r^0, \omega_r^0, t_0, t) \; ; \quad s = 1, \cdot\cdot, n \tag{17.3}$$

where $q_r^0 = q_r(t_0)$ and $\omega_r^0 = \dot{q}_r(t_0)$. Then the Lagrangian will be of the form $L = L(q_r^0, \omega_r^0, t_0, t)$ and from Eqn. (17.2) the principal function will be of the form

$$S = S(q_r^0, \omega_r^0, t_0, t_1) \tag{17.4}$$

We want, instead, to express S in terms of boundary conditions

$$S = S(q_r^0, q_r^1, t_0, t_1) \tag{17.5}$$

where $q_r^1 = q_r(t_1)$. We see that the solution may be thought of as a $2n$ parameter family of functions, the parameters being the q_r^0 and the ω_r^0. Alternatively, the solutions may be parameterized by the q_r^0 and the q_r^1, and we now proceed to replace the dependence on the ω_r^0 by dependence on the q_r^1. In effect, this replaces a point-slope specification of the solution curves by a point-point specification (see Fig. 17-2).

From Eqn. (17.3),

$$q_s^1 = \rho_s(q_r^0, \omega_r^0, t_0, t_1) \; ; \quad s = 1, \cdot\cdot, n$$

If the Jacobian of this transformation is non-zero, this relation may be inverted to give

$$\omega_s^0 = \psi_s(q_r^0, q_r^1, t_0, t_1) \; ; \quad s = 1, \cdot\cdot, n \tag{17.6}$$

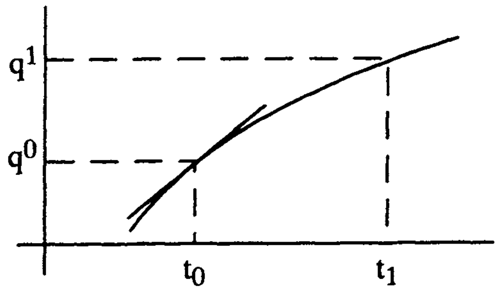


Fig. 17-2

Substitution of these relations in Eqns. (17.4) gives Eqns. (17.5).

Variation of the Principal Function. First, fix t_0 and t_1 and vary the q_s^1, which also varies the q_s^0, as shown on Fig. 17-3 for one q. Since t_0 and t_1 are fixed, Eqn. (17.1) applies

$$\begin{aligned}\delta S &= \delta \int_{t_0}^{t_1} L dt = \int_{t_0}^{t_1} \delta L dt \\ &= \sum p_r^1 \delta q_r^1 - \sum p_r^0 \delta q_r^0 \end{aligned} \tag{17.7}$$

Further, by Eqn. (17.5) with t_0 and t_1 fixed,

$$\delta S = \sum_r \frac{\partial S}{\partial q_r^0} \delta q_r^0 + \sum_r \frac{\partial S}{\partial q_r^1} \delta q_r^1 \tag{17.8}$$

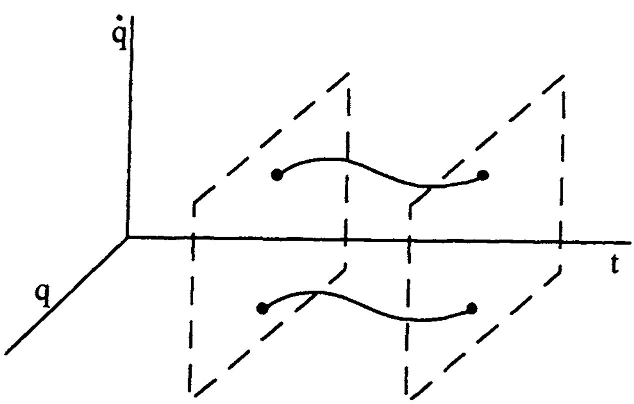


Fig. 17-3

Comparing Eqns. (17.7) and (17.8) and recalling that the δq_r^0 and the δq_r^1 are being regarded as independent, their coefficients must be equal,

$$p_r^0 = -\frac{\partial S}{\partial q_r^0} \,; \quad p_r^1 = \frac{\partial S}{\partial q_r^1} \,; \quad r = 1, \cdot\cdot, n \tag{17.9}$$

From these equations, we see that if S is known, all the integrals of the motion are known; that is, the dynamics problem is completely solved. Indeed, the first set of Eqns. (17.9) provides the q_r^1 in terms of the q_r^0 and p_r^0, and t_0 and t_1. Together, they provide the solution (i.e. all the integrals) of Hamilton's equations. We note that if $L \neq L(t)$ then the q_r are functions of time only of the form $(t - t_0)$ and hence S is a function of time only of the form $(t_1 - t_0)$.

Next, fix the q_r^0 and ω_r^0 and vary t_1 (and hence also the q_r^1). From Eqn. (17.2),

$$L_1 = \frac{dS}{dt_1} = \frac{\partial S}{\partial t_1} + \sum_r \frac{\partial S}{\partial q_r^1} \frac{\partial q_r^1}{\partial t_1} \tag{17.10}$$

Thus, using the second set of Eqns. (17.9),

$$\frac{\partial S}{\partial t_1} = L_1 - \sum_r p_r^1 \omega_r^1 = -H_1 \tag{17.11}$$

Similarly, it may be shown that

$$\frac{\partial S}{\partial t_0} = H_0 \tag{17.12}$$

Finally, then, from Eqns. (17.7), (17.11), and (17.12) the *total variation* in S due to variations in all the $2n + 2$ arguments of S is

$$dS = \sum_r p_r^1 dq_r^1 - \sum_r p_r^0 dq_r^0 - H_1 dt_1 + H_0 dt_0 \tag{17.13}$$

Thus the transformation $(q_r^0, p_r^0) \to (q_r^1, p_r^1)$ is a contact transformation (CT) with generating function S.

Remarks

1. The goal has been to construct a unique trajectory through any two points in the event (q_r, t) space; if this can be done, S exists.

2. We have started by assuming that the integrals of motion, Eqns. (17.3), are known. Thus none of what we have done indicates how to find S. We have shown only that if S can be found, the dynamics problem is solved. We will turn to the problem of finding S in Section 17.2. Thus S is another descriptive function, but one with an important difference from L and H.

Example. Consider a particle of unit mass moving in a plane under constant gravity (Fig. 17-4). The solution is known to be

$$\begin{aligned} x &= x_0 + u_0(t - t_0) \\ y &= y_0 + v_0(t - t_0) - \frac{1}{2}g(t - t_0)^2 \end{aligned}$$

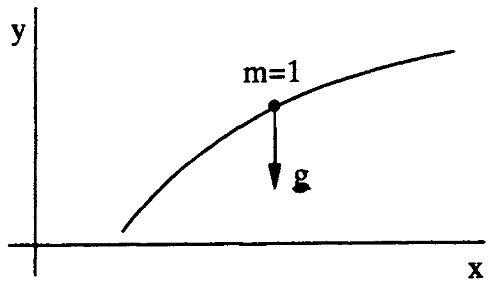


Fig. 17-4

where $x_0 = x(t_0)$, $y_0 = y(t_0)$, $u_0 = \dot{x}(t_0)$ and $v_0 = \dot{y}(t_0)$. From this solution, S may be computed directly from Eqn. (17.2) as follows

$$\begin{aligned} L &= T - V = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - gy \\ &= \frac{1}{2}\left(u_0^2 + (v_0 - g(t - t_0))^2\right) - gy_0 - gv_0(t - t_0) + \frac{1}{2}g^2(t - t_0)^2 \\ S &= \int_{t_0}^{t_1} L dt = S(x_0, u_0, y_0, v_0, t_0, t_1) \end{aligned}$$

This corresponds to Eqn. (17.4). Next we do the inversion, Eqn. (17.6):

$$u_0 = \frac{x_1 - x_0}{t_1 - t_0}; \qquad v_0 = \frac{y_1 - y_0 + \frac{1}{2}g(t_1 - t_0)^2}{t_1 - t_0}$$

Substitution of these into the expression for S gives (the details are left as an exercise)

$$\begin{aligned} S(x_0, x_1, y_0, y_1, t_0, t_1) &= \frac{(x_1 - x_0)^2 + (y_1 - y_0)^2}{2(t_1 - t_0)} \\ &\quad - \frac{1}{2} g(t_1 - t_0)(y_1 - y_0) - \frac{1}{24} g^2 (t_1 - t_0)^3 \end{aligned}$$

Now apply the first set of Eqns. (17.9) to get

$$\begin{aligned} u_0 &= -\frac{\partial S}{\partial x_0} = \frac{x_1 - x_0}{t_1 - t_0} \\ v_0 &= -\frac{\partial S}{\partial y_0} = \frac{y_1 - y_0}{t_1 - t_0} + \frac{1}{2} g(t_1 - t_0) \end{aligned}$$

This is the solution we started out with; this demonstrates that if S is known, then the solution (all integrals of the motion) is readily obtained. Clearly, this is valid for all cases except the trivial one, $t_1 - t_0 = 0$.

Example – Harmonic Oscillator. The equation of motion and its solution are

$$\ddot{x} + n^2 x = 0$$

$$x = x_0 \cos n(t - t_0) + \frac{u_0}{n} \sin n(t - t_0)$$

Computing S as before

$$L = \frac{1}{2}\dot{x}^2 - \frac{1}{2} n^2 x^2$$

$$S(x_0, u_0, t_0, t_1) = \frac{1}{2} \int_{t_0}^{t_1} (\dot{x}^2 - n^2 x^2) dt$$

We can solve for $u_0 = u_0(x_0, x_1, t_0, t_1)$ uniquely provided $n(t_1 - t_0) \neq r\pi$, r an integer (see Fig. 17-5). Under this restriction, the result of putting S in the form of Eqn. (17.5) is

$$S = \frac{1}{2} n(x_1^2 + x_0^2) \cot n(t_1 - t_0) - \frac{n x_1 x_0}{\sin n(t_1 - t_0)}$$

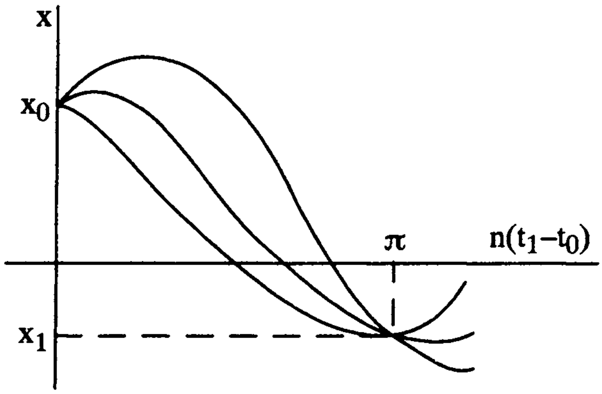


Fig. 17-5

so that from Eqns. (17.9)

$$u_0 = -\frac{\partial S}{\partial x_0} = -nx_0 \cot n(t_1 - t_0) + \frac{nx_1}{\sin n(t_1 - t_0)}$$

which is the solution we started out with.

17.2 Hamilton-Jacobi Theorem

The Hamilton-Jacobi Equation. The solution of Eqns. (17.9) gives the q_r^1 and p_r^1 as functions of the $2n$ parameters q_r^0 and p_r^0. It is often convenient, however, to use other parameters α_r and β_r related to the q_r^0 and p_r^0 by a HCT (see Section 16.3). We seek such a transformation that leaves S invariant:

$$S(q_r^0, q_r^1, t_0, t_1) = \tilde{S}(\alpha_r, q_r^1, t_0, t_1) \tag{17.14}$$

Using Eqns. (17.13) and (16.19) results in

$$\begin{aligned} d\tilde{S} &= dS = \sum p_r^1 dq_r^1 - \sum p_r^0 dq_r^0 - H_1 dt_1 + H_0 dt_0 \\ &= \sum p_r^1 dq_r^1 - \sum \beta_r d\alpha_r - H_1 dt_1 + H_0 dt_0 \end{aligned} \tag{17.15}$$

so that

$$\frac{\partial \tilde{S}}{\partial \alpha_r} = -\beta_r \tag{17.16}$$

These equations are the solutions to Lagrange's equations. Also,

$$\frac{\partial \tilde{S}}{\partial q_r^1} = p_r^1 \tag{17.17}$$

Previously, we have shown that if we could find the principal function, S, then the solution to the dynamics problem is easily obtained. We now turn to the task of finding an equation for S. Let α_r, β_r define a HCT; that is, from Eqn. (16.19),

$$\sum_r \beta_r d\alpha_r = \sum_r p_r^0 dq_r^0 \tag{17.18}$$

In this section, we shall take $t_0 = 0$, write $S = \tilde{S}$, and suppress the superscript 1; thus

$$S = \tilde{S}(q_r^1, \alpha_r, t_0, t_1) = S(q_r, \alpha_r, t) \tag{17.19}$$

With this new notation, Eqn. (17.15) becomes

$$dS = \sum_r p_r dq_r - \sum_r \beta_r d\alpha_r - H dt \tag{17.20}$$

so that

$$\frac{\partial S}{\partial q_r} = p_r\ ; \quad r = 1, \cdot\cdot, n \tag{17.21}$$

$$\frac{\partial S}{\partial \alpha_r} = -\beta_r\ ; \quad r = 1, \cdot\cdot, n \tag{17.22}$$

$$\frac{\partial S}{\partial t} = -H \tag{17.23}$$

where $H = H(q_r, p_r, t)$. Substituting Eqn. (17.21) into (17.23), we arrive at

$$\frac{\partial S}{\partial t} + H(q_r,\ p_r,\ t) = 0$$

$$\frac{\partial S}{\partial t} + H\left(q_r,\ \frac{\partial S}{\partial q_r},\ t\right) = 0$$

$$\frac{\partial S}{\partial t} + H\left(q_1, \cdot\cdot, q_n, \frac{\partial S}{\partial q_1}, \cdot\cdot, \frac{\partial S}{\partial q_n}, t\right) = 0 \tag{17.24}$$

This first-order, non-linear, partial differential equation is known as the *Hamilton-Jacobi equation*,[1] or sometimes as Hamilton's equation.

We know that the principal function, Eqn. (17.19), is a solution (complete integral) of Eqn. (17.24). The solutions of Eqn. (17.24), however, are not unique, raising the question of which of the solutions solve the dynamics problem. The following theorem establishes that *any* solution of Eqn. (17.24) also satisfies Eqns. (17.21) and (17.22), and thus solves the problem.

Hamilton-Jacobi Theorem. If $S = S(q_r, \alpha_r, t)$ is a complete integral of Eqn. (17.24) then the integral's of Hamilton's equations are given by Eqns. (17.21) and (17.22). Hence we have replaced the problem of solving a $2n$-order system of ordinary differential equations (Hamilton's equations) by the problem of solving one first order partial differential equation (the Hamilton-Jacobi equation).

The proof proceeds as follows. By definition, a complete integral of Eqn. (17.24) is a function of class C^2 containing n arbitrary constants $\alpha_1, \cdot\cdot, \alpha_n$ such that

$$\begin{vmatrix} \dfrac{\partial^2 S}{\partial q_1 \partial \alpha_1} & \cdots & \dfrac{\partial^2 S}{\partial q_1 \partial \alpha_n} \\ \vdots & & \vdots \\ \dfrac{\partial^2 S}{\partial q_n \partial \alpha_1} & \cdots & \dfrac{\partial^2 S}{\partial q_n \partial \alpha_n} \end{vmatrix} \neq 0 \tag{17.25}$$

Regard the q_r and α_r as independent parameters and differentiate Eqn. (17.24) w.r.t. α_1:

$$\frac{\partial^2 S}{\partial \alpha_1 \partial t} + \sum_r \frac{\partial H}{\partial p_r} \frac{\partial^2 S}{\partial \alpha_1 \partial q_r} = 0 \tag{17.26}$$

where Eqn. (17.21) was used. Also, from Eqns. (17.21) and (17.22),

$$\frac{\partial S}{\partial \alpha_1} = -\beta_1$$

$$\frac{\partial^2 S}{\partial t \partial \alpha_1} = -\frac{\partial \beta_1}{\partial t} = -\sum_r \frac{\partial \beta_1}{\partial q_r} \frac{\partial q_r}{\partial t}$$

$$\frac{\partial^2 S}{\partial t \partial \alpha_1} + \sum \frac{\partial^2 S}{\partial q_r \partial \alpha_1} \frac{\partial q_r}{\partial t} = 0 \tag{17.27}$$

Since $S \in C^2$,

$$\frac{\partial^2 S}{\partial t \partial \alpha_1} = \frac{\partial^2 S}{\partial \alpha_1 \partial t_1}\,; \qquad \frac{\partial^2 S}{\partial \alpha_1 \partial q_r} = \frac{\partial^2 S}{\partial q_r \partial \alpha_1}$$

so that Eqns. (17.26) and (17.27) combine to give

$$\sum_r \frac{\partial^2 S}{\partial q_r \partial \alpha_1}\left[\frac{\partial q_r}{\partial t} - \frac{\partial H}{\partial p_r}\right] = 0 \tag{17.28}$$

If this procedure is repeated for $\alpha_2, \cdot\cdot, \alpha_n$, the following matrix equation results

$$\left\| \frac{\partial^2 S}{\partial q \partial \alpha} \right\| \, \left| \frac{\partial q}{\partial t} - \frac{\partial H}{\partial p} \right| = 0 \tag{17.29}$$

The first of these factors in an $n \times n$ matrix and the second is $n \times 1$. In view of Eqn. (17.25), Eqn. (17.29) implies

$$\frac{\partial q_r}{\partial t} = \frac{\partial H}{\partial p_r}\,; \quad r = 1, \cdot\cdot, n \tag{17.30}$$

which are the first n of Hamilton's equations. Note that we have written $\partial q_r/\partial t$ here instead of dq_r/dt because we are considering the family of trajectories generated by independently varying α, β, and t, and not the time rate of change along a trajectory.

To get the other n of Hamilton's equations, we proceed much as before. Differentiate Eqn. (17.24) w.r.t. q_1 and use Eqns. (17.21) to obtain

$$\frac{\partial^2 S}{\partial q_1 \partial t} + \frac{\partial H}{\partial q_1} + \sum_r \frac{\partial^2 S}{\partial q_1 \partial q_r}\frac{\partial H}{\partial p_r} = 0 \tag{17.31}$$

Also, from Eqns. (17.21) and (17.22),

$$\frac{\partial p_1}{\partial t} = \frac{\partial^2 S}{\partial t \partial q_1} + \sum_r \frac{\partial^2 S}{\partial q_r \partial q_1}\frac{\partial q_r}{\partial t} \tag{17.32}$$

Combining these two equations, repeating this for $q_2, \cdot\cdot, q_n$, and forming a matrix equation as before, we arrive at

$$\frac{\partial p_r}{\partial t} = -\frac{\partial H}{\partial q_r}\,; \quad r = 1, \cdot\cdot, n \tag{17.33}$$

and the theorem is proved.

Historical Remarks. Hamilton, born in Ireland, in 1805, was the ultimate child prodigy. Attracted to foreign languages as a child, by the age of 10 he was proficient in writing Latin, Greek, Hebrew, Italian, French, Arabic, and Sanskrit, and was learning a half a dozen others. He then became interested in mathematics, teaching himself the known mathematics of the time, by the age of 17. He was by then a student at Trinity College, Dublin, and had started his revolutionary research in optics.

Hamilton's goal was to bring the theory of optics to the same "state of perfection" that Lagrange had brought dynamics. By the age of 22 his work on optics was complete; it succeeded in resolving the most outstanding problem of mathematical physics of his time – unifying the particle and wave concepts of light into one elegant, comprehensive theory. Already he was called "the first mathematician of the age" and it was said that "a second Newton has arrived". At this time, while still a student, he was appointed professor of astronomy at Trinity, not having even applied for the position.

Hamilton then turned his attention back to dynamics, applying the methods he had developed in optics. In his *First Essay on a General Method in Dynamics* (1834) he introduced the "characteristic function", $\int_0^t 2T dt$, clearly motivated by the Principle of Least Action, and used it to formulate the dynamics problem. In this work he also introduced the principal function, $S = \int_0^t L\, dt$. In the *Second Essay on a General Method in Dynamics* (1835), he derived both what we now call the Hamilton-Jacobi equation, Eqn. (17.24) and Hamilton's canonical equations, Eqns. (15.11). He fully realized the importance of finding the function S as well as the difficulty in doing so (he gave an approximation method). As Hamilton stated in his paper, in the impersonal style then in vogue:

> "Professor Hamilton's solution of this long celebrated problem contains, indeed, one unknown function, namely, the *principal function* S, to the search and the study of which he has reduced mathematical dynamics. This function must not be confounded with that so beautifully conceived by Lagrange for the more simple and elegant expression of the known differential equations. Lagrange's function *states*, Mr. Hamilton's function would *solve* the problem."

Beginning in his late 20's, Hamilton suffered severe psychological problems. He became reclusive, alcoholic, and irregular in his eating and sleeping habits. As a consequence, he was not productive during these years. Later in life he spent all of his mathematical energies on the development of quaternians, which he regarded as his greatest achievement.

During his life, Hamilton was awarded every honor possible to a scientist, including being knighted and being named the first foreign member of the U.S. Academy of Sciences. When he died at age 61, his study was found piled high with mathematical papers, interspersed with plates of partially finished meals.

Jacobi's life was apparently relatively settled. He was born in Prussia in 1804 and spent most of his professional life as a professor (he was by all accounts an excellent teacher). His main contributions to dynamics were the proof of the Hamilton-Jacobi theorem and putting Hamilton-Jacobi theory into its modern form. These contributions were given in a series of lectures in 1842 and 1843, which were not published until 1866. Jacobi was a first-rate mathematician and he is perhaps best known for his contributions outside of dynamics, specifically to the fields of elliptic functions, solution of algebraic equations, number theory, and differential equations.

Hamiltonian dynamics has had a far-reaching impact on all of mathematics and physical science. As Bell states, "it is the aim of many workers in particular branches of theoretical physics to sum up the whole of a theory in a Hamiltonian principle." Most remarkable, is that when, about 100 years ago, experiments began to reveal the nature of the motion of atomic particles, the tools of Hamiltonian dynamics (Hamilton-Jacobi equation, canonical equations, contact transformations, and Poisson brackets) proved to be ideal as the basis for the modern theory of quantum mechanics.

17.3 Integration of the Hamilton-Jacobi Equation

Natural Systems. Consider a natural system. For such a system

$$H(q_r, \dot{q}_r) = T(q_r, \dot{q}_r) + V(q_r) = h = \text{constant} \qquad (17.34)$$

We see by direct substitution that in this case a solution of Eqn. (17.24) is of the form

$$S = -ht + K \tag{17.35}$$

where

$$K = K(q_1, \cdot\cdot, q_n, h, \alpha_2, \cdot\cdot, \alpha_n)$$

and where we have taken $\alpha_1 = h$. The function $K(\cdot)$ is called *Hamilton's characteristic function.*[2] Substitution of Eqn. (17.35) into (17.24) gives

$$H\left(q_r, \frac{\partial K}{\partial q_r}\right) = h \tag{17.36}$$

By the Hamilton-Jacobi Theorem, the integrals of motion are given by Eqns. (17.21) and (17.22); the first of Eqns. (17.22) yields:

$$\frac{\partial S}{\partial \alpha_1} = \frac{\partial S}{\partial h} = -\beta_1$$

$$-t + \frac{\partial K}{\partial h} = -\beta_1 = -t_0$$

$$\frac{\partial K}{\partial h} = t - t_0 \tag{17.37}$$

where t_0 is written in place of β_1. The rest of the equations are

$$\frac{\partial K}{\partial \alpha_r} = -\beta_r \ ; \quad r = 2, \cdot\cdot, n \tag{17.38}$$

$$\frac{\partial K}{\partial q_r} = p_r \ ; \quad r = 1, \cdot\cdot, n \tag{17.39}$$

The remaining problem is to find the function K. Note that Eqns. (17.38) determine the path in configuration space, and Eqn. (17.37) then gives time elapsed along the path.

Natural System with Ignorable Coordinate. Now, in addition, suppose q_n is ignorable with corresponding momentum integral $p_n = \gamma =$ constant. Then

$$H(q_1, \cdot\cdot, q_{n-1}, \dot{q}_1, \cdot\cdot, \dot{q}_n) = T(q_1, \cdot\cdot, q_{n-1}, \dot{q}_1, \cdot\cdot, \dot{q}_n) + V(q_1, \cdot\cdot, q_{n-1})$$

In this case we write

$$K = \gamma q_n + K' \tag{17.40}$$

where

$$K' = K'(q_1, \cdot\cdot, q_{n-1}, h, \alpha_2, \cdot\cdot, \alpha_{n-1}, \gamma)$$

where we have taken $\alpha_n = \gamma$. The last of Eqns. (17.22) gives

$$\frac{\partial S}{\partial \alpha_n} = -\beta_n$$

$$q_n + \frac{\partial K'}{\partial \gamma} = -\beta_n = q_n^0$$

where we have taken $\beta_n = -q_n^0$. Thus from Eqns. (17.22) the first n integrals are given by

$$\begin{aligned} \frac{\partial K'}{\partial h} &= t - t_0 \\ \frac{\partial K'}{\partial \alpha_r} &= -\beta_r \; ; \quad r = 2, \cdot\cdot, n-1 \\ \frac{\partial K'}{\partial \gamma} &= q_n^0 - q_n \end{aligned} \tag{17.41}$$

Equations (17.21) give the other n integrals:

$$\begin{aligned} \frac{\partial K'}{\partial q_r} &= p_r \; ; \quad r = 1, \cdot\cdot, n-1 \\ \frac{\partial K'}{\partial q_n} &= p_n = \gamma \end{aligned} \tag{17.42}$$

The remaining problem is now to find the function K'.

17.4 Examples

Example. We now return to the two examples of Section 17.1. Consider again a particle of unit mass in 2-D motion in a uniform gravitational field (Fig. 17-4). We have

$$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - gy$$

$$p_x = \frac{\partial L}{\partial \dot{x}} = \dot{x} \; ; \qquad p_y = \frac{\partial L}{\partial \dot{y}} = \dot{y}$$

$$H = \sum_{r=1}^{2} p_r \dot{q}_r - L = \dot{x}^2 + \dot{y}^2 - \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + gy$$
$$= \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + gy = T + V = \frac{1}{2}(p_x^2 + p_y^2) + gy$$

We see that x is ignorable. Hamilton's equation is

$$\frac{\partial S}{\partial t} + H\left(x,\ y,\ \frac{\partial S}{\partial x},\ \frac{\partial S}{\partial y}\right) = 0$$
$$\frac{\partial S}{\partial t} + \frac{1}{2}\left(\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2\right) + gy = 0$$

Since $H = h =$ constant and x is ignorable, Eqns. (17.35) and (17.40) apply:

$$S = -ht + \gamma x + K'(y)$$

Substitution into Hamilton's equation gives

$$-h + \frac{1}{2}\left(\gamma^2 + \left(\frac{\partial K'}{\partial y}\right)^2\right) + gy = 0$$

Letting $gk = h - \frac{1}{2}\gamma^2$, the solution of this equation is

$$K' = \int_y^k \sqrt{2g(k-\eta)}\, d\eta$$
$$K' = \sqrt{2g}\int_0^{k-y} \sqrt{\nu}\, d\nu$$

Thus

$$S = -\left(\frac{1}{2}\gamma^2 + gk\right)t + \gamma x + \sqrt{2g}\int_0^{k-y}\sqrt{\nu}\, d\nu$$

Now apply Eqns. (17.21) and (17.22)

$$\frac{\partial S}{\partial x} = \gamma = p_x = \dot{x}$$
$$\frac{\partial S}{\partial y} = -\sqrt{2g(k-y)} = p_y = \dot{y}$$
$$\frac{\partial S}{\partial \gamma} = -\gamma t + x = -\beta_1$$
$$\frac{\partial S}{\partial k} = -gt + \sqrt{2g(k-y)} = -\beta_2$$

The last two of these may be written as

$$x + \beta_1 = \gamma t$$
$$2g(k - y) = \beta_2^2 + g^2t^2 - 2\beta_2 gt$$

If a projectile is launched at position (x_0, y_0) at time $t_0 = 0$ with velocity components (u_0, v_0), these equations give at time t_0

$$\gamma = u_0$$
$$-\sqrt{2g(k - y_0)} = v_0$$
$$x_0 + \beta_1 = 0$$
$$2g(k - y_0) = \beta_2^2$$

which give

$$\gamma = u_0\,, \qquad \beta_1 = -x_0$$
$$k = y_0 + \frac{v_0^2}{2g}\,, \qquad \beta_2 = v_0$$

so that the solution may be written as

$$x - x_0 = u_0 t$$
$$2g(y_0 - y) = g^2t^2 - 2v_0 gt$$

which is the well-known solution to this problem. We note that h is the energy integral and k is the maximum height of the projectile. The problem has been completely solved.[3]

Example – Harmonic Oscillator. For this problem,

$$T = \frac{1}{2}\dot{x}^2\,, \quad V = \frac{1}{2}n^2x^2\,, \quad p = \frac{\partial L}{\partial \dot{x}} = \dot{x}$$
$$H = \frac{1}{2}\dot{x}^2 + \frac{1}{2}n^2x^2 = \frac{1}{2}(p^2 + n^2x^2) = h$$

Thus Hamilton's equation is

$$\frac{\partial S}{\partial t} + \frac{1}{2}\left(\left(\frac{\partial S}{\partial x}\right)^2 + n^2x^2\right) = 0$$

We try a solution of the form

$$S = -\frac{1}{2}n^2\alpha^2 t + \rho(x)$$

Substitution gives

$$-\frac{1}{2}n^2\alpha^2 + \frac{1}{2}\left(\left(\frac{\partial \rho}{\partial x}\right)^2 + n^2x^2\right) = 0$$

$$\rho = n\int_0^x \sqrt{\alpha^2 - \eta^2}\, d\eta$$

where η is a dummy integration variable. Then

$$S = -\frac{1}{2}n^2\alpha^2 t + n\int_0^x \sqrt{\alpha^2 - \eta^2}\, d\eta$$

Equation (17.22) gives the solution of the problem as:

$$\frac{\partial S}{\partial \alpha} = -\beta$$

$$-n^2\alpha t + n\alpha\int_0^x (\alpha^2 - \eta^2)^{-1/2} d\eta = -\beta$$

where Liebnitz' rule for differentiating under the integral sign has been used; this may be written in the more familiar form[3]

$$x = \alpha \sin n(t - t_0)$$

where $\beta = n^2\alpha t_0$. The constants α and t_0 may be expressed in terms of initial conditions if desired.

In Section 17.1, we found (by starting with a known solution) a different function S that satisfies Hamilton's equation for this problem than the one found here. This shows that the solution of Hamilton's partial differential equation is not unique. Both functions, however, provide the complete solution to the problem.

17.5 Separable Systems

Separability. In this section we continue to consider natural systems. In Section 17.3, we gave partial solutions of the Hamilton-Jacobi equation by writing the principal function as the sum of two or more parts. For example, in Eqn. (17.35) S was written as the sum of a function depending only on t and a function depending only on the q_r. Such a separation is always possible when $H \neq H(t)$. More generally, it may be possible to write S as a sum of functions, each containing just one of the q_i or just t. In this case we say the problem is *completely separable.* Both of the examples of Section 17.4 were completely separable.

In practice, the Hamilton-Jacobi equation is only useful when there is some degree of separability. Some problems, for example the famous three-body problem, are not separable. For other problems, separability depends on the choice of coordinates. For example, the central force problem is not separable in rectangular coordinates but is in polar (because in the latter case one coordinate is ignorable). A classic and important special case of complete separability is that of linear systems written in terms of modal coordinates.

Conditions for Separability. General conditions for complete separability are not known, but Pars (who devotes two chapters to the subject of separability), gives some results for systems for which the kinetic energy contains only squared terms:

$$T = \frac{1}{2}\sum_r a_r \dot{q}_r^2 \tag{17.43}$$

Not surprisingly, the same systems for which Lagrange's equations are separable (see Section 8.4) are separable in the Hamiltonian sense. Thus Liouville systems, defined by Eqns. (8.4), are completely separable.

The most general separable system of the type Eqn. (17.43) is given by Stackel's theorem, not stated here.

Solution of Separable Systems. General methods have been developed for solving completely separable systems; see for example Pars, Goldstein, or McCuskey. These methods depend on the theory of contact transformations, developed in the previous chapter. They will not be reviewed here.

As a practical matter, the most common case of separability occurs when there are ignorable coordinates. We have already seen that if a coordinate, say q_n, is ignorable, then a partial separation occurs, as expressed by Eqn. (17.40). In general, if coordinates $q_m, \cdot\cdot, q_n$ are ignorable then the characteristic function may be written as

$$K = \alpha_m q_m + \cdots + \alpha_n q_n + K'(q_1, \cdot\cdot, q_{m-1}, h, \alpha_2, \cdot\cdot, \alpha_n) \tag{17.44}$$

where the α_i are constants. In particular, if all coordinates but q_1 are ignorable, the problem is completely separable, because then

$$K = \alpha_2 q_2 + \cdots + \alpha_n q_n + K'(q_1, h, \alpha_2, \cdots, \alpha_n) \tag{17.45}$$

In this case the Hamilton-Jacobi equation reduces to an equation in q_1, which is always reducible to quadratures; the problem has been completely solved.

Example.[4] As an example of a non-trivial problem, consider again the heavy symmetrical top analyzed in Section 11.2 and shown on Fig. 11-4. From Eqns. (8.24) and (11.10),

$$p_\theta = I\dot{\theta}$$

$$p_\phi = I\dot{\phi}\sin^2\theta + J(\dot{\psi} + \dot{\phi}\cos\theta)\cos\theta$$

$$p_\psi = J(\dot{\psi} + \dot{\phi}\cos\theta)$$

Using Eqns. (11.11) and (15.15), we arrive at

$$H = \frac{1}{2}\left[\frac{p_\theta^2}{I} + \frac{(p_\phi - p_\psi\cos\theta)^2}{I\sin^2\theta} + \frac{p_\psi^2}{J}\right] + mg\ell\cos\theta$$

First, we see that $H \neq H(t)$ so that Eqns. (17.35) and (17.36) apply. Second, we see that ϕ and ψ are ignorable so that K has the form

$$K = \alpha_2\phi + \alpha_3\psi + K'(\theta)$$

The problem is thus completely separable.

The Hamilton-Jacobi equation, Eqn. (17.36), is

$$\frac{1}{2I}\left(\frac{\partial K}{\partial\theta}\right)^2 + \frac{1}{2I\sin^2\theta}\left(\frac{\partial K}{\partial\phi} - \frac{\partial K}{\partial\psi}\cos\theta\right)^2 + \frac{1}{2J}\left(\frac{\partial K}{\partial\psi}\right)^2$$
$$+ mg\ell\cos\theta = h$$

Substituting for K, we obtain

$$\frac{1}{2I}\left(\frac{dK'}{d\theta}\right)^2 + \frac{1}{2I\sin^2\theta}(\alpha_2 - \alpha_3\cos\theta)^2 + \frac{1}{2J}\alpha_3^2 + mg\ell\cos\theta = h$$

so that

$$\frac{dK'}{d\theta} = \sqrt{F(\theta)}$$

and

$$K' = \int \sqrt{F(\theta)}d\theta$$

where

$$F(\theta) = 2Ih - \frac{I}{J}\alpha_3^2 - 2Img\ell\cos\theta - \frac{1}{\sin^2\theta}(\alpha_2 - \alpha_3\cos\theta)^2$$

Thus

$$K = \alpha_2\phi + \alpha_3\psi + \int \sqrt{F(\theta)}d\theta$$

and S is given by Eqn. (17.35).

Now apply Eqns. (17.22); the first of these is given by Eqn. (17.37):

$$\int \frac{Id\theta}{\sqrt{F(\theta)}} = t - t_0$$

and the other two are

$$\phi - \int \frac{(\alpha_2 - \alpha_3\cos\theta)d\theta}{\sin^2\theta\sqrt{F(\theta)}} = -\beta_2$$

$$\psi - \frac{I}{J}\alpha_3\int \frac{d\theta}{\sqrt{F(\theta)}} + \int \frac{(\alpha_2 - \alpha_3\cos\theta)\cos\theta d\theta}{\sin^2\theta\sqrt{F(\theta)}} = -\beta_3$$

The constants h, α_2, α_3 may be determined by initial conditions.

Notes

1 The same equation plays a central role in the subject of *dynamic programming*, where it is called the Hamilton-Jacobi-Bellman equation.

2 It may be shown that K is equivalent to the action integral, Eqn. (4.40), see Goldstein.

3 The details of the solution to this problem are left as an exercise.

4 McCuskey.

PROBLEMS

Solve the following three problems by the Hamilton-Jacobi method.

17/1. Problem 4/2.

17/2. Problem 6/7.

17/3. Problem 10/1.

17/4. Show that linear systems written in terms of modal coordinates are completely separable.

17/5. Fill in the details of the solution to the first example of Section 17.4.

17/6. Fill in the details of the solution to the second example of Section 17.4.

17/7. Fill in the details of the solution to the example of Section 17.5.

Chapter 18

Approximation Methods

18.1 Variation of Constants

Remarks. It is clear that obtaining solutions (either closed form or in quadratures) of dynamics problems is difficult for all but the simplest problems. In this chapter, approximation methods are introduced which are based on the Hamiltonian formulation of dynamics, beginning with the method called the variation of constants.

General Case. Consider a general system of first order ordinary differential equations:

$$\dot{x}_i = f_i(x_s, t) = f_i(x_1, \cdot\cdot, x_n, t) \; ; \quad i = 1, \cdot\cdot, n \tag{18.1}$$

with solution

$$x_i = \rho_i(c_s, t) = \rho_i(c_1, \cdot\cdot, c_n, t) \tag{18.2}$$

Now suppose the system is altered or perturbed so that

$$\dot{x}_i = f_i(x_s, t) + g_i(x_s, t) \; ; \quad i = 1, \cdot\cdot, n \tag{18.3}$$

To account for the fact that the solution will change, introduce new variables by replacing the constants c_i by functions of time $\gamma_i(t)$:

$$x_i = \rho_i(\gamma_s, t) = \rho_i(\gamma_1, \cdot\cdot, \gamma_n, t) \; ; \quad i = i, \cdot\cdot, n \tag{18.4}$$

and assume that

$$\frac{\partial(\rho_1, \cdot\cdot, \rho_n)}{\partial(\gamma_1, \cdot\cdot, \gamma_n)} \neq 0 \tag{18.5}$$

We seek differential equations for the γ_s such that the solution of Eqn. (18.3) is in the same form as Eqn. (18.2), that is

$$x_i = \rho_i(\gamma_s(t), t) \; ; \qquad i = 1, \cdot\cdot, n \tag{18.6}$$

where

$$\gamma_i(t) = \psi_i(b_s, t) = \psi_i(b_1, \cdot\cdot, b_n, t) \; ; \qquad i = 1, \cdot\cdot, n$$

The b_i are constants to be determined by, for example, initial conditions.

From Eqns. (18.1) and (18.2),

$$f_i(x_s, t) = \dot{x}_i = \frac{dx_i}{dt} = \frac{d\rho_i}{dt} = \frac{\partial \rho_i}{\partial t} \; ; \qquad i = 1, \cdot\cdot, n \tag{18.7}$$

From Eqn. (18.3), using Eqn. (18.4),

$$f_i(x_s, t) + g_i(x_s, t) = \dot{x}_i = \frac{\partial \rho_i}{\partial t} + \sum_j \frac{\partial \rho_i}{\partial \gamma_j}\dot{\gamma}_j \; ; \qquad i = 1, \cdot\cdot, n \tag{18.8}$$

Comparing Eqns. (18.7) and (18.8):

$$g_i = \sum_j \frac{\partial \rho_i}{\partial \gamma_j}\dot{\gamma}_j \; ; \qquad i = 1, \cdot\cdot, n \tag{18.9}$$

By Eqn. (18.5), the matrix in Eqn. (18.9) is invertible, giving

$$\dot{\gamma}_j = \sum_i \left[\frac{\partial \rho_i}{\partial \gamma_j}\right]^{-1} g_i \; ; \qquad i = 1, \cdot\cdot, n \tag{18.10}$$

This is the equation for the variation of the constants that was sought.

Interpretation. The method just described is called the *variation of constants* or the variation of parameters. The main feature is that we let the constants of integration of the unperturbed system vary to get the solution of the perturbed system. Figure 18-1 provides a way of looking at this. The surfaces of constant c_i are solutions of the unperturbed system, $x_r = \rho_r(c_i, t)$. The solution of the perturbed system, $x_r = \rho_r(\gamma_i(t), t)$, crosses these surfaces, changing the value of c_i as time evolves.

One application of the method has been to orbital mechanics, where the γ_i are called *oscillating elements.* For example, the unperturbed motion of a small body orbiting around a massive body is that of a Kepler ellipse for which the eccentricity, ϵ, is a constant. If the motion

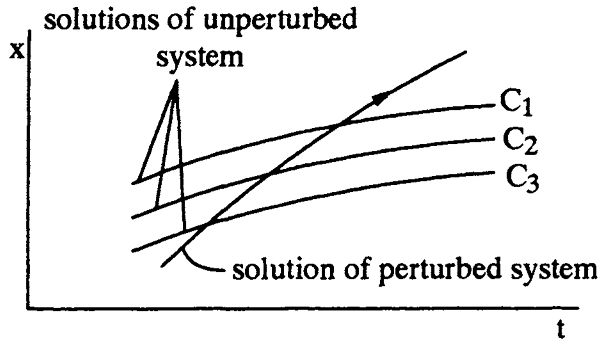


Fig. 18-1

is perturbed (due to the presence of a third body or to air resistance, for example), then the eccentricity will no longer be constant, say $\gamma = \epsilon(t)$. The orbiting body is now instantaneously on a certain Kepler ellipse but an instant later it will be on another one (Fig. 18-2).

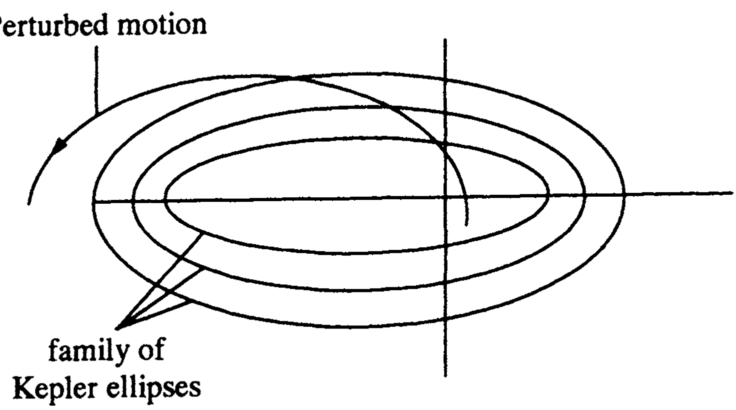


Fig. 18-2

Variation of Constants in Hamilton's Equations. Now suppose, in fact, that Eqns. (18.1) are Hamilton's equations, Eqns. (15.11),

$$\dot{q}_r = \frac{\partial H}{\partial p_r}\,; \qquad \dot{p}_r = -\frac{\partial H}{\partial q_r}\,; \quad r = 1, \cdot\cdot, n \tag{18.11}$$

with solution

$$q_r = F_r(c_1, \cdot\cdot, c_{2n}, t)\,; \qquad p_r = G_r(c_1, \cdot\cdot, c_{2n}, t)\,; \quad r = 1, \cdot\cdot, n \tag{18.12}$$

Suppose that the equations are perturbed

$$\dot{q}_r = \frac{\partial H}{\partial p_r}\ ; \qquad \dot{p}_r = -\frac{\partial H}{\partial q_r} + R_r\ ; \quad r = 1, \cdot\cdot, n \tag{18.13}$$

where the R_r are C^2 functions of the q_r, p_r, t. Note that only the second set of equations is perturbed because only these equations contain the dynamics of the system; the first set are direct consequences of the definition of the p_r. To account for the perturbation we let the constants of the unperturbed solution vary:

$$q_r = F_r(\gamma_1, \cdot\cdot, \gamma_{2n}, t)\ ; \qquad p_r = G_r(\gamma_1, \cdot\cdot, \gamma_{2n}, t)\ ; \quad r = 1, \cdot\cdot, n \tag{18.14}$$

Also let

$$R_r(q_s, p_s, t) = R_r^*(\gamma_i, t)$$

We seek solutions of Eqns. (18.13) in the form

$$q_r(t) = F_r(\gamma_s(t), t)\ ; \qquad p_r(t) = G_r(\gamma_s(t), t)\ ; \quad r = 1, \cdot\cdot, n \tag{18.15}$$

We proceed as previously. For the unperturbed motion,

$$\begin{aligned} \frac{\partial H}{\partial p_r} &= \dot{q}_r = \frac{dq_r}{dt} = \frac{\partial F_r}{\partial t}\ ; \quad r = 1, \cdot\cdot, n \\ -\frac{\partial H}{\partial q_r} &= \dot{p}_r = \frac{dp_r}{dt} = \frac{\partial G_r}{\partial t}\ ; \quad r = 1, \cdot\cdot, n \end{aligned} \tag{18.16}$$

and for the perturbed system

$$\begin{aligned} \frac{\partial H}{\partial p_r} &= \frac{\partial F_r}{\partial t} + \sum_{s=1}^{2n} \frac{\partial F_r}{\partial \gamma_s}\dot{\gamma}_s\ ; \quad r = 1, \cdot\cdot, n \\ -\frac{\partial H}{\partial q_r} &= \frac{\partial G_r}{\partial t} + \sum_{s=1}^{2n} \frac{\partial G_r}{\partial \gamma_s}\dot{\gamma}_s - R_r^*\ ; \quad r = 1, \cdot\cdot, n \end{aligned} \tag{18.17}$$

Comparing Eqns. (18.16) and (18.17) gives

$$\begin{aligned} \sum_s \frac{\partial F_r}{\partial \gamma_s}\dot{\gamma}_s &= 0\ ; \quad r = 1, \cdot\cdot, n \\ \sum_s \frac{\partial G_r}{\partial \gamma_s}\dot{\gamma}_s &= R_r^*\ ; \quad r = 1, \cdot\cdot, n \end{aligned} \tag{18.18}$$

Solution of these linear ordinary differential equations gives the functions $\gamma_i(t)$.

Alternative Form. An alternative method of obtaining the $\gamma_i(t)$ is available. From Eqn. (18.14), arbitrary virtual displacements satisfy

$$\begin{aligned} \delta q_r &= \sum_{j=1}^{2n} \frac{\partial F_r}{\partial \gamma_j}\delta\gamma_j \; ; \quad r = 1,\cdot\cdot,n \\ \delta p_r &= \sum_{j=1}^{2n} \frac{\partial G_r}{\partial \gamma_j}\delta\gamma_j \; ; \quad r = 1,\cdot\cdot,n \end{aligned} \tag{18.19}$$

Now multiply the first set of Eqns. (18.18) by the second set of Eqns. (18.19) and the second of Eqns. (18.18) by the first of Eqns. (18.19); sum each from 1 to n, and subtract one from the other; the result is

$$\begin{aligned} \sum_{j=1}^{2n}\sum_{i=1}^{2n}\left[\sum_{r=1}^{n}\left(\frac{\partial F_r}{\partial \gamma_j}\frac{\partial G_r}{\partial \gamma_i} - \frac{\partial G_r}{\partial \gamma_j}\frac{\partial F_r}{\partial \gamma_i}\right)\dot{\gamma}_i\delta\gamma_j\right] \\ = \sum_{j=1}^{2n}\sum_{r=1}^{n} R_r^* \frac{\partial F_r}{\partial \gamma_j}\delta\gamma_j \end{aligned} \tag{18.20}$$

Since the $\delta\gamma_j$ are arbitrary, each coefficient of this equation must vanish independently, giving

$$\sum_{i=1}^{2n} [\gamma_j, \gamma_i]\,\dot{\gamma}_i = \sum_{r=1}^{n} R_r^* \frac{\partial F_r}{\partial \gamma_j} \; ; \quad j = 1,\cdot\cdot,2n \tag{18.21}$$

where Eqn. (16.24) was used. Equations (18.21) are equivalent to Eqns. (18.18).

Special Case. If there exists a function $U(q_r)$ such that

$$R_r(q_r) = -\frac{\partial U}{\partial q_r} = R_r^*(\gamma, t) \; ; \quad r = 1,\cdot\cdot,n \tag{18.22}$$

then, using Eqns. (18.15),

$$\frac{\partial U}{\partial \gamma_j} = \sum_{r=1}^{n} \frac{\partial U}{\partial q_r}\frac{\partial q_r}{\partial \gamma_j} = -\sum_{r=1}^{n} R_r^* \frac{\partial F_r}{\partial \gamma_j} \; ; \quad j = 1,\cdot\cdot,2n \tag{18.23}$$

Consequently, Eqns. (18.21) become

$$\sum_{i=1}^{2n} [\gamma_j, \gamma_i]\dot{\gamma}_i = -\frac{\partial U}{\partial \gamma_j} \; ; \quad j = 1,\cdot\cdot,2n \tag{18.24}$$

Example. A particle of unit mass moves in a plane under constant gravity and some perturbing force $\underline{R} = R_x\hat{\imath} + R_y\hat{\jmath}$ (Fig. 18-3). Forming the Hamiltonian of the unperturbed system:

$$
\begin{aligned}
T &= \frac{1}{2}(\dot{x}^2 + \dot{y}^2) \; ; \quad V = gy \\
L &= T - V = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - gy \\
p_x &= \frac{\partial L}{\partial \dot{x}} = \dot{x} \; ; \quad p_y = \frac{\partial L}{\partial \dot{y}} = \dot{y} \\
H &= \sum_{r=1}^{2} p_r \dot{q}_r - L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + gy = \frac{1}{2}(p_x^2 + p_y^2) + gy
\end{aligned}
$$

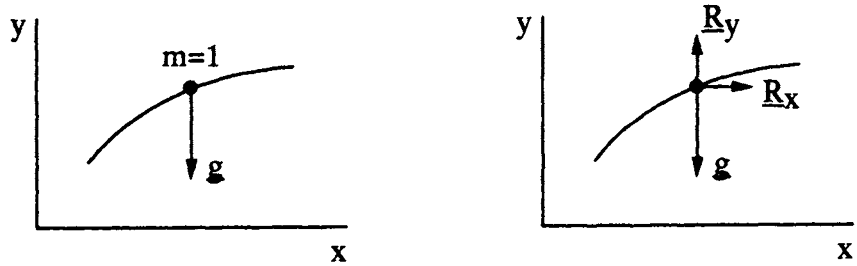


Fig. 18-3

Hamilton's equations are

$$
\dot{x} = \frac{\partial H}{\partial p_x} = p_x \; ; \qquad \dot{p}_x = -\frac{\partial H}{\partial x} = 0
$$

$$
\dot{y} = \frac{\partial H}{\partial p_y} = p_y \; ; \qquad \dot{p}_y = -\frac{\partial H}{\partial y} = -g
$$

The solution of these equations is

$$
\begin{aligned}
&p_x = c_1 \; ; \qquad p_y = -gt + c_2 \\
&x = c_1 t + c_3 \\
&y = -\tfrac{1}{2}gt^2 + c_2 t + c_4
\end{aligned}
$$

Now consider the perturbed motion; Eqns. (18.13) are

$$
\dot{q}_r = \frac{\partial H}{\partial p_r} \; ; \qquad \dot{p}_r = -\frac{\partial H}{\partial q_r} + R_r \; ; \quad r = 1, 2
$$

Next replace the constants c_1, c_2, c_3, c_4 in the unperturbed solution by γ_1, γ_2, γ_3, γ_4, the latter being functions of t; thus

$$\begin{aligned} x &= \gamma_1 t + \gamma_3 = F_1(\gamma_s, t) \\ y &= -\frac{1}{2}gt^2 + \gamma_2 t + \gamma_4 = F_2(\gamma_s, t) \\ p_x &= \gamma_1 = G_1(\gamma_s, t) \\ p_y &= -gt + \gamma_2 = G_2(\gamma_s, t) \end{aligned}$$

Computing partials,

$$\frac{\partial F_1}{\partial \gamma_1} = t\ ; \quad \frac{\partial F_1}{\partial \gamma_3} = 1\ ; \quad \frac{\partial F_2}{\partial \gamma_2} = t\ ; \quad \frac{\partial F_2}{\partial \gamma_4} = 1\ ; \quad \frac{\partial G_1}{\partial \gamma_1} = 1\ ;$$

$$\frac{\partial G_2}{\partial \gamma_2} = 1\ ; \quad \text{all others zero}$$

Equations (18.18) give the differential equations for the γ_i:

$$\begin{aligned} t\dot{\gamma}_1 + \dot{\gamma}_3 = 0\ ; &\qquad \dot{\gamma}_1 = R_x \\ t\dot{\gamma}_2 + \dot{\gamma}_4 = 0\ ; &\qquad \dot{\gamma}_2 = R_y \end{aligned}$$

If $R_x(t)$ and $R_y(t)$ are known functions, these equations may be solved for the γ_i in terms of four constants of integration; these constants may be determined, for example, from initial conditions $x(0)$, $y(0)$, $\dot{x}(0)$, $\dot{y}(0)$.

Remark. Note that this approach is not restricted to the case in which the perturbing force R_r is conservative. Also, it is applicable to the more general case in which the unperturbed problem is not conservative:

$$\dot{q}_r = \frac{\partial H}{\partial p_r}\ ; \qquad \dot{p}_r = -\frac{\partial H}{\partial q_r} + Q_r^{nc}\ ; \quad r = 1, \cdot\cdot, n \tag{18.25}$$

18.2 Variation of the Elements

The Method. Let H be the Hamiltonian of an unperturbed system for which we have the solution; specifically, suppose

$$S = S(q_1, \cdot\cdot, q_n,\ \alpha_1, \cdot\cdot, \alpha_n, t)$$

is a known complete integral of Hamilton's equation, Eqn. (17.24). Consider a transformation from (q_r, p_r) to (α_r, β_r) such that

$$p_r = \frac{\partial S}{\partial q_r}\ ; \qquad \beta_r = -\frac{\partial S}{\partial \alpha_r}\ ; \qquad r = 1, \cdot\cdot, n \tag{18.26}$$

We have chosen q_r, α_r as the independent variables (any combination of q_r, p_r, α_r, β_r could be chosen, provided that the Jacobian is nonzero); therefore

$$dS = \sum_r \frac{\partial S}{\partial q_r} dq_r + \sum_r \frac{\partial S}{\partial \alpha_r} d\alpha_r + \frac{\partial S}{\partial t} dt \qquad (18.27)$$

Using Eqns. (18.26) and (17.21) - (17.23),

$$\begin{aligned} \sum_r \beta_r d\alpha_r &= -\sum_r \frac{\partial S}{\partial \alpha_r} d\alpha_r = -dS + \sum_r \frac{\partial S}{\partial q_r} dq_r + \frac{\partial S}{\partial t} dt \\ &= \sum_r p_r dq_r - H dt - dS \end{aligned} \qquad (18.28)$$

Comparison of Eqn. (18.28) with (16.4) shows that this is a CT; thus a transformation satisfying Eqns. (18.26) is a CT with generating function S.

Now consider a perturbed problem, with Hamiltonian $H + K$, in which the α_r, β_r are allowed to vary. For this problem make the same transformation from (q_r, p_r) to (α_r, β_r). Then by Eqn. (16.39),

$$H^* = H + K + \frac{\partial S}{\partial t}$$

But $H = -\partial S/\partial t$ so that

$$H^* = K \qquad (18.29)$$

where K is expressed in the variables α_r, β_r. By Jacobi's theorem, Eqns. (16.38), the new equations of motion are

$$\dot{\alpha}_r = \frac{\partial H^*}{\partial \beta_r}\,; \qquad \dot{\beta}_r = -\frac{\partial H^*}{\partial \alpha_r}\,; \qquad r = 1, \cdot\cdot, n \qquad (18.30)$$

Remarks. For the unperturbed system, $K = H^* = 0$ and Eqns. (18.30) imply that the α_r, β_r are constants, as required. For the unperturbed system, S is the generating function for a transformation from the q_r, p_r to a set of constants α_r, β_r that completely solves the unperturbed problem. For this problem, the solution will be of the form

$$q_r = \rho_r(\alpha_r, \beta_r, t)\,; \qquad p_r = \rho_{r+n}(\alpha_r, \beta_r, t)\,; \qquad r = 1, \cdot\cdot, n \quad (18.31)$$

where the α_r, β_r are constants. For the perturbed problem with Hamiltonian $H + K$ the solution is still of the form of Eqns. (18.31) except that now the α_r, β_r are functions of time given by Eqns. (18.30).

This method differs from that of Section 18.1, the variation of constants, in that in the present method the constants that are allowed to vary satisfy themselves Hamilton's equations. The perturbed motion at any instant is one of the old motions whose elements are α_r, β_r but these elements are not now constants; in effect, K continually modifies the old motion.

The method of variation of elements also has been widely applied to orbital dynamics. For example, Pars has a detailed discussion of the variation of the elliptic elements in the two-body problem.

Example – Simple Pendulum. Consider a simple pendulum with mass 1 and length ℓ (Fig. 15-1). Letting $p = \dot{\theta}$ and $n^2 = g/\ell$, the exact Hamiltonian is

$$H = \frac{1}{2}p^2 + n^2(1 - \cos\theta)$$

Expanding about $\theta = 0$:

$$(1 - \cos\theta) = 1 - 1 + \frac{\theta^2}{2} - \frac{\theta^4}{4!} + \cdots = \frac{\theta^2}{2} - \frac{\theta^4}{24} + \cdots$$

Thus, approximately,

$$H = \frac{1}{2}p^2 + \frac{1}{2}n^2\theta^2 - \frac{1}{24}n^2\theta^4$$

We call this the second approximation; $\frac{1}{2}p^2 + \frac{1}{2}n^2\theta^2$ is the first approximation and $-\frac{1}{24}n^2\theta^4$ is the function K of the previous section.

First consider the unperturbed system; from Eqn. (17.24),

$$\frac{\partial S}{\partial t} + \frac{1}{2}\left(\left(\frac{\partial S}{\partial \theta}\right)^2 + n^2\theta^2\right) = 0$$

Using the methods of Section 17.3, the solution of this equation is

$$S = -n\alpha t + \int_0^{\theta} \sqrt{2n\alpha - n^2\rho^2 d\rho}$$

By the Hamilton-Jacobi theorem, Eqns. (17.21) and (17.22),

$$p = \frac{\partial S}{\partial \theta}\,; \qquad \beta = -\frac{\partial S}{\partial \alpha}$$

with solution

$$\begin{aligned} \theta &= \sqrt{\frac{2\alpha}{n}}\sin(nt-\beta) \\ p &= \sqrt{2n\alpha}\cos(nt-\beta) \end{aligned}$$

For the perturbed system (second approximation), α and β become functions of time. We have

$$\begin{aligned} H^* = K &= -\frac{1}{24}n^2\theta^4 = -\frac{1}{6}\alpha^2\sin^4(nt-\beta) \\ &= -\frac{1}{48}\alpha^2[3 - 4\cos 2(nt-\beta) + \cos 4(nt-\beta)] \end{aligned}$$

so that Eqns. (18.30) give

$$\begin{aligned} \dot{\alpha} &= \frac{\partial H^*}{\partial \beta} = \frac{1}{12}\alpha^2\,[2\sin 2(nt-\beta) - \sin 4(nt-\beta)] \\ \dot{\beta} &= -\frac{\partial H^*}{\partial \alpha} = \frac{1}{24}\alpha\,[3 - 4\cos 2(nt-\beta) + \cos 4(nt-\beta)] \end{aligned}$$

Since θ is small and θ is proportional to $\sqrt{\alpha}$, we ignore second and higher order terms in α to get the approximation:

$$\dot{\alpha} = 0\,; \qquad \dot{\beta} = \frac{1}{8}\alpha$$

which have solutions

$$\begin{aligned} \alpha &= \text{constant} \\ \beta &= \beta' + \frac{1}{8}\alpha t \end{aligned}$$

where β' is a constant. Let $A = \sqrt{2\alpha/n}$ denote the amplitude of the motion. Substituting the solutions for α and β into the expression for θ gives

$$\theta = A\sin\left[\left(n - \frac{1}{8}\alpha\right)t - \beta'\right] = A\sin\left[n\left(1 - \frac{A^2}{16}\right)t - \beta'\right]$$

Thus the amplitude is not altered but the period is; the new period τ is given approximately by:

$$\tau = \frac{2\pi}{n\left(1 - \frac{A^2}{16}\right)} = \frac{2\pi}{n}\left(1 + \frac{A^2}{16}\right)$$

The constants α and β' may be determined by initial conditions $\theta(0)$ and $\dot{\theta}(0)$.

18.3 Infinitesimal Contact Transformations

The Transformation. Consider a generating function

$$W = M + \sum_r q_r P_r - \sum_r Q_r P_r \tag{18.32}$$

where $M \in C^2(q_r, p_r, t)$. For the transformation from (q_r, p_r) to (Q_r, P_r) to be a CT, Eqn. (16.4) must hold; the variational version is

$$\sum_r P_r \delta Q_r = \sum_r p_r \delta q_r - \delta W \tag{18.33}$$

Substituting Eqn. (18.32) into (18.33) results in

$$\begin{aligned} \sum P_r \delta Q_r &= \sum_r p_r \delta q_r - \delta M - \sum_r q_r \delta P_r - \sum_r P_r \delta q_r \\ &\quad + \sum_r Q_r \delta P_r + \sum_r P_r \delta Q_r \\ \delta M &= \sum_r (p_r - P_r)\delta q_r + \sum_r (Q_r - q_r)\delta P_r \end{aligned}$$

so that

$$-\frac{\partial M}{\partial q_r} = P_r - p_r \; ; \qquad \frac{\partial M}{\partial P_r} = Q_r - q_r \tag{18.34}$$

Note that these resemble Hamilton's equations.

Now let

$$M = \epsilon\theta \tag{18.35}$$

where $\epsilon \ll 1$ and $\theta \in C^2(q_r, p_r, t)$. Using the first of Eqns. (18.34), we write

$$\theta(q_r, P_r, t) = \theta\left(q_r, p_r - \epsilon\frac{\partial\theta}{\partial q_r}, t\right) = \rho(q_r, p_r, t)$$

Hence, from Eqns. (18.34) and (18.35), to first order,

$$\frac{\partial M}{\partial P_r} = \epsilon\frac{\partial \theta}{\partial P_r} = \epsilon\frac{\partial \rho}{\partial p_s}\frac{\partial p_s}{\partial P_r} = \epsilon\frac{\partial \rho}{\partial p_r}\ ; \qquad r = 1, \cdot\cdot, n \tag{18.36}$$

Similarly,

$$\frac{\partial M}{\partial q_r} = \epsilon\frac{\partial \theta}{\partial q_r} = \epsilon\frac{\partial \rho}{\partial q_r}\ ; \qquad r = 1, \cdot\cdot, n \tag{18.37}$$

to first order. Combining Eqns. (18.34), (18.36) and (18.37):

$$Q_r - q_r = \epsilon\frac{\partial \rho}{\partial p_r}\ ; \qquad P_r - p_r = -\epsilon\frac{\partial \rho}{\partial q_r}\ ; \qquad r = 1, \cdot\cdot, n \tag{18.38}$$

which defines an *infinitesimal contact transformation* (ICT).

Interpretation. Equations (18.38) may be viewed as first order approximations to Hamilton's equations. To see this, take

$$\rho = H\ ; \qquad \epsilon = \Delta t$$

$$\Delta q_r = Q_r - q_r\ ; \qquad r = 1, \cdot\cdot, n$$

$$\Delta p_r = P_r - p_r\ ; \qquad r = 1, \cdot\cdot, n$$

Then Eqns. (18.38) become

$$\frac{\Delta q_r}{\Delta t} = \frac{\partial H}{\partial p_r}\ ; \qquad \frac{\Delta p_r}{\Delta t} = -\frac{\partial H}{\partial q_r}\ ; \qquad r = 1, \cdot\cdot, n \tag{18.39}$$

Thus (q_r, p_r) gets transformed to (Q_r, P_r) in time increment Δt and the motion may be viewed as a finite sequence of ICT's (Fig. 18-4).

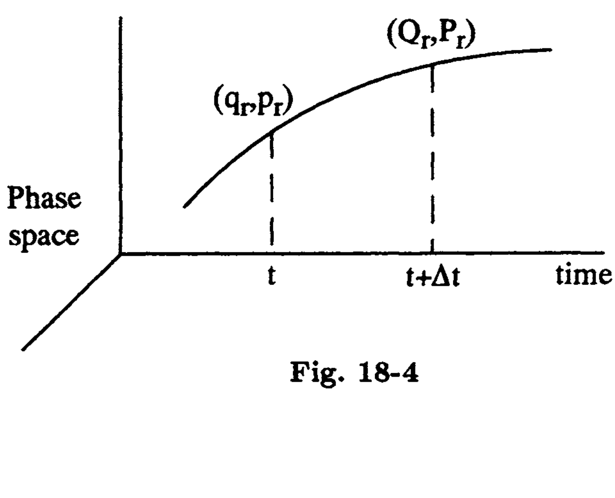


Fig. 18-4

The Hamiltonian. Let $f \in C^2(q_r, p_r, t)$ be any function and form the Taylor series expansion, holding t fixed and retaining only the first order terms; using Eqn. (18.38),

$$\begin{aligned} f(Q_r, P_r, t) - f(q_r, p_r, t) &= \sum_r (Q_r - q_r)\frac{\partial f}{\partial q_r} + \sum_r (P_r - p_r)\frac{\partial f}{\partial p_r} \\ &= \epsilon \sum_r \left(\frac{\partial f}{\partial q_r}\frac{\partial \rho}{\partial p_r} - \frac{\partial f}{\partial p_r}\frac{\partial \rho}{\partial q_r} \right) \\ &= \epsilon (f, \rho) \end{aligned} \tag{18.40}$$

where (f, ρ) is the Poisson bracket of f and ρ. Now identify $f(q_r, p_r, t)$ with $H(q_r, p_r, t)$. By Jacobi's theorem, Eqn. (16.39),

$$H^*(Q_r, P_r, t) = H(q_r, p_r, t) + \frac{\partial W}{\partial t} \tag{18.41}$$

with

$$\frac{\partial W}{\partial t} = \frac{\partial M}{\partial t} = \epsilon \frac{\partial \theta}{\partial t} = \epsilon \frac{\partial \rho}{\partial t} \tag{18.42}$$

Combining Eqns. (18.40) - (18.42) results in

$$H(Q_r, P_r, t) - H(q_r, p_r, t) = \epsilon (H, \rho)$$

$$H^*(Q_r, P_r, t) = H(Q_r, P_r, t) - \epsilon (H, \rho) + \epsilon \frac{\partial \rho}{\partial t}$$

$$H^*(Q_r, P_r, t) = H(Q_r, P_r, t) + \epsilon \left[\frac{\partial \rho}{\partial t} + (\rho, H) \right] \tag{18.43}$$

which is the Hamiltonian of the transformed system.

Recall from Section 15.5 that if $\rho(q_r, p_r, t)$ is an integral of the motion of the original system, then by Poisson's theorem

$$\frac{\partial \rho}{\partial t} + (\rho, H) = 0 \tag{18.44}$$

and thus in our case, if ρ is an integral,

$$H^*(Q_r, P_r, t) = H(Q_r, P_r, t) = H(q_r, p_r, t) \tag{18.45}$$

Hence an ICT with generating function an integral of the motion leaves H invariant. The converse of what we have just proved provides a means of getting candidate integrals of the motion. The procedure is to seek ICT's that leave H invariant; the generating function will be then an integral of the motion. In other words, Eqns. (18.43) and (18.45) would imply Eqn. (18.44).

Ignorable Coordinate. As an example of the above procedure, suppose q_i is ignorable; that is, H is independent of q_i. Then choose ρ such that $\rho = p_i$ so that

$$\frac{\partial \rho}{\partial p_r} = 0 \text{ for } r \neq i \, ; \qquad \frac{\partial \rho}{\partial p_i} = 1$$

$$\frac{\partial \rho}{\partial q_r} = 0 \, ; \qquad r = 1, \cdot\cdot, n$$

Equations (18.38) yield in this case

$$Q_r - q_r = \epsilon \frac{\partial \rho}{\partial p_r} = \epsilon \delta_{r_i} \, ; \qquad r = 1, \cdot\cdot, n$$

$$P_r - p_r = -\epsilon \frac{\partial \rho}{\partial q_r} = 0 \, ; \qquad r = 1, \cdot\cdot, n$$

which is

$$Q_r = q_r + \epsilon \delta_{r_i} \, ; \qquad P_r = p_r \, ; \qquad r = 1, \cdot\cdot, n$$

Since q_i is the only q_r to change and H does not depend on q_i, the transformation leaves H invariant and thus ρ is an integral of the motion,

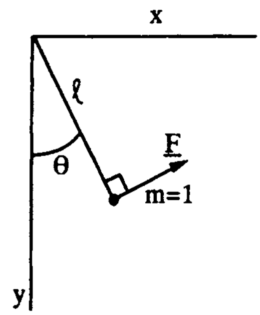
$$\rho = p_i = \text{constant}$$

which of course we already knew.

PROBLEMS

18/1. Fill in the details of the solution of the example of Section 18.1. Completely solve the problem in terms of initial conditions for the case in which R_x and R_y are both constants.

18/2. Shown is a simple pendulum with perturbing force $\underline{F}$. Find Eqns. (18.18) for this problem. Use the small angle, linearized solution as a starting point.



Problem 18/2

18/3. Fill in the details of the solution of the example of Section 18.2.

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Index