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AN INTRODUCTION TO THE THEORY OF PIEZOELECTRICITY

Jiashi Yang

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AN INTRODUCTION TO THE THEORY OF PIEZOELECTRICITY

by

JIASHI YANG
Department of Engineering Mechanics
University of Nebraska-Lincoln, U.S.A.

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Foreword

This book is based on lecture notes for a graduate course that has been offered at University of Nebraska-Lincoln on and off since 1998. The course is intended to provide graduate students with the basic aspects of the continuum modeling of electroelastic interactions in solids. A concise treatment of linear, nonlinear, static and dynamic theories and problems is presented. The emphasis is on formulation and understanding of problems useful in device applications rather than solution techniques of mathematical problems. The mathematics used in the book is minimal. The book is suitable for a one-semester graduate course on electroelasticity. It can also be used as a reference for researchers. I would like to take this opportunity to thank UNL for a Maude Hammond Fling Faculty Research Fellowship in 2003 for the preparation of the first draft of this book. I also wish to thank Ms. Deborah Derrick of the College of Engineering and Technology at UNL for editing assistance with the book, and Professor David Y. Gao of Virginia Polytechnic Institute and State University for recommending this book to Kluwer for publication in the series of Advances in Mechanics and Mathematics.

JSY
Lincoln, Nebraska
2004

Preface

Electroelastic materials exhibit electromechanical coupling. They experience mechanical deformations when placed in an electric field, and become electrically polarized under mechanical loads. Strictly speaking, piezoelectricity refers to linear electromechanical couplings only. Electrostriction may be the simplest nonlinear electromechanical coupling, where mechanical fields depend on electric fields quadratically in the simplest description. Electroelastic materials have been used for a long time to make many electromechanical devices. Examples include transducers for converting electrical energy to mechanical energy or vice versa, resonators and filters for frequency control and selection for telecommunication and precise timing and synchronization, and acoustic wave sensors.

Although most of the book is devoted to the linear theory of piezoelectricity, the book begins with a concise chapter on the nonlinear theory of electroelasticity. It is hoped that this will be helpful for a deeper understanding of the theory of piezoelectricity, because the linear theory is a linearization of the nonlinear theory about a natural state with zero fields. The presentation of the linear theory of piezoelectricity is rather independent so that readers who are not interested in nonlinear electroelasticity can begin directly with Section 2 of Chapter 2 on linear piezoelectricity.

Whereas the majority of books on elasticity treat static problems, the author believes that dynamic problems deserve more attention for piezoelectricity. Therefore, they occupy more space in this book. Chapter 3 is on linear statics and Chapters 4 and 5 are on linear dynamics. This is because in technological applications piezoelectric materials seem to be used in devices operating with vibration modes or propagating waves more than with static deformations. Chapters 2 to 5 form the core for a one-semester course on linear piezoelectricity.

Linear piezoelectricity assumes infinitesimal deviations from an ideal reference state in which there are no pre-existing mechanical and/or electric fields (initial or biasing fields). The presence of biasing fields makes a material apparently behave like a different material and renders the linear theory of piezoelectricity invalid. The behavior of electroelastic bodies under biasing fields can be described by the linear theory for infinitesimal incremental fields superposed on finite biasing fields, which is the subject of Chapter 6. The theory of the incremental fields is derived from the nonlinear theory of electroelasticity when the nonlinear theory is linearized about a bias.

Chapter 7 gives a brief presentation of nonlinear theory including the cubic effects of displacement gradient and electric potential gradient, linear nonlocal theory, linear theory of gradient effects of electrical variables, coupled thermal and dissipative effects, semiconduction, and dynamic theory with Maxwell equations.

The development of the theory of electroelasticity was strongly motivated and influenced by its applications in technology. A book on piezoelectricity does not seem to be complete without some discussion on the applications of the theory, which is given in Chapter 8. A piezoelectric gyroscope, a transformer, a pressure sensor, a temperature sensor, and a resonator are discussed in this chapter.

Throughout the book, effort has been made to present materials with mathematics that are necessary and minimal. Two-point Cartesian tensors with indices are assumed and are used from the very beginning, without which certain concepts of the nonlinear theory cannot be made fully clear. Some concepts from partial differential equations relevant to the well-postness of a boundary-value problem are helpful, but classical solution techniques of separation of variables and integral transforms, etc., are not necessary. Although most problems appear as boundary-value problems of partial differential equations, usually part of a solution is either known or can be guessed from physical reasoning. Therefore some solution techniques for ordinary differential equations are sufficient.

Many problems are analyzed in the book. Some exercise problems are also provided. The problems were chosen based on usefulness and simplicity. Most problems have applications in devices, and have closed-form solutions.

Due to the use of quite a few stress tensors and electric fields in nonlinear electroelasticity, a list of notation is provided in Appendix 1. Material constants used in the book are given in Appendix 2.

Chapter 1

NONLINEAR ELECTROELASTICITY FOR STRONG FIELDS

In this chapter we develop the nonlinear theory of electroelasticty for large deformations and strong electric fields. Readers who are only interested in linear theories may skip this chapter and begin with Chapter 2, Section 2. This chapter uses two-point Cartesian tensor notation, the summation convention for repeated tensor indices and the convention that a comma followed by an index denotes partial differentiation with respect to the coordinate associated with the index.

1. DEFORMATION AND MOTION OF A CONTINUUM

This section is on the kinematics of a deformable continuum. The section is not meant to be a complete treatment of the subject. Only results needed for the rest of the book are presented.

Consider a deformable continuum which, in the reference configuration at time t_0, occupies a region V with boundary surface S (see Figure 1.1-1). $\mathbf{N}$ is the unit exterior normal of S. In this state the body is free from deformation and fields. The position of a material point in this state may be denoted by a position vector $\mathbf{X} = X_K\mathbf{I}_K$ in a rectangular coordinate system X_K. X_K denotes the reference or material coordinates of the material point. They are a continuous labeling of material particles so that they are identifiable. At time t, the body occupies a region v with boundary surface s and exterior normal $\mathbf{n}$. The current position of the material point associated with $\mathbf{X}$ is given by $\mathbf{y} = y_k\mathbf{i}_k$, which denotes the present or spatial coordinates of the material point.

Since the coordinate systems are othogonal,

$$\mathbf{i}_k \cdot \mathbf{i}_l = \delta_{kl}, \quad \mathbf{I}_K \cdot \mathbf{I}_L = \delta_{KL}, \qquad (1.1\text{-}1)$$

where δ_{kl} and δ_{KL} are the Kronecker delta. In matrix notation,

$$[\delta_{kl}] = [\delta_{KL}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{1.1-2}$$

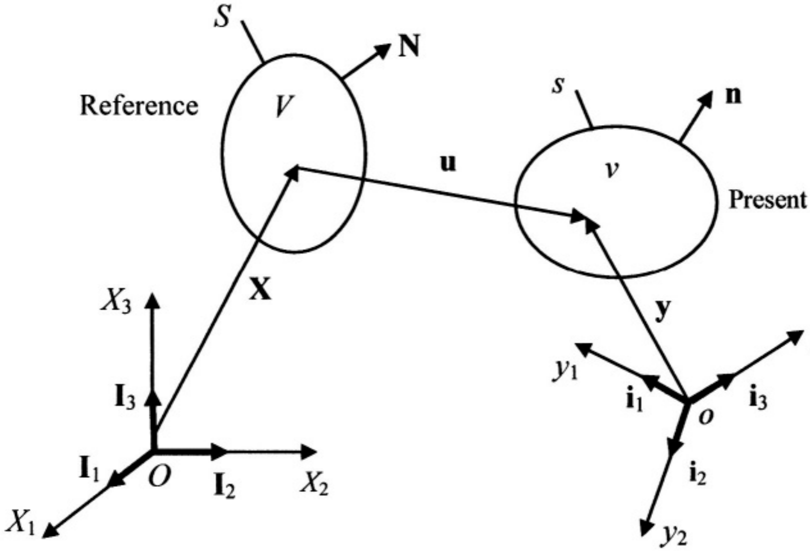

Figure 1.1-1. Motion of a continuum and coordinate systems.

The transformation coefficients (shifters) between the two coordinate systems are denoted by

$$\mathbf{i}_k \cdot \mathbf{I}_L = \delta_{kL}. \tag{1.1-3}$$

In the rest of this book the two coordinate systems are chosen to be coincident, i.e.,

$$o = O, \quad \mathbf{i}_1 = \mathbf{I}_1, \quad \mathbf{i}_2 = \mathbf{I}_2, \quad \mathbf{i}_3 = \mathbf{I}_3. \tag{1.1-4}$$

Then δ_{kL} becomes the Kronecker delta. A vector can be resolved into rectangular components in different coordinate systems. For example, we can also write

$$\mathbf{y} = y_K \mathbf{I}_K, \tag{1.1-5}$$

with

$$y_M = \delta_{Mi} y_i. \tag{1.1-6}$$

The motion of the body is described by

$$y_i = y_i(\mathbf{X}, t). \tag{1.1-7}$$

The displacement vector $\mathbf{u}$ of a material point is defined by

$$\mathbf{y} = \mathbf{X} + \mathbf{u}, \tag{1.1-8}$$

or

$$y_i = \delta_{iM}(X_M + u_M). \tag{1.1-9}$$

A material line element $d\mathbf{X}$ at t_0 deforms into the following line element at t:

$$dy_i\Big|_{t\ \text{fixed}} = y_{i,K}dX_K, \tag{1.1-10}$$

where the deformation gradient

$$y_{k,K} = \delta_{kK} + \delta_{kL}u_{L,K} \tag{1.1-11}$$

is a two-point tensor. The following determinant is called the Jacobian of the deformation:

$$\begin{aligned} J = \det(y_{k,K}) &= \varepsilon_{ijk}y_{i,1}y_{j,2}y_{k,3} = \varepsilon_{KLM}y_{1,K}y_{2,L}y_{3,M} \\ &= \frac{1}{6}\varepsilon_{klm}\varepsilon_{KLM}y_{k,K}y_{l,L}y_{m,M}, \end{aligned} \tag{1.1-12}$$

where ε_{klm} and ε_{KLM} are the permutation tensor, and

$$\varepsilon_{ijk} = \mathbf{i}_i \cdot (\mathbf{i}_j \times \mathbf{i}_k) = \begin{cases} 1 & i,j,k = 1,2,3; \quad 2,3,1; \quad 3,1,2, \\ -1 & i,j,k = 3,2,1; \quad 2,1,3; \quad 1,3,2, \\ 0 & \text{otherwise.} \end{cases} \tag{1.1-13}$$

The following relation exists (ε–δ identity):

$$\varepsilon_{ijk}\varepsilon_{pqr} = \begin{vmatrix} \delta_{ip} & \delta_{iq} & \delta_{ir} \\ \delta_{jp} & \delta_{jq} & \delta_{jr} \\ \delta_{kp} & \delta_{kq} & \delta_{kr} \end{vmatrix}. \tag{1.1-14}$$

As a special case, when $i = p$, then

$$\varepsilon_{ijk}\varepsilon_{iqr} = \delta_{jq}\delta_{kr} - \delta_{jr}\delta_{kq}. \tag{1.1-15}$$

With Equation (1.1 -14) it can be shown from (1.1-12) that

$$J = \frac{1}{6}\left[2\frac{\partial y_K}{\partial X_L}\frac{\partial y_L}{\partial X_M}\frac{\partial y_M}{\partial X_K} - 3\frac{\partial y_K}{\partial X_K}\frac{\partial y_L}{\partial X_M}\frac{\partial y_M}{\partial X_L} + \left(\frac{\partial y_M}{\partial X_M}\right)^3\right]. \tag{1.1-16}$$

It can be verified that for all L, M, and N the following is true:

$$\varepsilon_{ijk}y_{i,L}y_{j,M}y_{k,N} = J\varepsilon_{LMN}. \tag{1.1-17}$$

From Equation (1.1-17) the following can be shown:

$$\varepsilon_{ijk}y_{j,M}y_{k,N} = J\varepsilon_{LMN}X_{L,i}. \tag{1.1-18}$$

Proof: Multiplying both sides of (1.1-17) by $X_{L,r}$, we have

$$\varepsilon_{ijk}y_{i,L}X_{L,r}y_{j,M}y_{k,N} = J\varepsilon_{LMN}X_{L,r}. \tag{1.1-19}$$

Then

$$\varepsilon_{ijk}\delta_{ir}y_{j,M}y_{k,N} = J\varepsilon_{LMN}X_{L,r}, \tag{1.1-20}$$

$$\varepsilon_{rjk}y_{j,M}y_{k,N} = J\varepsilon_{LMN}X_{L,r}. \tag{1.1-21}$$

Replacing the index r by i gives (1.1-18).

The following relation can then be derived:

$$\varepsilon_{ijk}\varepsilon_{LMN}y_{j,M}y_{k,N} = 2JX_{L,i}. \tag{1.1-22}$$

Proof. Multiply both sides of (1.1-18) by ε_{PMN}

$$\begin{aligned}\varepsilon_{ijk}\varepsilon_{PMN}y_{j,M}y_{k,N} &= J\varepsilon_{LMN}\varepsilon_{PMN}X_{L,i} = J\varepsilon_{MNL}\varepsilon_{MNP}X_{L,i}\\ &= J(\delta_{NN}\delta_{LP} - \delta_{NP}\delta_{LN})X_{L,i} = J(3\delta_{LP} - \delta_{LP})X_{L,i}\\ &= 2JX_{P,i}.\end{aligned} \tag{1.1-23}$$

where (1.1-15) has been used. Replacing the index P by L gives (1.1-22).

The derivative of the Jacobian with respect to one of its elements is

$$\frac{\partial J}{\partial y_{i,L}} = JX_{L,i}. \tag{1.1-24}$$

Proof: From Equation (1.1-12)

$$\begin{aligned}6\frac{\partial J}{\partial y_{p,Q}} &= \varepsilon_{ijk}\varepsilon_{LMN}\delta_{ip}\delta_{LQ}y_{j,M}y_{k,N}\\ &+ \varepsilon_{ijk}\varepsilon_{LMN}y_{i,L}\delta_{jp}\delta_{MQ}y_{k,N} + \varepsilon_{ijk}\varepsilon_{LMN}y_{i,L}y_{j,M}\delta_{kp}\delta_{NQ}\\ &= \varepsilon_{pjk}\varepsilon_{QMN}y_{j,M}y_{k,N} + \varepsilon_{ipk}\varepsilon_{LQN}y_{i,L}y_{k,N} + \varepsilon_{ijp}\varepsilon_{LMQ}y_{i,L}y_{j,M}\\ &= 3\varepsilon_{pjk}\varepsilon_{QMN}y_{j,M}y_{k,N} = 3(2JX_{Q,p}),\end{aligned} \tag{1.1-25}$$

where (1.1-22) has been used.

With (1.1-22) we can also show that

$$(JX_{K,k})_{,K} = 0. \tag{1.1-26}$$

Proof: Differentiate both sides of (1.1-22) with respect to X_L

$$2(JX_{L,i})_{,L} = \varepsilon_{ijk}\varepsilon_{LMN}(y_{j,ML}y_{k,N} + y_{j,M}y_{k,NL}) = 0, \tag{1.1-27}$$

because

$$\varepsilon_{LMN}y_{j,ML} = 0, \quad \varepsilon_{LMN}y_{k,NL} = 0. \tag{1.1-28}$$

Similarly, the following is true:

$$(J^{-1}y_{k,K})_{,k} = 0. \tag{1.1-29}$$

The length of a material line element before and after deformation is given by

$$(dS)^2 = dX_K dX_K = \delta_{KL}dX_K dX_L, \tag{1.1-30}$$

and

$$(ds)^2 = dy_i dy_i = y_{i,K} dX_K y_{i,L} dX_L = C_{KL} dX_K dX_L, \tag{1.1-31}$$

where C_{KL} is the deformation tensor

$$C_{KL} = y_{k,K} y_{k,L} = C_{LK}. \tag{1.1-32}$$

Its inverse is

$$C_{KL}^{-1} = X_{K,k} X_{L,k} = C_{LK}^{-1}, \quad C_{KL}^{-1} C_{LM} = \delta_{KM}. \tag{1.1-33}$$

From Equation (1.1-32)

$$\det(C_{KL}) = J^2, \tag{1.1-34}$$

which defines J as a function of $\mathbf{C}$.

It can then be shown that

$$\frac{\partial J}{\partial C_{KL}} = \frac{1}{2} J C_{LK}^{-1}. \tag{1.1-35}$$

Proof:

$$\begin{aligned}
\frac{\partial J}{\partial y_{i,L}} &= \frac{\partial J}{\partial C_{KM}} \frac{\partial C_{KM}}{\partial y_{i,L}} = \frac{\partial J}{\partial C_{KM}} \frac{\partial}{\partial y_{i,L}} (y_{j,K} y_{j,M}) \\
&= \frac{\partial J}{\partial C_{KM}} (\delta_{ji} \delta_{KL} y_{j,M} + y_{j,K} \delta_{ji} \delta_{ML}) \\
&= \frac{\partial J}{\partial C_{KM}} (\delta_{KL} y_{i,M} + y_{i,K} \delta_{ML}) \\
&= \frac{\partial J}{\partial C_{LM}} y_{i,M} + \frac{\partial J}{\partial C_{KL}} y_{i,K} \\
&= \left(\frac{\partial J}{\partial C_{LK}} + \frac{\partial J}{\partial C_{KL}} \right) y_{i,K} = J X_{L,i},
\end{aligned} \tag{1.1-36}$$

where (1.1-24) has been used, and the components of $\mathbf{C}$ are treated as if they were independent in the partial differentiation. Equation (1.1-36) implies that

$$\frac{\partial J}{\partial C_{LK}} + \frac{\partial J}{\partial C_{KL}} = J X_{L,i} X_{K,i} = J C_{LK}^{-1}. \tag{1.1-37}$$

If J is written as a symmetric function of $\mathbf{C}$ in the sense that

$$\frac{\partial J}{\partial C_{LK}} = \frac{\partial J}{\partial C_{KL}}, \tag{1.1-38}$$

then Equation (1.1-35) is true.

The derivative of $\mathbf{C}^{-1}$ with respect to $\mathbf{C}$ is given by

$$\frac{\partial C_{KN}^{-1}}{\partial C_{LM}} = -\frac{1}{2}(C_{KL}^{-1}C_{MN}^{-1} + C_{KM}^{-1}C_{LN}^{-1}). \tag{1.1-39}$$

Proof: From Equation (1.1-33)$_2$, for a small variation of **C**,

$$\delta(C_{KL}^{-1}C_{LM}) = C_{LM}\frac{\partial C_{KL}^{-1}}{\partial C_{PQ}}\delta C_{PQ} + C_{KL}^{-1}\delta C_{LM} = 0, \tag{1.1-40}$$

where the components of **C** are treated as if they were independent in the partial differentiation. Multiply Equation (1.1-40) by C_{MN}^{-1}:

$$\frac{\partial C_{KN}^{-1}}{\partial C_{PQ}}\delta C_{PQ} + C_{KL}^{-1}C_{MN}^{-1}\delta C_{LM} = 0, \tag{1.1-41}$$

or

$$\begin{aligned}\left(\frac{\partial C_{KN}^{-1}}{\partial C_{LM}} + C_{KL}^{-1}C_{MN}^{-1}\right)\delta C_{LM} &= \left(\frac{\partial C_{KN}^{-1}}{\partial C_{11}} + C_{K1}^{-1}C_{1N}^{-1}\right)\delta C_{11} \\ &+ \left(\frac{\partial C_{KN}^{-1}}{\partial C_{12}} + C_{K1}^{-1}C_{2N}^{-1} + \frac{\partial C_{KN}^{-1}}{\partial C_{21}} + C_{K2}^{-1}C_{1N}^{-1}\right)\delta C_{12} + \cdots = 0.\end{aligned} \tag{1.1-42}$$

Hence

$$\frac{\partial C_{KN}^{-1}}{\partial C_{LM}} + \frac{\partial C_{KN}^{-1}}{\partial C_{ML}} = -C_{KL}^{-1}C_{MN}^{-1} - C_{KM}^{-1}C_{LN}^{-1}. \tag{1.1-43}$$

Equation (1.1-39) follows when $\mathbf{C}^{-1}$ is written as a symmetric function of **C** similar to (1.1-38).

From Equations (1.1-30) and (1.1-31):

$$(ds)^2 - (dS)^2 = (C_{KL} - \delta_{KL})dX_K dX_L = 2S_{KL}dX_K dX_L, \tag{1.1-44}$$

where the finite strain tensor is defined by

$$\begin{aligned} S_{KL} &= (C_{KL} - \delta_{KL})/2 \\ &= (u_{K,L} + u_{L,K} + u_{M,K}u_{M,L})/2 = S_{LK}. \end{aligned} \tag{1.1-45}$$

The unabbreviated form of (1.1-45) is given below:

$$\begin{aligned} S_{11} &= u_{1,1} + (u_{1,1}u_{1,1} + u_{2,1}u_{2,1} + u_{3,1}u_{3,1})/2, \\ S_{22} &= u_{2,2} + (u_{1,2}u_{1,2} + u_{2,2}u_{2,2} + u_{3,2}u_{3,2})/2, \\ S_{33} &= u_{3,3} + (u_{1,3}u_{1,3} + u_{2,3}u_{2,3} + u_{3,3}u_{3,3})/2, \\ S_{23} &= (u_{2,3} + u_{3,2} + u_{1,2}u_{1,3} + u_{2,2}u_{2,3} + u_{3,2}u_{3,3})/2, \\ S_{31} &= (u_{3,1} + u_{1,3} + u_{1,3}u_{1,1} + u_{2,3}u_{2,1} + u_{3,3}u_{3,1})/2, \\ S_{12} &= (u_{1,2} + u_{2,1} + u_{1,1}u_{1,2} + u_{2,1}u_{2,2} + u_{3,1}u_{3,2})/2. \end{aligned} \tag{1.1-46}$$

At the same material point consider two non-collinear material line elements $d\overset{(1)}{\mathbf{X}}$ and $d\overset{(2)}{\mathbf{X}}$ which deform into $d\overset{(1)}{\mathbf{y}}$ and $d\overset{(2)}{\mathbf{y}}$. The area of the parallelogram spanned by $d\overset{(1)}{\mathbf{X}}$ and $d\overset{(2)}{\mathbf{X}}$, and that by $d\overset{(1)}{\mathbf{y}}$ and $d\overset{(2)}{\mathbf{y}}$, can be represented by the following vectors, respectively:

$$N_L dA = dA_L = \varepsilon_{LMN} d\overset{(1)}{X}_M \, d\overset{(2)}{X}_N , \tag{1.1-47}$$

$$n_i da = da_i = \varepsilon_{ijk} d\overset{(1)}{y}_j \, d\overset{(2)}{y}_k . \tag{1.1-48}$$

They are related by

$$da_i = J X_{L,i} dA_L . \tag{1.1-49}$$

Proof:

$$\begin{aligned} da_i &= \varepsilon_{ijk} d\overset{(1)}{y}_j \, d\overset{(2)}{y}_k = \varepsilon_{ijk} y_{j,M} d\overset{(1)}{X}_M \, y_{k,N} d\overset{(2)}{X}_N \\ &= \varepsilon_{ijk} y_{j,M} y_{k,N} d\overset{(1)}{X}_M \, d\overset{(2)}{X}_N = J X_{L,i} \varepsilon_{LMN} d\overset{(1)}{X}_M \, d\overset{(2)}{X}_N \\ &= J X_{L,i} dA_L , \end{aligned} \tag{1.1-50}$$

where Equation (1.1-18) has been used.

At the same material point consider three non-coplanar material line elements $d\overset{(1)}{\mathbf{X}}$, $d\overset{(2)}{\mathbf{X}}$, and $d\overset{(3)}{\mathbf{X}}$ which deform into $d\overset{(1)}{\mathbf{y}}$, $d\overset{(2)}{\mathbf{y}}$, and $d\overset{(3)}{\mathbf{y}}$. The volume of the parallelepiped spanned by $d\overset{(1)}{\mathbf{X}}$, $d\overset{(2)}{\mathbf{X}}$ and $d\overset{(3)}{\mathbf{X}}$, and that by $d\overset{(1)}{\mathbf{y}}$, $d\overset{(2)}{\mathbf{y}}$, and $d\overset{(3)}{\mathbf{y}}$, are related by

$$dv = J dV . \tag{1.1-51}$$

Proof:

$$\begin{aligned} dv &= d\overset{(1)}{\mathbf{y}} \cdot (d\overset{(2)}{\mathbf{y}} \times d\overset{(3)}{\mathbf{y}}) = \varepsilon_{ijk} d\overset{(1)}{y}_i \, d\overset{(2)}{y}_j \, d\overset{(3)}{y}_k \\ &= \varepsilon_{ijk} y_{i,L} d\overset{(1)}{X}_L \, y_{j,M} d\overset{(2)}{X}_M \, y_{k,N} d\overset{(3)}{X}_N \\ &= \varepsilon_{ijk} y_{i,L} y_{j,M} y_{k,N} d\overset{(1)}{X}_L \, d\overset{(2)}{X}_M \, d\overset{(3)}{X}_N \\ &= J \varepsilon_{LMN} d\overset{(1)}{X}_L \, d\overset{(2)}{X}_M \, d\overset{(3)}{X}_N \\ &= J d\overset{(1)}{\mathbf{X}} \cdot (d\overset{(2)}{\mathbf{X}} \times d\overset{(3)}{\mathbf{X}}) = J dV , \end{aligned} \tag{1.1-52}$$

where Equation (1.1-17) has been used.

The velocity and acceleration of a material point are given by the following material time derivatives:

$$v_i = \frac{Dy_i}{Dt} = \dot{y}_i = \left.\frac{\partial y_i(\mathbf{X},t)}{\partial t}\right|_{\mathbf{X}\text{ fixed}},$$
$$\dot{v}_i = \frac{Dv_i}{Dt} = \left.\frac{\partial^2 y_i(\mathbf{X},t)}{\partial t^2}\right|_{\mathbf{X}\text{ fixed}}. \tag{1.1-53}$$

The deformation rate tensor d_{ij} and the spin tensor ω_{ij} are introduced by decomposing the velocity gradient into symmetric and anti-symmetric parts

$$\partial_i v_j = v_{j,i} = d_{ij} + \omega_{ij},$$
$$d_{ij} = \frac{1}{2}(v_{j,i} + v_{i,j}), \quad \omega_{ij} = \frac{1}{2}(v_{j,i} - v_{i,j}). \tag{1.1-54}$$

We also have

$$\frac{D}{Dt}(dy_i) = \frac{D}{Dt}\left(\frac{\partial y_i}{\partial X_K} dX_K\right) = \frac{\partial}{\partial X_K}\left(\frac{Dy_i}{Dt}\right) dX_K$$
$$= \frac{\partial}{\partial X_K}(v_i) dX_K = v_{i,j} y_{j,K} dX_K. \tag{1.1-55}$$

The strain rate and the deformation rate are related by

$$\dot{S}_{KL} = d_{ij} y_{i,K} y_{j,L}\ . \tag{1.1-56}$$

Proof:

$$\dot{S}_{KL} = \frac{1}{2}(\dot{y}_{i,K} y_{i,L} + y_{i,K}\dot{y}_{i,L}) = \frac{1}{2}(v_{i,K} y_{i,L} + y_{i,K} v_{i,L})$$
$$= \frac{1}{2}(v_{i,j} y_{j,K} y_{i,L} + y_{i,K} v_{i,j} y_{j,L})$$
$$= \frac{1}{2}(v_{j,i} y_{i,K} y_{j,L} + y_{i,K} v_{i,j} y_{j,L})$$
$$= \frac{1}{2}(v_{j,i} + v_{i,j}) y_{i,K} y_{j,L} = d_{ij} y_{i,K} y_{j,L}. \tag{1.1-57}$$

The material derivative of the Jacobian is

$$\dot{J} = J v_{k,k}\ . \tag{1.1-58}$$

Proof: From Equation (1.1-12)

$$\dot{J} = \frac{1}{6}\varepsilon_{klm}\varepsilon_{KLM}(v_{k,K}y_{l,L}y_{m,M} + y_{k,K}v_{l,L}y_{m,M} + y_{k,K}y_{l,L}v_{m,M})$$
$$= \frac{1}{2}\varepsilon_{klm}\varepsilon_{KLM}v_{k,K}y_{l,L}y_{m,M} = \frac{1}{2}v_{k,K}\varepsilon_{klm}\varepsilon_{KLM}y_{l,L}y_{m,M} \tag{1.1-59}$$
$$= \frac{1}{2}v_{k,K}2JX_{K,k} = Jv_{k,k},$$

where Equation (1.1-22) has been used.

The following expression will be useful later in the book:

$$\frac{D}{Dt}(X_{L,j}) = -v_{i,K}X_{K,j}X_{L,i}. \tag{1.1-60}$$

Proof: Since

$$y_{i,K}X_{K,j} = \delta_{ij}, \tag{1.1-61}$$

we have, upon differentiating both sides,

$$\dot{y}_{i,K}X_{K,j} + y_{i,K}\frac{D}{Dt}(X_{K,j}) = 0. \tag{1.1-62}$$

Then

$$y_{i,K}\frac{D}{Dt}(X_{K,j}) = -v_{i,K}X_{K,j}. \tag{1.1-63}$$

Multiplication of both sides of (1.1-63) by $X_{L,i}$ gives

$$\frac{D}{Dt}(X_{L,j}) = -v_{i,K}X_{K,j}X_{L,i}. \tag{1.1-64}$$

Problems

1.1-1. Show (1.1-15) from (1.1-14).

1.1-2. Show (1.1-16).

1.1-3. Show (1.1-45).

1.1-4. Show that $\dfrac{\partial J}{\partial S_{KL}} = 2\dfrac{\partial J}{\partial C_{KL}} = JC_{KL}^{-1}$.

2. GLOBAL BALANCE LAWS

This section summarizes the fundamental physical laws that govern the motion of an elastic dielectric. They are experimental laws and are postulated as the foundation for a continuum theory.

2.1 Polarization

When a dielectric is placed in an electric field, the electric charges in its molecules redistribute themselves microscopically, resulting in a macroscopic polarization. The microscopic charge redistribution occurs in different ways (see Figure 1.2-1).

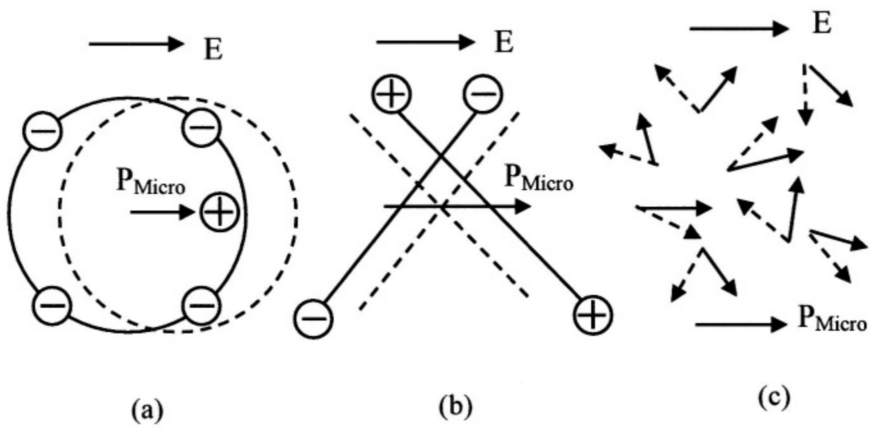

Figure 1.2-1. Microscopic polarization: (a) electronic, (b) ionic, (c) orientational.

At the macroscopic level the distinctions among different polarization mechanisms do not matter. A macroscopic polarization vector per unit present volume,

$$\mathbf{P} = \lim_{\Delta v \to 0} \frac{\sum_{\Delta v} \mathbf{P}_{\text{Micro}}}{\Delta v}, \tag{1.2-1}$$

is introduced which describes the macroscopic polarizing state of the material.

2.2 Piezoelectric Effects

Experiments show that in certain materials polarization can also be induced by mechanical loads. Figure 1.2-2(a) shows such a phenomenon called the direct piezoelectric effect. The induced polarization can be at an angle, e.g., perpendicular to the applied load, depending on the anisotropy of the material. When the load is reversed, so is the induced polarization. When a voltage is applied to a material possessing the direct piezoelectric effect, the material deforms. This is called the converse piezoelectric effect (see Figure 1.2-2(b)).

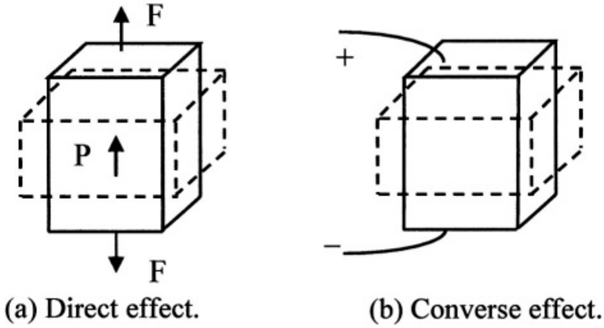

Figure 1.2-2. Macroscopic piezoelectric effects.

Whether a material is piezoelectric depends on its microscopic charge distribution. For example, the charge distribution in Figure 1.2-3(a), when deformed into Figure 1.2-3(b), results in a polarization.

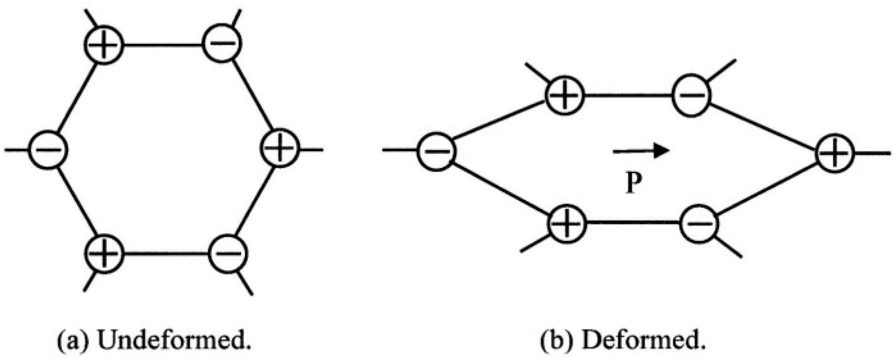

Figure 1.2-3. Origin of the direct piezoelectric effect.

2.3 Electric Body Force, Couple and Power

When a mechanically deformable and electrically polarizable material is subjected to an electric field, a differential element of the material experiences body force and couple due to the electric field. When such a material deforms and polarizes, the electric field also does work to the material. Fundamental to the development of the equations of electroelasticity is the derivation of the electric body force, couple, and power due to the electric field. This can be done by averaging fields associated with charged and interacting particles [1] or particles with internal degrees of freedom [2]. Tiersten [3] introduced a physical model of two mechanically and electrically interacting and interpenetrating continua to describe electric polarization macroscopically. One continuum is the lattice continuum which carries mass and positive charges. The other is the electronic continuum which is negatively charged and is without mass. Electric polarization is modeled by a small, relative displacement of the electronic continuum with respect to the lattice continuum. By systematic applications of the basic laws of physics to each continuum and combining the resulting equations, Tiersten [3] obtained the expressions for the electric body force $\mathbf{F}^E$, couple $\mathbf{C}^E$ and power w^E as

$$\begin{aligned} F_j^E &= \rho_e E_j + P_i E_{j,i}, \\ C_i^E &= \varepsilon_{ijk} P_j E_k, \\ w^E &= \rho E_i \dot{\pi}_i, \end{aligned} \tag{1.2-2}$$

where $\mathbf{E}$ is the electric field vector, ρ is the present mass density, ρ_e (a scalar) is the present free charge density, and $\pi_i = P_i / \rho$ is the polarization per unit mass. The presence of the mass density ρ in $(1.2\text{-}2)_3$ is not obvious. It is due to a relation between the density of the bound charge and mass density [3]. The problem at the end of this section is helpful for understanding (1.2-2).

2.4 Balance Laws

Let l be a closed curve. The continuum theory of electroelasticity postulates the following global balance laws in integral form:

$$\begin{aligned} &\int_s \mathbf{D} \cdot d\mathbf{a} = \int_v \rho_e dv, \\ &\int_l \mathbf{E} \cdot d\mathbf{l} = 0, \\ &\frac{D}{Dt}\int_v \rho dv = 0, \\ &\frac{D}{Dt}\int_v \rho \mathbf{v} dv = \int_v (\rho \mathbf{f} + \mathbf{F}^E) dv + \int_s \mathbf{t} da, \\ &\frac{D}{Dt}\int_v \mathbf{y} \times \rho \mathbf{v} dv = \int_v [\mathbf{y} \times (\rho \mathbf{f} + \mathbf{F}^E) + \mathbf{C}^E] dv + \int_s \mathbf{y} \times \mathbf{t} da, \\ &\frac{D}{Dt}\int_v \rho(\frac{1}{2}\mathbf{v} \cdot \mathbf{v} + e) dv = \int_v [(\rho \mathbf{f} + \mathbf{F}^E) \cdot \mathbf{v} + w^E] dv + \int_s \mathbf{t} \cdot \mathbf{v} da, \end{aligned} \tag{1.2-3}$$

where $\mathbf{D}$ is the electric displacement vector, $\mathbf{f}$ is the mechanical body force per unit mass, $\mathbf{t}$ is the surface traction on s, and e is the internal energy per unit mass. The equations in (1.2-3) are, respectively, Gauss's law (the charge equation), Faraday's law in quasistatic form, the conservation of mass, the conservation of linear momentum, the conservation of angular momentum, and the conservation of energy. In the above balance laws, the electric field appears to be static. This is the so-called quasistatic approximation [4]. The approximation is valid when we are considering phenomena at elastic wavelengths which are much shorter than electromagnetic wavelengths at the same frequency [4]. Quasistatic approximation can be considered as the lowest order approximation of the electrodynamic theory through a perturbation procedure [5], which will be shown in Chapter 7, Section 6 when discussing the dynamic theory. Within the quasistatic approximation, the electric field depends on time through coupling to the dynamic mechanical fields. The following relation exists among **D**, **E**, and **P**:

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}, \qquad (1.2\text{-}4)$$

where ε_0 is the permittivity of free space.

Problem

1.2-1. Derive expressions for the force, couple, and power on a single, stretchable dipole in an electric field.

3. LOCAL BALANCE LAWS

From Equation $(1.2\text{-}3)_1$, and using the divergence theorem, we can write

$$\int_v n_i D_i da = \int_v D_{i,i} dv = \int_v \rho_e dv, \qquad (1.3\text{-}1)$$

$$\int_v (D_{i,i} - \rho_e) dv = 0. \qquad (1.3\text{-}2)$$

Equation (1.3-2) holds for any v. Assume a continuous integrand, then

$$D_{i,i} - \rho_e = 0. \qquad (1.3\text{-}3)$$

From Equation $(1.2\text{-}3)_2$, with Stoke's theorem, the line integral along l can be converted to a surface integral over an area s whose boundary is l:

$$\int_l \mathbf{E} \cdot d\mathbf{l} = \int_s (\nabla \times \mathbf{E}) \cdot d\mathbf{a} = 0, \qquad (1.3\text{-}4)$$

which implies that

$$\nabla \times \mathbf{E} = \varepsilon_{ijk} E_{k,j} \mathbf{i}_i = 0. \qquad (1.3\text{-}5)$$

From Equation $(1.2\text{-}3)_3$, change the integral back to the reference configuration

$$\frac{D}{Dt}\int_v \rho dv = \frac{D}{Dt}\int_V \rho J dV = \int_V \frac{D}{Dt}(\rho J) dV$$
$$= \int_V (\dot{\rho} J + \rho \dot{J}) dV = \int_V (\dot{\rho} J + \rho J v_{i,i}) dV \tag{1.3-6}$$
$$= \int_v (\dot{\rho} + \rho v_{i,i}) dv = 0,$$

where Equation (1.1-58) has been used. Hence

$$\dot{\rho} + \rho v_{i,i} = 0. \tag{1.3-7}$$

With Equations (1.1-58) and (1.3-7) it can be shown that

$$\frac{D}{Dt}\int_v \rho[\quad] dv = \int_v \rho \frac{D[\quad]}{Dt} dv. \tag{1.3-8}$$

Proof: With the change of integration variables

$$\frac{D}{Dt}\int_v \rho[\quad] dv = \frac{D}{Dt}\int_V \rho[\quad] J dV = \int_V \frac{D}{Dt}(\rho[\quad] J) dV$$
$$= \int_V \left(\dot{\rho}[\quad] J + \rho \frac{D[\quad]}{Dt} J + \rho[\quad]\dot{J}\right) dV$$
$$= \int_V \left(-\rho v_{i,i}[\quad] J + \rho \frac{D[\quad]}{Dt} J + \rho[\quad] J v_{i,i}\right) dV \tag{1.3-9}$$
$$= \int_V \rho \frac{D[\quad]}{Dt} J dV = \int_v \rho \frac{D[\quad]}{Dt} dv.$$

The Cauchy stress tensor σ_{kj} can be introduced by

$$t_i = \sigma_{ji} n_j, \tag{1.3-10}$$

through the usual tetrahedron argument. Then from $(1.2\text{-}3)_4$, with (1.3-8) and the divergence theorem, the balance of linear momentum becomes

$$\int_v \rho \frac{Dv_i}{Dt} dv = \int_v (\rho f_i + F_i^E) dv + \int_s t_i da$$
$$= \int_v (\rho f_i + F_i^E) dv + \int_s \sigma_{ji} n_j da \tag{1.3-11}$$
$$= \int_v (\rho f_i + F_i^E) dv + \int_v \sigma_{ji,j} dv.$$

Hence

$$\sigma_{ji,j} + \rho f_i + F_i^E = \rho \dot{v}_i. \tag{1.3-12}$$

From $(1.2\text{-}3)_5$, the balance of angular momentum can be written as

$$\frac{D}{Dt}\int_v \varepsilon_{ijk} y_j \rho v_k dv$$
$$= \int_v [\varepsilon_{ijk} y_j (\rho f_k + F_k^E) + C_i^E] dv + \int_s \varepsilon_{ijk} y_j t_k da. \tag{1.3-13}$$

The term on the left-hand side can be written as

$$\frac{D}{Dt}\int_v \varepsilon_{ijk} y_j \rho v_k dv = \int_v \rho\varepsilon_{ijk} \frac{D}{Dt}(y_j v_k) dv$$
$$= \int_v \rho\varepsilon_{ijk}(\dot{y}_j v_k + y_j \dot{v}_k) dv \tag{1.3-14}$$
$$= \int_v \rho\varepsilon_{ijk}(v_j v_k + y_j \dot{v}_k) dv = \int_v \rho\varepsilon_{ijk} y_j \dot{v}_k dv.$$

The last term on the right-hand side can be written as

$$\int_s \varepsilon_{ijk} y_j t_k da = \int_s \varepsilon_{ijk} y_j \sigma_{lk} n_l da$$
$$= \int_v (\varepsilon_{ijk} y_j \sigma_{lk})_{,l} dv = \int_v \varepsilon_{ijk}(\delta_{jl}\sigma_{lk} + y_j \sigma_{lk,l}) dv \tag{1.3-15}$$
$$= \int_v \varepsilon_{ijk}(\sigma_{jk} + y_j \sigma_{lk,l}) dv.$$

Substituting Equations (1.3-14) and (1.3-15) back into (1.3-13), we obtain

$$\int_v \rho\varepsilon_{ijk} y_j \dot{v}_k dv = \int_v [\varepsilon_{ijk} y_j (\rho f_k + F_k^E) + C_i^E] dv$$
$$+ \int_v \varepsilon_{ijk}(\sigma_{jk} + y_j \sigma_{lk,l}) dv, \tag{1.3-16}$$

or

$$\int_v \varepsilon_{ijk} y_j (\rho\dot{v}_k - \rho f_k - F_k^E - \sigma_{lk,l}) dv$$
$$= \int_v (\varepsilon_{ijk}\sigma_{jk} + C_i^E) dv. \tag{1.3-17}$$

Hence

$$\int_v (\varepsilon_{ijk}\sigma_{jk} + C_i^E) dv = 0, \tag{1.3-18}$$

which implies that

$$\varepsilon_{ijk}\sigma_{jk} + C_i^E = 0, \tag{1.3-19}$$

or

$$\varepsilon_{ijk}(\sigma_{jk} + P_j E_k) = 0. \tag{1.3-20}$$

It will be proven convenient to introduce an electrostatic stress tensor σ_{ij}^E whose divergence yields the electric body force

$$\sigma_{ij,i}^E = F_j^E. \tag{1.3-21}$$

For the existence of such σ_{ij}^E consider

$$\begin{aligned}\sigma_{ij}^E &= D_i E_j - \frac{1}{2}\varepsilon_0 E_k E_k \delta_{ij} \\ &= P_i E_j + \varepsilon_0 (E_i E_j - \frac{1}{2} E_k E_k \delta_{ij}).\end{aligned} \tag{1.3-22}$$

We have

$$\sigma_{ij,i}^E = D_{i,i} E_j + P_i E_{j,i} = \rho_e E_j + P_i E_{j,i} = F_j^E, \tag{1.3-23}$$

where Equation (1.2-4) has been used. We note that σ_{ij}^E is not unique in the sense that there are other tensors that also satisfy (1.3-21). For example, adding a second rank tensor with zero divergence to the σ_{ij}^E in (1.3-22) will not affect (1.3-21). In this book we will use (1.3-22).

With σ_{ij}^E, the balance of linear momentum, Equation (1.3-12), can be written as

$$(\sigma_{ij} + \sigma_{ij}^E)_{,i} + \rho f_j = \rho \dot{v}_j. \tag{1.3-24}$$

The balance of angular momentum, Equation (1.3-20), can be written as

$$\varepsilon_{ijk}(\sigma_{jk} + P_j E_k) = \varepsilon_{ijk}(\sigma_{jk} + \sigma_{jk}^E) = 0, \tag{1.3-25}$$

which shows that the sum of the Cauchy stress tensor σ_{ij} and the electrostatic stress tensor σ_{ij}^E is symmetric, which we call the total stress tensor and denote it by τ_{ij}

$$\begin{aligned}\tau_{ij} &= \sigma_{ij} + \sigma_{ij}^E \\ &= \sigma_{ij} + P_i E_j + \varepsilon_0 (E_i E_j - \frac{1}{2} E_k E_k \delta_{ij}) = \tau_{ji}.\end{aligned} \tag{1.3-26}$$

τ_{ij} can also be decomposed into the sum of a symmetric tensor σ_{ij}^S and the symmetric Maxwell stress tensor σ_{ij}^M as follows:

$$\begin{aligned}\tau_{ij} &= \sigma_{ij}^S + \sigma_{ij}^M, \\ \sigma_{ij}^S &= \sigma_{ij} + P_i E_j = \sigma_{ji}^S, \\ \sigma_{ij}^M &= \varepsilon_0 (E_i E_j - \frac{1}{2} E_k E_k \delta_{ij}) = \sigma_{ji}^M.\end{aligned} \tag{1.3-27}$$

From Equation $(1.2\text{-}3)_6$, the conservation of energy is

$$\frac{D}{Dt}\int_v \rho\left(\frac{1}{2}v_iv_i + e\right)dv$$
$$= \int_v (\rho f_k + F_k^E)v_k + w^E]dv + \int_s t_k v_k da. \tag{1.3-28}$$

The left-hand side can be written as

$$\frac{D}{Dt}\int_v \rho\left(\frac{1}{2}v_iv_i + e\right)dv = \int_v \rho\frac{D}{Dt}\left(\frac{1}{2}v_iv_i + e\right)dv$$
$$= \int_v \rho(v_i\dot{v}_i + \dot{e})dv. \tag{1.3-29}$$

The last term on the right-hand side can be written as

$$\int_s t_k v_k da = \int_s \sigma_{lk} n_l v_k da$$
$$= \int_v (\sigma_{lk}v_k)_{,l}dv = \int_v (\sigma_{lk,l}v_k + \sigma_{lk}v_{k,l})dv. \tag{1.3-30}$$

Substituting (1.3-29) and (1.3-30) back into (1.3-28) gives

$$\int_v \rho(v_k\dot{v}_k + \dot{e})dv = \int_v (\rho f_k + F_k^E)v_k + w^E]dv$$
$$+ \int_v (\sigma_{lk,l}v_k + \sigma_{lk}v_{k,l})dv, \tag{1.3-31}$$

or

$$\int_v v_k(\rho\dot{v}_k - \rho f_k - F_k^E - \sigma_{lk,l})dv = \int_v (\sigma_{lk}v_{k,l} + w^E - \rho\dot{e}]dv. \tag{1.3-32}$$

With the equation of motion (1.3-12), the left-hand side of (1.3-32) vanishes, and what is left is

$$\int_v (\sigma_{lk}v_{k,l} + w^E - \rho\dot{e})dv = 0, \tag{1.3-33}$$

which implies that

$$\rho\dot{e} = \sigma_{ij}v_{j,i} + \rho E_i\dot{\pi}_i. \tag{1.3-34}$$

A free energy ψ can be introduced through the following Legendre transform:

$$\psi = e - E_i\pi_i. \tag{1.3-35}$$

Then

$$\dot{\psi} = \dot{e} - \dot{E}_i\pi_i - E_i\dot{\pi}_i. \tag{1.3-36}$$

Substitute Equation (1.3-36) into (1.3-34)

$$\rho\dot{\psi} = \sigma_{ij}v_{j,i} - P_i\dot{E}_i. \tag{1.3-37}$$

In summary, the local balance laws are

$$
\begin{aligned}
&D_{i,i} = \rho_e, \\
&\varepsilon_{ijk} E_{k,j} = 0, \\
&\dot{\rho} + \rho v_{i,i} = 0, \\
&\tau_{ij,i} + \rho f_j = \rho \dot{v}_j, \\
&\varepsilon_{ijk} \tau_{jk} = 0, \\
&\rho \dot{\psi} = \sigma_{ij} v_{j,i} - P_i \dot{E}_i .
\end{aligned}
\qquad (1.3\text{-}38)
$$

Since the electric field is quasistatic (curl-free), an electrostatic potential can be introduced such that

$$E_i = -\phi_{,i} . \qquad (1.3\text{-}39)$$

Then Equation (1.3-38)$_2$ is satisfied identically.

Problems

1.3-1. Calculate the stress in a plate capacitor by studying the Coulomb force between charges at the two major surfaces of the plate.

1.3-2. Show (1.3-23).

1.3-3. Show that $\int_v F_i^E dv = \int_s \sigma_{ji}^E n_j da$.

4. MATERIAL FORM OF THE BALANCE LAWS

Up to this point, all the equations have been written in terms of the present coordinates y_i. Since the reference coordinates of material points are known while the present coordinates are not, it is essential to have the equations written in terms of the reference coordinates X_K.

From Equation (1.2-3)$_1$,

$$\int_s D_i da_i = \int_v \rho_e dv . \qquad (1.4\text{-}1)$$

Change the integral back to the reference configuration

$$\int_S D_i J X_{L,i} dA_L = \int_V \rho_e J dV , \qquad (1.4\text{-}2)$$

or

$$\int_S D_i J X_{L,i} N_L dA = \int_V \rho_e J dV , \qquad (1.4\text{-}3)$$

where Equation (1.1-49) has been used. Use the divergence theorem

$$\int_V (D_i J X_{L,i})_{,L} dV = \int_V \rho_e J dV , \qquad (1.4\text{-}4)$$

which implies that

$$\mathcal{D}_{L,L} = \rho_E , \tag{1.4-5}$$

where $\mathcal{D}_L$ and ρ_E (a scalar) defined by

$$\begin{aligned} &\mathcal{D}_L = JX_{L,i}D_i, \quad D_i = J^{-1}y_{i,L}\mathcal{D}_L, \\ &\rho_E = \rho_e J, \end{aligned} \tag{1.4-6}$$

are the reference or material electric displacement and body free change per unit undeformed volume.

From Equation $(1.2\text{-}3)_2$

$$\begin{aligned} \int_l \mathbf{E} \cdot d\mathbf{l} &= \int_l E_i dl_i = \int_L E_i y_{i,K} dL_K \\ &= \int_S \varepsilon_{IJK} (E_i y_{i,K})_{,J} dA_I = 0, \end{aligned} \tag{1.4-7}$$

which implies that

$$\varepsilon_{IJK} \mathcal{E}_{K,J} = 0 , \tag{1.4-8}$$

where $\mathcal{E}_K$ defined by

$$\mathcal{E}_K = E_i y_{i,K} = -\phi_{,i} y_{i,K} = -\phi_{,K}, \quad E_i = \mathcal{E}_K X_{K,i}, \tag{1.4-9}$$

is the reference electric field.

From $(1.2\text{-}3)_3$, the total mass of the material body is a constant, which is the mass in the reference state

$$\int_v \rho dv = \int_V \rho_0 dV , \tag{1.4-10}$$

where ρ_0 is the mass density in the reference state. With changes in integration variables, the conservation of mass takes the following form:

$$\int_V \rho J dV = \int_V \rho_0 dV , \tag{1.4-11}$$

which implies that

$$\rho_0 = \rho J . \tag{1.4-12}$$

The balance of linear momentum can be written as

$$\begin{aligned} &\int_v \rho \dot{v}_j dv = \int_v (\rho f_j + F_j^E) dv + \int_s t_j da \\ &= \int_v (\rho f_j + \sigma_{ij,i}^E) dv + \int_s t_j da = \int_v \rho f_j dv + \int_s (t_j + \sigma_{ij}^E n_i) da \\ &= \int_v \rho f_j dv + \int_s (\sigma_{ij} n_i + \sigma_{ij}^E n_i) da \\ &= \int_v \rho f_j dv + \int_s \tau_{ij} n_i da = \int_v \rho f_j dv + \int_s \tau_{ij} da_i . \end{aligned} \tag{1.4-13}$$

The last term on the right-hand side of (1.4-13) can be written as

$$\int_s \tau_{ij} da_i = \int_S \tau_{ij} J X_{L,i} dA_L$$
$$= \int_S \tau_{ij} J X_{L,i} N_L dA = \int_V (\tau_{ij} J X_{L,i})_{,L} dV. \tag{1.4-14}$$

Substitute (1.4-14) into (1.4-13)

$$\int_V \rho \dot{v}_j J dV = \int_V \rho f_j J dV + \int_V (\tau_{ij} J X_{L,i})_{,L} dV, \tag{1.4-15}$$

which implies that

$$K_{Lj,L} + \rho_0 f_j = \rho_0 \dot{v}_j, \tag{1.4-16}$$

where K_{Lj} is defined by

$$K_{Lj} = J X_{L,i} \tau_{ij}, \quad \tau_{ij} = J^{-1} y_{i,L} K_{Lj}. \tag{1.4-17}$$

The conservation of angular momentum can be written as

$$\varepsilon_{kij} \tau_{ij} = \varepsilon_{kij} J^{-1} y_{i,L} K_{Lj} = 0, \tag{1.4-18}$$

which implies that

$$\varepsilon_{kij} y_{i,L} K_{Lj} = 0. \tag{1.4-19}$$

For the energy equation

$$\rho \dot{\psi} = \sigma_{ij} v_{j,i} - P_i \dot{E}_i, \tag{1.4-20}$$

we introduce a symmetric stress tensor T_{KL}^S and a material polarization vector $\mathcal{P}_K$ by

$$\sigma_{ij}^S = \sigma_{ij} + P_i E_j = J^{-1} y_{i,K} y_{j,L} T_{KL}^S, \quad T_{KL}^S = J X_{K,k} X_{L,l} \sigma_{kl}^S$$
$$P_i = J^{-1} y_{i,K} \mathcal{P}_K, \quad \mathcal{P}_K = J X_{K,k} P_k. \tag{1.4-21}$$

Then

$$\rho \dot{\psi} = \sigma_{ij}^S v_{j,i} - P_i E_j v_{j,i} - P_i \dot{E}_i$$
$$= J^{-1} y_{i,K} y_{j,L} T_{KL}^S v_{j,i} - P_i E_j v_{j,i} - J^{-1} y_{i,K} \mathcal{P}_K \frac{D}{Dt}(X_{L,i} \mathcal{E}_L)$$
$$= J^{-1} y_{i,K} y_{j,L} T_{KL}^S (d_{ij} + \omega_{ij}) - P_i E_j v_{j,i}$$
$$- J^{-1} y_{i,K} \mathcal{P}_K \left[\frac{D}{Dt}(X_{L,i}) \mathcal{E}_L + X_{L,i} \dot{\mathcal{E}}_L \right] \tag{1.4-22}$$
$$= J^{-1} y_{i,K} y_{j,L} T_{KL}^S d_{ij} - P_i E_j v_{j,i}$$
$$- J^{-1} y_{i,K} \mathcal{P}_K (-v_{j,M} X_{M,i} X_{L,j}) \mathcal{E}_L - J^{-1} y_{i,K} \mathcal{P}_K X_{L,i} \dot{\mathcal{E}}_L$$
$$= J^{-1} T_{KL}^S \dot{S}_{KL} - P_i E_j v_{j,i} + P_i v_{j,i} E_j - J^{-1} \mathcal{P}_K \dot{\mathcal{E}}_K,$$

where Equations (1.1-60) and (1.1-56) have been used. Hence

$$\rho_0\dot{\psi} = T^S_{KL}\dot{S}_{KL} - \mathcal{P}_K\dot{\mathcal{E}}_K\,. \tag{1.4-23}$$

In summary, the balance laws in material form are

$$\begin{aligned}
&\mathcal{D}_{K,K} = \rho_E\,,\\
&\varepsilon_{IJK}\mathcal{E}_{K,J} = 0,\\
&\rho_0 = \rho J,\\
&K_{Lk,L} + \rho_0 f_k = \rho_0\dot{v}_k\,,\\
&\varepsilon_{kij}y_{i,L}K_{Lj} = 0,\\
&\rho_0\dot{\psi} = T^S_{KL}\dot{S}_{KL} - \mathcal{P}_K\dot{\mathcal{E}}_K\,.
\end{aligned} \tag{1.4-24}$$

5. CONSTITUTIVE RELATIONS

For constitutive relations we start with the following forms as suggested by $(1.4\text{-}24)_6$:

$$\begin{aligned}
&\psi = \psi(S_{KL},\mathcal{E}_K),\\
&T^S_{KL} = T^S_{KL}(S_{KL},\mathcal{E}_K),\\
&\mathcal{P}_K = \mathcal{P}_K(S_{KL},\mathcal{E}_K).
\end{aligned} \tag{1.5-1}$$

Substitution of Equation (1.5-1) into $(1.4\text{-}25)_6$ gives

$$(T^S_{KL} - \rho_0\frac{\partial\psi}{\partial S_{KL}})\dot{S}_{KL} - (\mathcal{P}_K + \rho_0\frac{\partial\psi}{\partial\mathcal{E}_K})\dot{\mathcal{E}}_K = 0. \tag{1.5-2}$$

Since Equation (1.5-2) is linear in $\dot{S}_{KL}$ and $\dot{\mathcal{E}}_K$, for the inequality to hold for any $\dot{S}_{KL} = \dot{S}_{LK}$ and $\dot{\mathcal{E}}_K$, we must have

$$\begin{aligned}
&T^S_{KL} = \rho_0\frac{1}{2}\left(\frac{\partial\psi}{\partial S_{KL}} + \frac{\partial\psi}{\partial S_{LK}}\right) = \rho_0\frac{\partial\psi}{\partial S_{KL}},\\
&\mathcal{P}_K = -\rho_0\frac{\partial\psi}{\partial\mathcal{E}_K},
\end{aligned} \tag{1.5-3}$$

where the partial derivatives are taken as if the strain components were independent, and the free energy is written as a symmetric function of the strain tensor similar to (1.1-38). Then

$$K_{Lj} = JX_{L,i}\tau_{ij} = JX_{L,i}(\sigma^S_{ij} + \sigma^M_{ij})$$
$$= JX_{L,i}(J^{-1}y_{i,K}y_{j,M}T^S_{KM}) + JX_{L,i}\varepsilon_0(E_iE_j - \frac{1}{2}E_kE_k\delta_{ij})$$
$$= y_{j,M}T^S_{ML} + y_{j,K}JX_{K,k}X_{L,l}\varepsilon_0(E_kE_l - \frac{1}{2}E_mE_m\delta_{kl}) \qquad (1.5\text{-}4)$$
$$= y_{j,K}\rho_0\frac{\partial\psi}{\partial S_{KL}} + y_{j,K}T^M_{KL}$$
$$= F_{Lj} + M_{Lj},$$

where we have introduced

$$T^M_{KL} = JX_{K,k}X_{L,l}\varepsilon_0(E_kE_l - \frac{1}{2}E_mE_m\delta_{kl}),$$
$$F_{Lj} = JX_{L,i}\sigma^S_{ij} = y_{j,K}T^S_{KL} = y_{j,K}\rho_0\frac{\partial\psi}{\partial S_{KL}}, \qquad (1.5\text{-}5)$$
$$M_{Lj} = JX_{L,i}\sigma^M_{ij} = y_{j,K}T^M_{KL} = JX_{L,i}\varepsilon_0(E_iE_j - \frac{1}{2}E_kE_k\delta_{ij}).$$

In terms of the displacement vector, we have

$$F_{Lj} = y_{j,K}T^S_{KL} = (\delta_{jK} + u_{j,K})T^S_{KL}, \qquad (1.5\text{-}6)$$

which, in the unabbreviated notation, takes the following form:

$$F_{11} = T^S_{11}(1+u_{1,1}) + T^S_{12}u_{1,2} + T^S_{13}u_{1,3},$$
$$F_{21} = T^S_{21}(1+u_{1,1}) + T^S_{22}u_{1,2} + T^S_{23}u_{1,3},$$
$$F_{31} = T^S_{31}(1+u_{1,1}) + T^S_{32}u_{1,2} + T^S_{33}u_{1,3},$$
$$F_{12} = T^S_{11}u_{2,1} + T^S_{12}(1+u_{2,2}) + T^S_{13}u_{2,3},$$
$$F_{22} = T^S_{21}u_{2,1} + T^S_{22}(1+u_{2,2}) + T^S_{23}u_{2,3}, \qquad (1.5\text{-}7)$$
$$F_{32} = T^S_{31}u_{2,1} + T^S_{32}(1+u_{2,2}) + T^S_{33}u_{2,3},$$
$$F_{13} = T^S_{11}u_{3,1} + T^S_{12}u_{3,2} + T^S_{13}(1+u_{3,3}),$$
$$F_{23} = T^S_{21}u_{3,1} + T^S_{22}u_{3,2} + T^S_{23}(1+u_{3,3}),$$
$$F_{33} = T^S_{31}u_{3,1} + T^S_{32}u_{3,2} + T^S_{33}(1+u_{3,3}).$$

From

$$D_i = \varepsilon_0E_i + P_i, \qquad (1.5\text{-}8)$$

we have

$$\mathcal{D}_K = \varepsilon_0 J X_{K,i} D_i = \varepsilon_0 J X_{K,i} E_i + J X_{K,i} P_i$$
$$= \varepsilon_0 J C^{-1}_{KL} \mathcal{E}_L + \mathcal{P}_K = \varepsilon_0 J C^{-1}_{KL} \mathcal{E}_L - \rho_0 \frac{\partial \psi}{\partial \mathcal{E}_K}, \qquad (1.5\text{-}9)$$

where

$$\varepsilon_0 J X_{K,i} E_i = \varepsilon_0 J X_{K,i} \mathcal{E}_L X_{L,i} = \varepsilon_0 J C^{-1}_{KL} \mathcal{E}_L, \qquad (1.5\text{-}10)$$

has been used.

The free energy ψ that determines the constitutive relations of nonlinear electroelastic materials may be written as [6]

$$\rho_0 \psi(S_{KL}, \mathcal{E}_K)$$
$$= \frac{1}{2} \underset{2}{c}_{ABCD} S_{AB} S_{CD} - e_{ABC} \mathcal{E}_A S_{BC} - \frac{1}{2} \underset{2}{\chi}_{AB} \mathcal{E}_A \mathcal{E}_B$$
$$+ \frac{1}{6} \underset{3}{c}_{ABCDEF} S_{AB} S_{CD} S_{EF} + \frac{1}{2} \underset{1}{k}_{ABCDE} \mathcal{E}_A S_{BC} S_{DE}$$
$$- \frac{1}{2} b_{ABCD} \mathcal{E}_A \mathcal{E}_B S_{CD} - \frac{1}{6} \underset{3}{\chi}_{ABC} \mathcal{E}_A \mathcal{E}_B \mathcal{E}_C \qquad (1.5\text{-}11)$$
$$+ \frac{1}{24} \underset{4}{c}_{ABCDEFGH} S_{AB} S_{CD} S_{EF} S_{GH} + \frac{1}{6} \underset{2}{k}_{ABCDEFG} \mathcal{E}_A S_{BC} S_{DE} S_{FG}$$
$$+ \frac{1}{4} \underset{1}{a}_{ABCDEF} \mathcal{E}_A \mathcal{E}_B S_{CD} S_{EF} + \frac{1}{6} \underset{3}{k}_{ABCDE} \mathcal{E}_A \mathcal{E}_B \mathcal{E}_C S_{DE}$$
$$- \frac{1}{24} \underset{4}{\chi}_{ABCD} \mathcal{E}_A \mathcal{E}_B \mathcal{E}_C \mathcal{E}_D + \cdots,$$

where the material constants

$$\underset{2}{c}_{ABCD}, \quad e_{ABC}, \quad \underset{2}{\chi}_{AB},$$
$$\underset{3}{c}_{ABCDEF}, \quad \underset{1}{k}_{ABCDE}, \quad b_{ABCD}, \quad \underset{3}{\chi}_{ABC}, \qquad (1.5\text{-}12)$$
$$\underset{4}{c}_{ABCDEFGH}, \quad \underset{2}{k}_{ABCDEFG}, \quad \underset{1}{a}_{ABCDEF}, \quad \underset{3}{k}_{ABCDE}, \quad \underset{4}{\chi}_{ABCD},$$

are called the second-order elastic, piezoelectric, electric susceptibility, third-order elastic, first odd electroelastic, electrostrictive, third-order electric susceptibility, fourth-order elastic, second odd electroelastic, first even electroelastic, third odd electroelastic, and fourth-order electric susceptibility, respectively. These material constants are called the fundamental material constants. The second-order constants are responsible for linear material behaviors. The third- and higher-order material constants are related to nonlinear behaviors of materials. The structure of

$\psi(S_{KL}, \mathcal{E}_K)$ depends on material symmetry. Integrity bases for constructing a scalar function of a symmetric tensor and a vector for all crystal classes are known [7].

6. INITIAL-BOUNDARY-VALUE PROBLEM

Since

$$\mathcal{E}_K = -\phi_{,K}, \quad (1.6\text{-}1)$$

the curl-free equation $(1.4\text{-}24)_2$

$$\varepsilon_{IJK}\mathcal{E}_{K,J} = 0, \quad (1.6\text{-}2)$$

is automatically satisfied. The angular momentum equation $(1.4\text{-}24)_5$

$$\begin{aligned} \varepsilon_{kij} y_{i,L} K_{Lj} &= \varepsilon_{kij} y_{i,L} J X_{L,l} \tau_{lj} \\ &= \varepsilon_{kij} \delta_{il} J \tau_{lj} = J\varepsilon_{kij}\tau_{ij} = 0, \end{aligned} \quad (1.6\text{-}3)$$

is satisfied because τ_{ij} is symmetric. The present mass density ρ appears in the equation of conservation of mass $(14.\text{-}24)_3$ only, and therefore can be calculated from the equation after the deformation and the electric field have been found. The energy equation $(1.4\text{-}24)_6$ is satisfied by the forms of the constitutive relations. In summary, we need to solve the following equations:

$$\begin{aligned} K_{Lj,L} + \rho_0 f_j &= \rho_0 \dot{v}_j, \\ \mathcal{D}_{K,K} &= \rho_E, \end{aligned} \quad (1.6\text{-}4)$$

where

$$\begin{aligned} K_{Lj} &= y_{j,K}\rho_0 \frac{\partial \psi}{\partial S_{KL}} + JX_{L,i}\varepsilon_0 (E_i E_j - \frac{1}{2} E_k E_k \delta_{ij}), \\ \mathcal{D}_K &= \varepsilon_0 J C^{-1}_{KL} \mathcal{E}_L - \rho_0 \frac{\partial \psi}{\partial \mathcal{E}_K}, \end{aligned} \quad (1.6\text{-}5)$$

and

$$\begin{aligned} S_{KL} &= (y_{i,K} y_{i,L} - \delta_{KL})/2, \\ \mathcal{E}_K &= -\phi_{,K}. \end{aligned} \quad (1.6\text{-}6)$$

With successive substitutions from Equations (1.6-5) and (1.6-6), Equation (1.6-4) can be written as four equations for the four unknowns $y_i(X_L, t)$ and $\phi(X_L, t)$.

Consider an interface between medium 1 and medium 2. The unit normal **n** of the interface points from 1 to 2. A mechanical traction $\bar{t}_j$ acts on the interface. The surface free charge density on the interface is $\bar{\sigma}_e$.

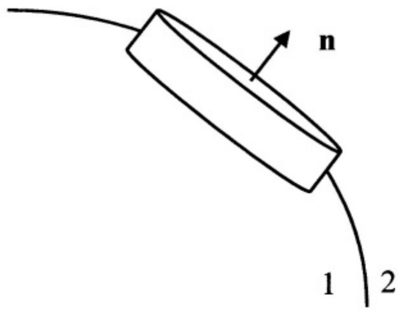

Figure 1.6-1. A material interface.

We want to derive interface jump or continuity conditions. Construct a pillbox on the interface as shown. Apply the balance of linear momentum (1.4-13) to the pillbox:

$$\tau_{ij}^{(2)} n_i - \tau_{ij}^{(1)} n_i + \bar{t}_j = 0 . \tag{1.6-7}$$

Apply Gauss's law of electrostatics to the pillbox:

$$D_i^{(2)} n_i - D_i^{(1)} n_i = \bar{\sigma}_e . \tag{1.6-8}$$

On the interface,

$$\tau_{ij} n_i da = \tau_{ij} da_i = \tau_{ij} J X_{L,i} dA_L = K_{Lj} dA_L = K_{Lj} N_L dA , \tag{1.6-9}$$

and

$$D_i n_i da = D_i da_i = D_i J X_{L,i} dA_L = \mathcal{D}_L dA_L = \mathcal{D}_L N_L dA . \tag{1.6-10}$$

Therefore Equations (1.6-7) and (1.6-8) can be written in the material form as

$$K_{Lj}^{(2)} N_L - K_{Lj}^{(1)} N_L + \bar{T}_j = 0 , \tag{1.6-11}$$

$$\mathcal{D}_L^{(2)} N_L - \mathcal{D}_L^{(1)} N_L = \bar{\sigma}_E , \tag{1.6-12}$$

where

$$\bar{T}_j = \bar{t}_j \frac{da}{dA}, \quad \bar{\sigma}_E = \bar{\sigma}_e \frac{da}{dA} . \tag{1.6-13}$$

For mechanical boundary conditions S is partitioned into S_y and S_T, on which motion (or displacement) and traction are prescribed, respectively. Electrically S is partitioned into S_ϕ and S_D with prescribed electric potential and surface free charge, respectively, and

$$S_y \cup S_T = S_\phi \cup S_D = S,$$
$$S_y \cap S_T = S_\phi \cap S_D = 0. \qquad (1.6\text{-}14)$$

Usual boundary value problems for an electroelastic body consist of (1.6-4)-(16.-6) and the following boundary conditions:

$$\begin{aligned} y_i &= \bar{y}_i \quad \text{on} \quad S_y, \\ \phi &= \bar{\phi} \quad \text{on} \quad S_\phi, \\ K_{Lk} N_L &= \bar{T}_k \quad \text{on} \quad S_T, \\ \mathcal{D}_K N_K &= -\bar{\sigma}_E \quad \text{on} \quad S_D, \end{aligned} \qquad (1.6\text{-}15)$$

where $\bar{y}_i$ and $\bar{\phi}$ are the prescribed boundary motion and potential, $\bar{T}_i$ is the surface traction per unit undeformed area, and $\bar{\sigma}_E$ is the surface free charge per unit undeformed area. For dynamic problems, initial conditions need to be added.

7. VARIATIONAL FORMULATION

Consider the following functional [8-10]:

$$\begin{aligned} \Pi(\mathbf{y},\phi) = \int_{t_0}^{t_1} dt \int_V \Big[& \frac{1}{2}\rho_0 \dot{y}_i \dot{y}_i - \rho_0 \psi(S_{KL}, \mathcal{E}_K) \\ & + \pi(S_{KL}, W_K) + \rho_0 f_i y_i - \rho_E \phi \Big] dV \\ & + \int_{t_0}^{t_1} dt \int_{S_T} \bar{T}_i y_i dS - \int_{t_0}^{t_1} dt \int_{S_D} \bar{\sigma}_E \phi dS, \end{aligned} \qquad (1.7\text{-}1)$$

where

$$\begin{aligned} \pi(S_{KL}, \mathcal{E}_K) &= \frac{1}{2}\varepsilon_0 J E_k E_k = \frac{1}{2}\varepsilon_0 J \phi_k \phi_{,k} \\ &= \frac{1}{2}\varepsilon_0 J X_{L,k} X_{M,k} \phi_{,L} \phi_{,M} = \frac{1}{2}\varepsilon_0 J C_{MN}^{-1} \mathcal{E}_M \mathcal{E}_N, \\ C_{KL} &= \delta_{KL} + 2 S_{KL}, \\ S_{KL} &= (y_{i,k} y_{i,L} - \delta_{KL})/2, \\ \mathcal{E}_L &= -\phi_{,L}. \end{aligned} \qquad (1.7\text{-}2)$$

The admissible y_i and ϕ for Π satisfy the following essential boundary conditions on S_y and S_ϕ:

$$\delta y_i \mid_{t=t_0} = 0, \quad \delta y_i \mid_{t=t_1} = 0, \quad \text{in} \quad V,$$
$$y_i = \bar{y}_i \quad \text{on} \quad S_y, \quad t_0 < t < t_1, \tag{1.7-3}$$
$$\phi = \bar{\phi} \quad \text{on} \quad S_\phi, \quad t_0 < t < t_1.$$

By straightforward differentiation, we have

$$\frac{\partial \pi}{\partial \mathcal{E}_K} = \varepsilon_0 J C_{KM}^{-1} \mathcal{E}_M = J X_{K,k} \varepsilon_0 E_k . \tag{1.7-4}$$

From Equation (1.1-39) and Problem 1.1-4, the other partial derivative of π is equal to

$$\begin{aligned}
\frac{\partial \pi}{\partial S_{KL}} &= 2\frac{\partial \pi}{\partial C_{KL}} = 2\left(\frac{1}{2}\right)\varepsilon_0\left(\frac{\partial J}{\partial C_{KL}} C_{MN}^{-1} + J\frac{\partial C_{MN}^{-1}}{\partial C_{KL}}\right)\mathcal{E}_M \mathcal{E}_N \\
&= \varepsilon_0\left[\frac{1}{2} J C_{KL}^{-1} C_{MN}^{-1} - J\frac{1}{2}\left(C_{MK}^{-1} C_{LN}^{-1} + C_{ML}^{-1} C_{KN}^{-1}\right)\right]\mathcal{E}_M \mathcal{E}_N \\
&= \varepsilon_0 \frac{1}{2} J (C_{KL}^{-1} C_{MN}^{-1} - C_{MK}^{-1} C_{LN}^{-1} - C_{ML}^{-1} C_{KN}^{-1}) \mathcal{E}_M \mathcal{E}_N \\
&= \varepsilon_0 \frac{1}{2} J (C_{KL}^{-1} C_{MN}^{-1} - 2 C_{MK}^{-1} C_{LN}^{-1}) \mathcal{E}_M \mathcal{E}_N \\
&= -J X_{K,k} X_{L,l} \varepsilon_0 (E_k E_l - E_m E_m \delta_{kl}/2) \\
&= -T_{KL}^M .
\end{aligned} \tag{1.7-5}$$

Then the first variation of Π is

$$\begin{aligned}
\delta \Pi = &\int_{t_0}^{t_1} dt \int_V \left[(K_{Li,L} + \rho_0 f_i - \rho_0 \ddot{y}_i)\delta y_i + (\mathcal{D}_{L,L} - \rho_E)\delta\phi\right] dV \\
&- \int_{t_0}^{t_1} dt \int_{S_T} (K_{Li} N_L - \bar{T}_i)\delta y_i dS \\
&- \int_{t_0}^{t_1} dt \int_{S_D} (\mathcal{D}_L N_L + \bar{\sigma}_E)\delta\phi dS.
\end{aligned} \tag{1.7-6}$$

Therefore the stationary condition of Π implies the following equations and natural boundary conditions:

$$\begin{aligned}
&K_{Lk,L} + \rho_0 f_k = \rho_0 \ddot{y}_k, \quad \text{in} \quad V, \\
&\mathcal{D}_{K,K} = \rho_E, \quad \text{in} \quad V, \\
&K_{Lk} N_L = \bar{T}_k, \quad \text{on} \quad S_T, \\
&\mathcal{D}_K N_K = -\bar{\sigma}_E, \quad \text{on} \quad S_D.
\end{aligned} \tag{1.7-7}$$

Problem

1.7-1. Show (1.7-6).

8. TOTAL STRESS FORMULATION

A more compact formulation will result if we introduce the following total energy density:

$$\rho_0\hat{\psi}(S_{KL},\mathcal{E}_K) = \rho_0\psi(S_{KL},\mathcal{E}_K) - \pi(S_{KL},\mathcal{E}_K). \tag{1.8-1}$$

Then the constitutive relations take the following form:

$$\begin{aligned}\rho_0\frac{\partial\hat{\psi}}{\partial S_{KL}} &= \rho_0\frac{\partial\psi}{\partial S_{KL}} - \frac{\partial\pi}{\partial S_{KL}} = T^S_{KL} + T^M_{KL} = \hat{T}_{KL},\\ \rho_0\frac{\partial\hat{\psi}}{\partial \mathcal{E}_K} &= \rho_0\frac{\partial\psi}{\partial \mathcal{E}_K} - \frac{\partial\pi}{\partial \mathcal{E}_K} = -\mathcal{P}_K - \varepsilon_0 J C^{-1}_{KL}\mathcal{E}_L = -\mathcal{D}_K,\end{aligned} \tag{1.8-2}$$

where we have introduced a total stress tensor $\hat{T}_{KL}$ in material form. In terms of the two-point total stress tensor K_{Lj}, the constitutive relations are

$$K_{Lj} = y_{j,K}\hat{T}_{KL} = y_{j,K}\rho_0\frac{\partial\hat{\psi}}{\partial S_{KL}}, \tag{1.8-3}$$

$$\mathcal{D}_K = \varepsilon_0 J C^{-1}_{KL}\mathcal{E}_L + \mathcal{P}_K = -\rho_0\frac{\partial\hat{\psi}}{\partial\mathcal{E}_K}. \tag{1.8-4}$$

The variational functional in (1.7-1) becomes

$$\begin{aligned}\Pi(\mathbf{y},\phi) = &\int_{t_0}^{t} dt\int_V \left[\frac{1}{2}\rho_0\dot{y}_i\dot{y}_i - \rho_0\hat{\psi}(S_{KL},\mathcal{E}_K)\right.\\ &\left. + \rho_0 f_i y_i - \rho_E\phi\right]dV\\ &+ \int_{t_0}^{t} dt\int_{S_T}\overline{T}_i y_i dS - \int_{t_0}^{t} dt\int_{S_D}\overline{\sigma}_E\phi dS.\end{aligned} \tag{1.8-5}$$

The following expressions will be useful in Chapter 6.

$$\frac{\partial^2\pi}{\partial\mathcal{E}_K\partial\mathcal{E}_L} = \varepsilon_0 J C^{-1}_{KL}, \tag{1.8-6}$$

$$\frac{\partial^2\pi}{\partial S_{KL}\partial\mathcal{E}_M} = \varepsilon_0\frac{1}{2}J(C^{-1}_{KL}C^{-1}_{MN} - C^{-1}_{MK}C^{-1}_{LN} - C^{-1}_{ML}C^{-1}_{KN})\mathcal{E}_N, \tag{1.8-7}$$

$$\frac{\partial^2 \pi}{\partial S_{KL} \partial S_{RS}} = \varepsilon_0 \frac{1}{2} \frac{\partial J}{\partial S_{RS}} (C_{KL}^{-1} C_{MN}^{-1} - 2C_{MK}^{-1} C_{LN}^{-1}) \mathcal{E}_M \mathcal{E}_N$$

$$+ \frac{\varepsilon_0 J}{2} \left(\frac{\partial C_{KL}^{-1}}{\partial S_{RS}} C_{MN}^{-1} + C_{KL}^{-1} \frac{\partial C_{MN}^{-1}}{\partial S_{RS}} - 2 \frac{\partial C_{MK}^{-1}}{\partial S_{RS}} C_{LN}^{-1} - 2C_{MK}^{-1} \frac{\partial C_{LN}^{-1}}{\partial S_{RS}} \right) \mathcal{E}_M \mathcal{E}_N$$

$$= \varepsilon_0 \frac{\partial J}{\partial C_{RS}} (C_{KL}^{-1} C_{MN}^{-1} - 2C_{MK}^{-1} C_{LN}^{-1}) \mathcal{E}_M \mathcal{E}_N$$

$$+ \varepsilon_0 J \left(\frac{\partial C_{KL}^{-1}}{\partial C_{RS}} C_{MN}^{-1} + C_{KL}^{-1} \frac{\partial C_{MN}^{-1}}{\partial C_{RS}} - 2 \frac{\partial C_{MK}^{-1}}{\partial C_{RS}} C_{LN}^{-1} - 2C_{MK}^{-1} \frac{\partial C_{LN}^{-1}}{\partial C_{RS}} \right) \mathcal{E}_M \mathcal{E}_N$$

$$= \varepsilon_0 \frac{1}{2} J C_{RS}^{-1} (C_{KL}^{-1} C_{MN}^{-1} - 2C_{MK}^{-1} C_{LN}^{-1}) \mathcal{E}_M \mathcal{E}_N$$

$$+ \varepsilon_0 J \Big[-\frac{1}{2} (C_{KR}^{-1} C_{SL}^{-1} + C_{KS}^{-1} C_{RL}^{-1}) C_{MN}^{-1} - \frac{1}{2} C_{KL}^{-1} (C_{MR}^{-1} C_{SN}^{-1} + C_{MS}^{-1} C_{RN}^{-1})$$

$$+ (C_{MR}^{-1} C_{SK}^{-1} + C_{MS}^{-1} C_{RK}^{-1}) C_{LN}^{-1} + C_{MK}^{-1} (C_{LR}^{-1} C_{SN}^{-1} + C_{LS}^{-1} C_{RN}^{-1}) \Big] \mathcal{E}_M \mathcal{E}_N$$

$$= \varepsilon_0 J \Big[\frac{1}{2} C_{RS}^{-1} C_{KL}^{-1} C_{MN}^{-1} - C_{RS}^{-1} C_{MK}^{-1} C_{LN}^{-1}$$

$$- \frac{1}{2} C_{KR}^{-1} C_{SL}^{-1} C_{MN}^{-1} - \frac{1}{2} C_{KS}^{-1} C_{RL}^{-1} C_{MN}^{-1} - \frac{1}{2} C_{KL}^{-1} C_{MR}^{-1} C_{SN}^{-1} - \frac{1}{2} C_{KL}^{-1} C_{MS}^{-1} C_{RN}^{-1}$$

$$+ C_{MR}^{-1} C_{SK}^{-1} C_{LN}^{-1} + C_{MS}^{-1} C_{RK}^{-1} C_{LN}^{-1} + C_{MK}^{-1} C_{LR}^{-1} C_{SN}^{-1} + C_{MK}^{-1} C_{LS}^{-1} C_{RN}^{-1}) \Big] \mathcal{E}_M \mathcal{E}_N . \tag{1.8-8}$$

Chapter 2

LINEAR PIEZOELECTRICITY FOR INFINITESIMAL FIELDS

In this chapter we specialize the nonlinear equations in Chapter 1 to the case of infinitesimal deformations and fields, which results in the linear theory of piezoelectricity. A few theoretical aspects of the linear theory are also discussed.

1. LINEARIZATION

In this section we reduce the nonlinear electroelastic equations in the previous chapter to the linear theory of piezoelectricity for infinitesimal deformation and fields. We consider small amplitude motions of an electroelastic body around its reference state due to small mechanical and electrical loads. It is assumed that the displacement gradient is infinitesimal in the following sense that

$$\| u_{i,K} \| << 1, \tag{2.1-1}$$

under some norm, e.g., $\| u_{i,K} \| = \max | u_{i,K} |$. It is also assumed that the electric potential gradient $\phi_{,K}$ is infinitesimal. We neglect powers of $u_{i,M}$ and $\phi_{,K}$ higher than the first as well as their products in all expressions. The linear terms themselves are also dropped in comparison with any finite quantity such the Kronecker delta or 1. Under (2.1-1),

$$\begin{aligned} \frac{\partial u_i}{\partial X_K} &= \frac{\partial u_i}{\partial y_k} y_{k,K} = \frac{\partial u_i}{\partial y_k}(\delta_{kK} + u_{k,K}) \cong \frac{\partial u_i}{\partial y_k}\delta_{kK}, \\ \phi_{,K} &= \phi_{,i} y_{i,K} \cong \phi_{,i}\delta_{iK}, \end{aligned} \tag{2.1-2}$$

which implies that, to the first order of approximation, the displacement and potential gradients calculated from the material and spatial coordinates are numerically equal. Therefore, within the linear theory, there is no need to distinguish capital and lowercase indices. Only lowercase indices will be used in the linear theory. The material time derivative of an infinitesimal field variable $f(\mathbf{y},t)$ is simply the partial derivative with respect to t:

$$\frac{Df}{Dt} = \left.\frac{\partial f}{\partial t}\right|_{\text{X fixed}} = \left.\frac{\partial f}{\partial t}\right|_{\text{y fixed}} + \left.\frac{\partial f}{\partial y_i}\right|_{t\text{ fixed}} \left.\frac{\partial y_i}{\partial t}\right|_{\text{X fixed}}$$
$$= \left.\frac{\partial f}{\partial t}\right|_{\text{y fixed}} + v_i \frac{\partial f}{\partial y_i} \cong \left.\frac{\partial f}{\partial t}\right|_{\text{y fixed}}. \tag{2.1-3}$$

For the finite strain tensor

$$S_{KL} = \frac{1}{2}(u_{L,K} + u_{K,L} + u_{M,K}u_{M,L}) \cong \frac{1}{2}(u_{L,K} + u_{K,L}). \tag{2.1-4}$$

In the linear theory, the infinitesimal strain tensor will be denoted by

$$S_{kl} = \frac{1}{2}(u_{l,k} + u_{k,l}). \tag{2.1-5}$$

The material electric field becomes

$$\mathcal{E}_K = E_i y_{i,K} \cong E_i \delta_{iK} \to E_k. \tag{2.1-6}$$

Similarly,

$$\sigma_{ij}^E \cong 0, \quad \sigma_{ij}^M \cong 0, \quad \sigma_{ij} \cong \sigma_{ij}^S \cong \tau_{ij},$$
$$M_{Lj} \cong 0, \quad K_{Lj} \cong F_{Lj} \cong \delta_{Ki}\sigma_{ij}, \quad T_{KL}^S \cong \delta_{Ki}\delta_{Lj}\sigma_{ij}, \tag{2.1-7}$$
$$\mathcal{P}_K \to P_k, \quad \mathcal{D}_K \to D_k.$$

Since the various stress tensors are either approximately zero (quadratic in the infinitesimal gradients) or about the same, we will use T_{ij} to denote the stress tensor that is linear in the infinitesimal gradients. This is according to the IEEE Standard on Piezoelectricity [11]. Our notation for the rest of the linear theory will also follow the IEEE Standard. Then

$$\sigma_{ij} \cong \sigma_{ij}^S \cong \tau_{ij} \to T_{ij},$$
$$K_{Lj} \cong F_{Lj} \cong \delta_{Li}\sigma_{ij} \to T_{lj}, \tag{2.1-8}$$
$$T_{KL}^S \cong \delta_{Ki}\delta_{Lj}\sigma_{ij} \to T_{kl}.$$

For small fields the total free energy can be approximated by

$$\rho_0\hat{\psi}(S_{KL}, \mathcal{E}_K) = \rho_0\psi(S_{KL}, \mathcal{E}_K) - \frac{1}{2}\varepsilon_0 J E_k E_k$$
$$\cong \frac{1}{2}c_{2ABCD} S_{AB}S_{CD} - e_{ABC}\mathcal{E}_A S_{BC} - \frac{1}{2}\chi_{2AB}\mathcal{E}_A\mathcal{E}_B - \frac{1}{2}\varepsilon_0 J E_k E_k \tag{2.1-9}$$
$$\to \frac{1}{2}c_{ijkl}^E S_{ij}S_{kl} - e_{ijk}E_i S_{jk} - \frac{1}{2}\varepsilon_{ij}^S E_i E_j = H(S_{kl}, E_k),$$

where

$$\varepsilon^S_{ij} = \chi_{2\,ij} + \varepsilon_0 \delta_{ij} . \tag{2.1-10}$$

The superscript E in c^E_{ijkl} indicates that the independent electric constitutive variable is the electric field $\mathbf{E}$. The superscript S in ε^S_{ij} indicates that the mechanical constitutive variable is the strain tensor $\mathbf{S}$. We have also denoted the total free energy of the linear theory by H which is usually called the electric enthalpy. The constitutive relations generated by H are

$$\begin{aligned} T_{ij} &= \frac{\partial H}{\partial S_{ij}} = c^E_{ijkl} S_{kl} - e_{kij} E_k , \\ D_i &= -\frac{\partial H}{\partial E_i} = e_{ikl} S_{kl} + \varepsilon^S_{ik} E_k . \end{aligned} \tag{2.1-11}$$

Hence $\mathbf{T}$, $\mathbf{D}$ and $\mathbf{P}$ are also infinitesimal. The material constants in Equation (2.1-11) have the following symmetries:

$$\begin{aligned} c^E_{ijkl} &= c^E_{jikl} = c^E_{klij} , \\ e_{kij} &= e_{kji} , \\ \varepsilon^S_{ij} &= \varepsilon^S_{ji} . \end{aligned} \tag{2.1-12}$$

We also assume that the elastic and dielectric material tensors are positive-definite in the following sense:

$$\begin{aligned} & c^E_{ijkl} S_{ij} S_{kl} \ge 0, \quad \text{for any} \quad S_{ij} = S_{ji} , \\ & \quad \text{and} \quad c^E_{ijkl} S_{ij} S_{kl} = 0 \quad \Rightarrow \quad S_{ij} = 0, \\ & \varepsilon^S_{ij} E_i E_j \ge 0, \quad \text{for any} \quad E_i , \\ & \quad \text{and} \quad \varepsilon^S_{ij} E_i E_j = 0 \quad \Rightarrow \quad E_i = 0. \end{aligned} \tag{2.1-13}$$

The total internal energy density per unit volume can be obtained from H by a Legendre transform, given as

$$U(\mathbf{S},\mathbf{D}) = H(\mathbf{S},\mathbf{E}(\mathbf{S},\mathbf{D})) + \mathbf{E}(\mathbf{S},\mathbf{D}) \cdot \mathbf{D} . \tag{2.1-14}$$

Constitutive relations in the following form then follow:

$$\mathbf{T} = \frac{\partial U}{\partial \mathbf{S}} , \quad \mathbf{E} = \frac{\partial U}{\partial \mathbf{D}} , \tag{2.1-15}$$

or

$$\begin{aligned} T_{ij} &= c^D_{ijkl} S_{kl} - h_{kij} D_k , \\ E_i &= -h_{ikl} S_{kl} + \beta^S_{ik} D_k . \end{aligned} \tag{2.1-16}$$

It can be shown that U is positive-definite:

$$U = H + E_i D_i$$
$$= \frac{1}{2} c_{ijkl}^E S_{ij} S_{kl} - e_{ijk} E_i S_{jk} - \frac{1}{2} \varepsilon_{ij}^S E_i E_j + E_i (e_{ikl} S_{kl} + \varepsilon_{ik}^S E_k) \quad (2.1\text{-}17)$$
$$= \frac{1}{2} c_{ijkl}^E S_{ij} S_{kl} + \frac{1}{2} \varepsilon_{ij}^S E_i E_j \geq 0.$$

For small fields, the total internal energy density U per unit volume and the internal energy density e per unit mass in the previous chapter are related by

$$\begin{aligned} U &= H + E_i D_i = \rho_0 \hat{\psi} + E_i D_i \\ &= \rho_0 \psi - \pi + E_i D_i = \rho_0 \psi - \frac{1}{2} \varepsilon_0 E_i E_i + E_i D_i \\ &= \rho_0 e - E_i P_i - \frac{1}{2} \varepsilon_0 E_i E_i + E_i (\varepsilon_0 E_i + P_i) \\ &= \rho_0 e + \frac{1}{2} \varepsilon_0 E_i E_i = \rho_0 e + \pi. \end{aligned} \quad (2.1\text{-}18)$$

Similar to (2.1-11) and (2.1-16), linear constitutive relations can also be written as [11]

$$\begin{aligned} S_{ij} &= s_{ijkl}^E T_{kl} + d_{kij} E_k, \\ D_i &= d_{ikl} T_{kl} + \varepsilon_{ik}^T E_k, \end{aligned} \quad (2.1\text{-}19)$$

and

$$\begin{aligned} S_{ij} &= s_{ijkl}^D T_{kl} + g_{kij} D_k, \\ E_i &= -g_{ikl} T_{kl} + \beta_{ik}^T D_k. \end{aligned} \quad (2.1\text{-}20)$$

The equations of motion and the charge equation become

$$\begin{aligned} T_{ji,j} + \rho f_i &= \rho \ddot{u}_i, \\ D_{i,i} &= \rho_e, \end{aligned} \quad (2.1\text{-}21)$$

in which the difference between the reference and present mass and charge densities can be ignored. The body force $\mathbf{f}$ and charge ρ_e are infinitesimal.

Within linear theory, the conservation of mass and the relation between the reference and present charge densities take the following form:

$$\begin{aligned} \rho_0 &\cong \rho(1 + u_{k,k}), \\ \rho_E &\cong \rho_e. \end{aligned} \quad (2.1\text{-}22)$$

The surface loads are also infinitesimal. Hence

$$\bar{T}_j \cong \bar{t}_j, \quad \bar{\sigma}_E \cong \bar{\sigma}_e, \tag{2.1-23}$$

and

$$K_{Lj}N_L \cong T_{ij}n_i, \quad \Delta_L N_L \cong D_i n_i. \tag{2.1-24}$$

Problem

2.1-1. Show (2.1-15).

2. BOUNDARY-VALUE PROBLEM

2.1 Displacement-Potential Formulation

In summary, the linear theory of piezoelectricity consists of the equations of motion and charge

$$T_{ji,j} + \rho f_i = \rho \ddot{u}_i, \quad D_{i,i} = \rho_e \ , \tag{2.2-1}$$

constitutive relations

$$T_{ij} = c_{ijkl}S_{kl} - e_{kij}E_k, \quad D_i = e_{ijk}S_{jk} + \varepsilon_{ij}E_j \ , \tag{2.2-2}$$

and the strain-displacement and electric field-potential relations

$$S_{ij} = (u_{i,j} + u_{j,i})/2, \quad E_i = -\phi_{,i} \ , \tag{2.2-3}$$

where **u** is the mechanical displacement vector, **T** is the stress tensor, **S** is the strain tensor, **E** is the electric field, **D** is the electric displacement, ϕ is the electric potential, ρ is the known reference mass density (or ρ_0 in the previous chapter), ρ_e is the body free charge density, and **f** is the body force per unit mass. The coefficients c_{ijkl}, e_{kij} and ε_{ij} are the elastic, piezoelectric and dielectric constants. We have neglected the superscripts in the material constants. With successive substitutions from Equations (2.2-2) and (2.2-3), Equation (2.2-1) can be written as four equations for **u** and ϕ

$$\begin{aligned} c_{ijkl}u_{k,lj} + e_{kij}\phi_{,kj} + \rho f_i &= \rho \ddot{u}_i, \\ e_{ikl}u_{k,li} - \varepsilon_{ij}\phi_{,ij} &= \rho_e. \end{aligned} \tag{2.2-4}$$

2.2 Boundary-Value Problem

Let the region occupied by the piezoelectric body be V and its boundary surface be S as shown in Figure 2.2-1. For linear piezoelectricity we use **x** as the independent spatial coordinates. Let the unit outward normal of S be **n**.

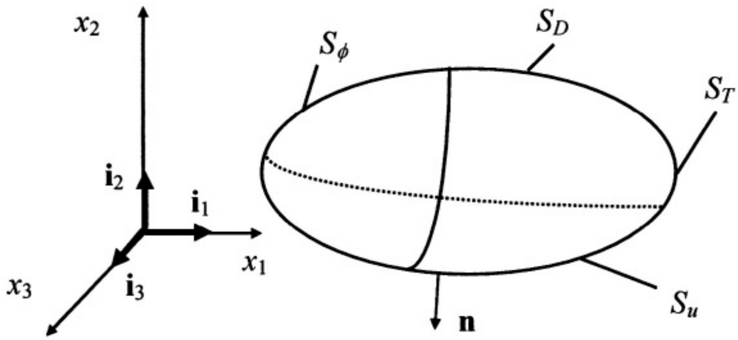

Figure 2.2-1. A piezoelectric body and partitions of its surface.

For boundary conditions we consider the following partitions of S:

$$S_u \cup S_T = S_\phi \cup S_D = S,$$
$$S_u \cap S_T = S_\phi \cap S_D = 0, \tag{2.2-5}$$

where S_u is the part of S on which the mechanical displacement is prescribed, and S_T is the part of S where the traction vector is prescribed. S_ϕ represents the part of S which is electroded where the electric potential is no more than a function of time, and S_D is the unelectroded part. For mechanical boundary conditions we have prescribed displacement $\bar{u}_i$

$$u_i = \bar{u}_i, \quad \text{on} \quad S_u, \tag{2.2-6}$$

and prescribed traction $\bar{t}_j$

$$T_{ij} n_i = \bar{t}_j, \quad \text{on} \quad S_T. \tag{2.2-7}$$

Electrically, on the electroded portion of S,

$$\phi = \bar{\phi}, \quad \text{on} \quad S_\phi, \tag{2.2-8}$$

where $\bar{\phi}$ does not vary spatially. On the unelectroded part of S, the charge condition can be written as

$$D_j n_j = -\bar{\sigma}_e, \quad \text{on} \quad S_D, \tag{2.2-9}$$

where $\overline{\sigma}_e$ is free charge density per unit surface area. In the above formulation, the mechanical effect of the electrode is neglected because we assume very thin electrodes.

On an electrode S_ϕ, the total free electric charge Q_e can be represented by

$$Q_e = \int_{S_\phi} -n_i D_i dS . \tag{2.2-10}$$

The electric current flowing out of the electrode is given by

$$i = -\dot{Q}_e . \tag{2.2-11}$$

Sometimes there are two (or more) electrodes on a body which are connected to an electric circuit. In this case, circuit equation(s) will need to be considered.

2.3 Principle of Superposition

The linearity of Equation (2.2-4) allows the superposition of solutions. Suppose the solutions under two different sets of loads of $\{\mathbf{f}^{(1)}, \rho_e^{(1)}\}$ and $\{\mathbf{f}^{(2)}, \rho_e^{(2)}\}$ are $\{\mathbf{u}^{(1)}, \phi^{(1)}\}$ and $\{\mathbf{u}^{(2)}, \phi^{(2)}\}$, respectively. Then under the combined load of $\{\mathbf{f}^{(1)} + \mathbf{f}^{(2)}, \rho_e^{(1)} + \rho_e^{(2)}\}$, the solution to (2.2-4) is $\{\mathbf{u}^{(1)} + \mathbf{u}^{(2)}, \phi^{(1)} + \phi^{(2)}\}$. This is called the principle of superposition and can be shown as

$$\begin{aligned}
& c_{ijkl}(u_k^{(1)} + u_k^{(2)})_{,lj} + e_{kij}(\phi^{(1)} + \phi^{(2)})_{,kj} \\
& \quad + \rho(f_i^{(1)} + f_i^{(2)}) - \rho \frac{\partial^2}{\partial t^2}(u_i^{(1)} + u_i^{(2)}) \\
& = c_{ijkl}u_{k,lj}^{(1)} + c_{ijkl}u_{k,lj}^{(2)} + e_{kij}\phi_{,kj}^{(1)} + \phi_{,kj}^{(2)} \\
& \quad + \rho f_i^{(1)} + \rho f_i^{(2)} - \rho \ddot{u}_i^{(1)} - \rho \ddot{u}_i^{(2)} \\
& = (c_{ijkl}u_{k,lj}^{(1)} + e_{kij}\phi_{,kj}^{(1)} + \rho f_i^{(1)} - \rho \ddot{u}_i^{(1)}) \\
& \quad + (c_{ijkl}u_{k,lj}^{(2)} + \phi_{,kj}^{(2)} + \rho f_i^{(2)} - \rho \ddot{u}_i^{(2)}) \\
& = 0 + 0 = 0,
\end{aligned} \tag{2.2-12}$$

and

$$
\begin{aligned}
&e_{ikl}(u_k^{(1)}+u_k^{(2)})_{,li}-\varepsilon_{ij}(\phi^{(1)}+\phi^{(2)})_{,ij}-(\rho_e^{(1)}+\rho_e^{(2)})\\
&=e_{ikl}u_{k,li}^{(1)}+e_{ikl}u_{k,li}^{(2)}-\varepsilon_{ij}\phi_{,ij}^{(1)}-\varepsilon_{ij}\phi_{,ij}^{(2)}-\rho_e^{(1)}-\rho_e^{(2)}\\
&=(e_{ikl}u_{k,li}^{(1)}-\varepsilon_{ij}\phi_{,ij}^{(1)}-\rho_e^{(1)})+(e_{ikl}u_{k,li}^{(2)}-\varepsilon_{ij}\phi_{,ij}^{(2)}-\rho_e^{(2)})\\
&=0+0=0.
\end{aligned}
\tag{2.2-13}
$$

The principle of superposition can be generalized to include boundary loads.

2.4 Compatibility

Since the six strain components are derived from three displacement components, it is natural to expect some relations among the strain components whether they are linear or nonlinear. The following can be verified by direct substitution:

$$
\begin{aligned}
S_{11,23}&=(S_{31,2}+S_{12,3}-S_{23,1})_{,1},\\
S_{22,31}&=(S_{12,3}+S_{23,1}-S_{31,2})_{,2},\\
S_{33,12}&=(S_{23,1}+S_{31,2}-S_{12,3})_{,3},\\
2S_{23,23}&=S_{22,33}+S_{33,22},\\
2S_{31,31}&=S_{33,11}+S_{11,33},\\
2S_{12,12}&=S_{11,22}+S_{22,11}.
\end{aligned}
\tag{2.2-14}
$$

Equations (2.2-14) are called compatibility conditions. The compatibility conditions are necessary conditions for the six strain components derived from three displacement components. They are also sufficient in the sense that for six strain components satisfying these compatibility conditions, there exist three displacement components from which the six strain components are derivable. The sufficiency of (2.2-14) is true over a simply-connected domain only. For a multiply-connected domain, some additional conditions are needed. The compatibility conditions are useful when solving a problem using stress components rather than displacement components as the primary unknowns. In more compact form, Equation (2.2-14) can be written as

$$
S_{ij,kl}+S_{kl,ij}-S_{ik,jl}-S_{jl,ik}=0, \tag{2.2-15}
$$

of which the six independent relations are (2.1-25), or

$$
\varepsilon_{ijk}\varepsilon_{lmn}S_{il,jm}=0. \tag{2.2-16}
$$

3. VARIATIONAL PRINCIPLES

3.1 Hamilton's Principle

The equations and boundary conditions of linear piezoelectricity can be derived from a variational principle. Consider [4]

$$\Pi(\mathbf{u},\phi) = \int_{t_0}^{t_1} dt \int_V \left[\frac{1}{2}\rho\dot{u}_i\dot{u}_i - H(\mathbf{S},\mathbf{E}) + \rho f_i u_i - \rho_e\phi\right]dV + \int_{t_0}^{t_1} dt \int_{S_T} \bar{t}_i u_i dS - \int_{t_0}^{t_1} dt \int_{S_D} \bar{\sigma}_e \phi dS, \tag{2.3-1}$$

where

$$S_{ij} = (u_{i,j} + u_{j,i})/2, \quad E_i = -\phi_{,i}. \tag{2.3-2}$$

$\mathbf{u}$ and ϕ are variationally admissible if they are smooth enough and satisfy

$$\begin{aligned} &\delta u_i\,|_{t_0} = \delta u_i\,|_{t_1} = 0, \quad \text{in} \quad V, \\ &u_i = \bar{u}_i, \quad \text{on} \quad S_u, \quad t_0 < t < t_1, \\ &\phi = \bar{\phi}, \quad \text{on} \quad S_\phi, \quad t_0 < t < t_1. \end{aligned} \tag{2.3-3}$$

The first variation of Π is

$$\delta\Pi = \int_{t_0}^{t_1} dt \int_V \left[(T_{ji,j} + \rho f_i - \rho\ddot{u}_i)\delta u_i + (D_{i,i} - \rho_e)\delta\phi\right]dV - \int_{t_0}^{t_1} dt \int_{S_T} (T_{ji}n_j - \bar{t}_i)\delta u_i dS - \int_{t_0}^{t_1} dt \int_{S_D} (D_i n_i + \bar{\sigma}_e)\delta\phi dS, \tag{2.3-4}$$

where we have denoted

$$\mathbf{T} = \frac{\partial H}{\partial \mathbf{S}}, \quad \mathbf{D} = -\frac{\partial H}{\partial \mathbf{E}}. \tag{2.3-5}$$

Therefore the stationary condition of Π is

$$\begin{aligned} &T_{ji,j} + \rho f_i = \rho\ddot{u}_i, \quad \text{in} \quad V, \quad t_0 < t < t_1, \\ &D_{i,i} = \rho_e, \quad \text{in} \quad V, \quad t_0 < t < t_1, \\ &T_{ji}n_j = \bar{t}_i, \quad \text{on} \quad S_T, \quad t_0 < t < t_1, \\ &D_i n_i = -\bar{\sigma}_e, \quad \text{on} \quad S_D, \quad t_0 < t < t_1. \end{aligned} \tag{2.3-6}$$

Hamilton's principle can be stated as: Among all the admissible $\{\mathbf{u}, \phi\}$, the one that also satisfies (2.3-6) makes Π stationary.

3.2 Mixed Variational Principles

If the functional in Equation (2.3-1) is viewed to be dependent on $\mathbf{u}$, ϕ, $\mathbf{S}$ and $\mathbf{E}$, then Equation (2.3-2) should be considered as constraints among the independent variables. These constraints, along with the boundary data in Equations $(2.3\text{-}3)_{2,3}$, can be removed by the method of Lagrange multipliers. Then the following variational functional will result [12]:

$$\begin{aligned}
&\Pi(\mathbf{u},\phi,\mathbf{S},\mathbf{T},\mathbf{E},\mathbf{D}) \\
&= \int_{t_0}^{t_1} dt \int_V \left\{ \frac{1}{2}\rho \dot{u}_i \dot{u}_i - H(\mathbf{S},\mathbf{E}) + T_{ij}\left[S_{ij} - \frac{1}{2}(u_{i,j} + u_{j,i})\right] \right. \\
&\quad \left. - D_i(E_i + \phi_{,i}) + \rho f_i u_i - \rho_e \phi \right\} dV \\
&\quad + \int_{t_0}^{t_1} dt \int_{S_u} T_{ji} n_j (u_i - \bar{u}_i) dS \\
&\quad + \int_{t_0}^{t_1} dt \int_{S_\phi} D_i n_i (\phi - \bar{\phi}) dS \\
&\quad + \int_{t_0}^{t_1} dt \int_{S_T} \bar{t}_i u_i dS - \int_{t_0}^{t_1} dt \int_{S_D} \bar{\sigma}_e \phi dS.
\end{aligned} \tag{2.3-7}$$

$\mathbf{u}$, ϕ, $\mathbf{S}$, $\mathbf{T}$, $\mathbf{E}$ and $\mathbf{D}$ are admissible if they are smooth enough and satisfy

$$\delta u_i \,|_{t_0} = \delta u_i \,|_{t_1} = 0, \quad \text{in} \quad V. \tag{2.3-8}$$

The first variation of Π is

$$\begin{aligned}
\delta\Pi = &\int_{t_0}^{t_1} dt \int_V \left\{ (T_{ji,j} + \rho f_i - \rho \ddot{u}_i)\delta u_i + (D_{i,i} - \rho_e)\delta\phi \right. \\
&+ \left[S_{ij} - \frac{1}{2}(u_{i,j} + u_{j,i})\right]\delta T_{ij} - (E_i + \phi_{,i})\delta D_i \\
&\left. + \left(T_{ij} - \frac{\partial H}{\partial S_{ij}}\right)\delta S_{ij} - \left(D_i + \frac{\partial H}{\partial E_i}\right)\delta E_i \right\} dV \\
&+ \int_{t_0}^{t_1} dt \int_{S_u} (u_i - \bar{u}_i)\delta T_{ji} n_j dS + \int_{t_0}^{t_1} dt \int_{S_\phi} (\phi - \bar{\phi})\delta D_i n_i dS \\
&- \int_{t_0}^{t_1} dt \int_{S_T} (T_{ji} n_j - \bar{t}_i)\delta u_i dS - \int_{t_0}^{t_1} dt \int_{S_D} (D_i n_i + \bar{\sigma}_e)\delta\phi dS.
\end{aligned} \tag{2.3-9}$$

Therefore the stationary condition of Π is

$$
\begin{aligned}
&T_{ji,j} + \rho f_i = \rho \ddot{u}_i, \quad D_{i,i} = \rho_e, \quad \text{in} \quad V, \quad t_0 < t < t_1, \\
&S_{ij} = (u_{i,j} + u_{j,i})/2, \quad E_i = -\phi_{,i}, \quad \text{in} \quad V, \quad t_0 < t < t_1, \\
&T_{ij} = \frac{\partial H}{\partial S_{ij}}, \quad D_i = -\frac{\partial H}{\partial E_i}, \quad \text{in} \quad V, \quad t_0 < t < t_1, \\
&u_i = \bar{u}_i, \quad \text{on} \quad S_u, \quad t_0 < t < t_1, \\
&T_{ji} n_j = \bar{t}_i, \quad \text{on} \quad S_T, \quad t_0 < t < t_1, \\
&\phi = \bar{\phi}_e, \quad \text{on} \quad S_\phi, \quad t_0 < t < t_1. \\
&D_i n_i = -\bar{\sigma}_e, \quad \text{on} \quad S_D, \quad t_0 < t < t_1.
\end{aligned}
\tag{2.3-10}
$$

Hence, among all the admissible {**u**, ϕ , **S, T, E, D**}, the one that also satisfies Equation (2.3-10) makes Π(**u,** ϕ **, S, T, E, D)** stationary. The functional in Equation (2.3-7) has all of the fields as independent variables. Its stationary condition yields all the equations and boundary conditions. Variational principles like this are called mixed or generalized variational principles.

3.3 Conservation Laws

From Noether's theorem on variational principles invariant under infinitesimal transformations, the following relations can be shown [13]:

$$
\begin{aligned}
&\frac{\partial}{\partial t}(\rho \dot{u}_j u_{j,i}) + \frac{\partial}{\partial x_k}(\Sigma \delta_{ik} - u_{j,i} T_{jk} - \phi_{,i} D_k) = 0, \\
&\frac{\partial}{\partial t} \varepsilon_{ijk} \rho (x_j \dot{u}_m u_{m,k} + \dot{u}_j u_k) \\
&\quad + \frac{\partial}{\partial x_k} \varepsilon_{imj} (x_m \Sigma \delta_{jk} - x_m u_{l,j} T_{lk} - x_m \phi_{,j} D_k + T_{jk} u_m) = 0, \\
&\frac{\partial}{\partial t}[\rho \dot{u}_j (u_j + x_m u_{j,m} + t\dot{u}_j) + t\Sigma] \\
&\quad + \frac{\partial}{\partial x_k}[x_k \Sigma - T_{jk}(u_j + x_m u_{j,m} + t\dot{u}_j) - D_k(\phi + x_m \phi_{,m} + t\dot{\phi})] = 0,
\end{aligned}
\tag{2.3-11}
$$

where

$$\Sigma = H(\mathbf{S},\mathbf{E}) - T_{ij}\left[S_{ij} - \frac{1}{2}(u_{i,j} + u_{j,i})\right] + D_i(E_i + \phi_{,i}) - \frac{1}{2}\rho\dot{u}_i\dot{u}_i. \tag{2.3-12}$$

Equation (2.3-11)$_{1,2,3}$ are obtained by the invariance of the functional in Equation (2.3-7) under translations, rotations, and scale changes, respectively. They can be verified by direct differentiation. The relations in Equation (2.3-11) are in divergence-free form and are called conservation laws. They can be transformed to path-independent integrals by the divergence theorem.

Problems

2.3-1. Show (2.3-4).
2.3-2. Show (2.3-9)
2.3-3. Study the conservation laws for linear, static piezoelectricity [13].

4. UNIQUENESS

4.1 Poynting's Theorem

We begin with the rate of change of the total internal energy density, given as

$$\begin{aligned}
\dot{U} &= \frac{\partial U}{\partial S_{ij}}\dot{S}_{ij} + \frac{\partial U}{\partial D_i}\dot{D}_i = T_{ij}\dot{S}_{ij} + E_i\dot{D}_i \\
&= T_{ij}\dot{S}_{ij} + E_i\dot{D}_i = T_{ij}\dot{u}_{i,j} - \phi_{,i}\dot{D}_i \\
&= (T_{ij}\dot{u}_i)_{,j} - T_{ij,j}\dot{u}_i - (\phi\,\dot{D}_i)_{,i} + \phi\dot{D}_{i,i} \\
&= (T_{ij}\dot{u}_i)_{,j} - (\rho\ddot{u}_i - \rho f_i)\dot{u}_i - (\phi\,\dot{D}_i)_{,i} + \phi\dot{\rho}_e \\
&= (T_{ji}\dot{u}_j)_{,i} - \frac{\partial}{\partial t}\left(\frac{1}{2}\rho\dot{u}_i\dot{u}_i\right) + \rho f_i\dot{u}_i - (\phi\,\dot{D}_i)_{,i} + \phi\dot{\rho}_e,
\end{aligned} \tag{2.4-1}$$

where (2.2-1) has been used. Therefore,

$$\frac{\partial}{\partial t}(T + U) = \rho f_i\dot{u}_i + \phi\dot{\rho}_e - (\phi\,\dot{D}_i - T_{ji}\dot{u}_j)_{,i}, \tag{2.4-2}$$

where

$$T = \frac{1}{2}\rho \dot{u}_i \dot{u}_i \tag{2.4-3}$$

is the kinetic energy density, and $\phi \dot{D}_i$ is the quasistatic Poynting vector. Equation (2.4-2) may be considered as a generalized version of Poynting's theorem in electromagnetics.

4.2 Energy Integral

Integration of (2.4-2) over V gives

$$\begin{aligned} \frac{\partial}{\partial t}\int_V (T+U)dV &= \int_V \rho(f_i\dot{u}_i + \phi\dot{\rho}_e)dV \\ &+ \int_{S_u} T_{ji}n_j\dot{\bar{u}}_i dS + \int_{S_T} \bar{t}_i\dot{u}_i dS \\ &- \int_{S_\phi} \dot{D}_i n_i \bar{\phi} dS + \int_{S_D} \dot{\bar{\sigma}}_e \phi dS, \end{aligned} \tag{2.4-4}$$

where Equations (2.2-6) through (2.2-9) have been used. Integrating Equation (2.4-4) from t_0 to t, we obtain

$$\begin{aligned} \int_V (T+U)\big|_t dV &= \int_V (T+U)\big|_{t_0} dV \\ &+ \int_{t_0}^t dt \int_V \rho(f_i\dot{u}_i + \phi\dot{\rho}_e)dV \\ &+ \int_{t_0}^t dt \int_{S_u} T_{ji}n_j\dot{\bar{u}}_i dS + \int_{t_0}^t dt \int_{S_T} \bar{t}_i\dot{u}_i dS \\ &- \int_{t_0}^t dt \int_{S_\phi} \dot{D}_i n_i \bar{\phi} dS + \int_{t_0}^t dt \int_{S_D} \dot{\bar{\sigma}}_e \phi dS. \end{aligned} \tag{2.4-5}$$

Equation (2.4-5) is called the energy integral which states that the energy at time t is the energy at time t_0 plus the work done to the body from t_0 to t.

4.3 Uniqueness

Consider two solutions to the following initial-boundary value problem:

$$
\begin{aligned}
&T_{ji,j} + \rho f_i = \rho \ddot{u}_i, \quad \text{in} \quad V, \quad t > t_0, \\
&D_{i,i} = \rho_e, \quad \text{in} \quad V, \quad t > t_0, \\
&T_{ij} = c_{ijkl} S_{kl} - e_{kij} E_k, \quad \text{in} \quad V, \quad t > t_0, \\
&D_i = e_{ijk} S_{jk} + \varepsilon_{ij} E_j, \quad \text{in} \quad V, \quad t > t_0, \\
&S_{ij} = (u_{i,j} + u_{j,i})/2, \quad \text{in} \quad V, \quad t > t_0, \\
&E_i = -\phi_{,i}, \quad \text{in} \quad V, \quad t > t_0, \\
&u_i = \overline{u}_i, \quad \text{on} \quad S_u, \quad t > t_0, \\
&T_{ji} n_j = \overline{t}_i, \quad \text{on} \quad S_T, \quad t > t_0, \\
&\phi = \overline{\phi}, \quad \text{on} \quad S_\phi, \quad t > t_0, \\
&D_i n_i = -\overline{\sigma}_e, \quad \text{on} \quad S_D, \quad t > t_0, \\
&u_i = u_i^0, \quad \text{in} \quad V, \quad t = t_0, \\
&\dot{u}_i = v_i^0, \quad \text{in} \quad V, \quad t = t_0.
\end{aligned} \tag{2.4-6}
$$

From the principle of superposition, the difference of the two solutions satisfies the homogeneous version of (2.4-6). Let $\mathbf{u}^*$, ϕ^*, $\mathbf{S}^*$, $\mathbf{T}^*$, $\mathbf{E}^*$, and $\mathbf{D}^*$ denote the difference of the corresponding fields and apply (2.4-5) to it. The initial energy and the external work for the difference are zero. Then the energy integral (2.4-5) implies that, for the difference, at any $t > t_0$

$$\int_V \left(T^* + U^*\right)\Big|_t dV = 0, \quad t > t_0. \tag{2.4-7}$$

Since both T and U are nonnegative,

$$U^* = 0, \quad T^* = 0, \quad \text{in} \quad V, \quad t > t_0. \tag{2.4-8}$$

From the positive-definiteness of T and U,

$$\mathbf{S}^* = 0, \quad \mathbf{E}^* = 0, \quad \dot{\mathbf{u}}^* = 0, \quad \text{in} \quad V, \quad t > t_0. \tag{2.4-9}$$

Hence the two solutions are identical to within a static rigid body displacement and a constant potential.

5. OTHER FORMULATIONS

5.1 Four-Vector Formulation

Let us define the four-space coordinate system [14]

$$x_p = \{x_i, t\}, \tag{2.5-1}$$

and the four-vector

$$U_p = \{u_i, \phi\}, \tag{2.5-2}$$

where subscripts p, q, r, s will be assumed to run 1 to 4. Also define the second-rank four-tensor

$$\rho_{pq} = \begin{cases} \rho\delta_{pq}, & p,q = 1,2,3, \\ 0, & p,q = 4, \end{cases} = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & \rho & 0 & 0 \\ 0 & 0 & \rho & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{2.5-3}$$

and the fourth-rank four-tensor M_{pqrs}, where

$$\begin{aligned} &M_{ijkl} = c_{ijkl}, \quad M_{4jkl} = e_{jkl}, \quad M_{ijk4} = e_{kij}, \\ &M_{4jk4} = -\varepsilon_{jk}, \quad M_{p44s} = -\rho_{ps}, \end{aligned} \tag{2.5-4}$$

and all other components of $M_{pqrs} = 0$. Then

$$\begin{aligned} (U_{p,q}M_{pqrl})_{,r} &= (U_{i,j}M_{ijrl} + U_{4,j}M_{4jrl} + U_{i,4}M_{i4rl} + U_{4,4}M_{44rl})_{,r} \\ &= (U_{i,j}M_{ijkl} + U_{4,j}M_{4jkl} + U_{i,4}M_{i4kl} + U_{4,4}M_{44kl})_{,k} \\ &\quad + (U_{i,j}M_{ij4l} + U_{4,j}M_{4j4l} + U_{i,4}M_{i44l} + U_{4,4}M_{444l})_{,4} \\ &= (u_{i,j}c_{ijkl} + \phi_{,j}e_{jkl})_{,k} + (-\dot{u}_i\rho_{il})_{,4} \\ &= c_{ijkl}u_{i,jk} + e_{jkl}\phi_{,jk} - \rho\ddot{u}_l, \end{aligned} \tag{2.5-5}$$

and

$$\begin{aligned} &(U_{p,q}M_{pqr4})_{,r} \\ &= (U_{i,j}M_{ijr4} + U_{4,j}M_{4jr4} + U_{i,4}M_{i4r4} + U_{4,4}M_{44r4})_{,r} \\ &= (U_{i,j}M_{ijk4} + U_{4,j}M_{4jk4} + U_{i,4}M_{i4k4} + U_{4,4}M_{44k4})_{,k} \\ &\quad + (U_{i,j}M_{ij44} + U_{4,j}M_{4j44} + U_{i,4}M_{i444} + U_{4,4}M_{4444})_{,4} \\ &= (u_{i,j}e_{kij} - \phi_{,j}\varepsilon_{jk})_{,k} \\ &= u_{i,jk}e_{kij} - \phi_{,jk}\varepsilon_{jk}. \end{aligned} \tag{2.5-6}$$

Therefore,

$$(U_{p,q}M_{pqrs})_{,r} = 0 \tag{2.5-7}$$

yields the homogeneous equation of motion and the charge equation.

5.2 Vector Potential Formulation

Consider the case when there is no body charge. Since the divergence of **D** vanishes, we can introduce a vector potential ψ_i by

$$D_i = \frac{1}{2}\varepsilon_{ijk}\psi_{k,j}, \tag{2.5-8}$$

which satisfies the divergence-free condition on **D**. Corresponding to vector **D**, we introduce an anti-symmetric tensor by [15]

$$D_i = \frac{1}{2}\varepsilon_{ijk}\mathsf{D}_{jk}, \tag{2.5-9}$$

which, when substituted into (2.5-8), yields

$$\mathsf{D}_{ij} = \frac{1}{2}(\psi_{j,i} - \psi_{i,j}). \tag{2.5-10}$$

Similarly, for the electric field **E**, we introduce an anti-symmetric tensor by

$$E_i = \frac{1}{2}\varepsilon_{ijk}\mathsf{E}_{jk}. \tag{2.5-11}$$

Then the curl-free condition on **E** takes the following form:

$$\mathsf{E}_{ij,i} = 0. \tag{2.5-12}$$

In summary, the equations for this formulation are

$$\begin{gathered} T_{ij,i} = \rho\ddot{u}_j, \quad \mathsf{E}_{ij,i} = 0, \\ T_{ij} = \frac{\partial U}{\partial S_{ij}}, \quad \mathsf{E}_{ij} = \frac{\partial U}{\partial \mathsf{D}_{ij}}, \\ S_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad \mathsf{D}_{ij} = \frac{1}{2}(\psi_{j,i} - \psi_{i,j}). \end{gathered} \tag{2.5-13}$$

Note that in this formulation the internal energy U is used, which is positive definite.

6. CURVILINEAR COORDINATES

Cylindrical and spherical shapes are often used in piezoelectric devices. To analyze these devices, it is usually convenient to use cylindrical or spherical coordinates.

6.1 Cylindrical Coordinates

The cylindrical coordinates (r, θ, z) are defined by

$$x_1 = r\cos\theta, \quad x_2 = r\sin\theta, \quad x_3 = z. \tag{2.6-1}$$

In cylindrical coordinates we have the strain-displacement relation

$$S_{rr} = u_{r,r}, \quad S_{\theta\theta} = \frac{1}{r}u_{\theta,\theta} + \frac{u_r}{r}, \quad S_{zz} = u_{z,z},$$
$$2S_{r\theta} = u_{\theta,r} + \frac{1}{r}u_{r,\theta} - \frac{u_\theta}{r}, \quad 2S_{\theta z} = \frac{1}{r}u_{z,\theta} + u_{\theta,z}, \tag{2.6-2}$$
$$2S_{zr} = u_{r,z} + u_{z,r}.$$

The electric field-potential relation is given by

$$E_r = -\phi_{,r}, \quad E_\theta = -\frac{1}{r}\phi_{,\theta}, \quad E_z = -\phi_{,z}. \tag{2.6-3}$$

The equations of motion are

$$\frac{\partial T_{rr}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta r}}{\partial \theta} + \frac{\partial T_{zr}}{\partial z} + \frac{T_{rr} - T_{\theta\theta}}{r} + \rho f_r = \rho \ddot{u}_r,$$
$$\frac{\partial T_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial \theta} + \frac{\partial T_{z\theta}}{\partial z} + \frac{2}{r}T_{r\theta} + \rho f_\theta = \rho \ddot{u}_\theta, \tag{2.6-4}$$
$$\frac{\partial T_{rz}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta z}}{\partial \theta} + \frac{\partial T_{zz}}{\partial z} + \frac{1}{r}T_{rz} + \rho f_z = \rho \ddot{u}_z.$$

The electrostatic charge equation is

$$\frac{1}{r}(rD_r)_{,r} + \frac{1}{r}D_{\theta,\theta} + D_{z,z} = \rho_e. \tag{2.6-5}$$

6.2 Spherical Coordinates

The spherical coordinates (r, θ, φ) are defined by

$$x_1 = r\sin\theta\cos\varphi, \quad x_2 = r\sin\theta\sin\varphi, \quad x_3 = r\cos\theta. \tag{2.6-6}$$

In spherical coordinates we have the strain-displacement relation

$$S_{rr} = \frac{\partial u_r}{\partial r}, \quad S_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r},$$
$$S_{\varphi\varphi} = \frac{1}{r\sin\theta}\frac{\partial u_\varphi}{\partial \varphi} + \frac{u_r}{r} + \frac{u_\theta}{r}\cot\theta,$$
$$2S_{r\theta} = \frac{\partial u_\theta}{\partial r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r}, \tag{2.6-7}$$
$$2S_{\theta\varphi} = \frac{1}{r}\frac{\partial u_\varphi}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial \varphi} - \frac{u_\varphi}{r}\cot\theta,$$
$$2S_{\varphi r} = \frac{1}{r\sin\theta}\frac{\partial u_r}{\partial \varphi} + \frac{\partial u_\varphi}{\partial r} - \frac{u_\varphi}{r}.$$

The electric field-potential relation is

$$E_r = -\frac{\partial\phi}{\partial r}, \quad E_\theta = -\frac{1}{r}\frac{\partial\phi}{\partial\theta}, \quad E_\varphi = -\frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\varphi}. \tag{2.6-8}$$

The equations of motion are

$$\begin{aligned}
&\frac{\partial T_{rr}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta r}}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial T_{\varphi r}}{\partial\varphi} \\
&\quad + \frac{1}{r}(2T_{rr} - T_{\theta\theta} - T_{\varphi\varphi} + T_{\theta r}\cot\theta) + \rho f_r = \rho\ddot{u}_r, \\
&\frac{\partial T_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial T_{\varphi\theta}}{\partial\varphi} \\
&\quad + \frac{1}{r}[3T_{r\theta} + (T_{\theta\theta} - T_{\varphi\varphi})\cot\theta] + \rho f_\theta = \rho\ddot{u}_\theta, \\
&\frac{\partial T_{r\varphi}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta\varphi}}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial T_{\varphi\varphi}}{\partial\varphi} \\
&\quad + \frac{1}{r}(3T_{r\varphi} + 2T_{\theta\varphi}\cot\theta) + \rho f_z = \rho\ddot{u}_\varphi.
\end{aligned} \tag{2.6-9}$$

The electrostatic charge equation is

$$r^2\frac{\partial}{\partial r}(r^2 D_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(D_\theta \sin\varphi) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}D_\varphi = \rho_e. \tag{2.6-10}$$

7. COMPACT MATRIX NOTATION

We now introduce a compact matrix notation [11]. This notation consists of replacing pairs of indices ij or kl by single indices p or q, where i, j, k and l take the values of 1, 2, and 3, and p and q take the values 1, 2, 3, 4, 5, and 6 according to

$$\begin{array}{lcccccc}
ij \text{ or } kl: & 11 & 22 & 33 & 23 \text{ or } 32 & 31 \text{ or } 13 & 12 \text{ or } 21 \\
p \text{ or } q: & 1 & 2 & 3 & 4 & 5 & 6
\end{array}. \tag{2.7-1}$$

Thus

$$c_{ijkl} \to c_{pq}, \quad e_{ikl} \to e_{ip}, \quad T_{ij} \to T_p. \tag{2.7-2}$$

For the strain tensor, we introduce S_p such that

$$\begin{aligned}
&S_1 = S_{11}, \quad S_2 = S_{22}, \quad S_3 = S_{33}, \\
&S_4 = 2S_{23}, \quad S_5 = 2S_{31}, \quad S_6 = 2S_{12}.
\end{aligned} \tag{2.7-3}$$

The constitutive relations in (2.1-11) can then be written as

$$T_p = c^E_{pq} S_q - e_{kp} E_k,$$
$$D_i = e_{iq} S_q + \varepsilon^S_{ik} E_k. \tag{2.7-4}$$

In matrix form, Equation (2.7-4) becomes

$$\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{Bmatrix} = \begin{pmatrix} c^E_{11} & c^E_{12} & c^E_{13} & c^E_{14} & c^E_{15} & c^E_{16} \\ c^E_{21} & c^E_{22} & c^E_{23} & c^E_{24} & c^E_{25} & c^E_{26} \\ c^E_{31} & c^E_{32} & c^E_{33} & c^E_{34} & c^E_{35} & c^E_{36} \\ c^E_{41} & c^E_{42} & c^E_{43} & c^E_{44} & c^E_{45} & c^E_{46} \\ c^E_{51} & c^E_{52} & c^E_{53} & c^E_{54} & c^E_{55} & c^E_{56} \\ c^E_{61} & c^E_{62} & c^E_{63} & c^E_{64} & c^E_{65} & c^E_{66} \end{pmatrix} \begin{Bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{Bmatrix} - \begin{pmatrix} e_{11} & e_{21} & e_{31} \\ e_{12} & e_{22} & e_{32} \\ e_{13} & e_{23} & e_{33} \\ e_{14} & e_{24} & e_{34} \\ e_{15} & e_{25} & e_{35} \\ e_{16} & e_{26} & e_{36} \end{pmatrix} \begin{Bmatrix} E_1 \\ E_2 \\ E_3 \end{Bmatrix},$$

$$\begin{Bmatrix} D_1 \\ D_2 \\ D_3 \end{Bmatrix} = \begin{bmatrix} e_{11} & e_{12} & e_{13} & e_{14} & e_{15} & e_{16} \\ e_{21} & e_{22} & e_{23} & e_{24} & e_{25} & e_{26} \\ e_{31} & e_{32} & e_{33} & e_{34} & e_{35} & e_{36} \end{bmatrix} \begin{Bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{Bmatrix} + \begin{pmatrix} \varepsilon^S_{11} & \varepsilon^S_{12} & \varepsilon^S_{13} \\ \varepsilon^S_{21} & \varepsilon^S_{22} & \varepsilon^S_{22} \\ \varepsilon^S_{31} & \varepsilon^S_{32} & \varepsilon^S_{33} \end{pmatrix} \begin{Bmatrix} E_1 \\ E_2 \\ E_3 \end{Bmatrix}. \tag{2.7-5}$$

Similarly, Equations (2.1-16), (2.1-19) and (2.1-20) can also be written in matrix form. The matrices of the material constants in various expressions are related by [11]

$$c^E_{pr} s^E_{qr} = \delta_{pq}, \quad c^D_{pr} s^D_{qr} = \delta_{pq},$$
$$\beta^S_{ik} \varepsilon^S_{jk} = \delta_{ijq}, \quad \beta^T_{ik} \varepsilon^T_{jk} = \delta_{ijq},$$
$$c^D_{pq} = c^E_{pq} + e_{kp} h_{kq}, \quad s^D_{pq} = s^E_{pq} - d_{kp} g_{kq},$$
$$\varepsilon^T_{ij} = \varepsilon^S_{ij} + d_{iq} e_{jq}, \quad \beta^T_{ij} = \beta^S_{ij} - g_{iq} h_{jq}, \tag{2.7-6}$$
$$e_{ip} = d_{iq} c^E_{pq}, \quad d_{ip} = \varepsilon^T_{ik} g_{kp},$$
$$g_{ip} = \beta^T_{ik} d_{kp}, \quad h_{ip} = g_{iq} c^D_{qp}.$$

As an example, some of the relations in (2.7-6) are shown below. In matrix-vector notation (2.7-4) can be written as

$$\{T\} = [c^E]\{S\} - [e]^T \{E\},$$
$$\{D\} = [e]\{S\} + [\varepsilon^S]\{E\}. \tag{2.7-7}$$

From $(2.7\text{-}7)_1$,

$$[c^E]\{S\} = \{T\} + [e]^T\{E\}. \tag{2.7-8}$$

Multiplication of both sides of (2.7-8) by the inverse of $[c^E]$ yields

$$\{S\} = [c^E]^{-1}\{T\} + [c^E]^{-1}[e]^T\{E\}. \tag{2.7-9}$$

Substituting Equation (2.7-9) into $(2.7\text{-}7)_2$ gives

$$\begin{aligned}\{D\} &= [e]([c^E]^{-1}\{T\} + [c^E]^{-1}[e]^T\{E\}) + [\varepsilon^S]\{E\} \\ &= [e][c^E]^{-1}\{T\} + ([e][c^E]^{-1}[e]^T + [\varepsilon^S])\{E\}.\end{aligned} \tag{2.7-10}$$

Compare Equations (2.7-9) and (2.7-10) with (2.1-19) which is rewritten in matrix form below:

$$\begin{aligned}\{S\} &= [s^E]\{T\} + [d]^T\{E\}, \\ \{D\} &= [d]\{T\} + [\varepsilon^T]\{E\},\end{aligned} \tag{2.7-11}$$

we identify

$$\begin{aligned}&[s^E] = [c^E]^{-1}, \quad [d] = [e][c^E]^{-1}, \\ &[\varepsilon^T] = [\varepsilon^S] + [e][c^E]^{-1}[e]^T.\end{aligned} \tag{2.7-12}$$

8. POLARIZED CERAMICS

Polarized ceramics are transversely isotropic. Let **a**, a constant unit vector, represent the direction of the axis of rotational symmetry or the poling direction of the ceramics. For linear constitutive relations we need a quadratic electric enthalpy function *H*. For transversely isotropic materials, a quadratic *H* is a function of the following invariants of degrees one and two [16] (higher degree invariants are not included):

$$\begin{aligned}&I_1 = \mathbf{a}\cdot\mathbf{S}\cdot\mathbf{a}, \quad I_2 = \mathrm{tr}\mathbf{S}, \quad I_3 = \mathbf{a}\cdot\mathbf{E}, \\ &II_1 = \mathbf{a}\cdot\mathbf{S}^2\cdot\mathbf{a}, \quad II_2 = \mathrm{tr}\mathbf{S}^2, \\ &II_3 = \mathbf{E}\cdot\mathbf{E}, \quad II_4 = \mathbf{a}\cdot\mathbf{S}\cdot\mathbf{E} + \mathbf{E}\cdot\mathbf{S}\cdot\mathbf{a}.\end{aligned} \tag{2.8-1}$$

A complete quadratic function of the above seven invariants can be written as [16]

$$\begin{aligned}H = {} & c_1 I_1^2 + c_2 I_2^2 + c_3 I_1 I_2 + c_4 II_1 + c_5 II_2 \\ & + \varepsilon_1 I_3^2 + \varepsilon_2 II_3 \\ & + e_1 I_1 I_3 + e_2 I_2 I_3 + e_3 I_4,\end{aligned} \tag{2.8-2}$$

where c_1, c_2, c_3, c_4, and c_5 are elastic constants, ε_1 and ε_2 are dielectric constants, and e_1, e_2, and e_3 are piezoelectric constants. Differentiation of Equation (2.8-2) yields

$$\begin{aligned}\mathbf{T} &= \frac{\partial H}{\partial \mathbf{S}} = \frac{\partial H}{\partial I_1}\mathbf{a}\otimes\mathbf{a} + \frac{\partial H}{\partial I_2}\mathbf{1} + \frac{\partial H}{\partial II_1}(\mathbf{a}\otimes\mathbf{S}\cdot\mathbf{a} + \mathbf{a}\cdot\mathbf{S}\otimes\mathbf{a}) \\ &\quad + 2\frac{\partial H}{\partial II_2}\mathbf{S} + \frac{\partial H}{\partial II_4}(\mathbf{a}\otimes\mathbf{E} + \mathbf{E}\otimes\mathbf{a}) \\ &= (2c_1I_1 + c_3I_2 + e_1I_3)\mathbf{a}\otimes\mathbf{a} + (2c_2I_2 + c_3I_1 + e_2I_3)\mathbf{1} \\ &\quad + c_4(\mathbf{a}\otimes\mathbf{S}\cdot\mathbf{a} + \mathbf{a}\cdot\mathbf{S}\otimes\mathbf{a}) + 2c_5\mathbf{S} + e_3(\mathbf{a}\otimes\mathbf{E} + \mathbf{E}\otimes\mathbf{a}),\end{aligned} \tag{2.8-3}$$

and

$$\begin{aligned}\mathbf{D} &= -\frac{\partial H}{\partial \mathbf{E}} = -\frac{\partial H}{\partial I_3}\mathbf{a} - 2\frac{\partial H}{\partial II_3}\mathbf{E} - 2\frac{\partial H}{\partial II_4}\mathbf{S}\cdot\mathbf{a} \\ &= -(2\varepsilon_1I_3 + e_1I_1 + e_2I_2)\mathbf{a} - 2\varepsilon_2\mathbf{E} - 2e_3\mathbf{S}\cdot\mathbf{a}.\end{aligned} \tag{2.8-4}$$

Let $\mathbf{a} = \mathbf{i}_3$, and rearrange (2.8-3) and (2.8-4) in the form of (2.7-5). The following matrices will result:

$$\begin{pmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{21} & c_{11} & c_{13} & 0 & 0 & 0 \\ c_{31} & c_{31} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & e_{15} & 0 \\ 0 & 0 & 0 & e_{15} & 0 & 0 \\ e_{31} & e_{31} & e_{33} & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{11} & 0 \\ 0 & 0 & \varepsilon_{33} \end{pmatrix}, \tag{2.8-5}$$

where $c_{66} = (c_{11} - c_{12})/2$. The matrices in Equation (2.8-5) have the same structures as those of crystals class C_{6v} (or 6mm). The elements of the matrices in (2.8-5) are related to the material constants in (2.8-2) by

$$
\begin{aligned}
&c_1 = c_{11} - 2c_{13} + c_{33} - 4c_{44}, \quad c_2 = c_{12}/2, \\
&c_3 = c_{13} - c_{12}, \quad c_4 = -c_{11} + c_{12} + 2c_{44}, \quad c_5 = (c_{11} - c_{12})/2, \\
&\varepsilon_1 = (\varepsilon_{11} - \varepsilon_{22})/2, \quad \varepsilon_2 = -\varepsilon_{11}/2, \\
&e_1 = e_{31} + 2e_{15} - e_{33}, \quad e_2 = -e_{31}, \quad e_3 = -e_{15}.
\end{aligned}
\tag{2.8-6}
$$

With Equation (2.8-5), the constitutive relations of ceramics poled in the x_3 direction take the following form:

$$
\begin{aligned}
T_{11} &= c_{11}u_{1,1} + c_{12}u_{2,2} + c_{13}u_{3,3} + e_{31}\phi_{,3}, \\
T_{22} &= c_{12}u_{1,1} + c_{22}u_{2,2} + c_{13}u_{3,3} + e_{31}\phi_{,3}, \\
T_{33} &= c_{13}u_{1,1} + c_{13}u_{2,2} + c_{33}u_{3,3} + e_{33}\phi_{,3}, \\
T_{23} &= c_{44}(u_{2,3} + u_{3,2}) + e_{15}\phi_{,2}, \\
T_{31} &= c_{44}(u_{3,1} + u_{1,3}) + e_{15}\phi_{,1}, \\
T_{12} &= c_{66}(u_{1,2} + u_{2,1}),
\end{aligned}
\tag{2.8-7}
$$

and

$$
\begin{aligned}
D_1 &= e_{15}(u_{3,1} + u_{1,3}) - \varepsilon_{11}\phi_{,1}, \\
D_2 &= e_{15}(u_{2,3} + u_{3,2}) - \varepsilon_{11}\phi_{,2}, \\
D_3 &= e_{31}(u_{1,1} + u_{2,2}) + e_{33}u_{3,3} - \varepsilon_{33}\phi_{,3}.
\end{aligned}
\tag{2.8-8}
$$

The equations of motion and charge are

$$
\begin{aligned}
&c_{11}u_{1,11} + (c_{12} + c_{66})u_{2,12} + (c_{13} + c_{44})u_{3,13} + c_{66}u_{1,22} \\
&\quad + c_{44}u_{1,33} + (e_{31} + e_{15})\phi_{,13} = \rho\ddot{u}_1, \\
&c_{66}u_{2,11} + (c_{12} + c_{66})u_{1,12} + c_{11}u_{2,22} + (c_{13} + c_{44})u_{3,23} \\
&\quad + c_{44}u_{2,33} + (e_{31} + e_{15})\phi_{,23} = \rho\ddot{u}_2, \\
&c_{44}u_{3,11} + (c_{44} + c_{13})u_{1,31} + c_{44}u_{3,22} + (c_{13} + c_{44})u_{2,23} \\
&\quad + c_{33}u_{3,33} + e_{15}(\phi_{,11} + \phi_{,22}) + e_{33}\phi_{,33} = \rho\ddot{u}_3, \\
&e_{15}u_{3,11} + (e_{15} + e_{31})u_{1,13} + e_{15}u_{3,22} + (e_{15} + e_{31})u_{2,32} \\
&\quad + e_{31}u_{3,33} - \varepsilon_{11}(\phi_{,11} + \phi_{,22}) - \varepsilon_{33}\phi_{,33} = 0.
\end{aligned}
\tag{2.8-9}
$$

Sometimes a piezoelectric device is heterogeneous with ceramics poled in different directions in different parts. In this case it is not possible to orient the x_3 axis along different poling directions unless a few local coordinate systems are introduced. Therefore, material matrices of ceramics poled along other axes are useful. They can be obtained from the matrices in (2.8-5) by rotating rows and columns properly. For ceramics poled in the x_1 direction, we have

$$\begin{pmatrix} c_{33} & c_{13} & c_{13} & 0 & 0 & 0 \\ c_{13} & c_{11} & c_{12} & 0 & 0 & 0 \\ c_{13} & c_{12} & c_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{66} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{44} \end{pmatrix},$$

$$\begin{pmatrix} e_{33} & e_{31} & e_{31} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e_{15} \\ 0 & 0 & 0 & 0 & e_{15} & 0 \end{pmatrix}, \begin{pmatrix} \varepsilon_{33} & 0 & 0 \\ 0 & \varepsilon_{11} & 0 \\ 0 & 0 & \varepsilon_{11} \end{pmatrix}. \qquad (2.8\text{-}10)$$

For ceramics poled in the x_2 direction, we obtain

$$\begin{pmatrix} c_{11} & c_{13} & c_{12} & 0 & 0 & 0 \\ c_{13} & c_{33} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{13} & c_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{66} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{44} \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & e_{15} \\ e_{31} & e_{33} & e_{31} & 0 & 0 & 0 \\ 0 & 0 & 0 & e_{15} & 0 & 0 \end{pmatrix}, \begin{pmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{33} & 0 \\ 0 & 0 & \varepsilon_{11} \end{pmatrix}. \qquad (2.8\text{-}11)$$

9. QUARTZ AND LANGASITE

Quartz is probably the most widely used piezoelectric crystal. It belongs to crystal class 32 (or D_3). Langasite and some of its isomorphs (langanite and langatate) are emerging piezoelectric crystals which have stronger piezoelectric coupling than quartz and also belong to crystal class 32. For such a crystal with x_3 a trigonal axis and x_1 a diagonal axis, the material matrices are

$$\begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} & 0 & 0 \\ c_{21} & c_{11} & c_{13} & -c_{14} & 0 & 0 \\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\ c_{14} & -c_{14} & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & c_{14} \\ 0 & 0 & 0 & 0 & c_{14} & c_{66} \end{pmatrix},$$

$$\begin{pmatrix} e_{11} & -e_{11} & 0 & e_{14} & 0 & 0 \\ 0 & 0 & 0 & 0 & -e_{14} & -e_{11} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{11} & 0 \\ 0 & 0 & \varepsilon_{33} \end{pmatrix}. \qquad (2.9\text{-}1)$$

The independent material constants are $6 + 2 + 2 = 10$.

Quartz plates are often used to make devices. Plates taken from a bulk crystal at different orientations are referred to as plates of different cuts. A particular cut is specified by two angles, φ and θ, with respect to the crystal axes (X,Y,Z).

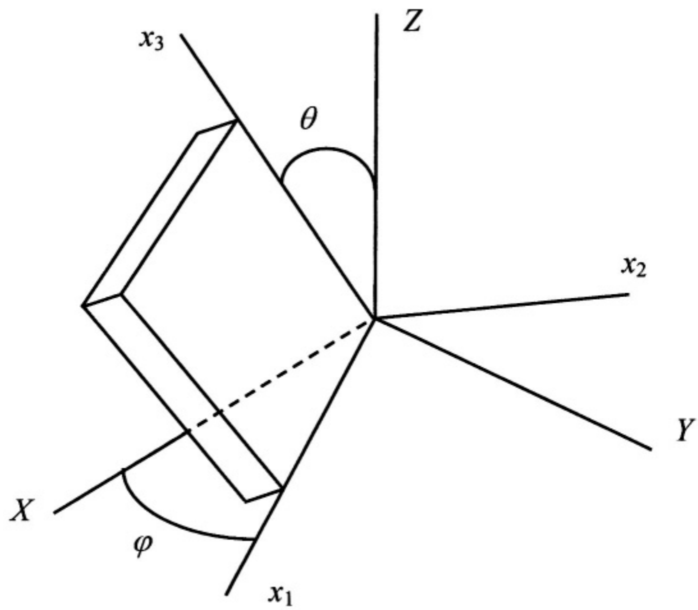

Figure 2.9-1. A quartz plate cut from a bulk crystal.

Plates of different cuts have different material matrices with respect to coordinate systems in and normal to the plane of the plates. One class of cuts of quartz plates, called rotated Y-cuts, has $\varphi = 0$ and is particularly useful in device applications. Rotated Y-cut quartz exhibits monoclinic symmetry of class 2 (or C_2) in a coordinate system (x_1,x_2) in and normal to the plane of the plate. Therefore we list the equations for monoclinic crystals below which are useful for studying quartz devices. For monoclinic crystals, with the diagonal axis along the x_1 axis,

$$\begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} & 0 & 0 \\ c_{21} & c_{22} & c_{23} & c_{24} & 0 & 0 \\ c_{31} & c_{32} & c_{33} & c_{34} & 0 & 0 \\ c_{41} & c_{42} & c_{43} & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & c_{56} \\ 0 & 0 & 0 & 0 & c_{65} & c_{66} \end{pmatrix},$$

$$\begin{pmatrix} e_{11} & e_{12} & e_{13} & e_{14} & 0 & 0 \\ 0 & 0 & 0 & 0 & e_{25} & e_{26} \\ 0 & 0 & 0 & 0 & e_{35} & e_{36} \end{pmatrix}, \begin{pmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{11} & \varepsilon_{23} \\ 0 & \varepsilon_{32} & \varepsilon_{33} \end{pmatrix}. \tag{2.9-2}$$

The constitutive relations are

$$\begin{aligned} T_{11} &= c_{11}u_{1,1} + c_{12}u_{2,2} + c_{13}u_{3,3} + c_{14}(u_{2,3} + u_{3,2}) + e_{11}\phi_{,1}, \\ T_{22} &= c_{12}u_{1,1} + c_{22}u_{2,2} + c_{23}u_{3,3} + c_{24}(u_{2,3} + u_{3,2}) + e_{12}\phi_{,1}, \\ T_{33} &= c_{13}u_{1,1} + c_{23}u_{2,2} + c_{33}u_{3,3} + c_{34}(u_{2,3} + u_{3,2}) + e_{13}\phi_{,1}, \\ T_{23} &= c_{14}u_{1,1} + c_{24}u_{2,2} + c_{34}u_{3,3} + c_{44}(u_{2,3} + u_{3,2}) + e_{14}\phi_{,1}, \\ T_{31} &= c_{55}(u_{3,1} + u_{1,3}) + c_{56}(u_{1,2} + u_{2,1}) + e_{25}\phi_{,2} + e_{35}\phi_{,3}, \\ T_{12} &= c_{56}(u_{3,1} + u_{1,3}) + c_{66}(u_{1,2} + u_{2,1}) + e_{26}\phi_{,2} + e_{36}\phi_{,3}, \end{aligned} \tag{2.9-3}$$

and

$$\begin{aligned} D_1 &= e_{11}u_{1,1} + e_{12}u_{2,2} + e_{13}u_{3,3} + e_{14}(u_{2,3} + u_{3,2}) - \varepsilon_{11}\phi_{,1}, \\ D_2 &= e_{25}(u_{3,1} + u_{1,3}) + e_{26}(u_{1,2} + u_{2,1}) - \varepsilon_{22}\phi_{,2} - \varepsilon_{23}\phi_{,3}, \\ D_3 &= e_{35}(u_{3,1} + u_{1,3}) + e_{36}(u_{1,2} + u_{2,1}) - \varepsilon_{23}\phi_{,2} - \varepsilon_{33}\phi_{,3}. \end{aligned} \tag{2.9-4}$$

The equations of motion and charge are

$$\begin{aligned}
&c_{11}u_{1,11}+(c_{12}+c_{66})u_{2,12}+(c_{13}+c_{55})u_{3,13}+(c_{14}+c_{56})u_{2,13}\\
&\quad+(c_{14}+c_{56})u_{3,12}+2c_{56}u_{1,23}+c_{66}u_{1,22}+c_{55}u_{1,33}\\
&\quad+e_{11}\phi_{,11}+e_{26}\phi_{,22}+(e_{36}+e_{25})\phi_{,23}+e_{35}\phi_{,33}=\rho\ddot{u}_1,\\
&c_{56}u_{3,11}+(c_{56}+c_{14})u_{1,13}+(c_{66}+c_{12})u_{1,12}+c_{66}u_{2,11}\\
&\quad+c_{22}u_{2,22}+(c_{23}+c_{44})u_{3,23}+2c_{24}u_{2,23}+c_{24}u_{3,22}\\
&\quad+c_{34}u_{3,33}+c_{44}u_{2,33}+(e_{26}+e_{12})\phi_{,12}+(e_{36}+e_{14})\phi_{,13}=\rho\ddot{u}_2,\\
&c_{55}u_{3,11}+(c_{55}+c_{13})u_{1,13}+(c_{56}+c_{14})u_{1,12}+c_{56}u_{2,11}\\
&\quad+c_{24}u_{2,22}+2c_{34}u_{3,23}+(c_{44}+c_{23})u_{2,23}+c_{44}u_{3,22}\\
&\quad+c_{33}u_{3,33}+c_{34}u_{2,33}+(e_{25}+e_{14})\phi_{,12}+(e_{35}+e_{13})\phi_{,13}=\rho\ddot{u}_3,\\
&e_{11}u_{1,11}+(e_{12}+e_{26})u_{2,12}+(e_{13}+e_{35})u_{3,13}+(e_{14}+e_{36})u_{2,13}\\
&\quad+(e_{14}+e_{25})u_{3,12}+(e_{25}+e_{36})u_{1,23}+e_{26}u_{1,22}\\
&\quad+e_{35}u_{1,33}-\varepsilon_{11}\phi_{,11}-\varepsilon_{22}\phi_{,22}-2\varepsilon_{23}\phi_{,23}-\varepsilon_{33}\phi_{,33}=0.
\end{aligned}\tag{2.9-5}$$

For rotated Y-cut quartz, motions with only one displacement component, u_1, are particularly useful in device applications. Consider

$$\begin{aligned}
&u_1=u_1(x_2,x_3,t),\quad u_2=u_3=0,\\
&\phi=\phi(x_2,x_3,t).
\end{aligned}\tag{2.9-6}$$

Equation (2.9-6) yields the following non-vanishing components of strain, electric field, stress, and electric displacement:

$$\begin{aligned}
&S_5=u_{1,3},\quad S_6=u_{1,2},\\
&E_2=-\phi_{,2},\quad E_3=-\phi_{,3},\\
&T_{31}=c_{55}u_{1,3}+c_{56}u_{1,2}+e_{25}\phi_{,2}+e_{35}\phi_{,3},\\
&T_{21}=c_{56}u_{1,3}+c_{66}u_{1,2}+e_{26}\phi_{,2}+e_{36}\phi_{,3},\\
&D_2=e_{25}u_{1,3}+e_{26}u_{1,2}-\varepsilon_{22}\phi_{,2}-\varepsilon_{23}\phi_{,3},\\
&D_3=e_{35}u_{1,3}+e_{36}u_{1,2}-\varepsilon_{23}\phi_{,2}-\varepsilon_{33}\phi_{,3}.
\end{aligned}\tag{2.9-7}$$

The equations left to be satisfied by u_1 and ϕ are

$$
\begin{aligned}
&c_{66}u_{1,22}+c_{55}u_{1,33}+2c_{56}u_{1,23}\\
&\quad+e_{26}\phi_{,22}+e_{35}\phi_{,33}+(e_{25}+e_{36})\phi_{,23}=\rho\ddot{u}_1,\\
&e_{26}u_{1,22}+e_{35}u_{1,33}+(e_{25}+e_{36})u_{1,23}\\
&\quad-\varepsilon_{22}\phi_{,22}-\varepsilon_{33}\phi_{,33}-2\varepsilon_{23}\phi_{,23}=0.
\end{aligned}
\tag{2.9-8}
$$

10. LITHIUM NIOBATE AND LITHIUM TANTALATE

Lithium niobate and lithium tantalate have stronger piezoelectric coupling than quartz. For these two crystals the crystal class is $C_{3v} = 3m$. The material matrices are

$$
\begin{pmatrix}
c_{11} & c_{12} & c_{13} & c_{14} & 0 & 0\\
c_{21} & c_{11} & c_{13} & -c_{14} & 0 & 0\\
c_{13} & c_{13} & c_{33} & 0 & 0 & 0\\
c_{14} & -c_{14} & 0 & c_{44} & 0 & 0\\
0 & 0 & 0 & 0 & c_{44} & c_{14}\\
0 & 0 & 0 & 0 & c_{14} & c_{66}
\end{pmatrix},
$$

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & e_{15} & -e_{22}\\
-e_{22} & e_{22} & 0 & e_{15} & 0 & 0\\
e_{31} & e_{32} & e_{33} & 0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
\varepsilon_{11} & 0 & 0\\
0 & \varepsilon_{11} & 0\\
0 & 0 & \varepsilon_{33}
\end{pmatrix}.
\tag{2.10-1}
$$

When a rotated Y-cut is formed, the material apparently has m-monoclinic symmetry with the following matrices

$$
\begin{pmatrix}
c_{11} & c_{12} & c_{13} & c_{14} & 0 & 0\\
c_{21} & c_{22} & c_{23} & c_{24} & 0 & 0\\
c_{31} & c_{32} & c_{33} & c_{34} & 0 & 0\\
c_{41} & c_{42} & c_{43} & c_{44} & 0 & 0\\
0 & 0 & 0 & 0 & c_{55} & c_{56}\\
0 & 0 & 0 & 0 & c_{65} & c_{66}
\end{pmatrix},
$$

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & e_{15} & e_{16}\\
e_{21} & e_{22} & e_{23} & e_{24} & 0 & 0\\
e_{31} & e_{32} & e_{33} & e_{34} & 0 & 0
\end{pmatrix},
\begin{pmatrix}
\varepsilon_{11} & 0 & 0\\
0 & \varepsilon_{11} & \varepsilon_{23}\\
0 & \varepsilon_{32} & \varepsilon_{33}
\end{pmatrix}.
\tag{2.10-2}
$$

The constitutive relations are

$$
\begin{aligned}
T_{11} &= c_{11}u_{1,1} + c_{12}u_{2,2} + c_{13}u_{3,3} + c_{14}(u_{2,3} + u_{3,2}) + e_{21}\phi_{,2} + e_{31}\phi_{,3}, \\
T_{22} &= c_{12}u_{1,1} + c_{22}u_{2,2} + c_{23}u_{3,3} + c_{24}(u_{2,3} + u_{3,2}) + e_{22}\phi_{,2} + e_{32}\phi_{,3}, \\
T_{33} &= c_{13}u_{1,1} + c_{23}u_{2,2} + c_{33}u_{3,3} + c_{34}(u_{2,3} + u_{3,2}) + e_{23}\phi_{,2} + e_{33}\phi_{,3}, \\
T_{23} &= c_{14}u_{1,1} + c_{24}u_{2,2} + c_{34}u_{3,3} + c_{44}(u_{2,3} + u_{3,2}) + e_{24}\phi_{,2} + e_{34}\phi_{,3}, \\
T_{31} &= c_{55}(u_{3,1} + u_{1,3}) + c_{56}(u_{1,2} + u_{2,1}) + e_{15}\phi_{,1}, \\
T_{12} &= c_{56}(u_{3,1} + u_{1,3}) + c_{66}(u_{1,2} + u_{2,1}) + e_{16}\phi_{,1},
\end{aligned}
\tag{2.10-3}
$$

and

$$
\begin{aligned}
D_1 &= e_{15}(u_{1,3} + u_{3,1}) + e_{16}(u_{1,2} + u_{2,1}) - \varepsilon_{11}\phi_{,1}, \\
D_2 &= e_{21}u_{1,1} + e_{22}u_{2,2} + e_{23}u_{3,3} + e_{24}(u_{2,3} + u_{3,2}) - \varepsilon_{22}\phi_{,2} - \varepsilon_{23}\phi_{,3}, \\
D_3 &= e_{31}u_{1,1} + e_{32}u_{2,2} + e_{33}u_{3,3} + e_{34}(u_{2,3} + u_{3,2}) - \varepsilon_{23}\phi_{,2} - \varepsilon_{33}\phi_{,3}.
\end{aligned}
\tag{2.10-4}
$$

The equations of motion and charge are

$$
\begin{aligned}
& c_{11}u_{1,11} + (c_{12} + c_{66})u_{2,12} + (c_{13} + c_{55})u_{3,13} + (c_{14} + c_{56})u_{2,13} \\
& \quad + (c_{14} + c_{56})u_{3,12} + 2c_{56}u_{1,23} + c_{66}u_{1,22} + c_{55}u_{1,33} \\
& \quad + (e_{21} + e_{16})\phi_{,12} + (e_{31} + e_{15})\phi_{,13} = \rho\ddot{u}_1, \\
& c_{56}u_{3,11} + (c_{56} + c_{14})u_{1,13} + (c_{66} + c_{12})u_{1,12} + c_{66}u_{2,11} + c_{22}u_{2,22} \\
& \quad + (c_{23} + c_{44})u_{3,23} + 2c_{24}u_{2,23} + c_{24}u_{3,22} + c_{34}u_{3,33} + c_{44}u_{2,33} \\
& \quad + e_{16}\phi_{,11} + e_{22}\phi_{,22} + (e_{32} + e_{24})\phi_{,23} + e_{34}\phi_{,33} = \rho\ddot{u}_2, \\
& c_{55}u_{3,11} + (c_{55} + c_{13})u_{1,13} + (c_{56} + c_{14})u_{1,12} + c_{56}u_{2,11} + c_{24}u_{2,22} \\
& \quad + 2c_{34}u_{3,23} + (c_{44} + c_{23})u_{2,23} + c_{44}u_{3,22} + c_{33}u_{3,33} + c_{34}u_{2,33} \\
& \quad + e_{15}\phi_{,11} + e_{24}\phi_{,22} + (e_{34} + e_{23})\phi_{,23} + e_{33}\phi_{,33} = \rho\ddot{u}_3, \\
& (e_{15} + e_{31})u_{1,31} + e_{15}u_{3,11} + (e_{16} + e_{21})u_{1,12} + e_{16}u_{2,11} + e_{22}u_{2,22} \\
& \quad + (e_{23} + e_{34})u_{3,23} + (e_{24} + e_{32})u_{2,23} + e_{24}u_{3,22} + e_{33}u_{3,33} + e_{34}u_{2,33} \\
& \quad - \varepsilon_{11}\phi_{,11} - \varepsilon_{22}\phi_{,22} - 2\varepsilon_{23}\phi_{,23} - \varepsilon_{33}\phi_{,33} = 0.
\end{aligned}
\tag{2.10-5}
$$

Chapter 3

STATIC PROBLEMS

In this chapter, a few solutions to the static equations of linear piezoelectricity are presented. Some simple, useful deformation modes are considered in Sections 1 to 5. The concept of electromechanical coupling factor is introduced in Sections 1 to 3. Sections 6 to 13 are on anti-plane deformations of polarized ceramics. Real piezoelectric materials more or less have some conductivity. This conductivity tends to neutralize the electric field in a piezoelectric material. In static problems, an electric field can be maintained by an applied voltage. Sometimes conductivity needs to be considered [17].

1. EXTENSION OF A CERAMIC ROD

Consider a cylindrical rod of length L made from polarized ceramics with axial poling. The cross-section of the rod can be arbitrary. The lateral surface of the rod is traction-free and is unelectroded. The two end faces are under a uniform normal traction p, but there is no tangential traction. Electrically the two end faces are electroded with a circuit between the electrodes, which can be switched on or off. Two cases of open and shorted electrodes will be considered.

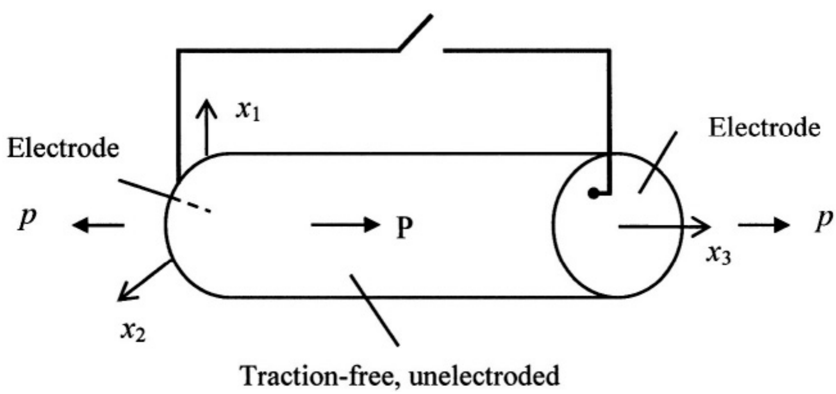

Figure 3.1-1. An axially poled ceramic rod.

1.1 Boundary-Value Problem

The boundary-value problem is:

$$
\begin{aligned}
&T_{ji,j} = 0, \quad D_{i,i} = 0, \quad \text{in} \quad V, \\
&S_{ij} = s^{E}_{ijkl} T_{kl} + D_{kij} E_k, \quad D_i = d_{ikl} T_{kl} + \varepsilon^{T}_{ik} E_k, \quad \text{in} \quad V, \\
&\varepsilon_{ijk} \varepsilon_{lmn} S_{il,jm} = 0, \quad \varepsilon_{ijk} E_{k,j} = 0, \quad \text{in} \quad V, \\
&T_{ji} n_j = 0, \quad D_i n_i = 0, \quad \text{on the lateral surface,} \\
&T_{31} = 0, \quad T_{32} = 0, \quad T_{33} = p, \quad E_1 = E_2 = 0, \quad x_3 = 0, L, \\
&\phi(x_3 = 0) = \phi(x_3 = L), \quad \text{if the end faces are shorted,} \\
&\text{or} \quad \int D_3 dA = 0, \quad x_3 = 0, L, \quad \text{if the end faces are open,}
\end{aligned}
\tag{3.1-1}
$$

where we have chosen the stress components and the electric displacement components as the primary unknowns. Many of these components are known on the lateral surface, and it is easy to guess what they are like inside the cylinder. Since many components of **T** will vanish, it is convenient to use constitutive relations with **T** as the independent constitutive variable. In this formulation the compatibility conditions on strains and the curl-free condition on the electric field have to be satisfied. As suggested by the boundary conditions on the lateral surface we consider the following **T** and **D** fields

$$
\begin{aligned}
&T_{33} = p, \quad \text{all other } T_{ij} = 0, \\
&D_3 = \text{constant}, \quad D_1 = D_2 = 0,
\end{aligned}
\tag{3.1-2}
$$

which satisfy the equation of motion and the charge equation. Since the **T** and **D** fields are constants, the constitutive relations imply that the **S** and **E** fields are also constants. Therefore the compatibility conditions on **S** and the curl-free condition on **E** are satisfied. (3.1-2) also satisfies the boundary conditions on the lateral surface and the mechanical boundary conditions on the end faces. From the constitutive relations

$$
\begin{aligned}
&S_{23} = S_{31} = S_{12} = 0, \\
&S_{33} = s^{E}_{33} p + d_{33} E_3, \quad S_{11} = S_{22} = s^{E}_{13} p + d_{31} E_3, \\
&E_1 = E_2 = 0, \quad D_3 = d_{33} p + \varepsilon^{T}_{33} E_3.
\end{aligned}
\tag{3.1-3}
$$

Hence the electrical boundary conditions of $E_1 = E_2 = 0$ (constant electric potential on an electrode) on the end electrodes are also satisfied. We consider two cases as follows.

1.2 Shorted Electrodes

In this case there is no potential difference between the end electrodes. Since E_3 is constant along the rod, we must have

$$E_3 = 0, \tag{3.1-4}$$

which implies that

$$D_3 = d_{33}p, \quad S_{33} = s_{33}^E p. \tag{3.1-5}$$

The mechanical work done to the rod per unit volume during the static extensional process is

$$W_1 = \frac{1}{2}T_{33}S_{33} = \frac{1}{2}s_{33}^E p^2. \tag{3.1-6}$$

1.3 Open Electrodes

In this case there is no net charge on the end electrodes. Since D_3 is constant over a cross-section, we must have

$$D_3 = 0, \tag{3.1-7}$$

which implies that

$$\begin{aligned} E_3 &= -\frac{d_{33}}{\varepsilon_{33}^T}p, \\ S_{33} &= s_{33}^E p - d_{33}\frac{d_{33}}{\varepsilon_{33}^T}p = s_{33}^E\left(1 - \frac{d_{33}^2}{\varepsilon_{33}^T s_{33}^E}\right)p. \end{aligned} \tag{3.1-8}$$

The mechanical work done to the rod per unit volume is

$$W_2 = \frac{1}{2}T_{33}S_{33} = \frac{1}{2}s_{33}^E\left(1 - \frac{d_{33}^2}{\varepsilon_{33}^T s_{33}^E}\right)p^2. \tag{3.1-9}$$

1.4 Electromechanical Coupling Factor

Since

$$\frac{d_{33}^2}{\varepsilon_{33}^T s_{33}^E} > 0, \tag{3.1-10}$$

we have

$$W_1 > W_2. \tag{3.1-11}$$

Therefore the rod appears to be stiffer when the electrodes are open and an axial electric field is produced. This is called the piezoelectric stiffening effect. The following ratio is called the longitudinal electromechanical coupling factor for the extension of a ceramic rod with axial poling, and is denoted by

$$(k_{33}^l)^2 = \frac{W_1 - W_2}{W_1} = \frac{d_{33}^2}{\varepsilon_{33}^T s_{33}^E}. \tag{3.1-12}$$

For PZT-5H, a common ceramic, from the material constants in Appendix 2,

$$(k_{33}^l)^2 = \frac{(593\times10^{-12})^2}{(3400\times8.85\times10^{-12})(20.7\times10^{-12})} = 0.56, \tag{3.1-13}$$

$$k_{33}^l = 0.75,$$

which is typical for ceramics. Graphically W_1, W_2 and their difference are represented by areas in the following figure.

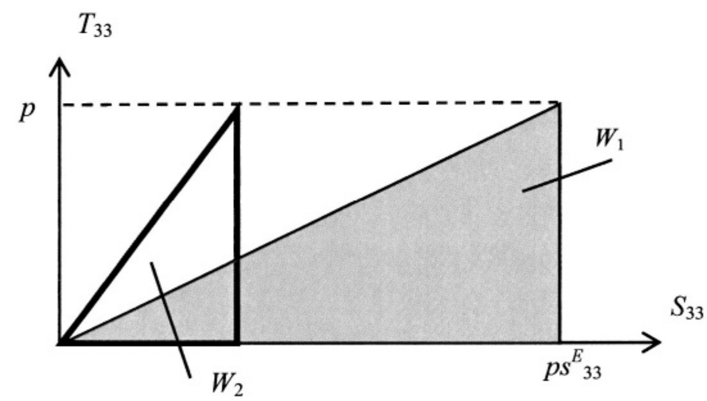

Figure 3.1-2. Work done to the ceramic rod per unit volume along different paths.

2. THICKNESS-STRETCH OF A CERAMIC PLATE

Consider an unbounded ceramic plate poled in the thickness direction. The major surfaces of the plate are under a normal traction p and are electroded. Two cases of shorted and open electrodes will be considered. The traction-produced charge or voltage on the electrodes can be used to detect the pressure electrically.

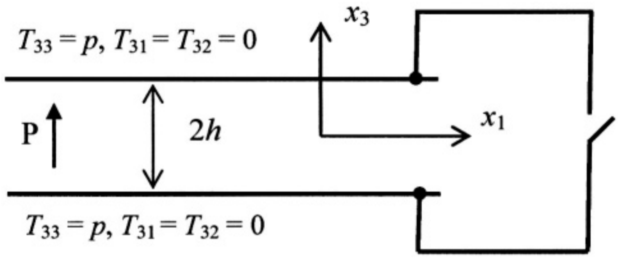

Figure 3.2-1. An electroded ceramic plate under mechanical loads.

2.1 Boundary-Value Problem

The boundary-value problem is:

$$
\begin{aligned}
&T_{ji,j}=0, \quad D_{i,i}=0, \quad \text{in} \quad V, \\
&T_{ij}=c_{ijkl}S_{kl}-e_{kij}E_k, \quad D_i=e_{ikl}S_{kl}+\varepsilon_{ik}E_k, \quad \text{in} \quad V, \\
&S_{ij}=(u_{i,j}+u_{j,i})/2, \quad E_i=-\phi_{,i}, \quad \text{in} \quad V, \\
&T_{ji}n_j=p\delta_{3i}, \quad x_3=\pm h, \\
&\phi(x_3=h)=\phi(x_3=-h), \quad \text{if the electrodes are shorted,} \\
&\text{or} \quad D_3(x_3=\pm h)=0, \quad \text{if the electrodes are open.}
\end{aligned}
\qquad (3.2\text{-}1)
$$

Consider the possibility of the following displacement and potential fields:

$$u_3=u_3(x_3), \quad u_1=u_2=0, \quad \phi=\phi(x_3), \qquad (3.2\text{-}2)$$

The nontrivial components of strain, electric field, stress, and electric displacement are

$$S_{33}=u_{3,3}, \quad E_3=-\phi_{,3}, \qquad (3.2\text{-}3)$$

and

$$
\begin{aligned}
T_{11}&=T_{22}=c_{13}u_{3,3}+e_{31}\phi_{,3}, \\
T_{33}&=c_{33}u_{3,3}+e_{33}\phi_{,3}, \\
D_3&=e_{33}u_{3,3}-\varepsilon_{33}\phi_{,3}.
\end{aligned}
\qquad (3.2\text{-}4)
$$

The equation of motion and the charge equation require that

$$
\begin{aligned}
T_{33,3}&=c_{33}u_{3,33}+e_{33}\phi_{,33}=0, \\
D_{3,3}&=e_{33}u_{3,33}-\varepsilon_{33}\phi_{,33}=0.
\end{aligned}
\qquad (3.2\text{-}5)
$$

Hence

$$u_{3,33}=0, \quad \phi_{,33}=0, \qquad (3.2\text{-}6)$$

unless

$$k_{33}^2 = \frac{e_{33}^2}{\varepsilon_{33} c_{33}} = 1, \tag{3.2-7}$$

which we do not consider because usually $k_{33}^2 < 1$. Equation (3.2-5) implies that all the strain, stress, electric field, and electric displacement components are constants.

2.2 Shorted Electrodes

Since the potential at the two electrodes are equal and E_3 is a constant, we must have

$$E_3 = 0. \tag{3.2-8}$$

The mechanical boundary conditions require that $T_{33} = p$. Then

$$S_3 = u_{3,3} = \frac{p}{c_{33}}, \quad T_{11} = T_{22} = \frac{c_{13}}{c_{33}} p, \quad D_3 = \frac{e_{33}}{c_{33}} p. \tag{3.2-9}$$

The work done to the plate per unit volume is

$$W_1 = \frac{1}{2} T_{33} S_{33} = \frac{p^2}{2c_{33}}. \tag{3.2-10}$$

2.3 Open Electrodes

In this case the boundary conditions require that

$$\begin{aligned} T_{33} &= c_{33} u_{3,3} + e_{33} \phi_{,3} = p, \\ D_3 &= e_{33} u_{3,3} - \varepsilon_{33} \phi_{,3} = 0, \end{aligned} \tag{3.2-11}$$

which imply that

$$S_3 = u_{3,3} = \frac{p}{c_{33}(1 + k_{33}^2)}, \quad E_3 = -\phi_{,3} = -\frac{e_{33}}{\varepsilon_{33} c_{33}(1 + k_{33}^3)} p. \tag{3.2-12}$$

The work done to the plate per unit volume is

$$W_2 = \frac{1}{2} T_{33} S_{33} = \frac{p^2}{2c_{33}(1 + k_{33}^2)}. \tag{3.2-13}$$

2.4 Electromechanical Coupling Factor

Clearly,

$$W_2 < W_1. \tag{3.2-14}$$

The electromechanical coupling factor for the thickness-stretch of a ceramic plate poled in the thickness direction is

$$(k_{33}^t)^2 = \frac{W_1 - W_2}{W_1} = 1 - \frac{1}{1 + k_{33}^2} = \frac{k_{33}^2}{1 + k_{33}^2}. \tag{3.2-15}$$

For PZT-7A, from the material constants in Appendix 2,

$$(k_{33}^t)^2 = \frac{(9.50)^2}{(235 \times 8.85 \times 10^{-12})(13.1 \times 10^{10})} = 0.33, \qquad k_{33}^t = 0.58. \tag{3.2-16}$$

3. THICKNESS-SHEAR OF A QUARTZ PLATE

Consider a quartz plate of rotated Y-cut. The major surfaces of the plate are electroded. A voltage V is applied across the plate thickness. Two cases of mechanical boundary conditions will be considered.

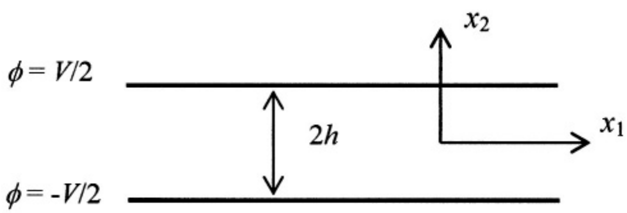

Figure 3.3-1. A quartz plate under a voltage *V*.

3.1 Boundary-Value Problem

The boundary-value problem is:

$$\begin{aligned}
&T_{ji,j} = 0, \quad D_{i,i} = 0, \quad \text{in} \quad V, \\
&T_{ij} = c_{ijkl} S_{kl} - e_{kij} E_k, \quad D_i = e_{ikl} S_{kl} + \varepsilon_{ik} E_k, \quad \text{in} \quad V, \\
&S_{ij} = (u_{i,j} + u_{j,i})/2, \quad E_i = -\phi_{,i}, \quad \text{in} \quad V, \\
&\phi = \pm V/2, \quad x_2 = \pm h, \\
&T_{ji} n_j = 0, \quad x_2 = \pm h, \quad \text{if the surfaces are traction-free,} \\
&\text{or} \quad u_j = 0, \quad x_2 = \pm h, \quad \text{if the surfaces are clamped (fixed).}
\end{aligned} \tag{3.3-1}$$

Consider the possibility of the following displacement and potential fields:

$$u_1 = u_1(x_2), \quad u_2 = u_3 = 0, \quad \phi = \phi(x_2). \tag{3.3-2}$$

The nontrivial components of the strain and the electric field are

$$2S_{12} = u_{1,2}, \quad E_2 = -\phi_{,2}. \tag{3.3-3}$$

From the equations in Section 9 of Chapter 2, the electric field, the nontrivial components of stress and electric displacement are

$$\begin{aligned} T_{31} &= c_{56}u_{1,2} + e_{25}\phi_{,2}, \quad T_{12} = c_{66}u_{1,2} + e_{26}\phi_{,2}, \\ D_2 &= e_{26}u_{1,2} - \varepsilon_{22}\phi_{,2}, \quad D_3 = e_{36}u_{1,2} - \varepsilon_{23}\phi_{,2}. \end{aligned} \tag{3.3-4}$$

The equation of motion and the charge equation require that

$$\begin{aligned} T_{21,2} &= c_{66}u_{1,22} + e_{26}\phi_{,22} = 0, \\ D_{2,2} &= e_{26}u_{1,22} - \varepsilon_{22}\phi_{,22} = 0. \end{aligned} \tag{3.3-5}$$

Hence

$$u_{1,22} = 0, \quad \phi_{,22} = 0, \tag{3.3-6}$$

unless

$$k_{26}^2 = \frac{e_{26}^2}{\varepsilon_{22}c_{66}} = 1, \tag{3.3-7}$$

which we do not consider because usually $k_{26}^2 < 1$. Equation (3.3-6) implies that all the strain, stress, electric field, and electric displacement components are constants. In particular,

$$E_2 = -\frac{V}{2h}. \tag{3.3-8}$$

3.2 Free Surfaces

We have

$$T_{12} = c_{66}u_{1,2} + e_{26}\phi_{,2} = 0. \tag{3.3-9}$$

Then

$$S_{12} = u_{1,2} = -\frac{e_{26}}{c_{66}}\frac{V}{2h},$$

$$T_{31} = -c_{56}\frac{e_{26}}{c_{66}}\frac{V}{2h} + e_{25}\frac{V}{2h} = (e_{25} - e_{26}\frac{c_{56}}{c_{66}})\frac{V}{2h},$$

$$D_2 = -e_{26}\frac{e_{26}}{c_{66}}\frac{V}{2h} - \varepsilon_{22}\frac{V}{2h} = -(1+k_{26}^2)\varepsilon_{22}\frac{V}{2h}, \tag{3.3-10}$$

$$D_3 = -e_{36}\frac{e_{26}}{c_{66}}\frac{V}{2h} - \varepsilon_{23}\frac{V}{2h} = -(\varepsilon_{23} + \frac{e_{36}e_{26}}{c_{66}})\frac{V}{2h}.$$

The free charge per unit area on the electrode at $x_2 = h$ is

$$\sigma_e = -D_2 = (1+k_{26}^2)\varepsilon_{22}\frac{V}{2h}. \tag{3.3-11}$$

Hence the capacitance of the plate per unit area is

$$\frac{\sigma_e}{V} = (1+k_{26}^2)\frac{\varepsilon_{22}}{2h}. \tag{3.3-12}$$

Equation (3.12) shows that the effect of piezoelectric coupling enhances the capacitance by a portion of k_{26}^2. The electrical energy stored in the plate capacitor per unit area is

$$W_1 = \frac{1}{2}\sigma_e V = \frac{1}{2}(1+k_{26}^2)\varepsilon_{22}\frac{V^2}{2h}. \tag{3.3-13}$$

3.3 Clamped Surfaces

In this case, since the strain S_{12} is a constant, u_1 must be a linear function of x_3. The displacement boundary conditions require this linear function to vanish at two points. Hence

$$u_3 = 0, \tag{3.3-14}$$

which implies that

$$S_{12} = u_{1,2} = 0, \quad T_{31} = e_{25}\frac{V}{2h}, \quad T_{12} = e_{26}\frac{V}{2h},$$

$$D_2 = -\varepsilon_{22}\frac{V}{2h}, \quad D_3 = -\varepsilon_{23}\frac{V}{2h}. \tag{3.3-15}$$

The free charge per unit area on the electrode at $x_2 = h$ is

$$\sigma_e = -D_2 = \varepsilon_{22}\frac{V}{2h}. \tag{3.3-16}$$

Hence the static capacitance of the plate per unit area is

$$\frac{\sigma_e}{V} = \frac{\varepsilon_{22}}{2h}. \tag{3.3-17}$$

The electric energy stored in the capacitor per unit area is

$$W_2 = \frac{1}{2}\sigma_e V = \frac{1}{2}\varepsilon_{22}\frac{V^2}{2h}. \tag{3.3-18}$$

3.4 Electromechanical Coupling Factor

The electromechanical coupling factor for a rotated Y-cut quartz plate in thickness-shear is then

$$\bar{k}_{26}^2 = \frac{W_1 - W_2}{W_1} = \frac{k_{26}^2}{1 + k_{26}^2}. \tag{3.3-19}$$

A rotated Y-cut of $\theta = 32.5^\circ$ is called an AT-cut and is widely used in devices. From the material constants in Appendix 2,

$$k_{26}^2 = \frac{0.095^2}{(39.8\times 10^{-12})(29.0\times 10^{9})} = 0.0078,$$
$$k_{26} = 0.088, \tag{3.3-20}$$

which is much smaller than that of polarized ceramics. Quartz is often used for signal processing in telecommunication or sensing rather than for power handling. Therefore a small electromechanical coupling coefficient is usually sufficient.

Problems

3.3-1. Study the thickness-stretch deformation of a ceramic plate with thickness poling due to a voltage across the plate thickness.

3.3-2. Study the thickness-shear deformation of a quartz plate due to tangential surface traction.

3.3-3. Study the thickness-shear deformation of a ceramic plate with in-plane poling under a voltage or tangential surface traction.

4. TORSION OF A CERAMIC CIRCULAR CYLINDER

Consider a circular cylinder of length L, inner radius a and outer radius b. The cylinder is made of ceramics with tangential poling. We choose (r,θ,z) to correspond to (2,3,1) so that the poling direction corresponds to 3. The lateral cylindrical surfaces are traction-free and are unelectroded. The end faces are electroded. The end electrodes can be either open or shorted. A torque M is applied.

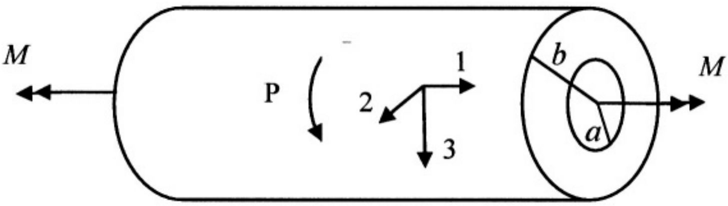

Figure 3.4-1. A circular cylinder in torsion.

4.1 Boundary-Value Problem

The boundary-value problem is:

$$T_{ji,j} = 0, \quad D_{i,i} = 0, \quad \text{in} \quad V,$$
$$T_{ij} = c_{ijkl}S_{kl} - e_{kij}E_k, \quad D_i = e_{ikl}S_{kl} + \varepsilon_{ik}E_k, \quad \text{in} \quad V,$$
$$S_{ij} = (u_{i,j} + u_{j,i})/2, \quad E_i = -\phi_{,i}, \quad \text{in} \quad V,$$
$$T_{ji}n_j = 0, \quad D_i n_i = 0, \quad r = a, b,$$
$$T_{zr} = 0, \quad T_{zz} = 0, \quad z = 0, L, \qquad (3.4\text{-}1)$$
$$\int_{a<r<b} (T_{z\theta}dA)r = M, \quad z = 0, L,$$
$$\phi = \text{constant}, \quad z = 0, L,$$
$$\phi(z=0) = \phi(z=L), \quad \text{if the electrodes are shorted,}$$
$$\int_{a<r<b} D_z dA = 0, \quad z = 0, L, \quad \text{if the electrodes are open.}$$

Consider the possibility of the following displacement and potential fields:

$$u_\theta = Arz + C, \quad u_r = u_z = 0, \quad \phi = -Bz, \qquad (3.4\text{-}2)$$

where A, C and B are undetermined constants. C represents a rigid body displacement which is taken to be zero. The nontrivial components of strain, electric field, stress, and electric displacement are

$$S_5 = 2S_{\theta z} = Ar, \quad E_1 = E_z = B, \qquad (3.4\text{-}3)$$

$$T_5 = T_{\theta z} = c_{44}Ar - e_{15}B, \quad D_1 = D_z = e_{15}Ar + \varepsilon_{11}B, \qquad (3.4\text{-}4)$$

thus the boundary conditions on the lateral surfaces are satisfied. The equation of motion and the charge equation are trivially satisfied. At the end faces $T_{zz} = T_{zr} = 0$, and the ϕ given in Equation (3.4-2) is a constant for fixed z. The torque at a cross-section is given by

$$\begin{aligned} \int_a^b T_{z\theta}(2\pi r dr) r &= \int_a^b (c_{44}Ar - e_{15}B)2\pi r^2 dr \\ &= \int_a^b (c_{44}A2\pi r^3 - e_{15}B2\pi r^2)dr \\ &= c_{44}AI_p - e_{15}B\frac{2\pi}{3}(b^3 - a^3) = M, \end{aligned} \tag{3.4-5}$$

where

$$I_p = \frac{\pi}{2}(b^4 - a^4) \tag{3.4-6}$$

is the polar moment of inertia of the cross-section about its center. The total charge on the electrode at $z = 0$ is represented by

$$\begin{aligned} Q_e &= \int_a^b D_z 2\pi r dr = \int_a^b (e_{15}Ar + \varepsilon_{11}B)2\pi r dr \\ &= \int_a^b (e_{15}A2\pi r^2 + \varepsilon_{11}B2\pi r)dr \\ &= e_{15}A\frac{2\pi}{3}(b^3 - a^3) + \varepsilon_{11}B\pi(b^2 - a^2). \end{aligned} \tag{3.4-7}$$

4.2 Shorted Electrodes

If the two end electrodes are shorted, we have $B = 0$ and

$$A = \frac{M}{c_{44}I_p}, \tag{3.4-8}$$

which is the same as the elasticity solution. There is no electric field in the cylinder. However, D_z does exist so the solution is not purely elastic.

4.3 Open Electrodes

If the end electrodes are open, we have $Q_e = 0$. From (3.4-5) and (3.4-7) we obtain

$$A = \frac{M}{c_{44}\left[I_p + k_{15}^2\left(\frac{2\pi}{3}\right)^2 \frac{(b^3 - a^3)^2}{\pi(b^2 - a^2)}\right]}. \tag{3.4-9}$$

The denominator of the right-hand side of Equation (3.4-9) represents a piezoelectrically stiffened torsional rigidity.

5. TANGENTIAL THICKNESS-SHEAR OF A CERAMIC CIRCULAR CYLINDER

Consider an infinite circular cylinder of inner radius a and out radius b. The cylinder is made of ceramics with tangential poling. We choose (r,θ,z) to correspond to (2,3,1) so that the poling direction corresponds to 3. The lateral cylindrical surfaces are unelectroded. $r = a$ is fixed. $r = b$ is under a shear stress τ.

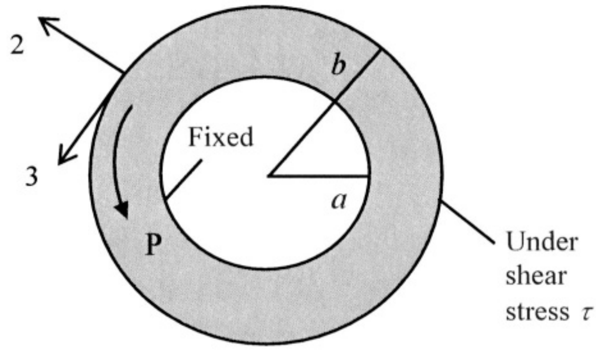

Figure 3.5-1. A circular cylinder with tangential poling.

The boundary-value problem is:

$$
\begin{aligned}
&T_{ji,j} = 0, \quad D_{i,i} = 0, \quad \text{in} \quad V,\\
&T_{ij} = c_{ijkl}S_{kl} - e_{kij}E_k, \quad D_i = e_{ikl}S_{kl} + \varepsilon_{ik}E_k, \quad \text{in} \quad V,\\
&S_{ij} = (u_{i,j} + u_{j,i})/2, \quad E_i = -\phi_{,i}, \quad \text{in} \quad V,\\
&u_i = 0, \quad r = a,\\
&T_{rr} = 0, \quad T_{r\theta} = \tau, \quad T_{rz} = 0, \quad r = b,\\
&D_r = 0, \quad r = a, b.
\end{aligned}
\tag{3.5-1}
$$

Consider the possibility of the following displacement and potential fields:

$$u_\theta = u_\theta(r), \quad u_r = u_z = 0, \quad \phi = \phi(r). \tag{3.5-2}$$

The nontrivial components of strain, electric field, stress, and electric displacement are

$$S_4 = 2S_{r\theta} = \frac{du_\theta}{dr} - \frac{u_\theta}{r}, \quad E_2 = E_r = -\frac{d\phi}{dr}, \tag{3.5-3}$$

$$T_4 = T_{r\theta} = c_{44}\left(\frac{du_\theta}{dr} - \frac{u_\theta}{r}\right) + e_{15}\frac{d\phi}{dr},$$
$$D_2 = D_r = e_{15}\left(\frac{du_\theta}{dr} - \frac{u_\theta}{r}\right) - \varepsilon_{11}\frac{d\phi}{dr}. \tag{3.5-4}$$

The stress components T_{rr} and T_{rz} vanish everywhere and on the lateral surfaces in particular. The equation of motion and the charge equation to be satisfied are

$$\frac{dT_{r\theta}}{dr} + \frac{2}{r}T_{r\theta} = 0, \quad \frac{1}{r}(rD_r)_{,r} = 0, \tag{3.5-5}$$

which can be integrated to give

$$T_{r\theta} = \frac{C_1}{r^2}, \quad D_r = \frac{C_2}{r}, \tag{3.5-6}$$

where C_1 and C_2 are integration constants. Hence

$$c_{44}\left(\frac{du_\theta}{dr} - \frac{u_\theta}{r}\right) + e_{15}\frac{d\phi}{dr} = \frac{C_1}{r^2},$$
$$e_{15}\left(\frac{du_\theta}{dr} - \frac{u_\theta}{r}\right) - \varepsilon_{11}\frac{d\phi}{dr} = \frac{C_2}{r}. \tag{3.5-7}$$

For D_r to vanish at $r = a$ and/or b, we must have $C_2 = 0$, and hence $D_r = 0$ everywhere. To satisfy the traction boundary condition at $r = b$, we have

$$C_1 = \tau b^2. \tag{3.5-8}$$

Therefore the only nonzero stress component is

$$T_{r\theta} = \tau\frac{b^2}{r^2}. \tag{3.5-9}$$

Problem

3.5-1. Determine the displacement and potential fields from $T_{r\theta}$ and D_r.

6. ANTI-PLANE PROBLEMS OF POLARIZED CERAMICS

We consider motions satisfying $\partial_3 = 0$ in ceramics poled in the x_3 direction. Then Equations (2.8-7) through (2.8-9) split into two uncoupled sets of equations. One set is with u_1 and u_2, which is not electrically coupled. These are called plane-strain problems. We consider the other set called anti-plane problems with

$$u_1 = u_2 = 0, \quad u_3 = u_3(x_1, x_2, t),$$
$$\phi = \phi(x_1, x_2, t). \tag{3.6-1}$$

The non-vanishing strain and electric field components are

$$\begin{Bmatrix} 2S_{13} \\ 2S_{23} \end{Bmatrix} = \nabla u, \quad \begin{Bmatrix} E_1 \\ E_2 \end{Bmatrix} = -\nabla \phi, \tag{3.6-2}$$

where

$$\nabla = \mathbf{i}_1 \partial_1 + \mathbf{i}_2 \partial_2 \tag{3.6-3}$$

is the two-dimensional gradient operator. The nontrivial components of T_{ij} and D_i are

$$\begin{Bmatrix} T_{13} \\ T_{23} \end{Bmatrix} = c\nabla u + e\nabla \phi,$$
$$\begin{Bmatrix} D_1 \\ D_2 \end{Bmatrix} = e\nabla u - \varepsilon \nabla \phi, \tag{3.6-4}$$

where we have denoted

$$c = c_{44}, \quad e = e_{15}, \quad \varepsilon = \varepsilon_{11}. \tag{3.6-5}$$

The nontrivial equation of motion and the charge equation take the following form [18]:

$$c\nabla^2 u + e\nabla^2 \phi + \rho f = \rho \ddot{u},$$
$$e\nabla^2 u - \varepsilon \nabla^2 \phi = \rho_e, \tag{3.6-6}$$

where $f = f_3$, and ∇^2 is the two-dimensional Laplacian

$$\nabla^2 = \partial_1^2 + \partial_2^2. \tag{3.6-7}$$

We introduce [18]

$$\psi = \phi - \frac{e}{\varepsilon} u, \tag{3.6-8}$$

and then

$$T_{23} = \bar{c}_{44} u_{3,2} + e_{15} \psi_{,2},$$
$$T_{31} = \bar{c}_{44} u_{3,1} + e_{15} \psi_{,1},$$
$$D_1 = -\varepsilon_{11} \psi_{,1},$$
$$D_2 = -\varepsilon_{11} \psi_{,2}, \tag{3.6-9}$$

and

$$\bar{c}\nabla^2 u + \rho f - \frac{e}{\varepsilon} \rho_e = \rho \ddot{u},$$
$$-\varepsilon \nabla^2 \psi = \rho_e, \tag{3.6-10}$$

where

$$\bar{c} = c + \frac{e^2}{\varepsilon} = c(1+k^2), \quad k^2 = \frac{e^2}{\varepsilon c}. \tag{3.6-11}$$

We note that in (3.6-10), u and ψ are decoupled. For static problems (3.6-6) can also be decoupled into

$$\begin{aligned} &\bar{c}\nabla^2 u + \rho f - \frac{e}{\varepsilon}\rho_e = 0, \\ &-\bar{\varepsilon}\nabla^2\phi = \rho_e + \frac{e}{c}\rho f, \end{aligned} \tag{3.6-12}$$

where

$$\bar{\varepsilon} = \varepsilon(1+k^2). \tag{3.6-13}$$

When there are no body source terms, (3.6-12) implies that

$$\nabla^2 u = 0, \quad \nabla^2\phi = 0. \tag{3.6-14}$$

In polar coordinates the general solution to (3.6-14) periodic in θ is

$$\begin{aligned} u &= l_0 + p_0 \ln r + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta)(l_n r^n + p_n r^{-n}), \\ \phi &= g_0 + h_0 \ln r + \sum_{n=1}^{\infty} (c_n \cos n\theta + d_n \sin n\theta)(g_n r^n + h_n r^{-n}), \end{aligned} \tag{3.6-15}$$

where l_n, p_n, a_n, b_n, g_n, h_n, c_n, and d_n are undetermined constants.

7. A SURFACE DISTRIBUTION OF ELECTRIC POTENTIAL

Consider a ceramic half-space poled along the x_3 direction. The surface is traction-free and a periodic potential is applied. The solution to this problem is useful for exciting or detecting surface waves.

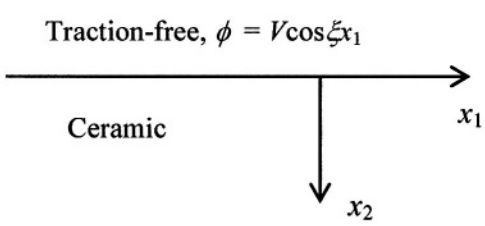

Figure 3.7-1. A ceramic half-space.

From the equations in Section 6, the boundary-value problem is:

$$\begin{aligned} &\nabla^2 u = 0, \quad \nabla^2 \phi = 0, \quad x_2 > 0, \\ &T_{23} = 0, \quad \phi = V\cos\xi x_1, \quad x_2 = 0, \\ &u, \phi \to 0, \quad x_2 \to +\infty. \end{aligned} \tag{3.7-1}$$

Consider the possibility of the following fields:

$$\begin{aligned} u &= A\exp(-\xi x_2)\cos\xi\, x_1, \\ \phi &= B\exp(-\xi x_2)\cos\xi\, x_1, \end{aligned} \tag{3.7-2}$$

which already satisfy the Laplace equations in (3.7-1). For boundary conditions we need

$$T_{23} = cu_{3,2} + e\phi_{,2} = (-cA\xi - eB\xi)\exp(-\xi x_2)\sin\xi x_1, \tag{3.7-3}$$

and the boundary conditions require

$$\begin{aligned} &c(-\xi)A\cos\xi x_1 + e(-\xi)B\cos\xi x_1 = 0, \\ &B\cos\xi\, x_1 = V\cos\xi\, x_1, \end{aligned} \tag{3.7-4}$$

which determines

$$A = -\frac{e}{c}V, \quad B = V. \tag{3.7-5}$$

Hence

$$\begin{aligned} u_3 &= -\frac{e}{c}V\exp(-\xi x_2)\cos\xi\, x_1, \\ \phi &= V\exp(-\xi x_2)\cos\xi\, x_1. \end{aligned} \tag{3.7-6}$$

8. A CIRCULAR HOLE UNDER AXI-SYMMETRIC LOADS

Consider a circular hole of radius R in an unbounded two-dimensional domain (Figure 3.8-1). The hole surface at $r = R$ is electroded with the electrode shown by the thick line in the figure. On the hole surface we apply a shear stress $T_{rz} = \tau$. We consider the case that the electrode is not connected to other objects. The surface charge density on the electrode is given to be σ_e. The problem is axi-symmetric.

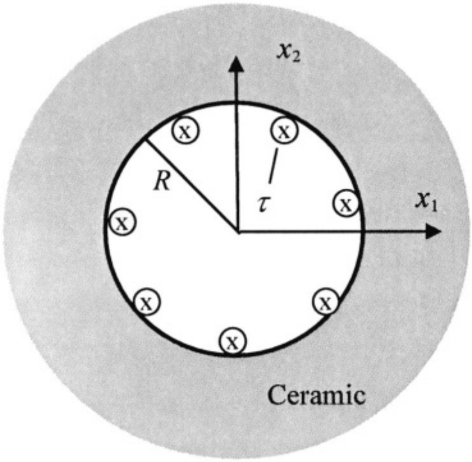

Figure 3.8-1. A circular hole under axi-symmetric loads.

From the equations in Section 6, the boundary-value problem is:

$$\begin{aligned} &\nabla^2 u = 0, \quad \nabla^2 \phi = 0, \quad r > R, \\ &T_{rz} = \tau, \quad D_r = \sigma_e, \quad r = R, \\ &T_{rz} \to 0, \quad D_r \to 0, \quad r \to \infty. \end{aligned} \tag{3.8-1}$$

In polar coordinates, for axi-symmetric problems, the Laplacian is given by

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}. \tag{3.8-2}$$

The general solution is

$$u = A_1 \ln r + A_2, \quad \phi = B_1 \ln r + B_2, \tag{3.8-3}$$

where A_1, A_2, B_1, and B_2 are undetermined constants. A_2 and B_2 represent a rigid body displacement and a constant in the electric potential and are immaterial to the problem we are considering. The stress and electric displacement are

$$T_{rz} = (cA_1 + eB_1)\frac{1}{r}, \quad D_r = (eA_1 - \varepsilon B_1)\frac{1}{r}, \tag{3.8-4}$$

which satisfy the boundary conditions at infinity. On the hole surface,

$$(cA_1 + eB_1)\frac{1}{R} = \tau, \quad (eA_1 - \varepsilon B_1)\frac{1}{R} = \sigma_e, \tag{3.8-5}$$

which determines

$$A_1 = \frac{1}{\bar{c}}\left(\tau + \frac{e}{\varepsilon}\sigma_e\right)R, \quad B_1 = \frac{1}{\bar{\varepsilon}}\left(\frac{e}{c}\tau - \sigma_e\right)R. \tag{3.8-6}$$

Hence

$$\begin{aligned} u &= \frac{1}{\bar{c}}(\tau + \frac{e}{\varepsilon}\sigma_e)R\ln\frac{r}{R} + C_1, \\ \phi &= \frac{1}{\bar{\varepsilon}}(-\sigma_e + \frac{e}{c}\tau)R\ln\frac{r}{R} + C_2, \\ T_{rz} &= \tau\frac{R}{r}, \quad D_r = \sigma_e\frac{R}{r}, \end{aligned} \tag{3.8-7}$$

where C_1 and C_2 are arbitrary constants. Now consider the limit of $R \to 0$ and at the same time $\tau \to \infty$ and $\sigma_e \to \infty$, such that

$$\tau 2\pi R \to F, \quad \sigma_e 2\pi R \to Q_e. \tag{3.8-8}$$

Then

$$\begin{aligned} u &= \frac{1}{2\pi\bar{c}}\left(F + \frac{e}{\varepsilon}Q_e\right)\ln\frac{r}{R} + C_1, \\ \phi &= \frac{1}{2\pi\bar{\varepsilon}}\left(\frac{e}{c}F - Q_e\right)\ln\frac{r}{R} + C_2, \\ T_{rz} &= \frac{F}{2\pi r}, \quad D_r = \frac{Q_e}{2\pi r}. \end{aligned} \tag{3.8-9}$$

Equations (3.8-9) represent the fields of a line force and a line charge at the origin. Mathematically they are the fundamental solution to the following problem according to (3.6-12):

$$\begin{aligned} &\bar{c}\nabla^2 u + (-F - \frac{e}{\varepsilon}Q_e)\delta(\mathbf{x}) = 0, \\ &-\bar{\varepsilon}\nabla^2\phi = (Q_e - \frac{e}{c}F)\delta(\mathbf{x}), \end{aligned} \tag{3.8-10}$$

where δ is the Dirac delta function. The solution given by (3.8-9) is unbounded when $r \to 0$. This is a typical failure of continuum mechanics in problems with a zero characteristic length. Continuum mechanics is valid only when the characteristic length in a problem is much larger than the microstructural characteristic length of matter. Equation (3.8-9) is valid sufficiently far away from the origin. At a point very close to the origin, the source can no longer be treated as a line source; therefore, its size has to be considered.

9. AXIAL THICKNESS-SHEAR OF A CIRCULAR CYLINDER

Consider a circular cylindrical shell of ceramics poled in the x_3 direction with an inner radius of R_1 and an outer radius of R_2 (see Figure 3.9-1). The inner and outer surfaces are electroded with electrodes shown by the thick lines in the figure. A voltage V is applied across the thickness. Mechanically the boundary surfaces are either traction-free or fixed. The problem is axi-symmetric.

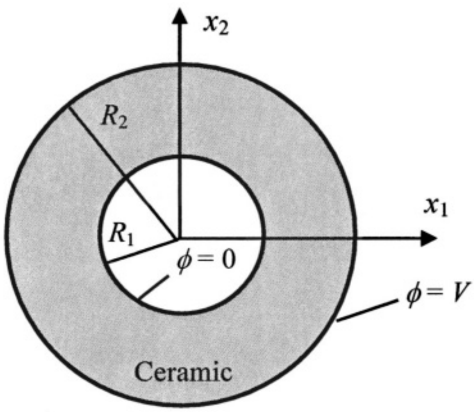

Figure 3.9-1. A. circular cylindrical ceramic shell as a capacitor.

From the equations in Section 6, the boundary-value problem is:

$$\begin{aligned}
&\nabla^2 u = 0, \quad \nabla^2\phi = 0, \quad R_1 < r < R_2, \\
&\phi = 0, V, \quad r = R_1, R_2, \\
&T_{rz} = 0, \quad r = R_1, R_2, \quad \text{if the surfaces are traction - free,} \\
&\text{or} \quad u = 0, \quad r = R_1, R_2, \quad \text{if the surfaces are fixed.}
\end{aligned} \tag{3.9-1}$$

In polar coordinates, for axi-symmetric problems, the Laplacian takes the following form:

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}. \tag{3.9-2}$$

The general solution is

$$u = A_1 \ln r + A_2, \quad \phi = B_1 \ln r + B_2, \tag{3.9-3}$$

where A_1, A_2, B_1, and B_2 are undetermined constants. The stress and electric displacements are

$$T_{rz} = (cA_1 + eB_1)\frac{1}{r}, \quad D_r = (eA_1 - \varepsilon B_1)\frac{1}{r}. \tag{3.9-4}$$

Consider traction-free surfaces

$$(cA_1 + eB_1)\frac{1}{R_1} = 0, \quad (cA_1 + eB_1)\frac{1}{R_2} = 0, \tag{3.9-5}$$

$$B_1 \ln R_1 + B_2 = 0, \quad B_1 \ln R_2 + B_2 = V,$$

which determines

$$A_1 = -\frac{e}{c}\frac{V}{\ln(R_2 / R_1)},$$

$$B_1 = \frac{V}{\ln(R_2 / R_1)}, \quad B_2 = -\frac{V}{\ln(R_2 / R_1)}\ln R_1. \tag{3.9-6}$$

Hence

$$\phi = \frac{V}{\ln(R_2 / R_1)}\ln\left(\frac{r}{R_1}\right), \quad u = -\frac{e}{c}\frac{V}{\ln R_2 / R_1}\ln\left(\frac{r}{R_1}\right) + C,$$

$$T_{rz} = 0, \quad D_r = -\bar{\varepsilon}\frac{V}{\ln(R_2 / R_1)}\frac{1}{r}, \tag{3.9-7}$$

where C is an arbitrary constant representing a rigid body displacement. The surface charge density on the electrode at $r = R_2$ is given by

$$\sigma_e = -D_r(r = R_2) = \bar{\varepsilon}\frac{V}{\ln(R_2 / R_1)}\frac{1}{R_2}. \tag{3.9-8}$$

The capacitance per unit length of the cylinder is

$$C_0 = \frac{2\pi R_2 \sigma_e}{V} = \frac{2\pi\bar{\varepsilon}}{\ln(R_2 / R_1)}. \tag{3.9-9}$$

Equation (3.9-9) shows that the effect of piezoelectric coupling on the capacitance is of the order of k^2 through $\bar{\varepsilon} = \varepsilon(1 + k^2)$.

Problem

3.9-1. Study the case when the cylindrical surfaces are fixed.

10. A CIRCULAR HOLE UNDER SHEAR

Consider a circular cylindrical hole or radius R in an unbounded ceramic poled in the x_3 direction. The hole surface is electroded and the electrode is grounded. The hole is under a uniform shear stress τ at $x_2 = \pm\infty$ (see Figure 3.10-1). Electrically $x_2 = \pm\infty$ are either open or shorted. When $x_2 = \pm\infty$ are electrically open, the problem is anti-symmetric about $x_2 = 0$.

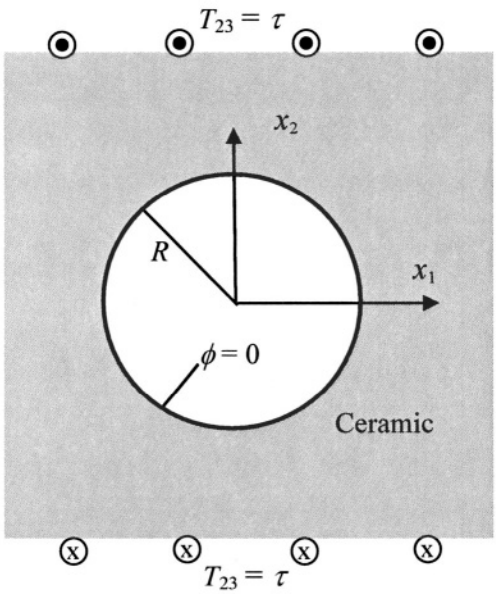

Figure 3.10-1. A circular hole under shear.

From the equations in Section 6, the boundary-value problem is:

$$\begin{aligned}
&\nabla^2 u = 0, \quad \nabla^2 \phi = 0, \quad r > R, \\
&T_{rz} = 0, \quad \phi = 0, \quad r = R, \\
&T_{23} \to \tau, \quad x_2 \to \pm\infty, \\
&D_2 \to 0, \quad x_2 \to \pm\infty, \quad \text{if } x_2 = \pm\infty \text{ are open,} \\
&\text{or} \quad E_2 \to 0, \quad x_2 \to \pm\infty, \quad \text{if } x_2 = \pm\infty \text{ are shorted.}
\end{aligned} \tag{3.10-1}$$

First we determine the fields when $|x_2|$ is very large. For mechanical fields we have

$$T_{23} = cu_{,2} + e\phi_{,2} = \tau. \tag{3.10-2}$$

We consider the case when $x_2 = \pm\infty$ are electrically open, such that for large $|x_2|$,

$$D_2 = eu_{,2} - \varepsilon\phi_{,2} = 0. \tag{3.10-3}$$

Equations (3.10-2) and (3.10-3) imply that for the far field

$$\begin{aligned}
u &= \frac{\tau}{\bar{c}} x_2 + C_1 = \frac{\tau}{\bar{c}} r\sin\theta, \\
\phi &= \frac{e}{\varepsilon}\frac{\tau}{\bar{c}} x_2 + C_2 = \frac{e}{\varepsilon}\frac{\tau}{\bar{c}} r\sin\theta,
\end{aligned} \tag{3.10-4}$$

where C_1 and C_2 are arbitrary constants and have been set to zero. In view of the far field solution, we look for solutions in the following form:

$$u(r,\theta) = u(r)\sin\theta, \quad \phi(r,\theta) = \phi(r)\sin\theta. \tag{3.10-5}$$

Substituting (3.10-5) into the Laplace equations in (3.10-1), we obtain

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2}\right)\sin\theta = 0,$$
$$\nabla^2 \phi = \left(\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} - \frac{\phi}{r^2}\right)\sin\theta = 0. \tag{3.10-6}$$

The general solution is then

$$u = (A_1 r + \frac{A_2}{r})\sin\theta, \quad \phi = (B_1 r + \frac{B_2}{r})\sin\theta, \tag{3.10-7}$$

where A_1, A_2, B_1, and B_2 are undetermined constants. For (3.10-7) to match the applied field at the far field, we must have

$$A_1 = \frac{\tau}{\bar{c}}, \quad B_1 = \frac{e\tau}{\varepsilon \bar{c}}. \tag{3.10-8}$$

The stress and electric displacement components are

$$T_{rz} = [(cA_1 + eB_1) - (cA_2 + eB_2)\frac{1}{r^2}]\sin\theta,$$
$$T_{\theta z} = [(cA_1 + eB_1) + (cA_2 + eB_2)\frac{1}{r^2}]\cos\theta,$$
$$D_r = [(eA_1 - \varepsilon B_1) - (eA_2 - \varepsilon B_2)\frac{1}{r^2}]\sin\theta,$$
$$D_\theta = [(eA_1 - \varepsilon B_1) + (eA_2 - \varepsilon B_2)\frac{1}{r^2}]\cos\theta. \tag{3.10-9}$$

At $r = R$ the boundary conditions require that

$$T_{rz}(R) = [(cA_1 + eB_1) - (cA_2 + eB_2)\frac{1}{R^2}]\sin\theta = 0,$$
$$\phi(R) = (B_1 R + \frac{B_2}{R})\sin\theta = 0, \tag{3.10-10}$$

which imply that

$$A_2 = \frac{1}{\bar{c}}(1 + 2k^2)R^2\tau, \quad B_2 = -\frac{e}{c\bar{\varepsilon}}R^2\tau. \tag{3.10-11}$$

The displacement and potential fields are

$$\phi = \frac{e\tau}{c\bar{\varepsilon}}(r - \frac{R^2}{r})\sin\theta,$$
$$u = \frac{\tau}{\bar{c}}[r + (1+2k^2)\frac{R^2}{r}]\sin\theta. \tag{3.10-12}$$

Problem

3.10-1. Study the case when $x_2 = \pm\infty$ are electrically shorted.

11. A CIRCULAR CYLINDER IN AN ELECTRIC FIELD

Consider an infinite circular cylinder of ceramics poled in the x_3 direction with radius R in a uniform electric field $\mathbf{E}^0 = E^0\mathbf{i}_1$ (Figure 3.11-1). The problem is symmetric about $x_2 = 0$ and is anti-symmetric about $x_1 = 0$.

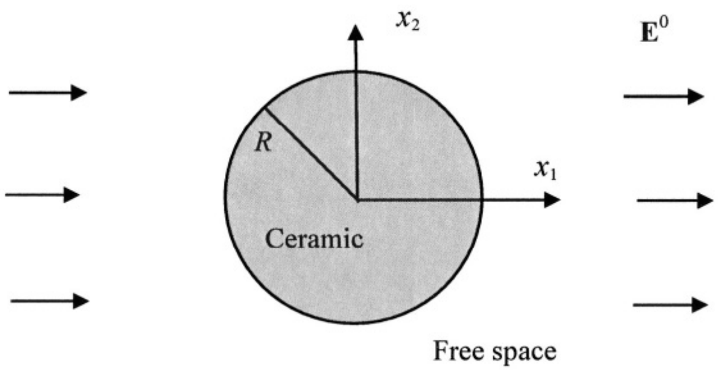

Figure 3.11-1. A circular cylinder in a uniform electric field.

From the equations in Section 6, the boundary-value problem is

$$\begin{aligned} &\nabla^2 u = 0, \quad \nabla^2\phi = 0, \quad r < R, \\ &\nabla^2\phi = 0, \quad r > R, \\ &u \text{ and } \phi \text{ are bounded}, \quad r = 0, \\ &T_{rz}(r = R) = 0, \\ &\phi(r = R^-) = \phi(r = R^+), \quad D_r(r = R^-) = D_r(r = R^+), \\ &\mathbf{E} \to \mathbf{E}^0, \quad r \to \infty. \end{aligned} \tag{3.11-1}$$

For large r, the fields are known to be

$$\phi = -E^0 x_1 = -E^0 r\cos\theta,$$
$$E_r = E_0\cos\theta, \quad E_\theta = -E_0\sin\theta. \tag{3.11-2}$$

In view of the far field solution, we look for solutions in the following form for the field in the free space:

$$\phi(r,\theta) = \phi(r)\cos\theta. \tag{3.11-3}$$

Substituting (3.11-2) into the Laplace equation gives

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}$$
$$= \left(\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} - \frac{\phi}{r^2}\right)\cos\theta = 0. \tag{3.11-4}$$

The general solution for the field in the free space is then

$$\phi = (C_1 r + C_2\frac{1}{r})\cos\theta,$$
$$E_r = -(C_1 - C_2\frac{1}{r^2})\cos\theta, \quad E_\theta = (C_1 + C_2\frac{1}{r^2})\sin\theta, \tag{3.11-5}$$

where C_1 and C_2 are undetermined constants. For the electric field in (3.11-5) to be equal to the applied field in (3.11-2) for large r, we must have

$$C_1 = -E^0. \tag{3.11-6}$$

Inside the cylinder we look for solutions in the following form:

$$u(r,\theta) = u(r)\cos\theta, \quad \phi(r,\theta) = \phi(r)\cos\theta. \tag{3.11-7}$$

Substituting (3.11-7) into the Laplace equations, we have

$$\nabla^2 u = \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2}\right)\cos\theta = 0,$$
$$\nabla^2\phi = \left(\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} - \frac{\phi}{r^2}\right)\cos\theta = 0. \tag{3.11-8}$$

The general solution is

$$u = (A_1 r + \frac{A_2}{r})\cos\theta, \quad \phi = (B_1 r + \frac{B_2}{r})\cos\theta, \tag{3.11-9}$$

where A_1, A_2, B_1, and B_2 are undetermined constants. For the boundedness of u and ϕ at the origin, we must have

$$A_2 = 0, \quad B_2 = 0, \tag{3.11-10}$$

and hence

$$u = A_1 r\cos\theta = A_1 x_1, \quad \phi = B_1 r\cos\theta = B_1 x_1. \tag{3.11-11}$$

The stress and electric displacement fields in the cylinder are

$$\begin{aligned} T_{rz} &= (cA_1 + eB_1)\cos\theta, \quad T_{\theta z} = -(cA_1 + eB_1)\sin\theta, \\ D_r &= (eA_1 - \varepsilon B_1)\cos\theta, \quad D_\theta = -(eA_1 - \varepsilon B_1)\sin\theta. \end{aligned} \tag{3.11-12}$$

We note that (3.11-11) and (3.11-12) represent uniform strain, stress, electric field, and electric displacement inside the cylinder. At $r = R$, the traction-free condition and the continuity of ϕ and D_r require that

$$\begin{aligned} &T_{rz}(r = R) = (cA_1 + eB_1)\cos\theta = 0, \\ &\phi(r = R^-) = RB_1\cos\theta \\ &\quad = (-E_0 R + \frac{C_2}{R})\cos\theta = \phi(r = R^+), \\ &D_r(r = R^-) = (eA_1 - \varepsilon B_1)\cos\theta \\ &\quad = \varepsilon_0(E_0 + \frac{C_2}{R^2})\cos\theta = D_r(r = R^+), \end{aligned} \tag{3.11-13}$$

which determines

$$C_2 = \frac{\bar{\varepsilon} - \varepsilon_0}{\bar{\varepsilon} + \varepsilon_0} E^0 R^2, \quad B_1 = -\frac{2\varepsilon_0}{\bar{\varepsilon} + \varepsilon_0} E^0, \quad A_1 = \frac{e}{c}\frac{2\varepsilon_0}{\bar{\varepsilon} + \varepsilon_0} E^0. \tag{3.11-14}$$

Then the electric field in the free space is given by

$$\begin{aligned} \phi &= (-r + \frac{\bar{\varepsilon} - \varepsilon_0}{\bar{\varepsilon} + \varepsilon_0}\frac{R^2}{r})E^0\cos\theta, \\ E_r &= (1 + \frac{\bar{\varepsilon} - \varepsilon_0}{\bar{\varepsilon} + \varepsilon_0}\frac{R^2}{r^2})E^0\cos\theta, \\ E_\theta &= (-1 + \frac{\bar{\varepsilon} - \varepsilon_0}{\bar{\varepsilon} + \varepsilon_0}\frac{R^2}{r^2})E_0\sin\theta, \end{aligned} \tag{3.11-15}$$

and the fields inside the cylinder are

$$\begin{aligned} &\phi = -\frac{2\varepsilon_0}{\bar{\varepsilon} + \varepsilon} E^0 r\cos\theta, \quad u = \frac{e}{c}\frac{2\varepsilon_0}{\bar{\varepsilon} + \varepsilon_0} E^0 r\cos\theta, \\ &T_{rz} = 0, \quad T_{\theta z} = 0, \\ &D_r = \bar{\varepsilon}\frac{2\varepsilon_0}{\bar{\varepsilon} + \varepsilon_0} E^0\cos\theta, \quad D_\theta = -\bar{\varepsilon}\frac{2\varepsilon_0}{\bar{\varepsilon} + \varepsilon_0} E^0\sin\theta. \end{aligned} \tag{3.11-16}$$

12. A SCREW DISLOCATION

Consider a screw dislocation at $\theta = \pi$ in a polar coordinate system (see Figure 3.12-1).

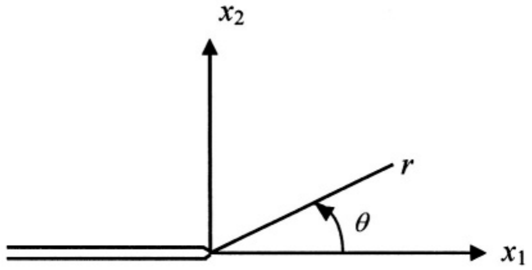

Figure 3.12-1. A screw dislocation.

From the equations in Section 6, the boundary-value problem is:

$$\begin{aligned} &\nabla^2 u = 0, \quad r > 0, \quad -\pi < \theta < \pi, \\ &\nabla^2 \phi = 0, \quad r > 0, \quad -\pi < \theta < \pi, \\ &u(r,\pi) - u(r,-\pi) = \delta, \\ &\phi(r,\pi) - \phi(r,-\pi) = V. \end{aligned} \tag{3.12-1}$$

We look for a solution in the following form:

$$u(r,\theta) = u(\theta), \quad \phi(r,\theta) = \phi(\theta). \tag{3.12-2}$$

Substitute (3.12-2) into the Laplace equations as follows:

$$\begin{aligned} &\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0, \\ &\nabla^2 \phi = \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = 0. \end{aligned} \tag{3.12-3}$$

The general solution is

$$u = A_1\theta + A_2, \quad \phi = B_1\theta + B_2, \tag{3.12-4}$$

where A_1, A_2, B_1, and B_2 are undetermined constants. From the boundary conditions in (3.12-1),

$$A_1 = \frac{\delta}{2\pi}, \quad B_1 = \frac{V}{2\pi}. \tag{3.12-5}$$

Hence

$$u = \frac{\delta}{2\pi}\theta + A_2, \quad \phi = \frac{V}{2\pi}\theta + B_2, \tag{3.12-6}$$

and

$$\begin{aligned}
&2S_{rz} = 0, \quad 2S_{\theta z} = \frac{1}{r}\frac{\delta}{2\pi}, \\
&E_r = 0, \quad E_\theta = -\frac{1}{r}\frac{V}{2\pi}, \\
&T_{rz} = cu_{,r} + e\phi_{,r} = 0, \\
&T_{\theta z} = c\frac{1}{r}u_{,\theta} + e\frac{1}{r}\phi_{,\theta} = c\frac{\delta}{2\pi r} + e\frac{V}{2\pi r}, \\
&D_r = eu_{,r} - \varepsilon\phi_{,r} = 0, \\
&D_\theta = e\frac{1}{r}u_{,\theta} - \varepsilon\frac{1}{r}\phi_{,\theta} = e\frac{\delta}{2\pi r} - \varepsilon\frac{V}{2\pi r}.
\end{aligned} \tag{3.12-7}$$

The singularity of the fields at the origin is an indication of the failure of continuum mechanics in problems with a zero characteristic length. Equation (3.12-7) is valid sufficiently far away from the origin only.

13. A CRACK

Consider a semi-infinite crack at $\theta = \pi$ in a polar coordinate system as shown in Figure 3.13-1.

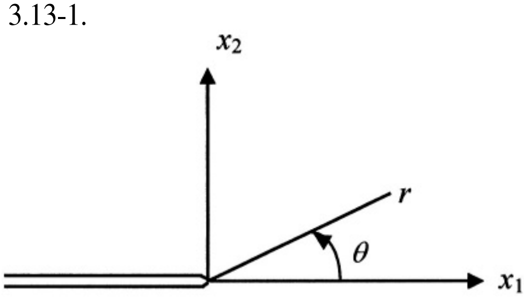

Figure 3.13-1. A semi-infinite crack.

From the equations in Section 6, the boundary-value problem is:

$$\begin{aligned}
&\nabla^2 u = 0, \quad r > 0, \quad -\pi < \theta < \pi, \\
&\nabla^2 \phi = 0, \quad r > 0, \quad -\pi < \theta < \pi, \\
&T_{\theta z}(r,-\pi) = 0, \quad T_{\theta z}(r,\pi) = 0, \\
&D_\theta(r,-\pi) = 0, \quad D_\theta(r,\pi) = 0.
\end{aligned} \tag{3.13-1}$$

Physically, the above boundary conditions indicate traction-free and unelectroded crack faces. We look for a solution in the following form:

$$u(r,\theta) = u(r)\sin\frac{\theta}{2}, \quad \phi(r,\theta) = \phi(r)\sin\frac{\theta}{2}. \tag{3.13-2}$$

Substitute (3.13-2) into the Laplace equations as follows:

$$\begin{aligned}
\nabla^2 u &= \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \\
&= \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{1}{4}\frac{u}{r^2}\right)\sin\frac{\theta}{2} = 0, \\
\nabla^2 \phi &= \left(\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} - \frac{1}{4}\frac{\phi}{r^2}\right)\sin\frac{\theta}{2} = 0.
\end{aligned} \tag{3.13-3}$$

The relevant solution is

$$u = A\sqrt{r}\sin\frac{\theta}{2}, \quad \phi = B\sqrt{r}\sin\frac{\theta}{2}, \tag{3.13-4}$$

where A and B are undetermined constants. The corresponding electromechanical fields are

$$\begin{aligned}
2S_{rz} &= \frac{A}{2\sqrt{r}}\sin\frac{\theta}{2}, \quad 2S_{\theta z} = \frac{A}{2\sqrt{r}}\cos\frac{\theta}{2}, \\
E_r &= -\frac{B}{2\sqrt{r}}\sin\frac{\theta}{2}, \quad E_\theta = -\frac{B}{2\sqrt{r}}\cos\frac{\theta}{2}, \\
T_{rz} &= \frac{cA + eB}{2\sqrt{r}}\sin\frac{\theta}{2}, \quad T_{\theta z} = \frac{cA + eB}{2\sqrt{r}}\cos\frac{\theta}{2}, \\
D_r &= \frac{eA - \varepsilon B}{2\sqrt{r}}\sin\frac{\theta}{2}, \quad D_\theta = \frac{eA - \varepsilon B}{2\sqrt{r}}\cos\frac{\theta}{2}.
\end{aligned} \tag{3.13-5}$$

The boundary conditions in Equation (3.13-1) are already satisfied and impose no more restrictions on the above fields. Note the singularity of the fields at the origin. The singularity is an indication of the failure of continuum mechanics in problems with a zero characteristic length. The solution is valid sufficiently far away from the crack tip only. As such the solution's usefulness is very limited because the behavior of a crack is mainly determined by the physics at the crack tip.

Chapter 4

VIBRATIONS OF FINITE BODIES

This chapter and Chapter 5 are on the linear dynamics of piezoelectrics. In this chapter we discuss time-harmonic vibrations of finite bodies, which are fundamental to device applications. Both free and forced vibrations are examined. Sections 1 to 5 present exact solutions from the three-dimensional equations. Section 6 provides some general results of the eigenvalue problem for the free vibration of a piezoelectric body. Sections 7 to 11 give approximate solutions of a few vibration problems that are very useful but do not allow simple, exact solutions. However, with some very accurate approximations, the problems can be solved very easily. Section 12 presents a special problem, i.e., frequency shifts of a piezoelectric body due to small amounts of mass added to its surface. This problem is particularly useful in sensor applications. It is treated by a perturbation method and a simple formula for frequency shifts is obtained.

1. THICKNESS-STRETCH VIBRATION OF A CERAMIC PLATE (THICKNESS EXCITATION)

Solutions to thickness vibrations of piezoelectric plates can be obtained in a general manner [19]. To simplify the algebra we discuss a few special cases in Sections 1 to 3. Consider a ceramic plate poled along the x_3 axis (see Figure 4.1-1). The plate is bounded by two planes at $x_3 = \pm h$ which are traction-free and electroded. A time-harmonic voltage is applied across the plate thickness.

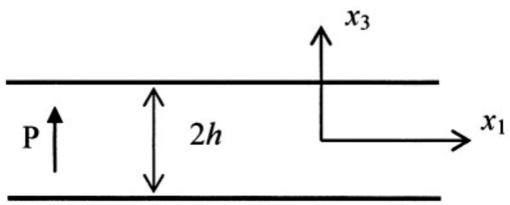

Figure 4.1-1. An electroded ceramic plate with thickness poling.

1.1 Boundary-Value Problem

The boundary-value problem is:

$$T_{ji,j} = \rho \ddot{u}_i, \quad D_{i,i} = 0, \quad \text{in} \quad V,$$
$$T_{ij} = c_{ijkl} S_{kl} - e_{kij} E_k, \quad D_i = e_{ikl} S_{kl} + \varepsilon_{ik} E_k, \quad \text{in} \quad V,$$
$$S_{ij} = (u_{i,j} + u_{j,i})/2, \quad E_i = -\phi_{,i}, \quad \text{in} \quad V, \qquad (4.1\text{-}1)$$
$$T_{3j} = 0, \quad x_3 = \pm h,$$
$$\phi(x_3 = h) - \phi(x_3 = -h) = V e^{i\omega t}.$$

Consider a possible solution in the following form:

$$u_3 = u_3(x_3) e^{i\omega t}, \quad u_1 = u_2 = 0, \quad \phi = \phi(x_3) e^{i\omega t}. \qquad (4.1\text{-}2)$$

The nontrivial components of strain and electric field are

$$S_{33} = u_{3,3}, \quad E_3 = -\phi_{,3}, \qquad (4.1\text{-}3)$$

where the time-harmonic factor has been dropped. The nontrivial stress and electric displacement components are

$$T_{11} = T_{22} = c_{13} u_{3,3} + e_{31} \phi_{,3}$$
$$T_{33} = c_{33} u_{3,3} + e_{33} \phi_{,3}, \qquad (4.1\text{-}4)$$
$$D_3 = e_{33} u_{3,3} - \varepsilon_{33} \phi_{,3}.$$

The equations to be satisfied are

$$c_{33} u_{3,33} + e_{33} \phi_{,33} = -\rho \omega^2 u_3,$$
$$e_{33} u_{3,33} - \varepsilon_{33} \phi_{,33} = 0. \qquad (4.1\text{-}5)$$

Equation $(4.1\text{-}5)_2$ can be integrated to yield

$$\phi = \frac{e_{33}}{\varepsilon_{33}} u_3 + B_1 x_3 + B_2, \qquad (4.1\text{-}6)$$

where B_1 and B_2 are integration constants, and B_2 is immaterial. Substitute Equation (4.1-6) into the expressions for T_{33}, D_3, and $(4.1\text{-}5)_1$:

$$T_{33} = \bar{c}_{33} u_{3,3} + e_{33} B_1, \quad D_3 = -\varepsilon_{33} B_1, \qquad (4.1\text{-}7)$$

$$\bar{c}_{33} u_{3,33} = -\rho \omega^2 u_3, \qquad (4.1\text{-}8)$$

where

$$\bar{c}_{33} = c_{33}(1 + k_{33}^2), \quad k_{33}^2 = \frac{e_{33}^2}{\varepsilon_{33} c_{33}}. \qquad (4.1\text{-}9)$$

The general solution to (4.1-8) and the corresponding expression for the electric potential are

$$u_3 = A_1 \sin \xi x_3 + A_2 \cos \xi x_3,$$

$$\phi = \frac{e_{33}}{\varepsilon_{33}} (A_1 \sin \xi x_3 + A_2 \cos \xi x_3) + B_1 x_3 + B_2, \tag{4.1-10}$$

where A_1 and A_2 are integration constants, and

$$\xi^2 = \frac{\rho}{\bar{c}_{33}} \omega^2. \tag{4.1-11}$$

The expression for stress is then

$$T_{33} = \bar{c}_{33} (A_1 \xi \cos \xi x_3 - A_2 \xi \sin \xi x_3) + e_{33} B_1. \tag{4.1-12}$$

The boundary conditions require that

$$\bar{c}_{33} A_1 \xi \cos \xi h - \bar{c}_{33} A_2 \xi \sin \xi h + e_{33} B_1 = 0,$$

$$\bar{c}_{33} A_1 \xi \cos \xi h + \bar{c}_{33} A_2 \xi \sin \xi h + e_{33} B_1 = 0, \tag{4.1-13}$$

$$2 \frac{e_{33}}{\varepsilon_{33}} A_1 \sin \xi h + 2 B_1 h = V,$$

or, add the first two, and subtract the first two from each other:

$$\bar{c}_{33} A_1 \xi \cos \xi h + e_{33} B_1 = 0,$$

$$\bar{c}_{33} A_2 \xi \sin \xi h = 0, \tag{4.1-14}$$

$$2 \frac{e_{33}}{\varepsilon_{33}} A_1 \sin \xi h + 2 B_1 h = V.$$

1.2 Free Vibration

Consider free vibrations with $V = 0$ first. Equation (4.1-14) decouples into two sets of equations.

1.2.1 Anti-Symmetric Modes

One set is called anti-symmetric modes for which

$$\bar{c}_{33} A_2 \xi \sin \xi h = 0. \tag{4.1-15}$$

Nontrivial solutions may exist if

$$\sin \xi h = 0, \tag{4.1-16}$$

or

$$\xi^{(n)} h = \frac{n\pi}{2}, \quad n = 0, 2, 4, 6, \cdots, \tag{4.1-17}$$

which determines the resonance frequencies

$$\omega^{(n)} = \frac{n\pi}{2h}\sqrt{\frac{\bar{c}_{33}}{\rho}}, \quad n = 0, 2, 4, 6, \cdots. \tag{4.1-18}$$

Equation (4.1-16) implies that $B_1 = 0$ and $A_1 = 0$. The corresponding modes are

$$u_3^{(n)} = \cos\xi^{(n)} x_3, \quad \phi^{(n)} = \frac{e_{33}}{\varepsilon_{33}} \cos\xi^{(n)} x_3, \tag{4.1-19}$$

where $n = 0$ is a rigid body mode.

1.2.1 Symmetric Modes

For symmetric modes

$$\begin{aligned} &\bar{c}_{33} A_1 \xi \cos\xi h + e_{33} B_1 = 0, \\ &2\frac{e_{33}}{\varepsilon_{33}} A_1 \sin\xi h + 2B_1 h = 0. \end{aligned} \tag{4.1-20}$$

The resonance frequencies are determined by

$$\begin{vmatrix} \bar{c}_{33}\xi \cos\xi h & e_{33} \\ \frac{e_{33}}{\varepsilon_{33}} \sin\xi h & h \end{vmatrix} = \bar{c}_{33}\xi h \cos\xi h - \frac{e_{33}^2}{\varepsilon_{33}} \sin\xi h = 0, \tag{4.1-21}$$

or

$$\tan\xi h = \frac{\xi h}{\bar{k}_{33}^2}, \tag{4.1-22}$$

where

$$\bar{k}_{33}^2 = \frac{e_{33}^2}{\varepsilon_{33}\bar{c}_{33}} = \frac{e_{33}^2}{\varepsilon_{33} c_{33}(1 + k_{33}^2)} = \frac{k_{33}^2}{1 + k_{33}^2} = (k_{33}^t)^2. \tag{4.1-23}$$

Equations (4.1-22) and (4.1-20) determine the resonance frequencies and modes. For symmetric modes, $A_2 = 0$.

1.3 Forced Vibration

Next consider forced vibrations. From Equation (4.1-14), $A_2 = 0$ which means that anti-symmetric modes are not excitable by a thickness electric field, and

$$A_1 = \frac{\begin{vmatrix} 0 & e_{33} \\ V & 2h \end{vmatrix}}{\begin{vmatrix} \bar{c}_{33}\xi\cos\xi h & e_{33} \\ 2\frac{e_{33}}{\varepsilon_{33}}\sin\xi h & 2h \end{vmatrix}} = \frac{-e_{33}V}{2\bar{c}_{33}\xi h\cos\xi h - 2\frac{e_{33}^2}{\varepsilon_{33}}\sin\xi h}, \tag{4.1-24}$$

$$B_1 = \frac{\begin{vmatrix} \bar{c}_{33}\xi\cos\xi h & 0 \\ 2\frac{e_{33}}{\varepsilon_{33}}\sin\xi h & V \end{vmatrix}}{\begin{vmatrix} \bar{c}_{33}\xi\cos\xi h & e_{33} \\ 2\frac{e_{33}}{\varepsilon_{33}}\sin\xi h & 2h \end{vmatrix}} = \frac{V\bar{c}_{33}\xi\cos\xi h}{2\bar{c}_{33}\xi h\cos\xi h - 2\frac{e_{33}^2}{\varepsilon_{33}}\sin\xi h}. \tag{4.1-25}$$

Hence

$$D_3 = -\varepsilon_{33}B_1 = -\varepsilon_{33}\frac{V}{2h}\frac{\xi h}{\xi h - \bar{k}_{33}^2\tan\xi h} = -\sigma_e, \tag{4.1-26}$$

where σ_e is the surface charge per unit area on the electrode at $x_3 = h$. The capacitance per unit area is

$$C = \frac{\sigma_e}{V} = \frac{\varepsilon_{33}}{2h}\frac{\xi h}{\xi h - \bar{k}_{33}^2\tan\xi h}. \tag{4.1-27}$$

We note the following limits:

$$\lim_{e_{33}\to 0} C = \frac{\varepsilon_{33}}{2h},$$
$$\lim_{\omega\to 0} C = \frac{\varepsilon_{33}}{2h}\frac{1}{1-\frac{k_{33}^2}{1+k_{33}^2}} = \frac{\varepsilon_{33}}{2h}(1+k_{33}^2) = C_0, \tag{4.1-28}$$

where C_0 is the static capacitance. The motional capacitance C_m is defined by

$$\begin{aligned} C_m = C - C_0 &= \frac{\varepsilon_{33}}{2h}\left[\frac{\xi h}{\xi h - \bar{k}_{33}^2\tan\xi h} - (1+k_{33}^2)\right] \\ &= \frac{\varepsilon_{33}}{2h}\frac{\xi h - (1+k_{33}^2)(\xi h - \bar{k}_{33}^2\tan\xi h)}{\xi h - \bar{k}_{33}^2\tan\xi h} \\ &= \frac{\varepsilon_{33}}{2h}\frac{-k_{33}^2\xi h + k_{33}^2\tan\xi h}{\xi h - \bar{k}_{33}^2\tan\xi h} = k_{33}^2\frac{\varepsilon_{33}}{2h}\frac{\tan\xi h - \xi h}{\xi h - \bar{k}_{33}^2\tan\xi h}. \end{aligned} \tag{4.1-29}$$

Note that C_m depends on electromechanical coupling.

Problem

4.1-1. Study thickness-shear vibration of a ceramic plate with in-plane poling under thickness excitation. Hint: Consider $u_1 = 0$, $u_2 = 0$, $u_3 = u_3(x_1,t)$, and $\phi = \phi(x_1,t)$.

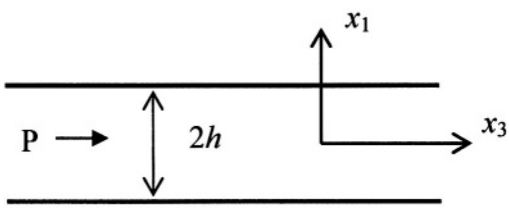

Figure 4.1-2. An electroded ceramic plate with in-plane poling.

2. THICKNESS-STRETCH VIBRATION OF A CERAMIC PLATE (LATERAL EXCITATION)

Consider a ceramic plate poled in the x_3 direction (Figure 4.2-1). The two major surfaces are traction-free and are unelectroded. A voltage is applied across $x_1 = \pm\infty$ and a uniform electric field $E_3(t) = Ee^{i\omega t}$ is produced.

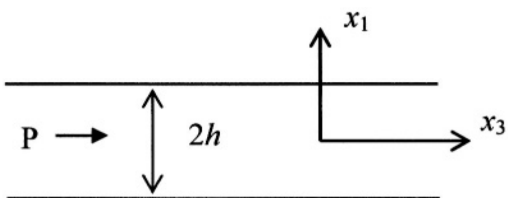

Figure 4.2-1. An unelectroded ceramic plate with in-plane poling.

2.1 Boundary-Value Problem

The boundary-value problem is:

$$
\begin{aligned}
&T_{ji,j} = \rho \ddot{u}_i, \quad D_{i,i} = 0, \quad \text{in} \quad V, \\
&T_{ij} = c_{ijkl} S_{kl} - e_{kij} E_k, \quad D_i = e_{ikl} S_{kl} + \varepsilon_{ik} E_k, \quad \text{in} \quad V, \\
&S_{ij} = (u_{i,j} + u_{j,i})/2, \quad E_i = -\phi_{,i}, \quad \text{in} \quad V, \\
&T_{1j} = 0, \quad D_1 = 0, \quad x_1 = \pm h, \\
&\phi = -x_3 E e^{i\omega t}, \quad \text{in} \quad V.
\end{aligned}
\tag{4.2-1}
$$

Consider the possibility of the following fields:

$$u_1 = u_1(x_1) e^{i\omega t}, \quad u_2 = u_3 = 0. \tag{4.2-2}$$

The nontrivial strain and electric field components are

$$S_{11} = u_{1,1}, \quad E_3 = E, \tag{4.2-3}$$

where the time-harmonic factor has been dropped. The nontrivial stress and electric displacement components are

$$
\begin{aligned}
T_{11} &= c_{11} u_{1,1} - e_{31} E, \\
T_{22} &= c_{21} u_{1,1} - e_{31} E, \\
T_{33} &= c_{31} u_{1,1} - e_{33} E, \\
D_3 &= e_{31} u_{1,1} + \varepsilon_{33} E.
\end{aligned}
\tag{4.2-4}
$$

The electrical boundary conditions and the charge equation are trivially satisfied. The equation of motion and the mechanical boundary conditions take the following form:

$$
\begin{aligned}
&c_{11} u_{1,11} = -\rho \omega^2 u_1, \quad -h < x_1 < h, \\
&c_{11} u_{1,1} - e_{31} E = 0, \quad x_1 = \pm h,
\end{aligned}
\tag{4.2-5}
$$

which shows that we effectively have an elastic plate driven by a surface traction. The general solution to $(4.2\text{-}5)_1$ is

$$u_1 = A_1 \sin \xi x_1 + A_2 \cos \xi x_1, \tag{4.2-6}$$

where A_1 and A_2 are integration constants, and

$$\xi^2 = \frac{\rho}{c_{11}} \omega^2. \tag{4.2-7}$$

Then the expression for the stress component relevant to the boundary conditions is

$$T_{11} = c_{11}(A_1 \xi \cos \xi x_2 - A_2 \xi \sin \xi x_2) - e_{31} E. \tag{4.2-8}$$

The boundary conditions require that

$$
\begin{aligned}
&c_{11}(A_1 \xi \cos \xi h - A_2 \xi \sin \xi h) - e_{31} E = 0, \\
&c_{11}(A_1 \xi \cos \xi h + A_2 \xi \sin \xi h) - e_{31} E = 0,
\end{aligned}
\tag{4.2-9}
$$

or, add and then subtract

$$c_{11}A_1\xi\cos\xi h = e_{31}E,$$
$$c_{11}A_2\xi\sin\xi h = 0. \tag{4.2-10}$$

2.2 Free Vibration

First consider free vibrations with $E = 0$. From $(4.2\text{-}10)_2$ nontrivial solutions may exist if

$$\sin\xi h = 0, \tag{4.2-11}$$

or

$$\xi^{(n)}h = \frac{n\pi}{2}, \quad n = 0, 2, 4, 6, \cdots, \tag{4.2-12}$$

which determines the following resonance frequencies

$$\omega^{(n)} = \frac{n\pi}{2h}\sqrt{\frac{c_{11}}{\rho}}, \quad n = 0, 2, 4, 6, \cdots. \tag{4.2-13}$$

Equation (4.2-11) implies that $A_1 = 0$. The corresponding modes are

$$u_1 = \cos\xi^{(n)}x_1, \tag{4.2-14}$$

which are called anti-symmetric modes, $n = 0$ represents a rigid body mode. For symmetric modes from $(4.2\text{-}10)_1$ ($E = 0$),

$$\cos\xi h = 0, \tag{4.2-15}$$

or

$$\xi^{(n)}h = \frac{n\pi}{2}, \quad n = 1, 3, 5, \cdots, \tag{4.2-16}$$

which determines the following resonance frequencies

$$\omega^{(n)} = \frac{n\pi}{2h}\sqrt{\frac{c_{11}}{\rho}}, \quad n = 1, 3, 5, \cdots. \tag{4.2-17}$$

Equation (4.2-15) implies that $A_2 = 0$. The corresponding modes are

$$u_1 = \sin\xi^{(n)}x_1. \tag{4.2-18}$$

2.3 Forced Vibration

For forced vibrations $A_2 = 0$ and from $(4.2\text{-}10)_1$,

$$A_1 = \frac{e_{31}}{c_{11}\xi h\cos\xi h}Eh. \tag{4.2-19}$$

The displacement field is

$$u_1 = \frac{e_{31}}{c_{11}\xi h \cos \xi h} E h \sin \xi x_1 e^{i\omega t} . \qquad (4.2\text{-}20)$$

Problem

4.2-1. Study the thickness-shear vibration of a ceramic plate with thickness poling under lateral excitation. Hint: Consider $u_1 = u_1(x_3, t)$, $u_2 = 0$, $u_3 = 0$, and $\phi = -x_1 E e^{i\omega t}$.

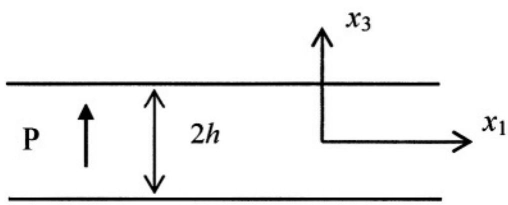

Figure 4.2-2. An unelectroded ceramic plate with thickness poling.

3. THICKNESS-SHEAR VIBRATION OF A QUARTZ PLATE (THICKNESS EXCITATION)

Consider a rotated Y-cut quartz plate. The two major surfaces are traction-free and are electroded, with a driving voltage across the thickness. This structure represents a widely used piezoelectric resonator.

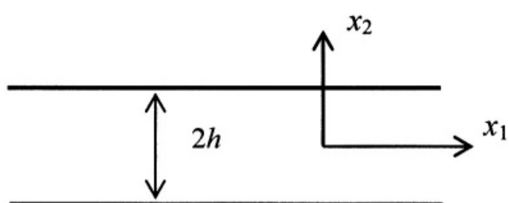

Figure 4.3-1. An electroded quartz plate.

3.1 Boundary-Value Problem

The boundary-value problem is:

$$T_{ji,j}=\rho\ddot{u}_i,\quad D_{i,i}=0,\quad \text{in}\quad V,$$
$$T_{ij}=c_{ijkl}S_{kl}-e_{kij}E_k,\quad D_i=e_{ikl}S_{kl}+\varepsilon_{ik}E_k,\quad \text{in}\quad V,$$
$$S_{ij}=(u_{i,j}+u_{j,i})/2,\quad E_i=-\phi_{,i},\quad \text{in}\quad V, \tag{4.3-1}$$
$$T_{2j}=0,\quad x_2=\pm h,$$
$$\phi(x_2=h)-\phi(x_2=-h)=Ve^{i\omega t}.$$

The problem is mathematically the same as the one in Section 1. Its solution can be obtained from that in Section 1 by changing notation. Because of the importance of this solution in applications, we solve this problem below so that this section can be used independently. Consider the possibility of the following displacement and potential fields:

$$u_1=u_1(x_2)e^{i\omega t},\quad u_2=u_3=0,\quad \phi=\phi(x_2)e^{i\omega t}. \tag{4.3-2}$$

The nontrivial components of strain, electric field, stress, and electric displacement are

$$2S_{12}=u_{1,2},\quad E_2=-\phi_{,2}, \tag{4.3-3}$$

and

$$T_{31}=c_{56}u_{1,2}+e_{25}\phi_{,2},\quad T_{12}=c_{66}u_{1,2}+e_{26}\phi_{,2},$$
$$D_2=e_{26}u_{1,2}-\varepsilon_{22}\phi_{,2},\quad D_3=e_{36}u_{1,2}-\varepsilon_{23}\phi_{,2}, \tag{4.3-4}$$

where the time-harmonic factor has been dropped. The equation of motion and the charge equation require that

$$T_{21,2}=c_{66}u_{1,22}+e_{26}\phi_{,22}=-\rho\omega^2u_1,$$
$$D_{2,2}=e_{26}u_{1,22}-\varepsilon_{22}\phi_{,22}=0. \tag{4.3-5}$$

Equation $(4.3\text{-}5)_2$ can be integrated to yield

$$\phi=\frac{e_{26}}{\varepsilon_{22}}u_1+B_1x_2+B_2, \tag{4.3-6}$$

where B_1 and B_2 are integration constants, and B_2 is immaterial. Substituting (4.3-6) into the expressions for T_{21}, D_2, and $(4.3\text{-}5)_1$ we obtain

$$T_{21}=\bar{c}_{66}u_{1,2}+e_{26}B_1,\quad D_2=-\varepsilon_{22}B_1, \tag{4.3-7}$$

$$\bar{c}_{66}u_{1,22}=-\rho\omega^2u_1, \tag{4.3-8}$$

where

$$\bar{c}_{66}=c_{66}(1+k_{26}^2),\quad k_{26}^2=\frac{e_{26}^2}{\varepsilon_{22}c_{66}}. \tag{4.3-9}$$

The general solution to (4.3-8) and the corresponding expression for the electric potential are

$$u_1 = A_1 \sin \xi x_2 + A_2 \cos \xi x_2,$$
$$\phi = \frac{e_{26}}{\varepsilon_{22}} (A_1 \sin \xi x_2 + A_2 \cos \xi x_2) + B_1 x_2 + B_2, \tag{4.3-10}$$

where A_1 and A_2 are integration constants, and

$$\xi^2 = \frac{\rho}{\bar{c}_{66}} \omega^2. \tag{4.3-11}$$

Then the expression for the stress component relevant to boundary conditions is

$$T_{21} = \bar{c}_{66} (A_1 \xi \cos \xi x_2 - A_2 \xi \sin \xi x_2) + e_{26} B_1. \tag{4.3-12}$$

The boundary conditions require that

$$\bar{c}_{66} A_1 \xi \cos \xi h - \bar{c}_{66} A_2 \xi \sin \xi h + e_{26} B_1 = 0,$$
$$\bar{c}_{66} A_1 \xi \cos \xi h + \bar{c}_{66} A_2 \xi \sin \xi h + e_{26} B_1 = 0, \tag{4.3-13}$$
$$2 \frac{e_{26}}{\varepsilon_{22}} A_1 \sin \xi h + 2 B_1 h = V,$$

or, add the first two, and subtract the first two from each other:

$$\bar{c}_{66} A_1 \xi \cos \xi h + e_{26} B_1 = 0,$$
$$\bar{c}_{66} A_2 \xi \sin \xi h = 0, \tag{4.3-14}$$
$$2 \frac{e_{26}}{\varepsilon_{22}} A_1 \sin \xi h + 2 B_1 h = V.$$

3.2 Free Vibration

First we consider free vibrations with $V = 0$. Equation (4.3-14) decouples into two sets of equations. For symmetric modes,

$$\bar{c}_{66} A_2 \xi \sin \xi h = 0. \tag{4.3-15}$$

Nontrivial solutions may exist if

$$\sin \xi h = 0, \tag{4.3-16}$$

or

$$\xi^{(n)} h = \frac{n\pi}{2}, \quad n = 0, 2, 4, 6, \cdots, \tag{4.3-17}$$

which determines the following resonance frequencies

$$\omega^{(n)} = \frac{n\pi}{2h} \sqrt{\frac{\bar{c}_{66}}{\rho}}, \quad n = 0, 2, 4, 6, \cdots. \tag{4.3-18}$$

Equation (4.3-16) implies that $B_1 = 0$ and $A_1 = 0$. The corresponding modes are

$$u_1 = \cos\xi^{(n)}x_2, \quad \phi = \frac{e_{26}}{\varepsilon_{22}}\cos\xi^{(n)}x_2, \tag{4.3-19}$$

where $n = 0$ represents a rigid body mode. For anti-symmetric modes,

$$\begin{aligned} &\bar{c}_{66}A_1\xi\cos\xi h + e_{26}B_1 = 0, \\ &2\frac{e_{26}}{\varepsilon_{22}}A_1\sin\xi h + 2B_1h = 0. \end{aligned} \tag{4.3-20}$$

The resonance frequencies are determined by

$$\begin{vmatrix} \bar{c}_{66}\xi\cos\xi h & e_{26} \\ \frac{e_{26}}{\varepsilon_{22}}\sin\xi h & h \end{vmatrix} = \bar{c}_{66}\xi h\cos\xi h - \frac{e_{26}^2}{\varepsilon_{22}}\sin\xi h = 0, \tag{4.3-21}$$

or

$$\tan\xi h = \frac{\xi h}{\bar{k}_{26}^2}, \tag{4.3-22}$$

where

$$\bar{k}_{26}^2 = \frac{e_{26}^2}{\varepsilon_{22}\bar{c}_{66}} = \frac{e_{26}^2}{\varepsilon_{22}c_{66}(1+k_{26}^2)} = \frac{k_{26}^2}{1+k_{26}^2}. \tag{4.3-23}$$

Equations (4.3-22) and (4.3-20) determine the resonance frequencies and modes. If the small piezoelectric coupling for quartz is neglected in (4.3-22), a set of frequencies similar to (4.3-17) with n equals odd numbers can be determined for a set of modes with sine dependence on the thickness coordinate. Static thickness-shear deformation and the first few thickness-shear modes in a plate are shown in Figure 4.3-2.

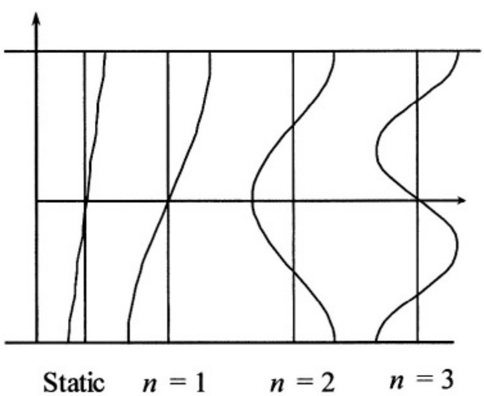

Figure 4.3-2. Thickness-shear deformation and modes in a plate.

3.3 Forced Vibration

For forced vibration we have $A_2 = 0$ and

$$A_1 = \frac{\begin{vmatrix} 0 & e_{26} \\ V & 2h \end{vmatrix}}{\begin{vmatrix} \bar{c}_{66}\xi\cos\xi h & e_{26} \\ 2\frac{e_{26}}{\varepsilon_{22}}\sin\xi h & 2h \end{vmatrix}} = \frac{-e_{26}V}{2\bar{c}_{66}\xi h\cos\xi h - 2\frac{e_{26}^2}{\varepsilon_{22}}\sin\xi h}, \quad (4.3\text{-}24)$$

$$B_1 = \frac{\begin{vmatrix} \bar{c}_{66}\xi\cos\xi h & 0 \\ 2\frac{e_{26}}{\varepsilon_{22}}\sin\xi h & V \end{vmatrix}}{\begin{vmatrix} \bar{c}_{66}\xi\cos\xi h & e_{26} \\ 2\frac{e_{26}}{\varepsilon_{22}}\sin\xi h & 2h \end{vmatrix}} = \frac{V\bar{c}_{66}\xi\cos\xi h}{2\bar{c}_{66}\xi h\cos\xi h - 2\frac{e_{26}^2}{\varepsilon_{22}}\sin\xi h}. \quad (4.3\text{-}25)$$

Hence

$$D_2 = -\varepsilon_{22}B_1 = -\varepsilon_{22}\frac{V}{2h}\frac{\xi h}{\xi h - \bar{k}_{26}^2\tan\xi h} = -\sigma_e, \quad (4.3\text{-}26)$$

where σ_e is the surface charge per unit area on the electrode at $x_2 = h$. The capacitance per unit area is

$$C = \frac{\sigma_e}{V} = \frac{\varepsilon_{22}}{2h}\frac{\xi h}{\xi h - \bar{k}_{26}^2\tan\xi h}. \quad (4.3\text{-}27)$$

We note the following limits:

$$\lim_{e_{26}\to 0} C = \frac{\varepsilon_{22}}{2h},$$

$$\lim_{\omega\to 0} C = \frac{\varepsilon_{22}}{2h}\frac{1}{1-\frac{k_{26}^2}{1+k_{26}^2}} = \frac{\varepsilon_{22}}{2h}(1+k_{26}^2). \quad (4.3\text{-}28)$$

3.4 Mechanical Effects of Electrodes

In certain applications, e.g., piezoelectric resonators, the electrodes cannot be treated as a constraint on the electric potential only, and its mechanical effects need to be considered. This may include the inertial effect of the electrode mass and the stiffness of the electrode. Consider a quartz plate with electrodes of unequal thickness on its two major faces as shown in Figure 4.3-3 [20].

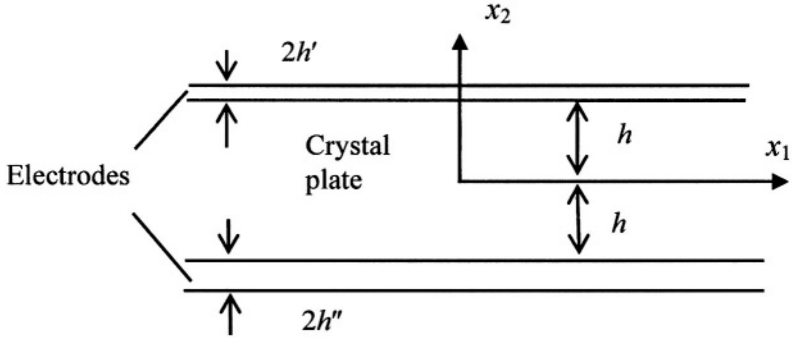

Figure 4.3-3. A quartz plate with electrodes of different thickness.

We are interested in free vibration frequencies. The governing equations are

$$\begin{aligned}
&T_{ji,j} = -\rho\omega^2 u_i, \quad D_{i,i} = 0, \quad -h < x_2 < h, \\
&T_{ij} = c_{ijkl}S_{kl} - e_{kij}E_k, \quad D_i = e_{ikl}S_{kl} + \varepsilon_{ik}E_k, \quad -h < x_2 < h, \\
&S_{ij} = (u_{i,j} + u_{j,i})/2, \quad E_i = -\phi_{,i}, \quad -h < x_2 < h, \\
&T_{ji,j} = -\rho'\omega^2 u_i, \quad -h - h'' < x_2 < -h, \quad h < x_2 < h + h', \\
&T_{ij} = c'_{ijkl}S_{kl}, \quad S_{ij} = (u_{i,j} + u_{j,i})/2, \\
&\quad -h - h'' < x_2 < -h, \quad h < x_2 < h + h',
\end{aligned} \tag{4.3-29}$$

where ρ' and c'_{ijkl} are the mass density and the elastic constants of the electrodes. The two electrodes are of the same isotropic material. The outer surfaces of the electrodes are traction-free. The electrodes are shorted. We have the following boundary and continuity conditions:

$$\begin{aligned}
&T_{2j} = 0, \quad x_2 = h + 2h', \quad x_2 = -h - 2h'', \\
&u_j(x_2 = h^-) = u_j(x_2 = h^+), \\
&T_{2j}(x_2 = h^-) = T_{2j}(x_2 = h^+), \\
&u_j(x_2 = -h^+) = u_j(x_2 = -h^-), \\
&T_{2j}(x_2 = -h^+) = T_{2j}(x_2 = -h^-), \\
&\phi(x_2 = h) = \phi(x_2 = -h).
\end{aligned} \tag{4.3-30}$$

Fields inside the plate are still given by (4.3-10), $(4.3\text{-}7)_2$, and (4.3-12).

For fields inside the electrodes, consider the upper electrode first:

$$T_{21,2} = c'_{66} u_{1,22} = -\rho' \omega^2 u_1 \,, \tag{4.3-31}$$

$$u_1 = A'_1 \sin \xi'(x_2 - h) + A'_2 \cos \xi'(x_2 - h), \tag{4.3-32}$$

$$T_{21} = c'_{66} [A'_1 \xi' \cos \xi'(x_2 - h) - A'_2 \xi' \sin \xi'(x_2 - h)], \tag{4.3-33}$$

where A'_1 and A'_2 are integration constants, and

$$(\xi')^2 = \frac{\rho'}{c'_{66}} \omega^2 . \tag{4.3-34}$$

Similarly, for the lower electrode we have

$$u_1 = A''_1 \sin \xi'(x_2 + h) + A''_2 \cos \xi'(x_2 + h) \tag{4.3-35}$$

$$T_{21} = c'_{66} [A''_1 \xi' \cos \xi'(x_2 + h) - A''_2 \xi' \sin \xi'(x_2 + h)], \tag{4.3-36}$$

where A''_1 and A''_2 are integration constants.

Substituting (4.3-10), $(4.3\text{-}7)_2$, (4.3-12), (4.3-32), (4.3-33), (4.3-35), and (4.3-36) into (4.3-30), we obtain

$$\begin{aligned}
& A_1 \sin \xi h + A_2 \cos \xi h = A'_2, \\
& -A_1 \sin \xi h + A_2 \cos \xi h = A''_2, \\
& \bar{c}_{66} \xi (A_1 \cos \xi h - A_2 \sin \xi h) + e_{26} B_1 = c'_{66} \xi' A'_1, \\
& \bar{c}_{66} \xi (A_1 \cos \xi h + A_2 \sin \xi) h + e_{26} B_1 = c'_{66} \xi' A''_1, \\
& A'_1 \cos \xi' 2h' - A'_2 \sin \xi 2h' = 0, \\
& A''_1 \cos \xi 2h'' + A''_2 \sin \xi 2h'' = 0, \\
& \frac{e_{26}}{\varepsilon_{22}} A_1 \sin \xi h + B_1 h = 0.
\end{aligned} \tag{4.3-37}$$

For nontrivial solutions of the undetermined constants, the determinant of the coefficient matrix of (4.3-37) has to vanish. This results in the following frequency equation:

$$\begin{aligned}
& \left(1 - \bar{k}_{26}^2 \frac{\tan \xi h}{\xi h}\right) \left[2 \tan \xi h + \sqrt{\frac{\rho' c'_{66}}{\rho \bar{c}_{66}}} \left(\tan \xi' 2h' + \tan \xi' 2h''\right) \right] \\
& = \sqrt{\frac{\rho' c'_{66}}{\rho \bar{c}_{66}}} \tan \xi h \left[\tan \xi h \left(\tan \xi' 2h' + \tan \xi' 2h''\right) \right. \\
& \qquad \left. + 2 \sqrt{\frac{\rho' c'_{66}}{\rho \bar{c}_{66}}} \tan \xi' 2h' \tan \xi' 2h'' \right].
\end{aligned} \tag{4.3-38}$$

We make the following observations from (4.3-38).

(i) In the limit of $h' \to 0$ and $h'' \to 0$, i.e., the mechanical effects of the electrodes are neglected, (4.3-38) reduces to

$$\left(1-\bar{k}_{26}^2\frac{\tan\xi h}{\xi h}\right)\tan\xi h = 0\,, \tag{4.3-39}$$

which is the frequency equation of both symmetric and anti-symmetric modes given in (4.3-16) and (4.3-22).

(ii) When $h' = h''$, i.e., the electrodes are of the same thickness, (4.3-38) reduces to

$$\begin{gathered}\left(1-\bar{k}_{26}^2\frac{\tan\xi h}{\xi h}-\sqrt{\frac{\rho' c'_{66}}{\rho\bar{c}_{66}}}\tan\xi h\tan\xi' 2h'\right)\\ \times\left(\tan\xi h+\sqrt{\frac{\rho' c'_{66}}{\rho\bar{c}_{66}}}\tan\xi' 2h'\right)=0.\end{gathered} \tag{4.3-40}$$

The first factor of (4.3-40) is the frequency equation for the anti-symmetric modes given in [21]. The second factor is for symmetric modes. For small h', i.e., very thin electrodes, we approximately have

$$\tan\xi' 2h' \cong \xi' 2h', \quad \sqrt{\frac{\rho' c'_{66}}{\rho\bar{c}_{66}}}\tan\xi' 2h' \cong R\xi h, \quad R=\frac{\rho' 2h'}{\rho h}. \tag{4.3-41}$$

In this case the first factor of (4.3-40) reduces to

$$\tan\xi h = \frac{\xi h}{\bar{k}_{26}^2 + R(\xi h)^2}\,, \tag{4.3-42}$$

which is the result given in [22]. Note that in Equation (4.3-42) the shear stiffness of the electrodes (c'_{66}) has disappeared. Only the mass effect of the electrodes is left and is represented by the mass ratio R.

(iii) For small h' and h'', i.e., thin and unequal electrodes, Equation (4.3-38) reduces to

$$\begin{gathered}\left(1-\bar{k}_{26}^2\frac{\tan\xi h}{\xi h}\right)\left[2\tan\xi h+(R'+R'')\xi h\right]\\ =\xi h\tan\xi h\left[(R'+R'')\tan\xi h+2R'R''\xi h\right],\end{gathered} \tag{4.3-43}$$

where we have denoted

$$R'=\frac{\rho' 2h'}{\rho h}, \quad R''=\frac{\rho' 2h''}{\rho h}. \tag{4.3-44}$$

To the lowest (first) order of the mass effect, the $R'R''$ term on the right-hand side of Equation (4.3-44) can be dropped.

Problem

4.3-1. When the electrodes are very thin, only the inertial effect of the electrode mass needs to be considered; its stiffness can be neglected. The boundary condition on an electroded surface is, according to Newton's 2nd law

$$-T_{ji}n_j = \rho'h'\ddot{u}_i = -\rho'h'\omega^2 u_i \,. \tag{4.3-45}$$

Use Equation (4.3-45) to study the anti-symmetric thickness-shear vibration of a quartz plate with electrodes of equal thickness and derive Equation (4.3-42).

4. TANGENTIAL THICKNESS-SHEAR VIBRATION OF A CIRCULAR CYLINDER

Consider an infinite circular cylinder of inner radius a and outer radius b. The cylinder is made of ceramics with tangential poling. We choose (r,θ,z) to correspond to (2,3,1) so that the poling direction corresponds to 3. The inner and outer surfaces are electroded. There is no load applied, and we are interested in free vibrations independent of θ.

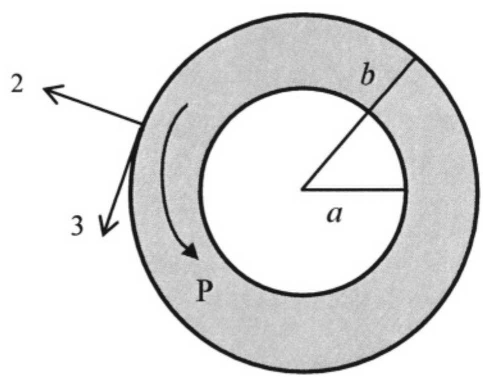

Figure 4.4-1. A circular cylinder with tangential poling.

The boundary-value problem is:

$$
\begin{aligned}
&T_{ji,j} = \rho\ddot{u}_i, \quad D_{i,i} = 0, \quad \text{in} \quad V, \\
&T_{ij} = c_{ijkl}S_{kl} - e_{kij}E_k, \quad D_i = e_{ikl}S_{kl} + \varepsilon_{ik}E_k, \quad \text{in} \quad V, \\
&S_{ij} = (u_{i,j} + u_{j,i})/2, \quad E_i = -\phi_{,i}, \quad \text{in} \quad V, \\
&T_{ji}n_j = 0, \quad r = a, b, \\
&\phi(r = a) = \phi(r = b), \quad \text{if the electrodes are shorted,} \\
&\text{or} \quad D_r = 0, \quad r = a, b, \quad \text{if the electrodes are open.}
\end{aligned} \tag{4.4-1}
$$

Consider the possibility of the following displacement and potential fields:

$$u_\theta = u_\theta(r)e^{i\omega t}, \quad u_r = u_z = 0, \quad \phi = \phi(r)e^{i\omega t}. \tag{4.4-2}$$

The nontrivial components of strain, electric field, stress, and electric displacement are

$$S_4 = 2S_{r\theta} = \frac{du_\theta}{dr} - \frac{u_\theta}{r}, \quad E_2 = E_r = -\frac{d\phi}{dr}, \tag{4.4-3}$$

$$
\begin{aligned}
T_4 = T_{r\theta} &= c_{44}\left(\frac{du_\theta}{dr} - \frac{u_\theta}{r}\right) + e_{15}\frac{d\phi}{dr}, \\
D_2 = D_r &= e_{15}\left(\frac{du_\theta}{dr} - \frac{u_\theta}{r}\right) - \varepsilon_{11}\frac{d\phi}{dr}.
\end{aligned} \tag{4.4-4}
$$

Thus on the boundary surfaces at $r = a$ and b there are no tangential electric fields. The electric potential assumes constant values on the electrodes as required. The stress components T_{rr} and T_{rz} vanish everywhere, particularly on the lateral surfaces. The equation of motion and the charge equation to be satisfied are

$$\frac{dT_{r\theta}}{dr} + \frac{2}{r}T_{r\theta} = -\rho\omega^2 u_\theta, \quad \frac{1}{r}(rD_r)_{,r} = 0. \tag{4.4-5}$$

Equation $(4.4\text{-}5)_2$ can be integrated as

$$D_r = e_{15}\frac{C_3}{r}, \tag{4.4-6}$$

where C_3 is an integration constant. Then, from $(4.4\text{-}4)_2$ we have

$$\phi_{,r} = \frac{e_{15}}{\varepsilon_{11}}\left(2S_{r\theta} - \frac{C_3}{r}\right). \tag{4.4-7}$$

Substitution of (4.4-7) into $(4.4\text{-}4)_1$ gives

$$T_{r\theta} = c_{44}\left(\frac{du_\theta}{dr} - \frac{u_\theta}{r}\right) + e_{15}\frac{e_{15}}{\varepsilon_{11}}\left(2S_{r\theta} - \frac{C_3}{r}\right)$$
$$= \bar{c}_{44}\left(\frac{du_\theta}{dr} - \frac{u_\theta}{r}\right) - \frac{e_{15}^2}{\varepsilon_{11}}\frac{C_3}{r}, \tag{4.4-8}$$

where

$$\bar{c}_{44} = c_{44}(1 + k_{15}^2), \quad k_{15}^2 = \frac{e_{15}^2}{\varepsilon_{11}c_{44}}. \tag{4.4-9}$$

Substitute (4.4-8) into $(4.4\text{-}5)_1$:

$$\bar{c}_{44}\left(\frac{d^2u_\theta}{dr^2} - \frac{1}{r}\frac{du_\theta}{dr} + \frac{1}{r^2}u_\theta\right) + \frac{e_{15}^2}{\varepsilon_{11}}\frac{C_3}{r^2}$$
$$+\frac{2}{r}\bar{c}_{44}\left(\frac{du_\theta}{dr} - \frac{u_\theta}{r}\right) - 2\frac{e_{15}^2}{\varepsilon_{11}}\frac{C_3}{r^2} = -\rho\omega^2 u_\theta, \tag{4.4-10}$$

or

$$\bar{c}_{44}\left(\frac{d^2u_\theta}{dr^2} + \frac{1}{r}\frac{du_\theta}{dr} - \frac{1}{r^2}u_\theta\right) + \rho\omega^2 u_\theta = \frac{e_{15}^2}{\varepsilon_{11}}\frac{C_3}{r^2}, \tag{4.4-11}$$

or

$$\frac{d^2u_\theta}{dr^2} + \frac{1}{r}\frac{du_\theta}{dr} - \frac{1}{r^2}u_\theta + \xi^2 u_\theta = \bar{k}_{15}^2\frac{C_3}{r^2}, \tag{4.4-12}$$

where

$$\xi^2 = \frac{\rho\omega^2}{\bar{c}_{44}}, \quad \bar{k}_{15}^2 = \frac{e_{15}^2}{\varepsilon_{11}\bar{c}_{44}} = \frac{e_{15}^2}{\varepsilon_{11}c_{44}(1 + k_{15}^2)} = \frac{k_{15}^2}{1 + k_{15}^2}. \tag{4.4-13}$$

Introduce a dimensionless variable $R = \xi r$. Equation (4.12) can be written as

$$\frac{d^2u_\theta}{dR^2} + \frac{1}{R}\frac{du_\theta}{dR} + \left(1 - \frac{1}{R^2}\right)u_\theta = \bar{k}_{15}^2\frac{C_3}{R^2}, \tag{4.4-14}$$

which is Bessel's equation of order one.

In the following we consider the case when the electrodes at $r = a$ and b are open. The electrical boundary conditions imply, through (4.4-6), that $C_3 = 0$. Then the general solution to (4.4-14) is

$$u_\theta = C_1J_1(R) + C_2Y_1(R) = C_1J_1(\xi r) + C_2Y_1(\xi r), \tag{4.4-15}$$

where J_1 and Y_1 are the first-order Bessel's functions of the first and second kind, respectively. From (4.4-8) the shear stress is

$$T_{r\theta} = \bar{c}_{44}\left(\frac{du_\theta}{dr} - \frac{u_\theta}{r}\right)$$

$$= \bar{c}_{44}[C_1 J_1'(\xi r)\xi + C_2 Y_1'(\xi r)\xi] - \bar{c}_{44}\frac{1}{r}[C_1 J_1(\xi r) + C_2 Y_1(\xi r)] \quad (4.4\text{-}16)$$

$$= C_1\bar{c}_{44}\xi\left[J_1'(\xi r) - \frac{J_1(\xi r)}{\xi r}\right] + C_2\bar{c}_{44}\xi\left[Y_1'(\xi r) - \frac{Y_1(\xi r)}{\xi r}\right].$$

The traction-free boundary conditions require that

$$C_1\bar{c}_{44}\xi\left[J_1'(\xi a) - \frac{J_1(\xi a)}{\xi a}\right] + C_2\bar{c}_{44}\xi\left[Y_1'(\xi a) - \frac{Y_1(\xi a)}{\xi a}\right] = 0,$$

$$C_1\bar{c}_{44}\xi\left[J_1'(\xi b) - \frac{J_1(\xi b)}{\xi b}\right] + C_2\bar{c}_{44}\xi\left[Y_1'(\xi b) - \frac{Y_1(\xi b)}{\xi b}\right] = 0. \quad (4.4\text{-}17)$$

The frequency equation is given by

$$\begin{vmatrix} J_1'(\xi a) - \dfrac{J_1(\xi a)}{\xi a} & Y_1'(\xi a) - \dfrac{Y_1(\xi a)}{\xi a} \\ J_1'(\xi b) - \dfrac{J_1(\xi b)}{\xi b} & Y_1'(\xi b) - \dfrac{Y_1(\xi b)}{\xi b} \end{vmatrix} = 0. \quad (4.4\text{-}18)$$

Problem

4.4-1. Study the tangential thickness-shear vibration of a circular cylinder of monoclinic crystals [23].

5. AXIAL THICKNESS-SHEAR VIBRATION OF A CIRCULAR CYLINDER

Consider an infinite circular cylinder of inner radius a and outer radius b. The cylinder is made of ceramics with axial poling along the x_3 direction. We choose (r,θ,z) to correspond to (1,2,3) so that the poling direction corresponds to 3. The inner and outer surfaces are electroded. There is no load applied, and we are interested in anti-plane axi-symmetric free vibrations [24].

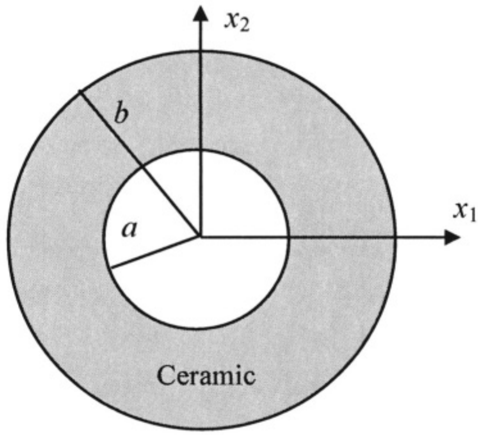

Figure 4.5-1. A circular cylindrical ceramic shell with axial poling.

5.1 Boundary-Value Problem

From Section 6 of Chapter 3, the boundary value problem is:

$$\begin{aligned}
&\bar{c}_{44}\nabla^2 u_z = \rho\ddot{u}_z, \quad a < r < b, \\
&\nabla^2\psi = 0, \quad a < r < b, \\
&u_z = 0, \quad r = a, b, \quad \text{if the cylindical surfaces are fixed,} \\
&\text{or} \quad T_{rz} = 0, \quad r = a, b, \quad \text{if the cylindical surfaces are free,} \\
&\phi(r = a) = \phi(r = b), \quad \text{if the electrodes are shorted,} \\
&\text{or} \quad D_r = 0, \quad r = a, b, \quad \text{if the electrodes are open,}
\end{aligned} \tag{4.5-1}$$

where ϕ and ψ are related by

$$\phi = \psi + \frac{e_{15}}{\varepsilon_{11}} u_z . \tag{4.5-2}$$

The stress and electric displacement components are

$$\begin{aligned}
T_{rz} &= \bar{c}_{44} u_{z,r} + e_{15}\psi_{,r}, \\
D_r &= -\varepsilon_{11}\psi_{,r}.
\end{aligned} \tag{4.5-3}$$

We look for solutions in the following form:

$$\begin{aligned}
u_z(r,t) &= u_z(r) e^{i\omega t}, \\
\psi(r,t) &= \psi(r) e^{i\omega t}.
\end{aligned} \tag{4.5-4}$$

The equations for u_z and ψ are

$$\nabla^2 u_z = \frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r}\frac{\partial u_z}{\partial r} = -\frac{\rho\omega^2}{\bar{c}_{44}} u_z,$$
$$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} = 0. \tag{4.5-5}$$

The general solution to (4.5-5) is

$$u_z = A_1 J_0(\xi r) + A_2 Y_0(\xi r),$$
$$\psi = A_3 \ln\frac{r}{b} + A_4, \tag{4.5-6}$$

where A_1, A_2, A_3 and A_4 are undetermined constants, J_0 and Y_0 are zero-order Bessel's functions of the first and second kind, and

$$\xi^2 = \frac{\rho\omega^2}{\bar{c}_{44}}. \tag{4.5-7}$$

Hence

$$\phi = \psi + \frac{e_{15}}{\varepsilon_{11}} u_z = \frac{e_{15}}{\varepsilon_{11}}[A_1 J_0(\xi r) + A_2 Y_0(\xi r)] + A_3 \ln\frac{r}{b} + A_4,$$
$$T_{rz} = -\bar{c}_{44}\xi[A_1 J_1(\xi r) + A_2 Y_1(\xi r)] + e_{15}\frac{A_3}{r}, \tag{4.5-8}$$
$$D_r = -\varepsilon_{11}\frac{A_3}{r},$$

where $J_0' = -J_1$ and $Y_0' = -Y_1$ have been used.

5.2 Clamped and Electroded Surfaces

First consider the case when the two cylindrical surfaces are fixed and the two electrodes are shorted. Then we have

$$u = 0, \quad r = a, b,$$
$$\phi = 0, \quad r = a, b, \tag{4.5-9}$$

which implies that

$$\psi = 0, \quad r = a, b. \tag{4.5-10}$$

Hence

$$A_3 = 0, \quad A_4 = 0, \tag{4.5-11}$$

and

$$\begin{vmatrix} J_0(\xi a) & Y_0(\xi a) \\ J_0(\xi b) & Y_0(\xi b) \end{vmatrix} = 0. \tag{4.5-12}$$

5.3 Free and Unelectroded Surfaces

Next consider the case

$$\begin{aligned} T_{rz} &= 0, \quad r = a, b, \\ D_r &= 0, \quad r = a, b. \end{aligned} \tag{4.5-13}$$

Then $A_3 = 0$ and

$$\begin{vmatrix} J_1(\xi a) & Y_1(\xi a) \\ J_1(\xi b) & Y_1(\xi b) \end{vmatrix} = 0. \tag{4.5-14}$$

5.4 Free and Electroded Surfaces

Finally, consider

$$\begin{aligned} T_{rz} &= 0, \quad r = a, b, \\ \phi &= 0, \quad r = a, b. \end{aligned} \tag{4.5-15}$$

It can be shown that

$$\begin{vmatrix} J_1(\xi a) & Y_1(\xi a) \\ J_1(\xi b) & Y_1(\xi b) \end{vmatrix} = \frac{\bar{k}_{15}^2}{\xi b \ln \frac{a}{b}} \begin{vmatrix} J_0(\xi a) - J_0(\xi b) & J_1(\xi a) - \frac{b}{a} J_1(\xi b) \\ Y_0(\xi a) - Y_0(\xi b) & Y_1(\xi a) - \frac{b}{a} Y_1(\xi b) \end{vmatrix}. \tag{4.5-16}$$

For large *x*, Bessel functions can be approximated by

$$\begin{aligned} J_\nu(x) &\cong \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right), \\ Y_\nu(x) &\cong \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right). \end{aligned} \tag{4.5-17}$$

Then it can be shown that for large *a* and *b*, (4.5-16) simplifies to

$$\sin\xi(b-a)=\frac{\bar{k}_{15}^2}{\xi b\ln\frac{a}{b}}\left[\left(1+\frac{b}{a}\right)\cos\xi(b-a)-2\sqrt{\frac{b}{a}}\right]. \tag{4.5-18}$$

Setting $2h = b\text{-}a$ and allowing $a,\ b\to\infty$, we have

$$\begin{aligned}
&\xi b\ln\frac{a}{b}=\xi b\ln\frac{b-2h}{b}=\xi b\ln\left(1-\frac{2h}{b}\right)\cong\xi b\left(-\frac{2h}{b}\right)=-\xi 2h,\\
&\sin\xi(b-a)=\sin(\xi 2h)\cong\xi 2h,\\
&\left(1+\frac{b}{a}\right)\cos\xi(b-a)-2\sqrt{\frac{b}{a}}\\
&\quad\cong 2\cos(\xi 2h)-2=-2[1-\cos(\xi 2h)]=-4\sin^2\xi h.
\end{aligned} \tag{4.5-19}$$

Then Equation (4.5-18) reduces to

$$\tan\xi h=\frac{\xi h}{\bar{k}_{15}^2}, \tag{4.5-20}$$

which is the frequency equation for the thickness-shear vibration of a ceramic plate with in-plane poling (see Problem 4.1-1).

Problems

4.5-1. Show (4.5-16).
4.5-2. Show (4.5-18).
4.5-3. Study the case of $u = 0$, $r = a, b$ and $D_r = 0$, $r = a, b$.
4.5-4. Study the axial thickness-shear vibration of a circular cylinder of monoclinic crystals [23].
4.5-5. Study vibrations of a ceramic wedge.

6. SOME GENERAL RESULTS

In this section we prove a few general properties of the eigenvalue problem for the free vibration of a piezoelectric body [25]. The free vibration of a piezoelectric body with frequency ω is governed by the differential equations

$$\begin{aligned}
&-c_{jikl}u_{k,lj}-e_{kji}\phi_{,kj}=\rho\omega^2u_i,\\
&-e_{ikl}u_{k,li}+\varepsilon_{ik}\phi_{,ki}=0.
\end{aligned} \tag{4.6-1}$$

6.1 Abstract Formulation

We introduce the following notation:

$$\lambda=\omega^2,\quad \mathbf{U}=\{u_i,\phi\},\quad \mathbf{V}=\{v_i,\psi\},$$
$$\mathbf{AU}=\{-c_{jikl}u_{k,lj}-e_{kji}\phi_{,kj},-e_{ikl}u_{k,li}+\varepsilon_{ik}\phi_{,ki}\},\tag{4.6-2}$$
$$\mathbf{BU}=\{\rho u_i,0\},$$

where **U** and **V** are four-vectors and **A** and **B** are operators. Then the eigenvalue problem for the free vibration of a piezoelectric body can be written as

$$\begin{aligned}&\mathbf{AU}=\lambda\mathbf{BU},\quad \text{in}\quad V,\\&u_i=0,\quad \text{on}\quad S_u,\\&T_{ji}(\mathbf{U})n_j=(c_{jikl}u_{k,l}+e_{kji}\phi_{,k})n_j=0,\quad \text{on}\quad S_T,\\&\phi=0,\quad \text{on}\quad S_\phi,\\&D_i(\mathbf{U})n_i=(e_{ikl}u_{k,l}-\varepsilon_{ik}\phi_{,k})n_i=0,\quad \text{on}\quad S_D,\end{aligned}\tag{4.6-3}$$

which is a homogeneous system. We are interested in nontrivial solutions of **U**. **A** and **B** are real but λ and **U** may be complex at this point. We note that for a nontrivial **U**, its first three components u_i have to be nontrivial, because $u_i = 0$ implies, through (4.6-3), that $\phi = 0$. For convenience we denote the collection of all **U** that are smooth enough and satisfy the boundary conditions in (4.6-3) by

$$H=\{\mathbf{U}\mid \mathbf{U}\text{ satisfies all boundary conditions in }(4.6\text{-}3)_{2\text{-}5}\}.\tag{4.6-4}$$

A scalar product over H is defined by

$$<\mathbf{U};\mathbf{V}>=\int_V(u_iv_i+\phi\psi)dV,\tag{4.6-5}$$

which has the following properties:

$$\begin{aligned}&<\mathbf{U};\mathbf{V}>=<\mathbf{V};\mathbf{U}>,\\&<\mathbf{U};\alpha\mathbf{V}+\beta\mathbf{W}>=\alpha<\mathbf{U};\mathbf{V}>+\beta<\mathbf{U};\mathbf{W}>,\end{aligned}\tag{4.6-6}$$

where α and β are scalars.

6.2 Self-Adjointness

For any **U**, **V** $\in H$

$$
\begin{aligned}
&< \mathbf{AU};\mathbf{V} >=< \{-c_{jikl}u_{k,lj} - e_{kji}\phi_{,kj}, -e_{ikl}u_{k,li} + \varepsilon_{ik}\phi_{,ki}\};\{v_i,\psi\} > \\
&= \int_V [(-c_{jikl}u_{k,lj} - e_{kji}\phi_{,kj})v_i + (-e_{ikl}u_{k,li} + \varepsilon_{ik}\phi_{,ki})\psi]dV \\
&= \int_S [-T_{ji}(\mathbf{U})n_j v_i - D_i(\mathbf{U})n_i\psi]dS \\
&\quad + \int_V [(c_{jikl}u_{k,l}v_{i,j} + e_{kji}\phi_{,k}v_{i,j} + e_{ikl}u_{k,l}\psi_{,i} - \varepsilon_{ik}\phi_{,k}\psi_{,i})dV \\
&= -\int_S [T_{ji}(\mathbf{U})n_j v_i + D_i(\mathbf{U})n_i\psi]dS \\
&\quad + \int_S [T_{kl}(\mathbf{V})n_l u_k + D_k(\mathbf{V})n_k\phi]dS \\
&\quad + \int_V [(-c_{klij}v_{i,jl} - e_{ikl}\psi_{,il})u_k + (-e_{kji}v_{i,jk} + \varepsilon_{ik}\psi_{,ki})\phi]dV \\
&= -\int_S [T_{ji}(\mathbf{U})n_j v_i + D_i(\mathbf{U})n_i\psi]dS \\
&\quad + \int_S [T_{kl}(\mathbf{V})n_l u_k + D_k(\mathbf{V})n_k\phi]dS \\
&\quad + < \mathbf{U};\mathbf{AV} >=< \mathbf{U};\mathbf{AV} >,
\end{aligned}
\tag{4.6-7}
$$

and

$$< \mathbf{BU};\mathbf{V} >=< \mathbf{U};\mathbf{BV} > . \tag{4.6-8}$$

Hence both **A** and **B** are self-adjoint on *H*. Equation (4.6-7) is called the reciprocal theorem in elasticity and Green's identity in mathematics.

6.3 Reality

Let λ be an eigenvalue and **U** the corresponding eigenvector. Hence

$$\mathbf{AU} = \lambda \mathbf{BU} . \tag{4.6-9}$$

Take complex conjugate

$$\mathbf{AU}^* = \lambda^* \mathbf{BU}^* , \tag{4.6-10}$$

where an asterisk means complex conjugate, and we have made use of the fact that **A** and **B** are real. Multiply (4.6-9) by $\mathbf{U}^*$ and (4.6-10) by **U** through the scalar product, and subtract the resulting equations:

$$0 = (\lambda - \lambda^*) < \mathbf{BU};\mathbf{U}^* > . \tag{4.6-11}$$

Since $< \mathbf{BU};\mathbf{U}^* >$ is strictly positive, we have

$$\lambda - \lambda^* = 0 , \tag{4.6-12}$$

or λ is real. Then let the real and imaginary parts of **U** be $\mathbf{U}^R$ and $\mathbf{U}^I$. Equation (4.6-9) can be written as

$$\mathbf{A}(\mathbf{U}^R + i\mathbf{U}^I) = \lambda \mathbf{B}(\mathbf{U}^R + i\mathbf{U}^I) , \tag{4.6-13}$$

which implies that

$$\mathbf{AU}^R = \lambda\mathbf{BU}^R, \quad \mathbf{AU}^I = \lambda\mathbf{BU}^I. \tag{4.6-14}$$

Equation (4.6-14) shows that $\mathbf{U}^R$ and $\mathbf{U}^I$ are also eigenvectors of λ. In the rest of this section, we will assume that the eigenvectors have been chosen as real.

6.4 Orthogonality

Let $\mathbf{U}^{(m)}$ and $\mathbf{U}^{(n)}$ be two eigenvectors corresponding to two distinct eigenvalues $\lambda^{(m)}$ and $\lambda^{(n)}$, respectively. Then

$$\begin{aligned} \mathbf{AU}^{(m)} &= \lambda^{(m)}\mathbf{BU}^{(m)}, \\ \mathbf{AU}^{(n)} &= \lambda^{(n)}\mathbf{BU}^{(n)}. \end{aligned} \tag{4.6-15}$$

Multiply $(4.6\text{-}15)_1$ by $\mathbf{U}^{(n)}$ and $(4.6\text{-}15)_2$ by $\mathbf{U}^{(m)}$ through the scalar product, and subtract the resulting equations

$$0 = (\lambda^{(m)} - \lambda^{(n)}) < \mathbf{BU}^{(m)}; \mathbf{U}^{(n)} >, \tag{4.6-16}$$

which implies that

$$< \mathbf{BU}^{(m)}; \mathbf{U}^{(n)} >= 0. \tag{4.6-17}$$

The multiplication of $(4.6\text{-}15)_1$ by $\mathbf{U}^{(n)}$ leads to

$$< \mathbf{AU}^{(m)}; \mathbf{U}^{(n)} >= 0. \tag{4.6-18}$$

Equations (4.6-17) and (4.6-18) are called the orthogonality conditions. In unabbreviated form they become

$$\begin{aligned} &< \mathbf{AU}^{(m)}; \mathbf{U}^{(n)} > \\ &= \int_V (c_{ijkl} u_{k,l}^{(m)} u_{i,j}^{(n)} + e_{kji} \phi_{,k}^{(m)} u_{i,j}^{(n)} \\ &\quad + e_{ikl} u_{k,l}^{(m)} \phi_{,i}^{(n)} - \varepsilon_{ik} \phi_{,k}^{(m)} \phi_{,i}^{(n)}) dV = 0, \\ &< \mathbf{BU}^{(m)}; \mathbf{U}^{(n)} >= \int_V \rho_0 u_i^{(m)} u_i^{(n)} dV = 0. \end{aligned} \tag{4.6-19}$$

6.5 Positivity

A subset of H consisting of $\mathbf{U}$ that also satisfies the charge equation is denoted by

$$H^* = \{\mathbf{U} \in H \mid \mathbf{U} \text{ real}, \ -e_{ikl} u_{k,li} + \varepsilon_{ik} \phi_{,ki} = 0 \ \text{in } V\}. \tag{4.6-20}$$

For any $\mathbf{U} \in H^*$

$$
\begin{aligned}
&< \mathbf{AU};\mathbf{U} >=< \{-c_{jikl}u_{k,lj} - e_{kji}\phi_{,kj}, -e_{ikl}u_{k,li} + \varepsilon_{ik}\phi_{,ki}\};\{u_i,\phi\} > \\
&= \int_V [(-c_{jikl}u_{k,lj} - e_{kji}\phi_{,kj})u_i + (-e_{ikl}u_{k,li} + \varepsilon_{ik}\phi_{,ki})\phi]dV \\
&= \int_S [-T_{ji}(\mathbf{U})n_j u_i - D_i(\mathbf{U})n_i\phi]dS \\
&\quad + \int_V [(c_{jikl}u_{k,l}u_{i,j} + e_{kji}\phi_{,k}u_{i,j} + e_{ikl}u_{k,l}\phi_{,i} - \varepsilon_{ik}\phi_{,k}\phi_{,i})dV \\
&= \int_V [c_{jikl}u_{k,l}u_{i,j} + \varepsilon_{ik}\phi_{,k}\phi_{,i} + 2(e_{ikl}u_{k,l}\phi_{,i} - \varepsilon_{ik}\phi_{,k}\phi_{,i})]dV \\
&= \int_V [c_{jikl}u_{k,l}u_{i,j} + \varepsilon_{ik}\phi_{,k}\phi_{,i} - 2(e_{ikl}u_{k,li} - \varepsilon_{ik}\phi_{,ki})\phi]dV \\
&\quad + \int_S 2(e_{ikl}u_{k,l} - \varepsilon_{ik}\phi_{,k})n_i\phi dS \\
&= \int_V (c_{jikl}u_{k,l}u_{i,j} + \varepsilon_{ik}\phi_{,k}\phi_{,i})dV \geq 0,
\end{aligned}
\tag{4.6-21}
$$

and

$$< \mathbf{BU};\mathbf{U} > \quad \geq 0. \tag{4.6-22}$$

Multiply (4.6-9) by $\mathbf{U}$

$$< \mathbf{AU};\mathbf{U} >= \lambda < \mathbf{BU};\mathbf{U} >, \tag{4.6-23}$$

which shows that λ is nonnegative.

6.6 Variational Formulation

Consider the following functional (Rayleigh quotient) of $\mathbf{U} \in H$

$$
\begin{aligned}
&\Pi(\mathbf{U}) = \frac{\Lambda(\mathbf{U})}{\Gamma(\mathbf{U})}, \\
&\Lambda(\mathbf{U}) =< \mathbf{AU};\mathbf{U} >, \quad \Gamma(\mathbf{U}) =< \mathbf{BU};\mathbf{U} >.
\end{aligned}
\tag{4.6-24}
$$

The first variation of Π is

$$\delta\Pi = \frac{\Gamma\delta\Lambda - \Lambda\delta\Gamma}{\Gamma^2} = \frac{\delta\Lambda - \Pi\delta\Gamma}{\Gamma}. \tag{4.6-25}$$

Therefore $\delta\Pi = 0$ implies that

$$\delta\Lambda - \Pi\delta\Gamma = 0. \tag{4.6-26}$$

From (4.6-24) we have

$$\begin{aligned}\delta\Lambda - \Pi\delta\Gamma &= <\mathbf{AU};\delta\mathbf{U}> + <\delta\mathbf{AU};\mathbf{U}> \\ &\quad -\Pi <\mathbf{BU};\delta\mathbf{U}> -\Pi <\delta\mathbf{BU};\mathbf{U}> \\ &= <\mathbf{AU};\delta\mathbf{U}> + <\mathbf{A}\delta\mathbf{U};\mathbf{U}> \\ &\quad -\Pi <\mathbf{BU};\delta\mathbf{U}> -\Pi <\mathbf{B}\delta\mathbf{U};\mathbf{U}> \\ &= <\mathbf{AU};\delta\mathbf{U}> + <\delta\mathbf{U};\mathbf{AU}> \\ &\quad -\Pi <\mathbf{BU};\delta\mathbf{U}> -\Pi <\delta\mathbf{U};\mathbf{BU}> \\ &= 2 <\mathbf{AU} - \Pi\mathbf{BU};\delta\mathbf{U}>,\end{aligned} \tag{4.6-27}$$

where the small variation $\delta\mathbf{U} \in H$. (4.6-27) implies that

$$\mathbf{AU} - \Pi\mathbf{BU} = 0\,. \tag{4.6-28}$$

Hence the $\mathbf{U}$ that makes $\delta\Pi = \mathbf{0}$ is an eigenvector of the eigenvalue Π.

6.7 Perturbation Based on Variational Formulation

Next we consider the case when **A** and **B** are slightly perturbed but are still self-adjoint, which causes small perturbations in λ and **U**:

$$(\mathbf{A} + \Delta\mathbf{A})(\mathbf{U} + \Delta\mathbf{U}) = (\lambda + \Delta\lambda)(\mathbf{B} + \Delta\mathbf{B})(\mathbf{U} + \Delta\mathbf{U})\,. \tag{4.6-29}$$

We are interested in an expression of $\Delta\lambda$ linear in $\Delta\mathbf{A}$ and $\Delta\mathbf{B}$. From (4.6-24),

$$\begin{aligned}\lambda + \Delta\lambda &= \frac{<(\mathbf{A} + \Delta\mathbf{A})(\mathbf{U} + \Delta\mathbf{U});\mathbf{U} + \Delta\mathbf{U}>}{<(\mathbf{B} + \Delta\mathbf{B})(\mathbf{U} + \Delta\mathbf{U});\mathbf{U} + \Delta\mathbf{U}>} \\ &\cong \frac{<\mathbf{AU} + (\Delta\mathbf{A})\mathbf{U} + \mathbf{A}(\Delta\mathbf{U});\mathbf{U} + \Delta\mathbf{U}>}{<\mathbf{BU} + (\Delta\mathbf{B})\mathbf{U} + \mathbf{B}(\Delta\mathbf{U});\mathbf{U} + \Delta\mathbf{U}>} \\ &\cong \frac{<\mathbf{AU};\mathbf{U}> + <\mathbf{AU};\Delta\mathbf{U}> + <(\Delta\mathbf{A})\mathbf{U} + \mathbf{A}(\Delta\mathbf{U});\mathbf{U}>}{<\mathbf{BU};\mathbf{U}> + <\mathbf{BU};\Delta\mathbf{U}> + <(\Delta\mathbf{B})\mathbf{U} + \mathbf{B}(\Delta\mathbf{U});\mathbf{U}>} \\ &= \frac{<\mathbf{AU};\mathbf{U}> +2 <\mathbf{AU};\Delta\mathbf{U}> + <(\Delta\mathbf{A})\mathbf{U};\mathbf{U}>}{<\mathbf{BU};\mathbf{U}> +2 <\mathbf{BU};\Delta\mathbf{U}> + <(\Delta\mathbf{B})\mathbf{U};\mathbf{U}>} \\ &\cong \frac{<\mathbf{AU};\mathbf{U}>}{<\mathbf{BU};\mathbf{U}>}\left(1 + \frac{2<\mathbf{AU};\Delta\mathbf{U}> + <(\Delta\mathbf{A})\mathbf{U};\mathbf{U}>}{<\mathbf{AU};\mathbf{U}>}\right) \\ &\quad \times\left(1 - \frac{2<\mathbf{BU};\Delta\mathbf{U}> + <(\Delta\mathbf{B})\mathbf{U};\mathbf{U}>}{<\mathbf{BU};\mathbf{U}>}\right) \cong\end{aligned}$$

$$\cong \frac{<\mathbf{AU};\mathbf{U}>}{<\mathbf{BU};\mathbf{U}>}\left(1+\frac{2<\mathbf{AU};\Delta\mathbf{U}>+<(\Delta\mathbf{A})\mathbf{U};\mathbf{U}>}{<\mathbf{AU};\mathbf{U}>}\right.$$
$$\left.-\frac{2<\mathbf{BU};\Delta\mathbf{U}>+<(\Delta\mathbf{B})\mathbf{U};\mathbf{U}>}{<\mathbf{BU};\mathbf{U}>}\right)$$
$$=\frac{<\mathbf{AU};\mathbf{U}>}{<\mathbf{BU};\mathbf{U}>}+\frac{2<\mathbf{AU};\Delta\mathbf{U}>+<(\Delta\mathbf{A})\mathbf{U};\mathbf{U}>}{<\mathbf{BU};\mathbf{U}>}$$
$$-\frac{<\mathbf{AU};\mathbf{U}>}{<\mathbf{BU};\mathbf{U}>}\frac{2<\mathbf{BU};\Delta\mathbf{U}>+<(\Delta\mathbf{B})\mathbf{U};\mathbf{U}>}{<\mathbf{BU};\mathbf{U}>} \tag{4.6-30}$$
$$=\lambda+\frac{2<\mathbf{AU}-\lambda\mathbf{BU};\Delta\mathbf{U}>+<(\Delta\mathbf{A})\mathbf{U};\mathbf{U}>-\lambda<(\Delta\mathbf{B})\mathbf{U};\mathbf{U}>}{<\mathbf{BU};\mathbf{U}>}$$
$$=\lambda+\frac{<(\Delta\mathbf{A})\mathbf{U};\mathbf{U}>-\lambda<(\Delta\mathbf{B})\mathbf{U};\mathbf{U}>}{<\mathbf{BU};\mathbf{U}>},$$

hence

$$\Delta\lambda=\frac{<(\Delta\mathbf{A})\mathbf{U};\mathbf{U}>-\lambda<(\Delta\mathbf{B})\mathbf{U};\mathbf{U}>}{<\mathbf{BU};\mathbf{U}>}. \tag{4.6-31}$$

6.8 Perturbation Based on Abstract Formulation

Equation (4.6-31) can also be obtained from the following perturbation procedure. Expand both sides of (4.6-29). The zero-order terms represent the unperturbed eigenvalue problem. The first-order terms are

$$\mathbf{A}(\Delta\mathbf{U})+(\Delta\mathbf{A})\mathbf{U}=\Delta\lambda\mathbf{BU}+\lambda(\Delta\mathbf{B})\mathbf{U}+\lambda\mathbf{B}(\Delta\mathbf{U}). \tag{4.6-32}$$

Multiply both sides by **U**:

$$<\mathbf{A}(\Delta\mathbf{U});\mathbf{U}>+<(\Delta\mathbf{A})\mathbf{U};\mathbf{U}>$$
$$=\Delta\lambda<\mathbf{BU};\mathbf{U}>+\lambda<(\Delta\mathbf{B})\mathbf{U};\mathbf{U}>+\lambda<\mathbf{B}(\Delta\mathbf{U});\mathbf{U}>, \tag{4.6-33}$$

or

$$<\Delta\mathbf{U};\mathbf{AU}>+<(\Delta\mathbf{A})\mathbf{U};\mathbf{U}>$$
$$=\Delta\lambda<\mathbf{BU};\mathbf{U}>+\lambda<(\Delta\mathbf{B})\mathbf{U};\mathbf{U}>+\lambda<\Delta\mathbf{U};\mathbf{BU}>. \tag{4.6-34}$$

The first term and the last term in (4.6-34) cancel, and what is left is

$$\Delta\lambda=\frac{<(\Delta\mathbf{A})\mathbf{U};\mathbf{U}>-\lambda<(\Delta\mathbf{B})\mathbf{U};\mathbf{U}>}{<\mathbf{BU};\mathbf{U}>}, \tag{4.6-35}$$

which is the same as (4.6-31). Note that in this perturbation procedure, no assumption regarding the self-adjointness of $\Delta\mathbf{A}$ and $\Delta\mathbf{B}$ was made.

7. EXTENSIONAL VIBRATION OF A THIN ROD

Consider a rectangular rod of length l, width w, and thickness t as shown in Figure 4.7-1, where $l >> w >> t$. We are interested in the low frequency extensional vibration of the rod [11]. By low frequency we mean that the wavelength of the vibration modes is much longer than the width and thickness of the rod.

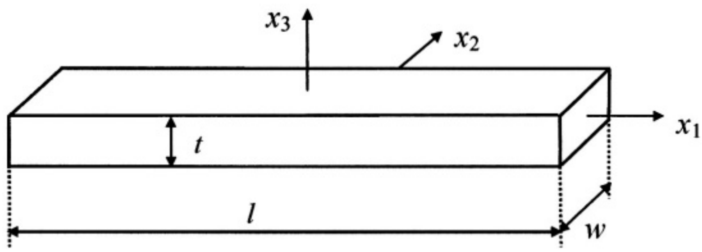

Figure 4.7-1. A piezoelectric rod with rectangular cross section.

As an approximation, it is appropriate to take the vanishing boundary stresses on the surfaces bounding the two small dimensions to vanish everywhere. Consequently

$$T_{11} = T_1(x_1, t), \text{ and all other } T_{ij} = 0. \tag{4.7-1}$$

If the surfaces of the area lw are fully electroded with a driving voltage V across the electrodes, the appropriate electrical conditions are

$$E_1 = E_2 = 0, \quad E_3 = -\frac{V}{t}. \tag{4.7-2}$$

The pertinent constitutive relations are

$$\begin{aligned} S_1 &= s_{11}T_1 + d_{31}E_3, \\ D_3 &= d_{31}T_1 + \varepsilon_{33}E_3. \end{aligned} \tag{4.7-3}$$

Equation $(4.7\text{-}3)_1$ can be inverted to give

$$T_1 = \frac{1}{s_{11}}S_1 - \frac{d_{31}}{s_{11}}E_3. \tag{4.7-4}$$

Then the differential equation of motion and boundary conditions are

$$\begin{aligned} &\frac{1}{s_{11}}u_{1,11} = \rho\ddot{u}_1, \quad -l/2 < x_1 < l/2, \\ &T_1 = \frac{1}{s_{11}}u_{1,1} + \frac{d_{31}}{s_{11}}\frac{V}{t} = 0, \quad x_1 = \pm l/2. \end{aligned} \tag{4.7-5}$$

Equations (4.7-5) show that the applied voltage effectively acts like two extensional end forces on the rod. For free vibrations, $V = 0$ and the electrodes are shorted. We look for free vibration solution in the form

$$u_1(x_1,t) = u_1(x_1)e^{i\omega t}. \tag{4.7-6}$$

Then the eigenvalue problem is

$$\begin{aligned} &u_{1,11} + \rho s_{11}\omega^2 u_1, \quad -l/2 < x_1 < l/2, \\ &u_{1,1} = 0, \quad x_1 = \pm l/2. \end{aligned} \tag{4.7-7}$$

The solution of $\omega = 0$ and u_1 = constant represents a rigid body mode. For the rest of the modes we try $u_1 = \sin kx_1$. Then, from $(4.7\text{-}7)_1$, $k = \omega\sqrt{\rho s_{11}}$. To satisfy $(4.7\text{-}7)_2$ we must have

$$\cos k\frac{l}{2} = 0, \quad \Rightarrow \quad k_{(n)}\frac{l}{2} = \frac{n\pi}{2}, \quad n = 1,3,5,\cdots, \tag{4.7-8}$$

or

$$\omega_{(n)}\sqrt{\rho s_{11}}\frac{l}{2} = \frac{n\pi}{2}, \quad \omega_{(n)} = \frac{n\pi}{l\sqrt{\rho s_{11}}}, \quad n = 1,3,5,\cdots. \tag{4.7-9}$$

Similarly, by considering $u_1 = \cos kx_1$, the following frequencies can be determined:

$$\omega_{(n)} = \frac{n\pi}{l\sqrt{\rho s_{11}}}, \quad n = 2,4,6,\cdots. \tag{4.7-10}$$

The frequencies in (4.7-9) and (4.7-10) are integral multiples of $\omega_{(1)}$ and are called harmonics. $\omega_{(1)}$ is called the fundamental and the rest are called the overtones.

If the surfaces of the area lt are fully electroded with a driving voltage V across the electrodes, the appropriate electrical conditions are

$$E_1 = 0, \quad D_3 = 0, \quad E_2 = -\frac{V}{w}. \tag{4.7-11}$$

The pertinent constitutive relations are

$$\begin{aligned} S_1 &= s_{11}T_1 + d_{21}E_2 + d_{31}E_3, \\ D_2 &= d_{21}T_1 + \varepsilon_{22}E_2 + \varepsilon_{23}E_3. \end{aligned} \tag{4.7-12}$$

From the boundary conditions on the areas of lw, we take the following to be approximately true everywhere:

$$D_3 = d_{31}T_1 + \varepsilon_{32}E_2 + \varepsilon_{33}E_3 = 0. \tag{4.7-13}$$

With (4.7-13), (4.7-12) can be written as

$$S_1 = \tilde{s}_{11}T_1 + \tilde{d}_{21}E_2,$$
$$D_2 = \tilde{d}_{21}T_1 + \tilde{\varepsilon}_{22}E_2, \tag{4.7-14}$$

where

$$\tilde{s}_{11} = s_{11} - d_{31}^2/\varepsilon_{33},$$
$$\tilde{d}_{21} = d_{21} - d_{31}\varepsilon_{23}/\varepsilon_{33}, \tag{4.7-15}$$
$$\tilde{\varepsilon}_{22} = \varepsilon_{22} - \varepsilon_{23}^2/\varepsilon_{33}.$$

If the surfaces of cross-sectional areas lw and lt are not electroded, the appropriate electrical conditions are

$$D_2 = D_3 = 0. \tag{4.7-16}$$

The pertinent constitutive relations are

$$S_1 = s_{11}T_1 + g_{11}D_1,$$
$$E_1 = -g_{11}T_1 + \beta_{11}D_1. \tag{4.7-17}$$

8. RADIAL VIBRATION OF A THIN RING

Axi-symmetric radial vibration can be set up in a thin ceramic ring (see Figure 4.8-1) with radial poling, electroded on its inner and outer surfaces [1].

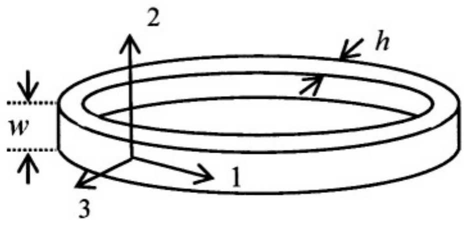

Figure 4.8-1. A ceramic ring with radial poling.

Let R be the mean radius, w the width and h the thickness of the ring. We assume $R >> w >> h$. In cylindrical coordinates, from the boundary conditions, we make the approximation that the following is true throughout the ring:

$$T_{\theta\theta} \neq 0, \quad \text{all other} \quad T_{ij} = 0,$$
$$E_\theta = E_z = 0. \tag{4.8-1}$$

Let (θ,z,r) correspond to (1,2,3). The radial electric field and the tangential strain are given by

$$S_1 = S_{\theta\theta} = \frac{u_r}{R}, \quad E_3 = E_r = -\frac{V}{h}. \tag{4.8-2}$$

The relevant constitutive relations are

$$\begin{aligned} S_1 &= S_{\theta\theta} = s_{11}T_{\theta\theta} + d_{31}E_r, \\ D_3 &= D_r = d_{31}T_{\theta\theta} + \varepsilon_{33}E_r, \end{aligned} \tag{4.8-3}$$

which can be solved to give

$$\begin{aligned} T_{\theta\theta} &= \frac{1}{s_{11}}\frac{u_r}{R} - \frac{d_{31}}{s_{11}}E_r, \\ D_r &= \frac{d_{31}}{s_{11}}\frac{u_r}{R} + \bar{\varepsilon}_{33}E_r, \end{aligned} \tag{4.8-4}$$

where

$$\bar{\varepsilon}_{33} = \varepsilon_{33} - d_{31}^2 / s_{11}. \tag{4.8-5}$$

The equation of motion takes the following form

$$-\frac{T_{\theta\theta}}{R} = \rho\ddot{u}_r. \tag{4.8-6}$$

Substitution of $(4.8\text{-}4)_1$ into (4.8-6) yields

$$-\frac{1}{s_{11}}\frac{u_r}{R^2} + \frac{d_{31}}{s_{11}R}E_r = \rho\ddot{u}_r. \tag{4.8-7}$$

For free vibrations $V = 0$ and

$$-\frac{1}{s_{11}}\frac{u_r}{R^2} = \rho\ddot{u}_r. \tag{4.8-8}$$

The resonance frequency is

$$\omega^2 = \frac{1}{\rho s_{11}R^2}. \tag{4.8-9}$$

9. RADIAL VIBRATION OF A THIN PLATE

A circular disk of a piezoelectric ceramic poled in the thickness direction is positioned in a coordinate system as shown in Figure 4.9-1. We consider axi-symmetric radial modes [26].

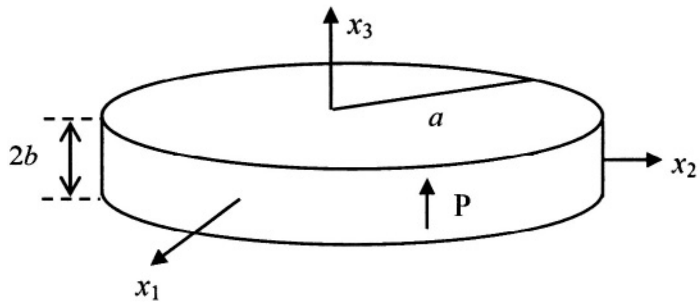

Figure 4.9-1. A circular ceramic plate with thickness poling.

The faces of the disk are traction-free and are completely coated with electrodes. The electrodes are connected to a voltage source of potential $Ve^{i\omega t}$. Under these circumstances, the boundary conditions at $x_3 = \pm b$ are

$$T_{3j} = 0, \quad x_3 = \pm b,$$
$$\phi = \pm \frac{V}{2} e^{i\omega t}, \quad x_3 = \pm b. \tag{4.9-1}$$

Since T_3, T_4, and T_5 vanish on both major surfaces of the plate and the plate is thin, these stresses cannot depart much from zero. Consequently they are assumed to vanish throughout. Thus we assume that

$$T_3 = T_4 = T_5 = 0. \tag{4.9-2}$$

Furthermore, since the plate is thin and has conducting surfaces,

$$E_1 = 0, \quad E_2 = 0, \quad E_3 = -\frac{V}{2b} e^{i\omega t}. \tag{4.9-3}$$

We consider radial modes with

$$u_\theta = 0, \quad \frac{\partial}{\partial \theta} = 0. \tag{4.9-4}$$

The constitutive relations are

$$\begin{aligned} T_{rr} &= c_{11}^p u_{r,r} + c_{12}^p u_r / r + e_{31}^p \phi_{,3}, \\ T_{\theta\theta} &= c_{11}^p u_r / r + c_{12}^p u_{r,r} + e_{31}^p \phi_{,3}, \\ T_{r\theta} &= 0, \\ D_3 &= e_{31}^p (u_{r,r} + u_r / r) - \varepsilon_{33}^p \phi_{,3}, \end{aligned} \tag{4.9-5}$$

where

$$c_{11}^p = c_{11} - c_{13}^2 / c_{33},$$
$$c_{12}^p = c_{12} - c_{13}^2 / c_{33},$$
$$e_{31}^p = e_{31} - e_{33}c_{13} / c_{33}, \tag{4.9-6}$$
$$\varepsilon_{33}^p = \varepsilon_{33} + e_{33}^2 / c_{33}$$

are the effective material constants for a thin plate after the relaxation of the normal stress in the thickness direction. The one remaining equation of motion in cylindrical coordinates is

$$\frac{\partial T_{rr}}{\partial r} + \frac{T_{rr} - T_{\theta\theta}}{r} + \rho f_r = \rho \ddot{u}_r . \tag{4.9-7}$$

Substitution from (4.9-5) for the stress components, we obtain

$$c_{11}^p (u_{,rr} + u_{r,r} / r - u_r / r^2) = \rho \ddot{u}_r , \tag{4.9-8}$$

which, since we are assuming a steady-state problem with frequency ω, becomes

$$u_{r,rr} + \frac{u_{r,r}}{r} + \left(\xi^2 - \frac{1}{r^2}\right) u_r = 0 , \tag{4.9-9}$$

where

$$\xi^2 = \frac{\omega^2}{(v^p)^2}, \quad (v^p)^2 = c_{11}^p / \rho . \tag{4.9-10}$$

Equation (4.9-9) can be written as Bessel's equation of order one. For a solid disk, the motion at the origin is zero and the general solution is

$$u_r = BJ_1(\xi r) e^{i\omega t} , \tag{4.9-11}$$

where J_1 is the first kind Bessel function of the first order. Equation (4.9-11) is subject to the boundary condition

$$T_{rr} = 0, \quad r = a , \tag{4.9-12}$$

hence (4.9-12) requires that

$$c_{11}^p B \frac{dJ_1}{dr}\bigg|_{r=a} + c_{12}^p B \frac{J_1}{a} = -e_{31}^p \frac{V}{2b} , \tag{4.9-13}$$

where, for convenience, the argument of the Bessel function is not written. From (4.9-13) B can be expressed in terms of V as follows:

$$B = \left[(1 - \sigma^p) \frac{J_1(\xi a)}{a} - \xi J_0(\xi a) \right]^{-1} \frac{e_{31}^p}{c_{11}^p} \frac{V}{2b} , \tag{4.9-14}$$

where

$$\frac{dJ_1(x)}{dx} = J_0(x) - \frac{J_1(x)}{x} \tag{4.9-15}$$

has been used and

$$\sigma^p = c_{12}^p / c_{11}^p, \tag{4.9-16}$$

which may be interpreted as a planar Poisson's ratio, since the material is isotropic in the plane normal to x_3. The total charge on the electrode at the bottom of the plate is given by

$$Q_e = \int_A D_3 dA = 2\pi \int_0^a D_3 r dr\,. \tag{4.9-17}$$

Substitution of (4.9-11) into $(4.9\text{-}5)_4$ and then into (4.9-17) yields

$$Q_e = 2\pi e_{31}^p aBJ_1(\xi a) - \pi\varepsilon_{33}^p Va^2 / 2b\,. \tag{4.9-18}$$

Hence we obtain for the current that flows to the resonator

$$I = \frac{dQ_e}{dt} = i\omega\left[\frac{2(k_{31}^p)^2 J_1(\xi a)}{(1-\sigma^p)J_1(\xi a) - \xi a J_0(\xi a)} - 1\right]\frac{\varepsilon_{33}^p \pi a^2 V}{2b}, \tag{4.9-19}$$

where

$$(k_{31}^p)^2 = \frac{(e_{31}^p)^2}{\varepsilon_{33}^p c_{11}^p}\,. \tag{4.9-20}$$

At mechanical resonance, the applied voltage can be zero, and from (4.9-13),

$$\left.\frac{dJ_1}{dr}\right|_{r=a} + \sigma^p \frac{J_1}{a} = 0\,. \tag{4.9-21}$$

Or, at the resonance frequency, the current goes to infinity. This condition is determined by setting the square bracketed factor in the denominator of (4.9-14) equal to zero. The resulting equation is

$$\frac{\xi a J_0(\xi a)}{J_1(\xi a)} = 1 - \sigma^p, \tag{4.9-22}$$

which can be brought into the same form as (4.9-21). The antiresonance frequency results when the current goes to zero. The resulting equation is

$$\frac{\xi a J_0(\xi a)}{J_1(\xi a)} = 1 - \sigma^p - 2(k_{31}^p)^2\,. \tag{4.9-23}$$

10. RADIAL VIBRATION OF A THIN CYLINDRICAL SHELL

In this section we analyze the axi-symmetric radial vibration of an unbounded thin ceramic circular cylindrical shell with radial poling, electroded on its inner and outer surfaces (see Figure 4.10-1). A voltage V is applied across the thickness. Let R be the mean radius, and h the thickness of the shell. By a thin shell we mean $R >> h$.

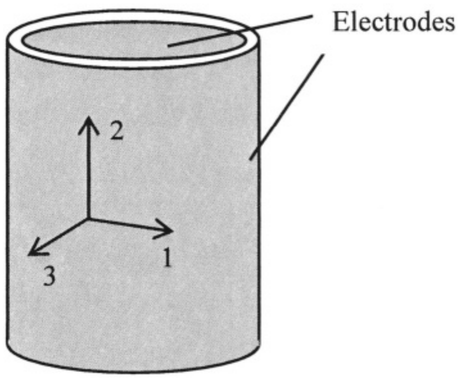

Figure 4.10-1. A thin ceramic circular cylindrical shell.

In cylindrical coordinates, the boundary conditions give

$$T_{rr} = T_{r\theta} = T_{rz} = 0, \quad E_\theta = E_z = 0, \tag{4.10-1}$$

which are taken to be approximately true throughout the shell. We consider motions independent of θ and z. By symmetry

$$T_{\theta z} = 0. \tag{4.10-2}$$

The tangential strain and radial electric field are given by

$$S_1 = S_{\theta\theta} = \frac{u_r}{R}, \quad E_3 = E_r = -\frac{V}{h}. \tag{4.10-3}$$

Let (θ,z,r) correspond to (1,2,3). From

$$S_2 = S_{zz} = s_{11}T_{zz} + s_{21}T_{\theta\theta} + d_{31}E_r = 0, \tag{4.10-4}$$

we solve for

$$T_{zz} = -\frac{s_{12}}{s_{11}}T_{\theta\theta} - \frac{d_{31}}{s_{11}}E_r. \tag{4.10-5}$$

Substituting (4.10-5) into the following constitutive relations

$$S_1 = S_{\theta\theta} = s_{11}T_{\theta\theta} + s_{12}T_{zz} + d_{31}E_r,$$
$$D_3 = D_r = d_{31}(T_{\theta\theta} + T_{zz}) + \varepsilon_{33}E_r, \tag{4.10-6}$$

we obtain

$$S_{\theta\theta} = \bar{s}_{11}T_{\theta\theta} + \bar{d}_{31}E_r,$$
$$D_r = \bar{d}_{31}T_{\theta\theta} + \bar{\varepsilon}_{33}E_r, \tag{4.10-7}$$

where

$$\bar{s}_{11} = s_{11} - \frac{s_{12}^2}{s_{11}}, \quad \bar{d}_{31} = d_{31} - \frac{d_{31}s_{12}}{s_{11}}, \quad \bar{\varepsilon}_{33} = \varepsilon_{33} - \frac{d_{31}^2}{s_{11}}. \tag{4.10-8}$$

Equation (4.10-7) can be inverted to give

$$T_{\theta\theta} = \frac{1}{\bar{s}_{11}}\frac{u_r}{R} - \frac{\bar{d}_{31}}{\bar{s}_{11}}E_r,$$
$$D_r = \frac{\bar{d}_{31}}{\bar{s}_{11}}\frac{u_r}{R} + (\bar{\varepsilon}_{33} - \frac{\bar{d}_{31}^2}{\bar{s}_{11}})E_r. \tag{4.10-9}$$

Substitution of $(4.10\text{-}9)_1$ into the following equation of motion

$$-\frac{T_{\theta\theta}}{R} = \rho\ddot{u}_r \tag{4.10-10}$$

yields

$$-\frac{1}{\bar{s}_{11}}\frac{u_r}{R^2} + \frac{\bar{d}_{31}}{\bar{s}_{11}R}E_r = \rho\ddot{u}_r. \tag{4.10-11}$$

For free vibrations, $V= 0$ and the resonance frequency is

$$\omega^2 = \frac{1}{\rho\bar{s}_{11}R^2}. \tag{4.10-12}$$

Problem

4.10-1. Study the forced vibration.

11. RADIAL VIBRATION OF A THIN SPHERICAL SHELL

Consider a thin spherical ceramic shell of mean radius R and thickness h with $R >> h$ (see Figure 4.11-1). The ceramic is poled in the thickness direction, with fully electroded inner and outer surfaces. Consider radial vibration of the shell [1].

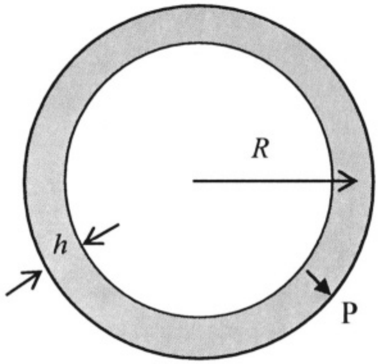

Figure 4.11-1. A spherical ceramic shell with radial poling.

In spherical coordinates the boundary conditions give

$$T_{rr} = T_{r\theta} = T_{r\varphi} = 0, \quad E_\theta = E_\varphi = 0, \tag{4.11-1}$$

which are taken to be valid approximately throughout the shell. For radial motions independent of θ and φ, by symmetry

$$T_{\theta\theta} = T_{\varphi\varphi}, \quad T_{\theta\varphi} = 0. \tag{4.11-2}$$

The relevant strain and electric field components are

$$S_{\theta\theta} = S_{\varphi\varphi} = \frac{u_r}{R}, \quad E_r = -\frac{V}{h}. \tag{4.11-3}$$

Let (r, θ, φ) correspond to (3,1,2) so that poling is along 3. The pertinent constitutive relations are

$$\begin{aligned} S_{\theta\theta} &= S_{\varphi\varphi} = s_{11}T_{\theta\theta} + s_{12}T_{\varphi\varphi} + d_{31}E_r, \\ D_r &= d_{31}(T_{\theta\theta} + T_{\varphi\varphi}) + \varepsilon_{33}E_r, \end{aligned} \tag{4.11-4}$$

which can be inverted to yield

$$\begin{aligned} T_{\theta\theta} &= T_{\varphi\varphi} = \frac{1}{s_{11}+s_{12}}\frac{u_r}{R} - \frac{d_{31}}{s_{11}+s_{12}}E_r, \\ D_r &= \frac{2d_{31}}{s_{11}+s_{12}}\frac{u_r}{R} + \hat{\varepsilon}_{33}E_r, \end{aligned} \tag{4.11-5}$$

where

$$\hat{\varepsilon}_{33} = \varepsilon_{33} - 2d_{31}^2/(s_{11}+s_{12}). \tag{4.11-6}$$

The relevant equation of motion is

$$\frac{-T_{\theta\theta} - T_{\varphi\varphi}}{R} = \rho\ddot{u}_r. \tag{4.11-7}$$

Substitute from (4.11-5)

$$-\frac{2}{s_{11}+s_{12}}\frac{u_r}{R^2} + \frac{2d_{31}}{s_{11}+s_{12}R}E_r = \rho\ddot{u}_r. \tag{4.11-8}$$

For free vibration, $V = 0$ and the resonance frequency is

$$\omega^2 = \frac{2}{\rho(s_{11}+s_{12})R^2}. \tag{4.11-9}$$

Problem

4.11-1. Study the forced vibration.

12. FREQUENCY SHIFTS DUE TO SURFACE ADDITIONAL MASS

In certain applications, we need to study shifts of resonance frequencies due to a small amount of mass added to the surface of a crystal. One example is the mass effect of a thin surface electrode on resonance frequencies. In addition, many chemical and biological acoustic wave sensors detect certain substances through the mass-frequency effect of the substances accumulated on the crystal surface by chemically or biologically active films. These situations can be modeled by a crystal with a thin film of thickness h' and mass density ρ' on part of the crystal surface (see Figure 4.12-1).

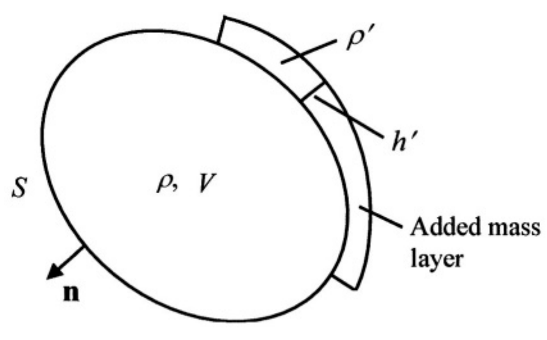

Figure 4.12-1. A crystal with a thin layer of additional mass on part of its surface.

The mass layer is assumed to be very thin. Only the inertial effect of the layer needs to be considered; its stiffness can be neglected. The boundary condition on the surface area with added mass is

$$-T_{ji}n_j = \rho' h' \ddot{u}_i = -\rho' h' \omega^2 u_i . \tag{4.12-1}$$

Then the eigenvalue problem for the resonance frequencies and modes of a crystal with surface added mass is

$$\begin{aligned}
&-c_{jikl}u_{k,lj} - e_{kji}\phi_{,kj} = \rho\lambda u_i, \quad \text{in} \quad V,\\
&-e_{ikl}u_{k,li} + \varepsilon_{ik}\phi_{,ki} = 0, \quad \text{in} \quad V,\\
&u_i = 0, \quad \text{on} \quad S_u,\\
&T_{ji}n_j = (c_{jikl}u_{k,l} + e_{kji}\phi_{,k})n_j = \varepsilon\lambda\rho' h' u_i, \quad \text{on} \quad S_T,\\
&\phi = 0, \quad \text{on} \quad S_\phi,\\
&D_i n_i = (e_{ikl}u_{k,l} - \varepsilon_{ik}\phi_{,k})n_i = 0, \quad \text{on} \quad S_D,
\end{aligned} \tag{4.12-2}$$

where we have denoted

$$\lambda = \omega^2 , \tag{4.12-3}$$

and we have artificially introduced a dimensionless number ε to show the smallness of the added mass. When $\varepsilon = 1$, $(4.12\text{-}2)_4$ becomes (4.12-1). In terms of the abstract notation in Section 6, Equation (4.12-2) can be written as

$$\begin{aligned}
&\mathbf{AU} = \lambda \mathbf{BU}, \quad \text{in} \quad V,\\
&u_i = 0, \quad \text{on} \quad S_u,\\
&T_{ji}(\mathbf{U})n_j = \varepsilon\lambda\rho' h' u_i, \quad \text{on} \quad S_T,\\
&\phi = 0, \quad \text{on} \quad S_\phi,\\
&D_i(\mathbf{U})n_i = 0, \quad \text{on} \quad S_D.
\end{aligned} \tag{4.12-4}$$

We make the following perturbation expansion [27]:

$$\begin{aligned}
&\lambda \cong \lambda^{(0)} + \varepsilon\lambda^{(1)},\\
&\mathbf{U} = \begin{Bmatrix} u_i \\ \phi \end{Bmatrix} \cong \begin{Bmatrix} u_i^{(0)} \\ \phi^{(0)} \end{Bmatrix} + \varepsilon \begin{Bmatrix} u_i^{(1)} \\ \phi^{(1)} \end{Bmatrix} = \mathbf{U}^{(0)} + \varepsilon \mathbf{U}^{(1)}.
\end{aligned} \tag{4.12-5}$$

Substituting (4.12-5) into (4.12-4), collecting terms of equal powers of ε, the following perturbation problems of successive orders can be obtained. Zero-order:

$$
\begin{aligned}
&-c_{jikl}u^{(0)}_{k,lj} - e_{kji}\phi^{(0)}_{,kj} = \rho\lambda^{(0)}u^{(0)}_i, \quad \text{in} \quad V, \\
&-e_{ikl}u^{(0)}_{k,li} + \varepsilon_{ik}\phi^{(0)}_{,ki} = 0, \quad \text{in} \quad V, \\
&u^{(0)}_i = 0, \quad \text{on} \quad S_u, \\
&(c_{jikl}u^{(0)}_{k,l} + e_{kji}\phi^{(0)}_{,k})n_j = 0, \quad \text{on} \quad S_T, \\
&\phi^{(0)} = 0, \quad \text{on} \quad S_\phi, \\
&(e_{ikl}u^{(0)}_{k,l} - \varepsilon_{ik}\phi^{(0)}_{,k})n_i = 0, \quad \text{on} \quad S_D.
\end{aligned}
\tag{4.12-6}
$$

The solution to the zero-order problem, $\lambda^{(0)}$ and $\mathbf{U}^{(0)}$, is assumed known. The first-order problem below is to be solved:

$$
\begin{aligned}
&-c_{jikl}u^{(1)}_{k,lj} - e_{kji}\phi^{(1)}_{,kj} = \rho\lambda^{(1)}u^{(0)}_i + \rho\lambda^{(0)}u^{(1)}_i, \quad \text{in} \quad V, \\
&-e_{ikl}u^{(1)}_{k,li} + \varepsilon_{ik}\phi^{(1)}_{,ki} = 0, \quad \text{in} \quad V, \\
&u^{(1)}_i = 0, \quad \text{on} \quad S_u, \\
&(c_{jikl}u^{(1)}_{k,l} + e_{kji}\phi^{(1)}_{,k})n_j = \rho'h'\lambda^{(0)}u^{(0)}_i, \quad \text{on} \quad S_T, \\
&\phi^{(1)} = 0, \quad \text{on} \quad S_\phi, \\
&(e_{ikl}u^{(1)}_{k,l} - \varepsilon_{ik}\phi^{(1)}_{,k})n_i = 0, \quad \text{on} \quad S_D.
\end{aligned}
\tag{4.12-7}
$$

The equations for the first-order problem can be written as

$$
\mathbf{AU}^{(1)} = \lambda^{(0)}\mathbf{BU}^{(1)} + \lambda^{(1)}\mathbf{BU}^{(0)}. \tag{4.12-8}
$$

Multiplying both sides of (4.12-8) by $\mathbf{U}^{(0)}$ gives

$$
<\mathbf{AU}^{(1)};\mathbf{U}^{(0)}> = \lambda^{(0)} <\mathbf{BU}^{(1)};\mathbf{U}^{(0)}> + \lambda^{(1)} <\mathbf{BU}^{(0)};\mathbf{U}^{(0)}>. \tag{4.12-9}
$$

From (4.6-7),

$$
\begin{aligned}
&<\mathbf{AU}^{(0)};\mathbf{U}^{(1)}> \\
&= -\int_S [T_{ji}(\mathbf{U}^{(0)})n_j u^{(1)}_i + D_i(\mathbf{U}^{(0)})n_i\phi^{(1)}]dS \\
&+ \int_S [T_{kl}(\mathbf{U}^{(1)})n_l u^{(0)}_k + D_k(\mathbf{U}^{(1)})n_k\phi^{(0)}]dS + <\mathbf{U}^{(0)};\mathbf{AU}^{(1)}>.
\end{aligned}
\tag{4.12-10}
$$

With (4.12-6) and (4.12-7), (4.12-10) becomes

$$
\begin{aligned}
&<\mathbf{AU}^{(0)};\mathbf{U}^{(1)}> \\
&= \int_{S_T} \rho'h'\lambda^{(0)}u^{(0)}_k u^{(0)}_k dS + <\mathbf{U}^{(0)};\mathbf{AU}^{(1)}>.
\end{aligned}
\tag{4.12-11}
$$

Substituting (4.12-11) into (4.12-9) yields

$$< \mathbf{AU}^{(0)};\mathbf{U}^{(1)} > - \int_{S_T} \rho' h' \lambda^{(0)} u_k^{(0)} u_k^{(0)} dS$$
$$= \lambda^{(0)} < \mathbf{BU}^{(1)};\mathbf{U}^{(0)} > + \lambda^{(1)} < \mathbf{BU}^{(0)};\mathbf{U}^{(0)} >, \qquad (4.12\text{-}12)$$

which can be further written as

$$< \mathbf{AU}^{(0)} - \lambda^{(0)}\mathbf{BU}^{(0)};\mathbf{U}^{(1)} > - \int_{S_T} \rho' h' \lambda^{(0)} u_k^{(0)} u_k^{(0)} dS$$
$$= \lambda^{(1)} < \mathbf{BU}^{(0)};\mathbf{U}^{(0)} >. \qquad (4.12\text{-}13)$$

With (4.12-6), from (4.12-13)

$$\lambda^{(1)} = -\frac{\int_{S_T} \rho' h' \lambda^{(0)} u_k^{(0)} u_k^{(0)} dS}{< \mathbf{BU}^{(0)};\mathbf{U}^{(0)} >} = -\lambda^{(0)} \frac{\int_{S_T} \rho' h' u_k^{(0)} u_k^{(0)} dS}{\int_V \rho u_i^{(0)} u_i^{(0)} dV}. \qquad (4.12\text{-}14)$$

The above expressions are for the eigenvalue $\lambda = \omega^2$. For ω we make the following expansion:

$$\omega \cong \omega^{(0)} + \varepsilon\omega^{(1)}. \qquad (4.12\text{-}15)$$

Then

$$\lambda = \omega^2 \cong (\omega^{(0)} + \varepsilon\omega^{(1)})^2$$
$$\cong (\omega^{(0)})^2 + 2\varepsilon\omega^{(0)}\omega^{(1)} \cong \lambda^{(0)} + \varepsilon\lambda^{(1)}. \qquad (4.12\text{-}16)$$

Hence

$$\frac{\varepsilon\omega^{(1)}}{\omega^{(0)}} \cong \frac{1}{2(\omega^{(0)})^2}\varepsilon\lambda^{(1)}$$
$$= -\frac{1}{2(\omega^{(0)})^2}\varepsilon\lambda^{(0)} \frac{\int_{S_T} \rho' h' u_k^{(0)} u_k^{(0)} dS}{\int_V \rho u_i^{(0)} u_i^{(0)} dV}. \qquad (4.12\text{-}17)$$

Finally, setting $\varepsilon = 1$ in (4.12-7), we obtain

$$\frac{\omega - \omega^{(0)}}{\omega^{(0)}} \cong -\frac{1}{2} \frac{\int_{S_T} \rho' h' u_k^{(0)} u_k^{(0)} dS}{\int_V \rho u_i^{(0)} u_i^{(0)} dV}. \qquad (4.12\text{-}18)$$

We make the following observations from (4.2-18):

(i) Clearly, we have $\omega - \omega^{(0)} \leq 0$. This shows that a small amount of mass added to the surface tends to lower the resonance frequencies, as expected. On the other hand, if a thin layer of material is removed from the surface, resonance frequencies increase.

(ii) Larger $\rho' h'$ causes more frequency shifts.

(iii) In an area where the surface displacement is large, the added mass has a larger effect on resonance frequencies.

(iv) If the additional mass is essentially a concentrated mass m at a point with Cartesian coordinates y_k on the surface (e.g., a local contamination), then (4.2-18) reduces to

$$\frac{\omega-\omega^{(0)}}{\omega^{(0)}} \cong -\frac{1}{2}\frac{m u_k^{(0)}(\mathbf{y}) u_k^{(0)}(\mathbf{y})}{\int_V \rho u_i^{(0)} u_i^{(0)} dV}. \tag{4.2-19}$$

(v) Obviously, S_T can be several disadjoint areas.

Problem

4.12-1. Use (4.12-18) to analyze Problem 4.3-1 [27].

Chapter 5

WAVES IN UNBOUNDED REGIONS

This chapter is on propagating waves in unbounded regions, which can be resolved into stationary waves. In unbounded regions, eigenvalue problems may have continuous rather than discrete spectra. Instead of resonance frequencies we are going to obtain dispersion relations. All solutions presented in this chapter are exact.

1. PLANE WAVES

First consider waves in a region without a boundary. The waves are governed by the following equations only without boundary conditions:

$$\begin{aligned} c_{ijkl}u_{k,lj} + e_{kij}\phi_{,kj} &= \rho\ddot{u}_i, \\ e_{ikl}u_{k,li} - \varepsilon_{ij}\phi_{,ij} &= 0. \end{aligned} \tag{5.1-1}$$

We are interested in the following plane wave:

$$\begin{aligned} u_k &= A_k f(\mathbf{n}\cdot\mathbf{x} - vt), \\ \phi &= Bf(\mathbf{n}\cdot\mathbf{x} - vt), \end{aligned} \tag{5.1-2}$$

where $\mathbf{A}$, B, $\mathbf{n}$, and v are constants, and f is an arbitrary function, $\mathbf{n}\cdot\mathbf{x} - vt$ is the phase of the wave. v is the phase velocity. $\mathbf{n}\cdot\mathbf{x} - vt$ = constant determines a wave front which is a plane with a normal $\mathbf{n}$. Differentiating (5.1-2), we obtain

$$\begin{aligned} &u_{k,l} = A_k f' n_l, \quad u_{k,li} = A_k f'' n_l n_i, \quad \ddot{u}_k = A_k f'' v^2, \\ &\phi_{,k} = Bf' n_k, \quad \phi_{,ki} = Bf'' n_k n_i. \end{aligned} \tag{5.1-3}$$

Substitution of (5.1-3) into (5.1-2) yields the following linear equations for $\mathbf{A}$ and B:

$$\begin{aligned} c_{ijkl}A_k n_l n_i + e_{kij}Bn_k n_i &= \rho v^2 A_j, \\ e_{ikl}A_k n_l n_i - \varepsilon_{ik}Bn_k n_i &= 0. \end{aligned} \tag{5.1-4}$$

For nontrivial solutions of **A** and/or B, the determinant of the coefficient matrix of (5.1-4) has to vanish. Equivalently, we proceed as follows. From $(5.1\text{-}4)_2$:

$$B = \frac{e_{ikl} n_l n_i}{\varepsilon_{pq} n_p n_q} A_k . \tag{5.1-5}$$

Substituting (5.1-5) into $(5.1\text{-}4)_1$ gives

$$c_{ijkl} A_k n_l n_i + e_{rsj} n_r n_s \frac{e_{ikl} A_k n_l n_i}{\varepsilon_{pq} n_p n_q} = \rho v^2 A_j , \tag{5.1-6}$$

which can be written as

$$(\Gamma_{jk} - \rho v^2 \delta_{jk}) A_k = 0 , \tag{5.1-7}$$

where

$$\Gamma_{jk} = c_{ijkl} n_l n_i + \frac{e_{rsj} n_r n_s e_{ilk} n_l n_i}{\varepsilon_{pq} n_p n_q} = \Gamma_{kj} \tag{5.1-8}$$

is the (piezoelectrically stiffened) acoustic tensor or Christoffel tensor. Equation (5.1-7) is an eigenvalue problem of the acoustic tensor. For nontrivial solutions of **A**, we must have

$$\det(\Gamma_{jk} - \rho v^2 \delta_{jk}) = 0 , \tag{5.1-9}$$

which is a polynomial equation of degree three for v^2. Since the acoustic tensor is real and symmetric, there are typically three real eigenvalues:

$$v^{(1)} ,\ v^{(2)} ,\ v^{(3)} , \tag{5.1-10}$$

and three corresponding eigenvectors that are orthogonal:

$$A_j^{(1)} ,\ A_j^{(2)} ,\ A_j^{(3)} . \tag{5.1-11}$$

Graphically, given a propagation direction **n**, there exist three plane waves with speeds $v^{(1)}$, $v^{(2)}$, and $v^{(3)}$. Their displacement vectors are perpendicular to each other as shown in Figure 5.1-1. In anisotropic materials, usually one of the waves, e.g., the one with $\mathbf{u}^{(1)}$, is roughly aligned with **n**. This is called a quasi-longitudinal wave. The other two waves have their displacement vectors roughly perpendicular to **n** and are called quasi-transverse waves. In materials with high symmetry, e.g. isotropic materials, one wave may be exactly longitudinal and the other two may be exactly transverse.

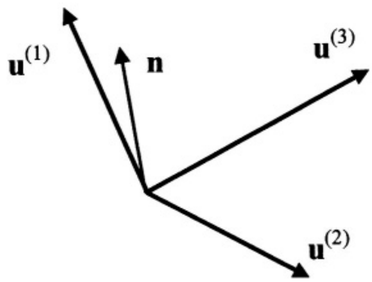

Figure 5.1-1. Plane waves propagating in a direction given by **n**.

Problems

5.1-1. Study plane waves propagating in the x_3 direction of polarized ceramics.
5.1-2. Study plane waves propagating in the x_2 direction of rotated Y-cut quartz.

2. REFLECTION AND REFRACTION

2.1 Reflection

In a semi-infinite medium, a wave solution needs to satisfy boundary conditions in addition to the differential equations in (5.1-1). As an example, consider an anti-plane (SH) wave with u_3 incident upon a plane boundary of a ceramic half-space poled in the x_3 direction as shown in Figure 5.2-1. The boundary is traction-free and is unelectroded. The electric field in the free space is neglected.

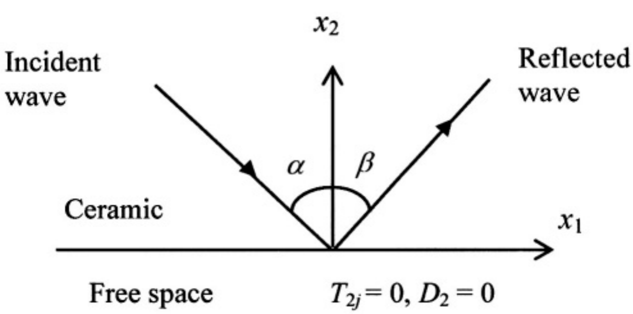

Figure 5.2-1. Incident and reflected waves at a plane boundary.

With the notation in Section 6 of Chapter 3, the incident and reflected waves together must satisfy the following equations and boundary conditions:

$$\begin{aligned} &\bar{c}\nabla^2 u = \rho \ddot{u}, \quad \nabla^2 \psi = 0, \quad x_2 > 0, \\ &T_{2j} = 0, \quad \psi_{,2} = 0, \quad x_2 = 0. \end{aligned} \tag{5.2-1}$$

The incident wave can be written as

$$\begin{aligned} &u = A\exp[i\xi(x_1 \sin\alpha - x_2\cos\alpha - vt)], \\ &\psi = 0, \end{aligned} \tag{5.2-2}$$

which is considered known. The incident wave alone can, in fact, satisfy the differential equations in $(5.2\text{-}1)_{1,2}$ provided that

$$\bar{c}(-\xi^2 \sin^2\alpha - \xi^2\cos^2\alpha) = \rho(-\xi^2 v^2), \tag{5.2-3}$$

or

$$v^2 = \frac{\bar{c}}{\rho} = v_T^2, \tag{5.2-4}$$

where v_T is the speed of a transverse wave propagating in a direction perpendicular to the poling direction in ceramics. The electric potential, the electric displacement, and the stress component needed for the boundary conditions are

$$\phi = \frac{e}{\varepsilon}u, \quad D_1 = D_2 = 0, \tag{5.2-5}$$

and

$$\begin{aligned} T_{23} &= \bar{c}u_{,2} + e\psi_{,2} \\ &= \bar{c}(-i\xi\cos\alpha)A\exp[i\xi(x_1\sin\alpha - x_2\cos\alpha - vt)]. \end{aligned} \tag{5.2-6}$$

Similarly, we write the reflected wave as

$$\begin{aligned} &u = B\exp[i\xi(x_1\sin\beta + x_2\cos\beta - vt)], \\ &\psi = 0, \end{aligned} \tag{5.2-7}$$

which satisfies (5.2-4) and is considered unknown. For boundary conditions, we need

$$T_{23} = \bar{c}(i\xi\cos\beta)B\exp[i\xi(x_1\sin\beta + x_2\cos\beta - vt)]. \tag{5.2-8}$$

The incident and reflected waves already satisfy the governing equations individually and so does their sum. At the boundary, the sum of the incident and reflected waves together has to satisfy

$$\begin{aligned} &\bar{c}(-i\xi\cos\alpha)A\exp[i\xi(x_1\sin\alpha - vt)] \\ &\quad + \bar{c}(i\xi\cos\beta)B\exp[i\xi(x_1\sin\beta - vt)] = 0, \end{aligned} \tag{5.2-9}$$

for any x_1 and any t. This implies that

$$\beta = \alpha, \quad B = A, \tag{5.2-10}$$

and thus determines the reflected wave.

2.2 Reflection and Refraction

At the interface between two semi-infinite media, an incident SH plane wave is reflected and refracted. As an example, consider two semi-infinite spaces of polarized ceramics as shown in Figure 5.2-2.

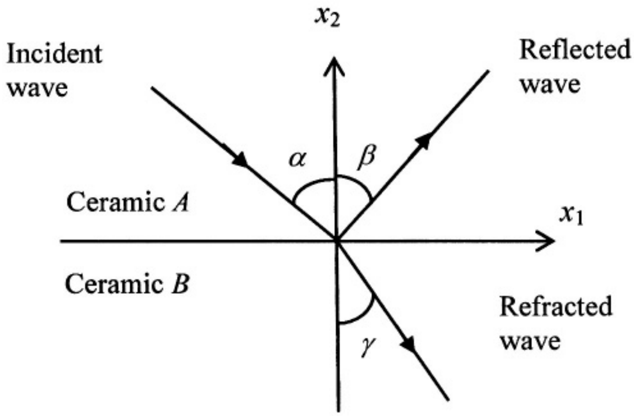

Figure 5.2-2. Incident, reflected, and refracted waves at a plane interface.

The incident and reflected waves together must satisfy the following equations:

$$\bar{c}_A \nabla^2 u = \rho_A \ddot{u}, \quad \nabla^2 \psi = 0, \quad x_2 > 0. \tag{5.2-11}$$

The refracted wave must satisfy

$$\bar{c}_B \nabla^2 u = \rho_B \ddot{u}, \quad \nabla^2 \psi = 0, \quad x_2 < 0. \tag{5.2-12}$$

The interface continuity conditions are

$$\begin{aligned} &u(x_2 = 0^+) = u(x_2 = 0^-), \\ &T_{23}(x_2 = 0^+) = T_{23}(x_2 = 0^-), \\ &\psi(x_2 = 0^+) = \psi(x_2 = 0^-), \\ &D_2(x_2 = 0^+) = D_2(x_2 = 0^-). \end{aligned} \tag{5.2-13}$$

The incident wave can be written as

$$u = A\exp[i\xi_A(x_1\sin\alpha - x_2\cos\alpha - v_A t)],$$
$$\psi = 0,\quad v_A^2 = \frac{\bar{c}_A}{\rho_A},\quad D_2 = 0, \tag{5.2-14}$$
$$T_{23} = \bar{c}_A(-i\xi_A\cos\alpha)A\exp[i\xi_A(x_1\sin\alpha - x_2\cos\alpha - v_A t)],$$

which is considered known. The reflected wave can be written as

$$u = B\exp[i\xi_A(x_1\sin\beta + x_2\cos\beta - v_A t)],$$
$$\psi = 0,\quad D_2 = 0, \tag{5.2-15}$$
$$T_{23} = \bar{c}_A(i\xi_A\cos\beta)B\exp[i\xi_A(x_1\sin\beta + x_2\cos\beta - v_A t)],$$

which is considered unknown. We write the refracted wave as

$$u = C\exp[i\xi_B(x_1\sin\gamma - x_2\cos\gamma - v_B t)],$$
$$\psi = 0,\quad v_B^2 = \frac{\bar{c}_B}{\rho_B},\quad D_2 = 0, \tag{5.2-16}$$
$$T_{23} = \bar{c}_B(-i\xi_B\cos\gamma)C\exp[i\xi_B(x_1\sin\gamma - x_2\cos\gamma - v_B t)].$$

Equation (5.2-11), (5.2-12), and $(5.2\text{-}13)_{3,4}$ are already satisfied. From $(5.2\text{-}13)_{1,2}$ we have

$$\begin{aligned} &A\exp[i\xi_A(x_1\sin\alpha - v_A t)] + B\exp[i\xi_A(x_1\sin\beta - v_A t)] \\ &\quad = C\exp[i\xi_B(x_1\sin\gamma - v_B t)], \\ &\bar{c}_A(-i\xi_A\cos\alpha)A\exp[i\xi_A(x_1\sin\alpha - v_A t)] \\ &\quad + \bar{c}_A(i\xi_A\cos\beta)B\exp[i\xi_A(x_1\sin\beta - v_A t)] \\ &\quad = \bar{c}_B(-i\xi_B\cos\gamma)C\exp[i\xi_B(x_1\sin\gamma - v_B t)]. \end{aligned} \tag{5.2-17}$$

Equation (5.2-17) can be satisfied if the following two sets of relations are true:

$$\begin{aligned} &\exp[i\xi_A(x_1\sin\alpha - v_A t)] \\ &\quad = \exp[i\xi_A(x_1\sin\beta - v_A t)] \\ &\quad = \exp[i\xi_B(x_1\sin\gamma - v_B t)], \end{aligned} \tag{5.2-18}$$

and

$$\begin{aligned} &A + B = C, \\ &\bar{c}_A\xi_A A\cos\alpha - \bar{c}_A\xi_A B\cos\beta = \bar{c}_B\xi_B C\cos\gamma. \end{aligned} \tag{5.2-19}$$

Next we analyze (5.2-18) and (5.2-19) separately. For (5.2-18) to be true for all t and all x_1, we have

$$\begin{aligned} &\xi_A v_A = \xi_B v_B, \\ &\xi_A\sin\alpha = \xi_A\sin\beta = \xi_B\sin\gamma. \end{aligned} \tag{5.2-20}$$

Hence

$$\frac{\xi_B}{\xi_A} = \frac{v_A}{v_B} = \sqrt{\frac{\bar{c}_A \rho_B}{\rho_A \bar{c}_B}} = \frac{1}{n}, \tag{5.2-21}$$

where n is the refractive index, and

$$\beta = \alpha, \quad \frac{\sin\gamma}{\sin\alpha} = \frac{\xi_A}{\xi_B} = \frac{v_B}{v_A} = n, \tag{5.2-22}$$

which is Snell's law. Then from (5.2-19) we solve for B and C:

$$\begin{aligned} \frac{B}{A} &= \frac{\rho_A \sin 2\alpha - \rho_B \sin 2\gamma}{\rho_A \sin 2\alpha + \rho_B \sin 2\gamma}, \\ \frac{C}{A} &= \frac{2\rho_A \sin 2\alpha}{\rho_A \sin 2\alpha + \rho_B \sin 2\gamma}. \end{aligned} \tag{5.2-23}$$

Thus the reflected and refracted waves are fully determined. As a special case consider normal incidence with $\alpha = 0$. From (5.2-22) $\beta = \gamma = 0$. (5.2-23) implies that

$$\begin{aligned} \frac{B}{A} &= \frac{\rho_A - n\rho_B}{\rho_A + n\rho_B} = \frac{\rho_A v_A - \rho_B v_B}{\rho_A v_A + \rho_B v_B}, \\ \frac{C}{A} &= \frac{2\rho_A v_A}{\rho_A v_A + \rho_B v_B}, \end{aligned} \tag{5.2-24}$$

where $\rho_A v_A$ or $\rho_B v_B$ is the acoustic impedance. When $\rho_A v_A = \rho_B v_B$, the wave is not reflected and total transmission occurs.

Problem

5.2-1. Study the reflection in Figure 5.2-1 when the boundary is fixed and unelectroded.

3. SURFACE WAVES

Surface waves in piezoelectrics have been used extensively to make surface acoustic wave (SAW) devices. In addition to Rayleigh waves, antiplane (SH) surface waves can also propagate in certain piezoelectrics. These waves rely on piezoelectric coupling and do not have an elastic counterpart. In this section, a surface wave in polarized ceramics called the Bleustein-Gulyaev wave is presented [18,28].

Consider a ceramic half-space as shown in Figure 5.3 -1. The ceramic is poled in the x_3 direction. We consider anti-plane motions.

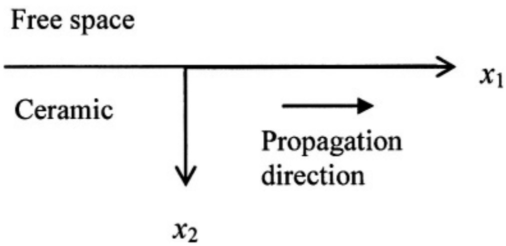

Figure 5.3-1. A ceramic half-space.

With the notation in Section 6 of Chapter 3, the governing equations are

$$\bar{c}\nabla^2 u = \rho\ddot{u}, \quad \nabla^2\psi = 0, \quad x_2 > 0. \tag{5.3-1}$$

Consider the possibility of solutions in the following form:

$$\begin{aligned}
u &= A\exp(-\xi_2 x_2)\cos(\xi_1 x_1 - \omega t), \\
\psi &= B\exp(-\xi_1 x_2)\cos(\xi_1 x_1 - \omega t), \\
T_{23} &= \bar{c}u_{,2} + e\psi_{,2} \\
&= -[\bar{c}A\xi_2\exp(-\xi_2 x_2) + eB\xi_1\exp(-\xi_1 x_2)]\cos(\xi_1 x_1 - \omega t), \\
\phi &= \psi + \frac{e}{\varepsilon}u \\
&= [B\exp(-\xi_1 x_2) + \frac{e}{\varepsilon}A\exp(-\xi_2 x_2)]\cos(\xi_1 x_1 - \omega t), \\
D_2 &= -\varepsilon\psi_{,2} = \varepsilon\xi_1 B\exp(-\xi_1 x_2)\cos(\xi_1 x_1 - \omega t),
\end{aligned} \tag{5.3-2}$$

where A and B are undetermined constants, and ξ_2 should be positive for decaying behavior away from the surface. Equation $(5.3\text{-}2)_2$ already satisfies $(5.3\text{-}1)_2$. For $(5.3\text{-}2)_1$ to satisfy $(5.3\text{-}1)_1$ we must have

$$\bar{c}(\xi_1^2 - \xi_2^2) = \rho\omega^2, \tag{5.3-3}$$

which determines

$$\begin{aligned}
\xi_2^2 &= \xi_1^2 - \frac{\rho\omega^2}{\bar{c}} = \xi_1^2(1 - \frac{v^2}{v_T^2}) > 0, \\
v^2 &= \frac{\omega^2}{\xi_1^2}, \quad v_T^2 = \frac{\bar{c}}{\rho}.
\end{aligned} \tag{5.3-4}$$

3.1 A Half-Space with an Electroded Surface

First we consider the case when the surface of the half-space is electroded and the electrode is grounded. The corresponding boundary conditions are

$$\begin{aligned} &T_{23} = 0, \quad x_2 = 0, \\ &\phi = 0, \quad x_2 = 0, \\ &u, \phi \to 0, \quad x_2 \to +\infty, \end{aligned} \tag{5.3-5}$$

or, in terms of u and ψ,

$$\begin{aligned} &T_{23} = \bar{c} u_{,2} + e\psi_{,2} = 0, \quad x_2 = 0, \\ &\phi = \psi + \frac{e}{\varepsilon} u = 0, \quad x_2 = 0, \\ &u, \psi \to 0, \quad x_2 \to +\infty. \end{aligned} \tag{5.3-6}$$

Substituting (5.3-2) into $(5.3\text{-}6)_{1,2}$:

$$\begin{aligned} &\bar{c} A \xi_2 + eB\xi_1 = 0, \\ &\frac{e}{\varepsilon} A + B = 0. \end{aligned} \tag{5.3-7}$$

For nontrivial solutions,

$$\begin{vmatrix} \bar{c}\xi_2 & e\xi_1 \\ e/\varepsilon & 1 \end{vmatrix} = \bar{c}\xi_2 - \frac{e^2}{\varepsilon}\xi_1 = 0, \tag{5.3-8}$$

or

$$\xi_2 = \bar{k}^2 \xi_1, \tag{5.3-9}$$

where

$$\bar{k}^2 = \frac{e^2}{\varepsilon \bar{c}}. \tag{5.3-10}$$

Substitution of (5.3-3) into (5.3-9) yields

$$\bar{c}(\xi_1^2 - \bar{k}^4 \xi_1^2) = \rho\omega^2, \tag{5.3-11}$$

from which the surface wave speed can be determined as

$$v^2 = \frac{\omega^2}{\xi_1^2} = \frac{\bar{c}}{\rho}(1 - \bar{k}^4) = v_T^2 (1 - \bar{k}^4) < v_T^2. \tag{5.3-12}$$

When $\bar{k} = 0$, we have $\xi_2 = 0$, and the wave is no longer a surface wave.

3.2 A Half-Space with an Unelectroded Surface

If the surface of the half-space is unelectroded, electric fields can also exist in the free space of $x_2 < 0$. Denoting the electric potential in the free space by $\hat{\phi}$, we have

$$\nabla^2\hat{\phi} = 0, \quad x_2 < 0. \tag{5.3-13}$$

The boundary and continuity conditions are

$$\begin{aligned} &T_{23} = 0, \quad \phi = \hat{\phi}, \quad D_2 = \hat{D}_2, \quad x_2 = 0, \\ &u_3, \phi \to 0, \quad x_2 \to +\infty, \\ &\hat{\phi} \to 0, \quad x_2 \to -\infty, \end{aligned} \tag{5.3-14}$$

or, in terms of u, ψ, and $\hat{\phi}$,

$$\begin{aligned} &T_{23} = \bar{c}u_{3,2} + e\psi_{,2} = 0, \quad x_2 = 0, \\ &\phi = \psi + \frac{e}{\varepsilon}u = \hat{\phi}, \quad x_2 = 0, \\ &-\varepsilon\psi_{,2} = -\varepsilon_0\hat{\phi}_{,2}, \quad x_2 = 0, \\ &u, \psi \to 0, \quad x_2 \to +\infty, \\ &\hat{\phi} \to 0, \quad x_2 \to -\infty. \end{aligned} \tag{5.3-15}$$

From (5.3-13), in the free space,

$$\begin{aligned} &\hat{\phi} = C\exp(\xi_1 x_2)\cos(\xi_1 x_1 - \omega t), \\ &D_2 = -\varepsilon_0\hat{\phi}_{,2} = -\varepsilon_0\xi_1 C\exp(\xi_1 x_2)\cos(\xi_1 x_1 - \omega t). \end{aligned} \tag{5.3-16}$$

Substituting (5.3-2) and (5.3-16) into $(5.3\text{-}15)_{1,2,3}$:

$$\begin{aligned} &\bar{c}(-A\xi_2) + e(-B\xi_1) = 0, \\ &-\varepsilon(-\xi_1 B) = -\varepsilon_0 C\xi_1, \\ &\frac{e}{\varepsilon}A + B = C. \end{aligned} \tag{5.3-17}$$

For nontrivial solutions,

$$\begin{vmatrix} \bar{c}\xi_2 & e\xi_1 & 0 \\ 0 & \varepsilon & \varepsilon_0 \\ e/\varepsilon & 1 & -1 \end{vmatrix} = -\bar{c}\xi_2\varepsilon + \frac{e^2}{\varepsilon}\varepsilon_0\xi_1 - \bar{c}\xi_2\varepsilon_0 = 0, \tag{5.3-18}$$

or

$$\xi_2 = \bar{k}^2 \xi_1 \frac{1}{1+\varepsilon/\varepsilon_0}. \tag{5.3-19}$$

Substitution of (5.3-3) into (5.3-19) yields the surface wave speed

$$v^2 = \frac{\omega^2}{\xi_1^2} = \frac{\bar{c}}{\rho}\left[1 - \frac{\bar{k}^4}{(1+\varepsilon/\varepsilon_0)^2}\right] = v_T^2\left[1 - \frac{\bar{k}^4}{(1+\varepsilon/\varepsilon_0)^2}\right] < v_T^2\,. \tag{5.3-20}$$

4. INTERFACE WAVES

Consider two half-spaces of polarized ceramics as shown in Figure 5.4-1. The ceramics are both poled along the $\pm x_3$ directions. We are interested in the possibility of a wave traveling near the interface between the two ceramics [29].

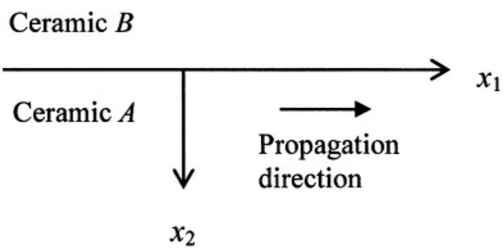

Figure 5.4-1. Two semi-infinite ceramic half-spaces.

The governing equations are

$$\begin{aligned} \bar{c}_A \nabla^2 u_A &= \rho_A \ddot{u}_A, \quad \nabla^2 \psi_A = 0, \quad x_2 > 0, \\ \bar{c}_A \nabla^2 u_B &= \rho_A \ddot{u}_B, \quad \nabla^2 \psi_B = 0, \quad x_2 > 0, \end{aligned} \tag{5.4-1}$$

where the subscripts A and B indicates quantities in ceramic A and ceramic B. For an interface wave, we require

$$\begin{aligned} u_A, \psi_A &\to 0, \quad x_2 \to +\infty, \\ u_B, \psi_B &\to 0, \quad x_2 \to -\infty. \end{aligned} \tag{5.4-2}$$

For $x_2 > 0$, the solutions to $(5.4\text{-}1)_1$ that satisfy $(5.4\text{-}2)_1$ can be written as

$$\begin{aligned} u_A &= U_A \exp(-\eta_A x_2)\cos(\xi x_1 - \omega t), \\ \psi_A &= \Psi_A \exp(-\xi x_2)\cos(\xi x_1 - \omega t), \end{aligned} \tag{5.4-3}$$

where

$$\eta_A^2 = \xi^2 - \frac{\rho_A \omega^2}{\bar{c}_A} = \xi^2\left(1 - \frac{v^2}{v_A^2}\right) > 0, \tag{5.4-4}$$

and

$$v = \frac{\omega}{\xi}, \quad v_A^2 = \frac{\bar{c}_A}{\rho_A}. \tag{5.4-5}$$

Similarly, for $x_2 < 0$, the solutions to $(5.4\text{-}1)_2$ that satisfy $(5.4\text{-}2)_2$ are

$$\begin{aligned} u_B &= U_B \exp(\eta_B x_2)\cos(\xi x_1 - \omega t), \\ \psi_B &= \Psi_B \exp(\xi x_2)\cos(\xi x_1 - \omega t), \end{aligned} \tag{5.4-6}$$

where

$$\eta_B^2 = \xi^2 - \frac{\rho_B \omega^2}{\bar{c}_B} = \xi^2\left(1 - \frac{v^2}{v_B^2}\right) > 0, \tag{5.4-7}$$

$$v_B^2 = \frac{\bar{c}_B}{\rho_B}. \tag{5.4-8}$$

For interface continuity conditions, we consider two situations separately.

4.1 An Electroded Interface

When the interface is a grounded electrode we have, at $x_2 = 0$,

$$\begin{aligned} &u_A = u_B, \quad T_A = T_B, \\ &\psi_A + \frac{e_A}{\varepsilon_A} u_A = 0, \quad \psi_B + \frac{e_B}{\varepsilon_B} u_B = 0. \end{aligned} \tag{5.4-9}$$

Substitution of (5.4-3) and (5.4-6) into (5.4-9) results in a system of linear homogenous equations for U_A, Ψ_A, U_B and Ψ_B. For nontrivial solutions, the determinant of the coefficient matrix has to vanish, which yields

$$\bar{c}_A\left(1 - \frac{v^2}{v_A^2}\right)^{1/2} + \bar{c}_B\left(1 - \frac{v^2}{v_B^2}\right)^{1/2} = \frac{e_A^2}{\varepsilon_A} + \frac{e_B^2}{\varepsilon_B}, \tag{5.4-10}$$

which determines the speed of the interface wave. As a special case, when medium B is free space with

$$\bar{c}_B = 0, \quad \bar{e}_B = 0, \quad \varepsilon_B = \varepsilon_0, \tag{5.4-11}$$

Equation (5.4-10) reduces to

$$v^2 = v_A^2\left[1 - \left(\frac{e_A^2}{\varepsilon_A \bar{c}_A}\right)^2\right], \tag{5.4-12}$$

which is the speed of the Bleustein-Gulyaev wave given by (5.3-12).

4.2 An Unelectroded Interface

When the interface is unelectroded, the continuity conditions at $x_2 = 0$ are

$$\begin{aligned} u_A &= u_B, \\ T_A &= T_B, \\ \psi_A + \frac{e_A}{\varepsilon_A} u_A &= \psi_B + \frac{e_B}{\varepsilon_B} u_B, \\ D_A &= D_B. \end{aligned} \tag{5.4-13}$$

Substituting (5.4-3) and (5.4-6) into (5.4-13), and requiring the determinant of the coefficient matrix to vanish, we obtain

$$\bar{c}_A \left(1 - \frac{v^2}{v_A^2}\right)^{1/2} + \bar{c}_B \left(1 - \frac{v^2}{v_B^2}\right)^{1/2} = \frac{(e_A / \varepsilon_A - e_B / \varepsilon_B)^2}{1/\varepsilon_A + 1/\varepsilon_B}. \tag{5.4-14}$$

We note that the right-hand side of (5.4-14) depends on the difference between the two materials. As a special case, when mediums B is free space with (5.4-11), Equation (5.4-14) reduces to

$$v^2 = v_A^2 \left[1 - \left(\frac{e_A^2}{\varepsilon_A \bar{c}_A (1 + \varepsilon_0 / \varepsilon_A)}\right)^2\right], \tag{5.4-15}$$

which is the speed of the Bleustein-Gulyaev wave given in (5.3-20).

Problems

5.4-1. Show (5.4-10).
5.4-2. Show (5.4-14).

5. WAVES IN A PLATE

Waves in plates are widely used to make bulk acoustic wave (BAW) devices. Consider a ceramic plate poled in the x_3 direction (see Figure 5.5-1). The major surfaces of the plate are traction-free and electroded, and the electrodes are grounded. We are interested in anti-plane waves in the plate [30].

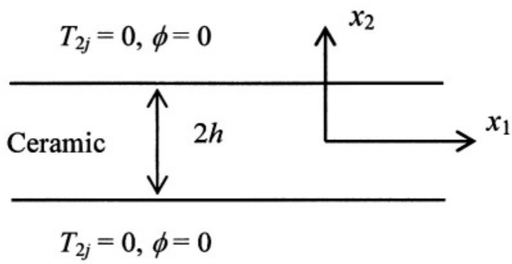

Figure 5.5-1. A ceramic plate.

With the notation in Section 6 of Chapter 3, the governing equations are

$$\bar{c}\nabla^2 u = \rho\ddot{u}, \quad \nabla^2\psi = 0, \quad -h < x_2 < h. \tag{5.5-1}$$

The boundary conditions are

$$\begin{aligned} &T_{23} = 0, \quad x_2 = \pm h, \\ &\phi = 0, \quad x_2 = \pm h, \end{aligned} \tag{5.5-2}$$

or, in terms of u and ψ

$$\begin{aligned} &\bar{c}u_{,2} + e\psi_{,2} = 0, \quad x_2 = \pm h, \\ &\psi + \frac{e}{\varepsilon}u = 0, \quad x_2 = \pm h. \end{aligned} \tag{5.5-3}$$

As we will show in the following paragraphs, there are two types of waves that can propagate in the plate. One type of waves is symmetric and the other is an anti-symmetric. We discuss them separately below.

5.1 Anti-Symmetric Waves

For anti-symmetric waves we consider the possibility of

$$\begin{aligned} &u = A\sin\xi_2 x_2 \cos(\xi_1 x_1 - \omega t), \\ &\psi = B\sinh\xi_1 x_2 \cos(\xi_1 x_1 - \omega t), \end{aligned} \tag{5.5-4}$$

where A and B are constants. For (5.5-4) to satisfy (5.5-1), we have

$$\begin{aligned} &\xi_2^2 = \frac{\rho\omega^2}{\bar{c}} - \xi_1^2 = \xi_1^2\left(\frac{v^2}{v_T^2} - 1\right), \\ &v^2 = \frac{\omega^2}{\xi_1^2}, \quad v_T^2 = \frac{\bar{c}}{\rho}. \end{aligned} \tag{5.5-5}$$

Substitution of (5.5-4) into (5.5-3) leads to

$$\bar{c}A\xi_2\cos\xi_2 h + eB\xi_1\cosh\xi_1 h = 0,$$
$$\frac{e}{\varepsilon}A\sin\xi_2 h + B\sinh\xi_1 h = 0, \tag{5.5-6}$$

which is a system of linear, homogeneous equations for A and B. For nontrivial solutions we must have

$$\frac{\tan\xi_2 h}{\tanh\xi_1 h} = \frac{\xi_2 h}{\xi_1 h}\frac{\varepsilon\bar{c}}{e^2}. \tag{5.5-7}$$

Equation (5.5-7) can be written as

$$\frac{\tan\frac{\pi}{2}(\Omega^2 - Z^2)^{1/2}}{\tanh\frac{\pi}{2}Z} = \frac{(\Omega^2 - Z^2)^{1/2}}{\bar{k}^2 Z}, \tag{5.5-8}$$

where the dimensionless frequency and the dimensionless wave number in the x_1 direction are defined by

$$\Omega^2 = \omega^2 \Big/ \left(\frac{\pi^2\bar{c}}{4\rho h^2}\right), \quad Z = \xi_1 \Big/ \left(\frac{\pi}{2h}\right). \tag{5.5-9}$$

In the limit when $Z \to 0$, Equation (5.5-8) reduces to

$$\tan\frac{\pi}{2}\Omega = \frac{\pi}{2}\frac{\Omega}{\bar{k}^2}, \tag{5.5-10}$$

which is the frequency equation for anti-symmetric thickness-shear modes in a ceramic plate (see Problem 4.1-1).

5.2 Symmetric Waves

For Symmetric waves we consider

$$u = A\cos\xi_2 x_2 \cos(\xi_1 x_1 - \omega t),$$
$$\psi = B\cosh\xi_1 x_2 \cos(\xi_1 x_1 - \omega t). \tag{5.5-11}$$

where A and B are constants. For (5.5-11) to satisfy (5.5-1), we have

$$\xi_2^2 = \frac{\rho\omega^2}{\bar{c}} - \xi_1^2 = \xi_1^2\left(\frac{v^2}{v_T^2} - 1\right). \tag{5.5-12}$$

Substitution of (5.5-11) into (5.5-3) leads to

$$-\bar{c}A\xi_2\sin\xi_2 h + eB\xi_1\sinh\xi_1 h = 0,$$
$$\frac{e}{\varepsilon}A\cos\xi_2 h + B\cosh\xi_1 h = 0. \tag{5.5-13}$$

For nontrivial solutions of *A* and/or *B*, we must have

$$\frac{\tan \xi_2 h}{\tanh \xi_1 h} = -\frac{\xi_1 h}{\xi_2 h}\frac{e^2}{\varepsilon \bar{c}}, \tag{5.5-14}$$

or

$$\frac{\tan \frac{\pi}{2}(\Omega^2 - Z^2)^{1/2}}{\tanh \frac{\pi}{2} Z} = -\frac{\bar{k}^2 Z}{(\Omega^2 - Z^2)^{1/2}}. \tag{5.5-15}$$

The first few branches of (5.5-8) and (5.5-15) are plotted in Figure 5.5-2. The dotted straight line represents Bleustein-Gulyaev waves in a half-space with an electroded surface, which is plotted for comparison. The frequency-wave number relations of the waves in Sections 1 to 4 are all linear relations or straight lines going through the origin. In other words, the speed of these waves does not depend on the wave number. They are called nondispersive waves. Nondispersive waves are usually associated with regions without a geometric characteristic length. The waves in plates discussed in this section have wave speed depending on the wave number, and are called dispersive waves. For dispersive waves the frequency-wave number relations are nonlinear as shown in the figure.

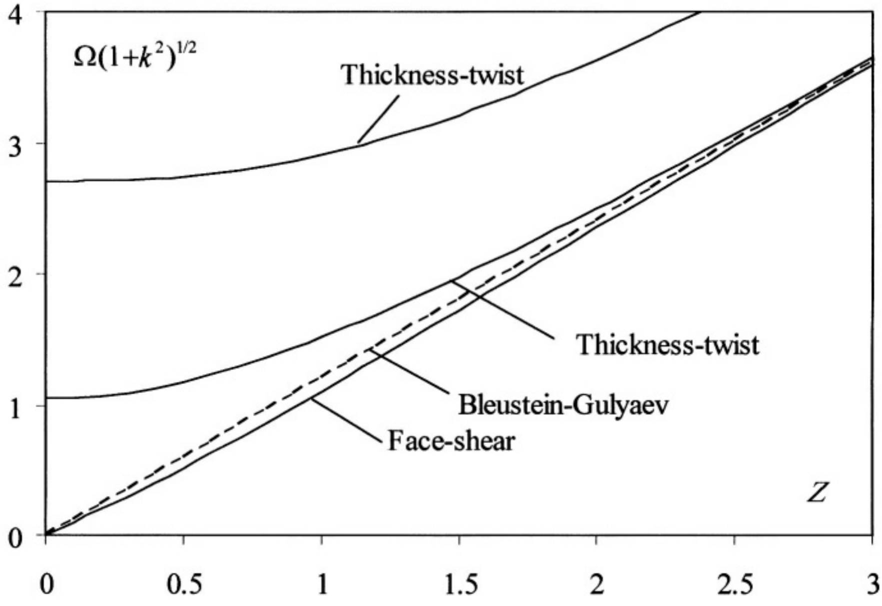

Figure 5.5-2. Dispersion relations for anti-plane waves in a plate.

Problem

5.5-1. Study the case when the plate is unelectroded [30].

6. WAVES IN A PLATE ON A SUBSTRATE

We now consider the possibility of a wave traveling in a metal plate on a ceramic half-space [31]. The ceramic is poled along the x_3 direction (see Figure 5.6-1).

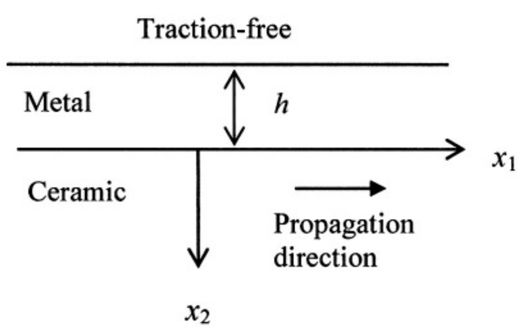

Figure 5.6-1. A metal plate on a ceramic half-space.

With the notation in Section 6 of Chapter 3, the governing equations are

$$\bar{c}\nabla^2 u = \rho\ddot{u}, \quad \nabla^2\psi = 0, \quad x_2 > 0,$$
$$\hat{c}\nabla^2 u = \hat{\rho}\ddot{u}, \quad -h < x_2 < 0, \tag{5.6-1}$$

where $\hat{\rho}$ and $\hat{c}$ are the mass density and shear constant of the metal plate. We look for solutions satisfying

$$u, \quad \psi \to 0, \quad x_2 \to +\infty. \tag{5.6-2}$$

For $x_2 > 0$, the solutions to $(5.6\text{-}1)_1$ satisfying (5.6-2) can be written as

$$u = A\exp(-\xi_2 x_2)\cos(\xi x_1 - \omega t),$$
$$\psi = B\exp(-\xi_1 x_2)\cos(\xi x_1 - \omega t), \tag{5.6-3}$$

where A and B are constants. For $(5.6\text{-}3)_1$ to satisfy $(5.6\text{-}1)_1$, the following must be true

$$\xi_2^2 = \xi_1^2 - \frac{\rho\omega^2}{\bar{c}} = \xi_1^2\left(1 - \frac{v^2}{v_T^2}\right) > 0, \tag{5.6-4}$$

where

$$v^2 = \frac{\omega^2}{\xi_1^2}, \quad v_T^2 = \frac{\bar{c}}{\rho}. \tag{5.6-5}$$

The electric potential and the stress component needed for the boundary conditions are

$$\begin{aligned} T_{23} &= \bar{c}u_{,2} + e\psi_{,2} \\ &= -[\bar{c}A\xi_2 \exp(-\xi_2 x_2) + eB\xi_1 \exp(-\xi_1 x_2)]\cos(\xi x_1 - \omega t), \\ \phi &= \psi + \frac{e}{\varepsilon}u \\ &= [\frac{e}{\varepsilon}A\exp(-\xi_2 x_2) + B\exp(-\xi_1 x_2)]\cos(\xi x_1 - \omega t). \end{aligned} \tag{5.6-6}$$

For $-h < x_2 < 0$, we write

$$u = (\hat{A}\cos\hat{\xi}_2 x_2 + \hat{B}\sin\hat{\xi}_2 x_2)\cos(\xi_1 x_1 - \omega t), \tag{5.6-7}$$

where

$$\hat{\xi}_2^2 = \frac{\hat{\rho}\omega^2}{\hat{c}} - \xi_1^2 = \xi_1^2\left(\frac{v^2}{\hat{v}_T^2} - 1\right), \tag{5.6-8}$$

and

$$\hat{v}_T^2 = \frac{\hat{c}}{\hat{\rho}}. \tag{5.6-9}$$

For boundary conditions, we need

$$T_{23} = \hat{c}u_{,2} = \hat{c}(-\hat{A}\hat{\xi}_2 \sin\hat{\xi}_2 x_2 + \hat{B}\hat{\xi}_2 \cos\hat{\xi}_2 x_2)\cos(\xi_1 x_1 - \omega t). \tag{5.6-10}$$

The continuity and boundary conditions are (except for a factor of $\cos(\xi_1 x_1 - \omega t)$)

$$\begin{aligned} &\phi(0^+) = \frac{e}{\varepsilon}A + B = 0, \quad u(0^+) = A = \hat{A} = u(0^-), \\ &T_{23}(0^+) = -\bar{c}A\xi_2 - eB\xi_1 = \hat{c}\hat{B}\hat{\xi}_2 = T_{23}(0^-), \\ &T_{23}(-h) = \hat{c}(\hat{A}\hat{\xi}_2 \sin\hat{\xi}_2 h + \hat{B}\hat{\xi}_2 \cos\hat{\xi}_2 h) = 0. \end{aligned} \tag{5.6-11}$$

Using $(5.6\text{-}11)_{1,2}$ to eliminate A and B, we obtain

$$-\bar{c}\hat{A}\xi_2 + e\frac{e}{\varepsilon}\hat{A}\xi_1 - \hat{c}\hat{B}\hat{\xi}_2 = 0,$$
$$\hat{A}\hat{\xi}_2 \sin\hat{\xi}_2 h + \hat{B}\hat{\xi}_2 \cos\hat{\xi}_2 h = 0. \quad (5.6\text{-}12)$$

For nontrivial solutions,

$$\begin{vmatrix} -\bar{c}\xi_2 + e\frac{e}{\varepsilon}\xi_1 & -\hat{c}\hat{\xi}_2 \\ \hat{\xi}_2 \sin\hat{\xi}_2 h & \hat{\xi}_2 \cos\hat{\xi}_2 h \end{vmatrix} = 0, \quad (5.6\text{-}13)$$

or

$$\frac{\xi_2}{\xi_1} - \frac{\hat{c}}{\bar{c}}\frac{\hat{\xi}_2}{\xi_1}\tan\hat{\xi}_2 h = \bar{k}^2. \quad (5.6\text{-}14)$$

Substituting from (5.6-4) and (5.6-8),

$$\sqrt{1-\frac{v^2}{v_T^2}} - \frac{\hat{c}}{\bar{c}}\sqrt{\frac{v^2}{\hat{v}_T^2}-1}\,\tan\left[\xi_1 h\sqrt{\frac{v^2}{\hat{v}_T^2}-1}\right] = \bar{k}^2, \quad (5.6\text{-}15)$$

which determines the surface wave speed v as a function of the wave number ξ_1, which is a dispersive wave. When the electromechanical coupling factor $\bar{k} = 0$, Equation (5.6-15) reduces to the dispersion relation for the well-known Love wave in an elastic layer on an elastic half-space. When $\hat{c} = 0$ or $h = 0$, Equation (5.6-15) reduces to (5.3-12) for the Bleustein-Gulyaev wave in a ceramic half-space with an electroded surface.

Problem

5.6-1. Study anti-plane waves in an elastic dielectric plate on a ceramic half-space (see Figure 5.6-2). The plate is electroded at $x_2 = -h$ and the electrode is grounded. What if the electrode is removed?

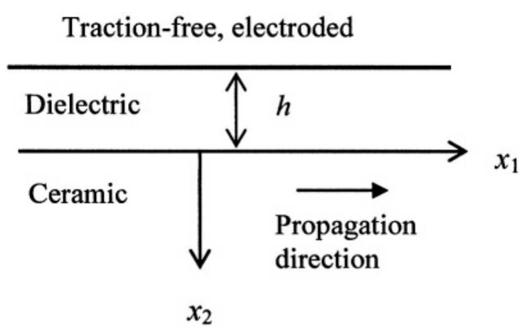

Figure 5.6-2. A dielectric plate on a ceramic half-space.

Answer [32]:

$$\left(1+\frac{\varepsilon}{\hat{\varepsilon}}\tan\xi_1 h\right)\left(\sqrt{1-\frac{v^2}{\hat{v}_T^2}}-\frac{\hat{c}}{\bar{c}}\sqrt{\frac{v^2}{\hat{v}_T^2}-1}\,\tan\xi_1 h\sqrt{\frac{v^2}{\hat{v}_T^2}-1}\right)=\bar{k}^2. \quad (5.6\text{-}16)$$

7. GAP WAVES

7.1 A Gap between Two Half-Spaces of Different Ceramics

Consider two piezoelectric half-spaces of polarized ceramics with a $2h$ gap between them (see Figure 5.7-1). The ceramics are poled in either the x_3 direction or its opposite. The two surfaces of the half-spaces at $x_2 = \pm h$ are traction-free and unelectroded. Acoustic waves in the two half-spaces can be coupled by the electric field in the gap. We consider surface waves in the half-spaces near $x_2 = \pm h$ [33].

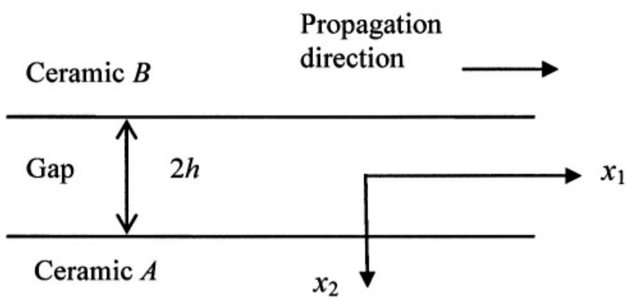

Figure 5.7-1. Two ceramic half-spaces with a gap.

The governing equations are

$$\begin{aligned}&\bar{c}_A\nabla^2 u_A=\rho_A\ddot{u}_A, \quad \nabla^2\psi_A=0, \quad x_2>h,\\&\nabla^2\phi=0, \quad -h<x_2<h,\\&\bar{c}_B\nabla^2 u_B=\rho_B\ddot{u}_B, \quad \nabla^2\psi_B=0, \quad x_2<-h,\end{aligned} \quad (5.7\text{-}1)$$

where the subscripts A and B indicate quantities in ceramic A and ceramic B. For waves near the gap, we require that

$$\begin{aligned}&u_A,\psi_A\to 0, \quad x_2\to+\infty,\\&u_B,\psi_B\to 0, \quad x_2\to-\infty.\end{aligned} \quad (5.7\text{-}2)$$

For $x_2 > h$, the solutions to (5.7-1)$_1$ that satisfy (5.7-2)$_1$ can be written as

$$\begin{aligned} u_A &= U_A \exp(-\eta_A x_2)\cos(\xi x_1 - \omega t), \\ \psi_A &= \Psi_A \exp(-\xi x_2)\cos(\xi x_1 - \omega t), \end{aligned} \tag{5.7-3}$$

where U_A and Ψ_A are undetermined constants,

$$\eta_A^2 = \xi^2 - \frac{\rho_A \omega^2}{\bar{c}_A} = \xi^2\left(1 - \frac{v^2}{v_A^2}\right) > 0, \tag{5.7-4}$$

and

$$v = \frac{\omega}{\xi}, \quad v_A^2 = \frac{\bar{c}_A}{\rho_A}. \tag{5.7-5}$$

For continuity conditions, we need T_{23} and D_2 in ceramic *A*, denoted by T_A and D_A:

$$\begin{aligned} T_A &= \bar{c}_A u_{A,2} + e_A \psi_{A,2} \\ &= -[\bar{c}_A \eta_A U_A \exp(-\eta_A x_2) + e_A \xi \Psi_A \exp(-\xi x_2)]\cos(\xi x_1 - \omega t), \\ D_A &= -\varepsilon_A \psi_{A,2} \\ &= \varepsilon_A \xi \Psi_A \exp(-\xi x_2)\cos(\xi x_1 - \omega t). \end{aligned} \tag{5.7-6}$$

Similarly, for $x_2 < -h$, the solutions to (5.7-1)$_3$ that satisfy (5.7-2)$_2$ are

$$\begin{aligned} u_B &= U_B \exp(\eta_B x_2)\cos(\xi x_1 - \omega t), \\ \psi_B &= \Psi_B \exp(\xi x_2)\cos(\xi x_1 - \omega t), \end{aligned} \tag{5.7-7}$$

where U_B and Ψ_B are undetermined constants,

$$\eta_B^2 = \xi^2 - \frac{\rho_B \omega^2}{\bar{c}_B} = \xi^2\left(1 - \frac{v^2}{v_B^2}\right) > 0, \tag{5.7-8}$$

and

$$v_B^2 = \frac{\bar{c}_B}{\rho_B}. \tag{5.7-9}$$

For continuity conditions, we need T_{23} and D_2 in ceramic *B*, denoted by T_B and D_B:

$$\begin{aligned}
T_B &= \bar{c}_B u_{B,2} + e_B \psi_{B,2} \\
&= [\bar{c}_B \eta_B U_B \exp(\eta_B x_2) + e_B \xi \Psi_B \exp(\xi x_2)] \cos(\xi x_1 - \omega t), \\
D_B &= -\varepsilon_B \psi_{B,2} \\
&= -\varepsilon_B \xi \Psi_B \exp(\xi x_2) \cos(\xi x_1 - \omega t).
\end{aligned} \tag{5.7-10}$$

The fields in the gap can be written as

$$\phi = (\Phi_1 \cosh \xi x_2 + \Phi_2 \sinh \xi x_2) \cos(\xi x_1 - \omega t), \tag{5.7-11}$$

where Φ_1 and Φ_2 are undetermined constants and $(5.7\text{-}1)_2$ is already satisfied. For continuity conditions, we need D_2 in the gap:

$$\begin{aligned}
D_2 &= -\varepsilon_0 \phi_{,2} \\
&= -\varepsilon_0 (\xi \Phi_1 \sinh \xi x_2 + \xi \Phi_2 \cosh \xi x_2) \cos(\xi x_1 - \omega t).
\end{aligned} \tag{5.7-12}$$

For interface continuity conditions, we impose

$$\begin{aligned}
&T_A = 0, \quad \psi_A + \frac{e_A}{\varepsilon_A} u_A = \phi, \quad D_A = D_2, \quad x_2 = h, \\
&T_B = 0, \quad \psi_B + \frac{e_B}{\varepsilon_B} u_B = \phi, \quad D_B = D_2, \quad x_2 = -h,
\end{aligned} \tag{5.7-13}$$

which implies that

$$\begin{aligned}
&\bar{c}_A \eta_A U_A \exp(-\eta_A h) + e_A \xi \Psi_A \exp(-\xi h) = 0, \\
&\Psi_A \exp(-\xi h) + \frac{e_A}{\varepsilon_A} U_A \exp(-\eta_A h) = \Phi_1 \cosh \xi h + \Phi_2 \sinh \xi h, \\
&\varepsilon_A \xi \Psi_A \exp(-\xi h) = -\varepsilon_0 (\xi \Phi_1 \sinh \xi h + \xi \Phi_2 \cosh \xi h), \\
&\bar{c}_B \eta_B U_B \exp(-\eta_B h) + e_B \xi \Psi_B \exp(-\xi h) = 0, \\
&\Psi_B \exp(-\xi h) + \frac{e_B}{\varepsilon_B} U_B \exp(-\eta_B h) = \Phi_1 \cosh \xi h - \Phi_2 \sinh \xi h, \\
&\varepsilon_B \xi \Psi_B \exp(-\xi h) = \varepsilon_0 (-\xi \Phi_1 \sinh \xi h + \xi \Phi_2 \cosh \xi h).
\end{aligned} \tag{5.7-14}$$

Equation (5.7-14) can be rearranged into

$$\begin{pmatrix} \bar{c}_A\eta_A e^{-\eta_A h} & e_A\xi e^{-\xi h} & 0 & 0 & 0 & 0 \\ \frac{e_A}{\varepsilon_A} e^{-\eta_A h} & e^{-\xi h} & -\cosh\xi h & -\sinh\xi h & 0 & 0 \\ 0 & \varepsilon_A\xi e^{-\xi h} & \varepsilon_0\xi\sinh\xi h & \varepsilon_0\xi\cosh\xi h & 0 & 0 \\ 0 & 0 & \varepsilon_0\xi\sinh\xi h & -\varepsilon_0\xi\cosh\xi h & 0 & \varepsilon_B\xi e^{-\xi h} \\ 0 & 0 & -\cosh\xi h & \sinh\xi h & \frac{e_B}{\varepsilon_B} e^{-\eta_B h} & e^{-\xi h} \\ 0 & 0 & 0 & 0 & \bar{c}_B\eta_B e^{-\eta_B h} & e_B\xi e^{-\xi h} \end{pmatrix} \begin{Bmatrix} U_A \\ \Psi_A \\ \Phi_1 \\ \Phi_2 \\ U_B \\ \Psi_B \end{Bmatrix} = 0$$

(5.7-15)

For nontrivial solutions the determinant of the coefficient matrix has to vanish, which yields the dispersion relation of the waves.

7.2 A Gap between Two Half-Spaces of the Same Ceramic

We examine the special case when the two ceramic half-spaces are of the same material:

$$\begin{aligned} &\bar{c}_A = \bar{c}_B = \bar{c}, \quad e_A = e_B = e, \\ &\varepsilon_A = \varepsilon_B = \varepsilon, \quad \rho_A = \rho_B = \rho, \\ &v_A = v_B = v_T. \end{aligned} \qquad (5.7\text{-}16)$$

Then the waves can be separated into symmetric and anti-symmetric ones.

7.2.1 Symmetric Waves

For symmetric waves we consider

$$U_A = U_B,\ \Psi_A = \Psi_B,\ \Phi_2 = 0, \qquad (5.7\text{-}17)$$

for which the last three equations of (5.7-15) become identical to the first three, and the dispersion relation assumes the following simple form:

$$\xi \begin{vmatrix} \bar{c}\eta & e\xi & 0 \\ \frac{e}{\varepsilon} & 1 & -\cosh\xi h \\ 0 & \varepsilon & \varepsilon_0\sinh\xi h \end{vmatrix} = 0, \qquad (5.7\text{-}18)$$

or

$$\cosh\xi h\begin{vmatrix}\bar{c}\eta & e\xi\\ 0 & \varepsilon\end{vmatrix}+\varepsilon_0\sinh\xi h\begin{vmatrix}\bar{c}\eta & e\xi\\ \dfrac{e}{\varepsilon} & 1\end{vmatrix}$$

$$=\varepsilon\bar{c}\eta\cosh\xi h+\varepsilon_0(\bar{c}\eta-\xi\frac{e^2}{\varepsilon})\sinh\xi h=0. \tag{5.7-19}$$

Equation (5.7-19) can be further written as

$$\tanh\xi h=\frac{\varepsilon\bar{c}\eta}{\varepsilon_0(\xi\dfrac{e^2}{\varepsilon}-\bar{c}\eta)}=\frac{n^2\eta}{\xi\bar{k}^2-\eta}, \tag{5.7-20}$$

where we have introduced the refractive index by

$$n^2=\frac{\varepsilon}{\varepsilon_0}. \tag{5.7-21}$$

With (5.7-4), Equation (5.7-20) takes the following form

$$\tanh\xi h=\frac{n^2\sqrt{1-\dfrac{v^2}{v_T^2}}}{\bar{k}^2-\sqrt{1-\dfrac{v^2}{v_T^2}}}. \tag{5.7-22}$$

Equation (5.7-22) shows that the wave is dispersive.

7.2.2 Anti-Symmetric Waves

For anti-symmetric waves we consider

$$U_A=-U_B,\ \Psi_A=-\Psi_B,\ \Phi_1=0. \tag{5.7-23}$$

Then the last three equations of (5.7-15) become identical to the first three, and the dispersion relation assumes the following simple form:

$$\xi\begin{vmatrix}\bar{c}\eta & e\xi & 0\\ \dfrac{e}{\varepsilon} & 1 & -\sinh\xi h\\ 0 & \varepsilon & \varepsilon_0\cosh\xi h\end{vmatrix}=0, \tag{5.7-24}$$

or

$$\sinh\xi h\begin{vmatrix}\bar{c}\eta & e\xi\\ 0 & \varepsilon\end{vmatrix}+\varepsilon_0\cosh\xi h\begin{vmatrix}\bar{c}\eta & e\xi\\ \dfrac{e}{\varepsilon} & 1\end{vmatrix} \tag{5.7-25}$$

$$=\varepsilon\bar{c}\eta\sinh\xi h+\varepsilon_0(\bar{c}\eta-\xi\frac{e^2}{\varepsilon})\cosh\xi h=0.$$

Equation (5.7-25) can be further written as

$$\tanh\xi h=\frac{\varepsilon_0(\xi\dfrac{e^2}{\varepsilon}-\bar{c}\eta)}{\varepsilon\bar{c}\eta}=\frac{\xi\bar{k}^2-\eta}{n^2\eta}. \tag{5.7-26}$$

With (5.7-4), Equation (5.7-26) takes the following form:

$$\tanh\xi h=\left(\frac{n^2\sqrt{1-\dfrac{v^2}{v_T^2}}}{\bar{k}^2-\sqrt{1-\dfrac{v^2}{v_T^2}}}\right)^{-1}. \tag{5.7-27}$$

We note that the right-hand side of (5.7-27) is the inverse of that of (5.7-22). Therefore for any $v < v_T$, one of the right-hand sides of (5.7-27) and (5.7-22) is smaller than 1. Since the range of a hyperbolic tangent function is between –1 and 1, a root for ξ can be found from either (5.7-27) or (5.7-22).

Problem

5.7-1. Study anti-plane waves propagating in a ceramic plate between two ceramic half-spaces.

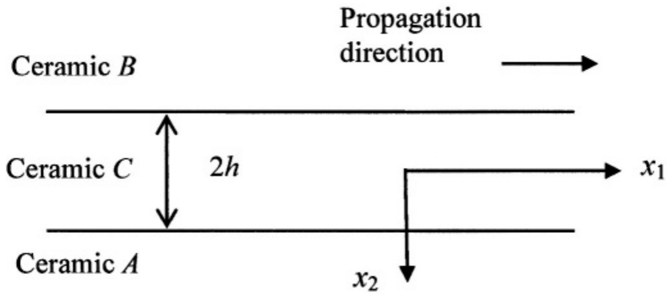

Figure 5.7-2. A ceramic plate between two ceramic half-spaces.

8. WAVES ON A CIRCULAR CYLINDRICAL SURFACE

Consider SH waves propagating on the surface of a circular cylinder [34] as shown in Figure 5.8-1. The cylinder is made of ceramics with axial poling along the x_3 direction. We choose (r,θ,z) to correspond to (1,2,3) so that the poling direction corresponds to 3.

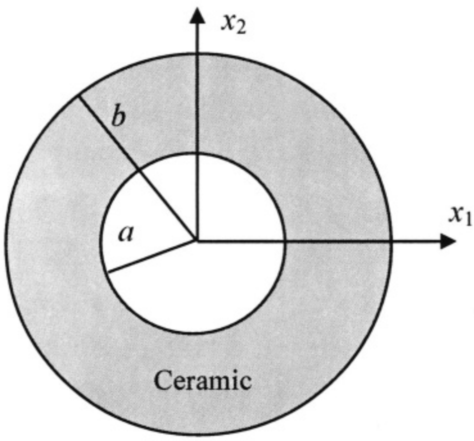

Figure 5.8-1. A circular cylinder of polarized ceramics.

From Section 6 of Chapter 3, the governing equations are

$$\begin{aligned} &\bar{c}_{44}\nabla^2 u_z = \rho \ddot{u}_z, \quad a < r < b, \\ &\nabla^2 \psi = 0, \quad a < r < b. \end{aligned} \tag{5.8-1}$$

The electric potential is related to ψ and u by

$$\phi = \psi + \frac{e_{15}}{\varepsilon_{11}} u_z . \tag{5.8-2}$$

The stress and electric displacement components relevant to boundary conditions are

$$\begin{aligned} &T_{rz} = \bar{c}_{44} u_{z,r} + e_{15}\psi_{,r}, \\ &D_r = -\varepsilon_{11}\psi_{,r}. \end{aligned} \tag{5.8-3}$$

In polar coordinates we have

$$v_T^2(\frac{\partial^2 u_z}{\partial r^2}+\frac{1}{r}\frac{\partial u_z}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2})=\ddot{u}_z,$$
$$\frac{\partial^2 \psi}{\partial r^2}+\frac{1}{r}\frac{\partial \psi}{\partial r}+\frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2}=0, \quad (5.8\text{-}4)$$

where

$$v_T^2=\frac{\bar{c}_{44}}{\rho}. \quad (5.8\text{-}5)$$

For waves propagating in the θ direction we consider

$$u_z(r,\theta,t)=u(r)\cos(\nu\theta-\omega t),$$
$$\psi(r,\theta,t)=\psi(r)\cos(\nu\theta-\omega t), \quad (5.8\text{-}6)$$

where ν is allowed to assume any real value that is greater than or equal to 1. Substitution of (5.8-6) into (5.8-4) results in

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+(\xi^2-\frac{\nu^2}{r^2})u=0,$$
$$\frac{\partial^2 \psi}{\partial r^2}+\frac{1}{r}\frac{\partial \psi}{\partial r}-\frac{\nu^2}{r^2}\psi=0, \quad (5.8\text{-}7)$$

where we have denoted

$$\xi=\frac{\omega}{v_T}. \quad (5.8\text{-}8)$$

Equation $(5.8\text{-}7)_1$ can be written as Bessel's equation of order ν. Then the general solution can be written as

$$u_z=[C_1J_\nu(\xi r)+C_2Y_\nu(\xi r)]\cos(\nu\theta-\omega t),$$
$$\psi=[C_3r^\nu+C_4r^{-\nu}]\cos(\nu\theta-\omega t), \quad (5.8\text{-}9)$$

where J_ν and Y_ν are the ν-th order Bessel functions of the first and second kind, and C_1 – C_4 are undetermined constants. From (5.8-9), (5.8-2) and (5.8-3) we obtain the following expressions that are needed for boundary conditions:

$$\phi=\{C_3r^\nu+C_4r^{-\nu}+\frac{e_{15}}{\varepsilon_{11}}[C_1J_\nu(\xi r)+C_2Y_\nu(\xi r)]\}\cos(\nu\theta-\omega t),$$
$$T_{rz}=\{\bar{c}_{44}[C_1\xi J'_\nu(\xi r)+C_2\xi Y'_\nu(\xi r)]$$
$$+e_{15}[C_3\nu r^{\nu-1}-C_4\nu r^{-\nu-1}]\}\cos(\nu\theta-\omega t), \quad (5.8\text{-}10)$$
$$D_r=-\varepsilon_{11}(C_3\nu r^{\nu-1}-C_4\nu r^{-\nu-1})\}\cos(\nu\theta-\omega t),$$

where a superimposed prime indicates differentiation with respect to the whole argument of a function. Consider a solid cylinder *(a* = 0 in Figure 5.8-

1). The surface at $r = b$ is traction-free and carries a very thin electrode of a perfect conductor. Since Y_ν and $r^{-\nu-1}$ are singular at the origin, terms associated with C_2 and C_4 have to be dropped. T_{rz} and ϕ should both vanish at $r = b$. This leads to the following two homogeneous, linear, algebraic equations for C_1 and C_3:

$$\begin{aligned} &\bar{c}_{44}C_1\xi J_\nu'(\xi b) + e_{15}C_3\nu b^{\nu-1} = 0, \\ &C_3 b^\nu + \frac{e_{15}}{\varepsilon_{11}}C_1 J_\nu(\xi b) = 0. \end{aligned} \tag{5.8-11}$$

For nontrivial solutions the determinant of the coefficient matrix has to vanish, which results in the following equation for ξ:

$$\begin{aligned} &\begin{vmatrix} \bar{c}_{44}\xi J_\nu'(\xi b) & e_{15}\nu b^{\nu-1} \\ \dfrac{e_{15}}{\varepsilon_{11}} J_\nu(\xi b) & b^\nu \end{vmatrix} \\ &= \bar{c}_{44}\xi J_\nu'(\xi b) b^\nu - \frac{e_{15}^2}{\varepsilon_{11}} J_\nu(\xi b)\nu b^{\nu-1} = 0, \end{aligned} \tag{5.8-12}$$

or,

$$\frac{\xi b J_\nu'(\xi b)}{\nu J_\nu(\xi b)} = \bar{k}_{15}^2. \tag{5.8-13}$$

Consider the special case when the wavelength is much smaller than the cylinder radius. Then the cylinder is effectively like a half-space for these short waves. It is convenient to introduce a surface wave number α and a surface wave speed v by

$$\alpha = \frac{\nu}{b}, \quad v = \frac{\omega}{\alpha}. \tag{5.8-14}$$

Consider the limit when $\nu \to \infty$ and $b \to \infty$ but α remains finite. We have

$$\xi b = \frac{\omega}{v_T} b = \frac{\omega}{v_T}\frac{b}{\nu}\nu = \frac{\omega}{v_T}\frac{1}{\alpha}\nu = \frac{v}{v_T}\nu. \tag{5.8-15}$$

With the following asymptotic expression due to Carlini for Bessel's functions of large orders:

$$J_\nu(\nu x) \to \frac{x^\nu \exp(\nu\sqrt{1-x^2})}{\sqrt{2\pi\nu}\,\sqrt[4]{1-x^2}\,(1+\sqrt{1-x^2})^\nu}, \quad \nu \to \infty, \quad 0 < x < 1, \tag{5.8-16}$$

Equation (5.8-13) reduces to

$$\sqrt{1-\frac{v^2}{v_T^2}} = \bar{k}_{15}^2, \tag{5.8-17}$$

which gives the surface wave speed in (5.3-12)

Problems

5.8-1. Show (5.8-17).
5.8-2. Study the case of an unelectroded cylinder [34].

9. ACOUSTIC WAVE GENERATION

We now consider the generation of SH acoustic waves in an elastic medium (see Figure 5.9-1). The piezoelectric layer is ceramics poled along the x_3 direction. It is driven by an alternating voltage. Waves are generated in the elastic half-space, propagating away from the ceramic plate.

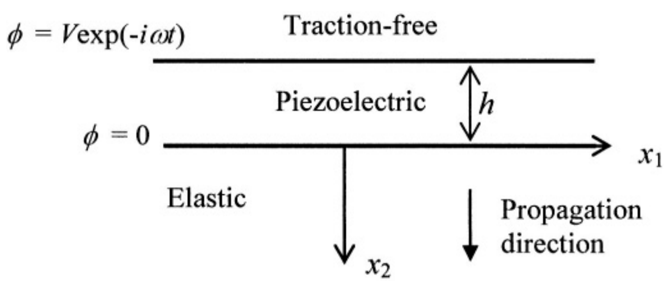

Figure 5.9-1. A ceramic plate on an elastic half-space.

With the notation in Section 6 of Chapter 3, the governing equations, the boundary and continuity conditions are

$$\begin{aligned}
&\bar{c}\nabla^2 u = \rho\ddot{u}, \quad \nabla^2\psi = 0, \quad \phi = \psi + \frac{e}{\varepsilon}u, \quad -h < x_2 < 0, \\
&\hat{c}\nabla^2 u = \hat{\rho}\ddot{u}, \quad x_2 > 0, \\
&T_{23}(x_2 = -h) = 0, \\
&\phi(x_2 = -h) - \phi(x_2 = 0) = V\exp(-i\omega t), \\
&u(x_2 = 0^-) = u(x_2 = 0^+), \quad T_{23}(x_2 = 0^-) = T_{23}(x_2 = 0^+), \\
&u \quad \text{outgoing}, \quad x_2 \to +\infty,
\end{aligned} \tag{5.9-1}$$

where $\hat{\rho}$ and $\hat{c}$ are the mass density and shear constant of the elastic medium. For $-h < x_2 < 0$, the solutions can be written as

$$u = (A_1 \cos \xi x_2 + A_2 \sin \xi x_2)\exp(-i\omega t),$$
$$\psi = (B_1 x_2 + B_2)\exp(-i\omega t), \tag{5.9-2}$$

where A_1, A_2, B_1 and B_2 are undetermined constants, and

$$\xi^2 = \frac{\rho\omega^2}{\bar{c}}. \tag{5.9-3}$$

For boundary and continuity conditions, we need

$$T_{23} = \bar{c}u_{,2} + e\psi_{,2}$$
$$= (-\bar{c}A_1\xi \sin \xi x_2 + \bar{c}A_2\xi \cos \xi x_2 + eB_1)\exp(-i\omega t),$$
$$\phi = \psi + \frac{e}{\varepsilon}u \tag{5.9-4}$$
$$= \left(\frac{e}{\varepsilon}A_1 \cos \xi x_2 + \frac{e}{\varepsilon}A_2 \sin \xi x_2 + B_1 x_2 + B_2\right)\exp(-i\omega t).$$

For $x_2 > 0$, the solution can be written as

$$u = \hat{A}\exp i(\hat{\xi}x_2 - \omega t), \tag{5.9-5}$$

where $\hat{A}$ is an undetermined constant, and

$$\hat{\xi}^2 = \frac{\hat{\rho}\omega^2}{\hat{c}}. \tag{5.9-6}$$

Equation (5.9-5) already satisfies the condition that the waves in the elastic medium are propagating away from the plate (radiation condition). For continuity conditions we need

$$T_{23} = \hat{c}u_{,2} = \hat{c}i\hat{\xi}\hat{A}\exp i(\hat{\xi}x_2 - \omega t). \tag{5.9-7}$$

The continuity and boundary conditions are (except for a factor of $\exp(-i\omega t)$)

$$\bar{c}A_1\xi \sin \xi h + \bar{c}A_2\xi \cos \xi h + eB_1 = 0,$$
$$\frac{e}{\varepsilon}A_1 \cos \xi h - \frac{e}{\varepsilon}A_2 \sin \xi h - B_1 h + B_2 - \frac{e}{\varepsilon}A_1 - B_2 = V, \tag{5.9-8}$$
$$A_1 = \hat{A},$$
$$\bar{c}A_2\xi + eB_1 = \hat{c}i\hat{\xi}\hat{A}.$$

Our main interest is to find the following:

$$\hat{A} = \frac{e}{\bar{c}}\frac{1 - \cos \xi h}{\Delta}, \tag{5.9-9}$$

where

$$\Delta = (1-\cos\xi h)\left[k^2(\cos\xi h - 1) - i\frac{\hat{c}}{c}\hat{\xi}h\right]$$
$$+\left(\sin\xi h + i\frac{\hat{c}\hat{\xi}}{\bar{c}\xi}\right)(\xi h - k^2\sin\xi h), \tag{5.9-10}$$
$$k^2 = \frac{e^2}{\varepsilon\bar{c}}.$$

It can be seen that $\hat{A}$ depends strongly on the driving frequency.

Problem

5.9-1. Study the dependence of $\hat{A}$ on the driving frequency.

Chapter 6

LINEAR EQUATIONS FOR SMALL FIELDS SUPERPOSED ON FINITE BIASING FIELDS

The theory of linear piezoelectricity assumes infinitesimal deviations from an ideal reference state of the material in which there are no pre-existing mechanical and/or electrical fields (initial or biasing fields). The presence of biasing fields makes a material apparently behave like a different material, and renders the linear theory of piezoelectricity invalid. The behavior of electroelastic bodies under biasing fields can be described by the theory for infinitesimal incremental fields superposed on finite biasing fields, which is a consequence of the nonlinear theory of electroelasticity. Knowledge of the behavior of electroelastic bodies under biasing fields is important in many applications including the buckling of thin electroelastic structures, frequency stability of piezoelectric resonators, acoustic wave sensors based on frequency shifts due to biasing fields, characterization of nonlinear electroelastic materials by propagation of small-amplitude waves in electroelastic bodies under biasing fields, and electrostrictive ceramics which operate under a biasing electric field. This chapter presents the theory for small fields superposed on biasing fields in an electroelastic body and some of its applications.

1. A NONLINEAR SPRING

The basic concept of small fields superposed on finite biasing fields can be well explained by a simple nonlinear spring. Consider the following spring-mass system (see Figure 6.1-1). When the spring is stretched by x, the force in the spring is $kx + k'x^2$, where k and k' are linear and nonlinear spring constants.

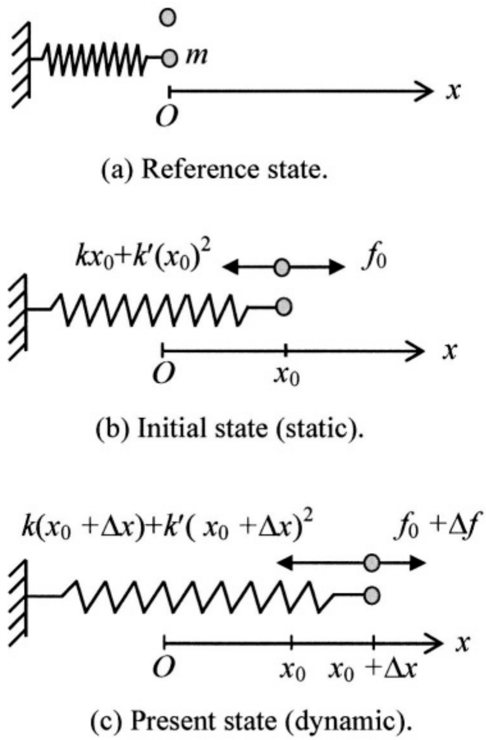

Figure 6.1-1. Reference, initial, and present states of a nonlinear spring-mass system.

The reference state in Figure 6.1-1 (a) is the natural state of the spring when there is no force and stretch in it. Under an initial, constant force f_0 the mass m is in equilibrium with an initial stretch x_0 in the spring (see Figure 6.1-1 (b)) such that

$$f_0 = kx_0 + k'x_0^2. \tag{6.1-1}$$

Then a small, dynamic, incremental force Δf is applied, and the mass is in small amplitude motion around x_0 with position $x_0 + \Delta x$ (see Figure 6.1-1 (c)). Since both Δf and Δx are small, we want to derive a linear relation between them. In the state in Figure 6.1-1 (c) the equation of motion for the mass is

$$\begin{aligned} m\frac{d^2}{dt^2}(x_0 + \Delta x) &= (f_0 + \Delta f) - [k(x_0 + \Delta x) + k'(x_0 + \Delta x)^2] \\ &= (f_0 + \Delta f) - [kx_0 + k\Delta x + k'x_0^2 + k'2x_0\Delta x + k'(\Delta x)^2] \\ &= (f_0 - kx_0 - k'x_0^2) + [\Delta f - k\Delta x - k'2x_0\Delta x - k'(\Delta x)^2]. \end{aligned} \tag{6.1-2}$$

Using (6.1-1) and the smallness of Δx,

$$m\frac{d^2}{dt^2}(\Delta x) = \Delta f - k\Delta x - k'2x_0\Delta x - k'(\Delta x)^2$$
$$\cong \Delta f - (k + k'2x_0)\Delta x \qquad (6.1\text{-}3)$$
$$= \Delta f - k^e\Delta x,$$

where

$$k^e = k + 2k'x_0 \qquad (6.1\text{-}4)$$

is an effective linear spring constant at the initial stretch x_0. Thus at the state with an initial stretch, the nonlinear spring responds to small, incremental changes like a linear spring with an effective linear spring constant k^e. It is important to note that k^e depends on x_0 and the nonlinear spring constant k'.

2. LINEARIZATION ABOUT A BIAS

The concept in the previous section can be generalized to an electroelastic body [35]. Consider the following three states of an electroelastic body (see Figure 6.2-1):

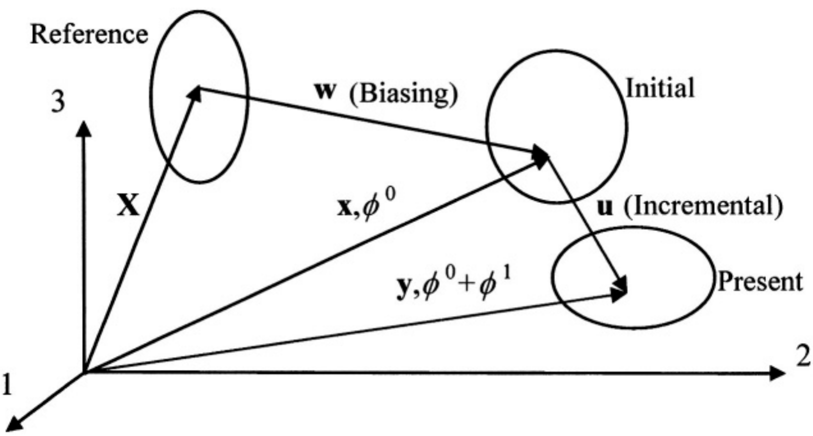

Figure 6.2-1. Reference, initial, and present configurations of an electroelastic body.

(i) The reference state: In this state the body is undeformed and free of electric fields. A generic point at this state is denoted by **X** with Cartesian coordinates X_K. The mass density is ρ_0.

(ii) The initial state: In this state the body is deformed finitely and statically, and carries finite static electric fields. The body is under the action of body force f_α^0, body charge ρ_E^0, prescribed surface position $\bar{x}_\alpha$, surface traction $\bar{T}_\alpha^0$, surface potential $\bar{\phi}^0$ and surface charge $\bar{\sigma}_e^0$. The deformation and fields at this configuration are the initial or biasing fields. The position of the material point associated with **X** is given by **x** = **x**(**X**) or $x_\gamma = x_\gamma(\mathbf{X})$, with strain S_{KL}^0. Greek indices are used for the initial configuration. The electric potential in this state is denoted by $\phi^0(\mathbf{X})$, with electric field $\mathcal{E}_K^0$. **x**(**X**) and $\phi^0(\mathbf{X})$ satisfy the following static equations of nonlinear electroelasticity:

$$\begin{aligned}
&S_{KL}^0 = (x_{\alpha,K}x_{\alpha,L} - \delta_{KL})/2, \quad \mathcal{E}_K^0 = -\phi_{,K}^0, \quad E_\alpha^0 = -\phi_{,\alpha}^0,\\
&T_{KL}^0 = \rho_0 \left.\frac{\partial \psi}{\partial S_{KL}}\right|_{S_{KL}^0,\mathcal{E}_K^0}, \quad \mathcal{P}_K^0 = -\rho_0 \left.\frac{\partial \psi}{\partial \mathcal{E}_K}\right|_{S_{KL}^0,\mathcal{E}_K^0},\\
&J^0 = \det(x_{\alpha,K}),\\
&K_{K\alpha}^0 = x_{\alpha,L}T_{KL}^0 + M_{K\alpha}^0, \quad \mathcal{D}_K^0 = \varepsilon_0 J^0 X_{K,\alpha}X_{L,\alpha}\mathcal{E}_L^0 + \mathcal{P}_K^0,\\
&M_{K\alpha}^0 = J^0 X_{K,\beta}\varepsilon_0(E_\beta^0 E_\alpha^0 - \frac{1}{2}E_\gamma^0 E_\gamma^0 \delta_{\beta\alpha}),\\
&K_{K\alpha,K}^0 + \rho_0 f_\alpha^0 = 0, \quad \mathcal{D}_{K,K}^0 = \rho_E^0.
\end{aligned} \tag{6.2-1}$$

(iii) The present state: In this state, time-dependent, small, incremental deformations and electric fields are applied to the deformed body at the initial state. The body is under the action of f_i, ρ_E, $\bar{y}_i$, $\bar{T}_i$, $\bar{\phi}$ and $\bar{\sigma}$. The final position of **X** is given by **y** = **y**(**X**,t), and the final electric potential is $\phi(\mathbf{X},t) = \phi^0(\mathbf{X}) + \phi^1(\mathbf{X},t)$. **y**(**X**,$t$) and $\phi(\mathbf{X},t)$ satisfy the dynamic equations of nonlinear electroelasticity:

$$\begin{aligned}
&S_{KL} = (y_{i,K}y_{i,L} - \delta_{KL})/2, \quad \mathcal{E}_K = -\phi_{,K}, \quad E_i = -\phi_{,i}\\
&T_{KL}^S = \rho_0 \left.\frac{\partial \psi}{\partial S_{KL}}\right|_{S_{KL},\mathcal{E}_K}, \quad \mathcal{P}_K = -\rho_0 \left.\frac{\partial \psi}{\partial \mathcal{E}_K}\right|_{S_{KL},\mathcal{E}_K},\\
&K_{Lj} = y_{j,K}T_{KL}^S + M_{Lj}, \quad \mathcal{D}_K = \varepsilon_0 J C_{KL}^{-1}\mathcal{E}_L + \mathcal{P}_K,\\
&M_{Lj} = JX_{L,i}\varepsilon_0(E_i E_j - \frac{1}{2}E_k E_k \delta_{ij}),\\
&K_{Lj,L} + \rho_0 f_j = \rho_0 \dot{v}_j, \quad \mathcal{D}_{K,K} = \rho_E.
\end{aligned} \tag{6.2-2}$$

2.1 Linearization of Differential Equations

Let the incremental displacement be $\mathbf{u}(\mathbf{X},t)$ (see Figure 6.2-1). $\mathbf{u}$ and ϕ^1 are assumed to be infinitesimal. We write $\mathbf{y}$ and ϕ as

$$\begin{aligned} y_i(\mathbf{X},t) &= \delta_{i\alpha}[x_\alpha(\mathbf{X},t) + \lambda u_\alpha(\mathbf{X},t)], \\ \phi(\mathbf{X},t) &= \phi^0(\mathbf{X},t) + \lambda\phi^1(\mathbf{X},t), \end{aligned} \tag{6.2-3}$$

where a dimensionless parameter λ is introduced to indicate the smallness of the incremental deformations and fields. In the following, terms quadratic in or of higher order of λ will be dropped. Substitution of (6.2-3) into (6.2-2) yields

$$\begin{aligned} S_{KL} &= (y_{i,K}y_{i,L} - \delta_{KL})/2 \cong S_{KL}^0 + \lambda S_{KL}^1, \\ \mathcal{E}_K &= -\phi_{,K} = \mathcal{E}_K^0 + \lambda \mathcal{E}_K^1, \end{aligned} \tag{6.2-4}$$

where

$$\begin{aligned} S_{KL}^1 &= (x_{\alpha,K}u_{\alpha,L} + x_{\alpha,L}u_{\alpha,K})/2, \\ \mathcal{E}_K^1 &= -\phi_{,K}^1, \end{aligned} \tag{6.2-5}$$

and

$$\begin{aligned} T_{KL}^S &\cong T_{KL}^0 + \lambda T_{KL}^1, \\ \mathcal{P}_K &\cong \mathcal{P}_K^0 + \lambda \mathcal{P}_K^1, \end{aligned} \tag{6.2-6}$$

where

$$\begin{aligned} T_{KL}^1 &= \rho_0 \frac{\partial^2 \psi}{\partial S_{KL}\partial S_{MN}}\bigg|_{S_{KL}^0,\mathcal{E}_K^0} S_{MN}^1 + \rho_0 \frac{\partial^2 \psi}{\partial \mathcal{E}_{KL}\partial \mathcal{E}_M}\bigg|_{S_{KL}^0,\mathcal{E}_K^0} \mathcal{E}_M^1, \\ \mathcal{P}_K^1 &= -\rho_0 \frac{\partial^2 \psi}{\partial \mathcal{E}_K\partial S_{MN}}\bigg|_{S_{KL}^0,\mathcal{E}_K^0} S_{MN}^1 - \rho_0 \frac{\partial^2 \psi}{\partial \mathcal{E}_K\partial \mathcal{E}_M}\bigg|_{S_{KL}^0,\mathcal{E}_K^0} \mathcal{E}_M^1. \end{aligned} \tag{6.2-7}$$

To the first order of λ,

$$\begin{aligned} X_{K,j} &= X_{K,\alpha}x_{\alpha,j} = X_{K,\alpha}(\delta_{i\alpha}y_i - u_\alpha)_{,j} \\ &= X_{K,\alpha}(\delta_{i\alpha}\delta_{i,j} - \lambda u_{\alpha,j}) = X_{K,\alpha}(\delta_{j\alpha} - \lambda u_{\alpha,L}X_{L,j}) \\ &= X_{K,\alpha}[\delta_{j\alpha} - \lambda u_{\alpha,L}X_{L,\beta}(\delta_{j\beta} - \lambda u_{\beta,M}X_{M,j})] \\ &\cong X_{K,\alpha}[\delta_{j\alpha} - \lambda u_{\alpha,L}X_{L,\beta}\delta_{j\beta}] \\ &= \delta_{j\alpha}X_{K,\alpha} - \lambda X_{K,\alpha}u_{\alpha,L}X_{L,\beta}\delta_{\beta j}. \end{aligned} \tag{6.2-8}$$

From (1.1-22),

$$\begin{aligned} JX_{K,j} &= \frac{1}{2}\varepsilon_{KLM}\varepsilon_{jlm}y_{l,L}y_{m,M} \\ &\cong J^0\delta_{j\alpha}X_{K,\alpha} + \lambda\varepsilon_{KLM}\varepsilon_{j\beta\gamma}x_{\beta,L}u_{\gamma,M} \\ &= J^0\delta_{j\alpha}X_{K,\alpha} + \lambda J^0\delta_{j\alpha}(X_{K,\alpha}X_{L,\gamma} - X_{K,\gamma}X_{L,\alpha})u_{\gamma,L}. \end{aligned} \tag{6.2-9}$$

For the electric field

$$\begin{aligned} E_i &= -\phi_{,i} = -\phi_{,K}X_{K,i} = \mathcal{E}_K X_{K,i} \\ &\cong \delta_{i\alpha}E^0_\alpha + \lambda\delta_{i\alpha}(\mathcal{E}^1_M X_{M,\alpha} - E^0_\beta u_{\beta,M}X_{M,\alpha}). \end{aligned} \tag{6.2-10}$$

Then the Maxwell stress tensor and $JC^{-1}_{KL}\mathcal{E}_L$ can be expanded as

$$\begin{aligned} M_{Ki} &= \delta_{i\alpha}(M^0_{K\alpha} + \lambda M^1_{K\alpha}), \\ JC^{-1}_{KL}\mathcal{E}_L &= (JC^{-1}_{KL}\mathcal{E}_L)^0 + \lambda(JC^{-1}_{KL}\mathcal{E}_L)^1, \end{aligned} \tag{6.2-11}$$

where

$$\begin{aligned} M^1_{K\alpha} &= g_{K\alpha L\gamma}u_{\gamma,L} - r_{LK\alpha}\mathcal{E}^1_L, \\ (JC^{-1}_{KL}\mathcal{E}_L)^1 &= r_{KL\gamma}u_{\gamma,L} + l_{KL}\mathcal{E}^1_L, \end{aligned} \tag{6.2-12}$$

and

$$\begin{aligned} g_{K\alpha L\gamma} &= \varepsilon_0 J^0[E^0_\alpha E^0_\beta(X_{K,\beta}X_{L,\gamma} - X_{K,\gamma}X_{L,\beta}) \\ &\quad - E^0_\alpha E^0_\gamma X_{K,\beta}X_{L,\beta} \\ &\quad + E^0_\beta E^0_\gamma(X_{K,\alpha}X_{L,\beta} - X_{K,\beta}X_{L,\alpha}) \\ &\quad + \frac{1}{2}E^0_\beta E^0_\beta(X_{K,\gamma}X_{L,\alpha} - X_{K,\alpha}X_{L,\gamma})], \\ r_{KL\gamma} &= \varepsilon_0 J^0(E^0_\alpha X_{K,\alpha}X_{L,\gamma} - E^0_\alpha X_{K,\gamma}X_{L,\alpha} - E^0_\gamma X_{K,\alpha}X_{L,\alpha}), \\ l_{KL} &= \varepsilon_0 J^0 X_{K,\alpha}X_{L,\alpha}. \end{aligned} \tag{6.2-13}$$

Then

$$\begin{aligned} K_{Ki} &= y_{i,L}T^S_{KL} + M_{Ki} \cong \delta_{i\alpha}(K^0_{K\alpha} + \lambda K^1_{K\alpha}), \\ \mathcal{D}_K &= \varepsilon_0 JC^{-1}_{KL}\mathcal{E}_L + \mathcal{P}_K \cong \mathcal{D}^0_K + \lambda\mathcal{D}^1_K, \end{aligned} \tag{6.2-14}$$

where

$$\begin{aligned} K^1_{K\alpha} &= u_{\alpha,L}T^0_{KL} + x_{\alpha,L}T^1_{KL} + M^1_{K\alpha}, \\ \mathcal{D}^1_K &= \varepsilon_0(JC^{-1}_{KL}E_L)^1 + \mathcal{P}^1_K. \end{aligned} \tag{6.2-15}$$

Equation (6.2-15) can be further written as

$$K^1_{L\gamma} = G_{L\gamma M\alpha} u_{\alpha,M} - R_{ML\gamma} \mathcal{E}^1_M, \qquad (6.2\text{-}16)$$
$$\mathcal{D}^1_K = R_{KL\gamma} u_{\gamma,L} + L_{KL} \mathcal{E}^1_L,$$

which shows that the incremental stress tensor and electric displacement vector depend linearly on the incremental displacement gradient and potential gradient. In (6.2-16),

$$G_{K\alpha L\gamma} = x_{\alpha,M}\, \rho_0 \left. \frac{\partial^2 \psi}{\partial S_{KM} \partial S_{LN}} \right|_{S^0_{KL}, \mathcal{E}^0_K} x_{\gamma,N}$$
$$+ T^0_{KL} \delta_{\alpha\gamma} + g_{K\alpha L\gamma} = G_{L\gamma K\alpha},$$
$$R_{KL\gamma} = -\rho_0 \left. \frac{\partial^2 \psi}{\partial \mathcal{E}_K \partial S_{ML}} \right|_{S^0_{KL}, \mathcal{E}^0_K} x_{\gamma,M} + r_{KL\gamma}, \qquad (6.2\text{-}17)$$
$$L_{KL} = -\rho_0 \left. \frac{\partial^2 \psi}{\partial \mathcal{E}_K \partial \mathcal{E}_L} \right|_{S^0_{KL}, \mathcal{E}^0_K} + l_{KL} = L_{LK}.$$

$G_{K\alpha L\gamma}$, $R_{KL\gamma}$, and L_{KL} are called the effective or apparent elastic, piezoelectric, and dielectric constants. They depend on the initial deformation $x_\alpha(\mathbf{X})$ and electric potential $\phi^0(\mathbf{X})$. Even when a material is considered linear, i.e., only the second-order material constants need to be considered, the effective material constants still show modifications by the biasing fields. The effective material constants in general have lower symmetry than the fundamental linear elastic, piezoelectric, and dielectric constants. This is called induced anisotropy or symmetry breaking. There can be as many as 45 independent components for $G_{K\alpha L\gamma}$, 27 independent components for $R_{KL\gamma}$, and 6 independent components for L_{KL}. Since the fields in the present configuration satisfy (6.2-2) and the biasing fields satisfy (6.2-1), we have

$$K^1_{K\alpha,K} + \rho_0 f^1_\alpha = \rho_0 \ddot{u}_\alpha, \qquad (6.2\text{-}18)$$
$$\mathcal{D}^1_{K,K} = \rho^1_E,$$

where f^1_α and ρ^1_E are determined from

$$f_i = \delta_{i\alpha}(f^0_\alpha + \lambda f^1_\alpha), \quad \rho_E = \rho^0_E + \lambda \rho^1_E. \qquad (6.2\text{-}19)$$

In the above derivation, λ can be set to 1 everywhere.

The boundary value problem for the incremental fields **u** and ϕ^1 consists of the following equations and boundary conditions:

$$
\begin{aligned}
&K^1_{K\alpha,K} + \rho_0 f^1_\alpha = \rho_0 \ddot{u}_\alpha, \quad \text{in} \quad V,\\
&\mathcal{D}^1_{K,K} = \rho^1_E, \quad \text{in} \quad V,\\
&K^1_{L\gamma} = G_{L\gamma M\alpha} u_{\alpha,M} - R_{ML\gamma} \mathcal{E}^1_M, \quad \text{in} \quad V,\\
&\mathcal{D}^1_K = R_{KL\gamma} u_{\gamma,L} + L_{KL} \mathcal{E}^1_L, \quad \text{in} \quad V,\\
&u_\alpha = \overline{u}_\alpha \quad \text{on} \quad S_y,\\
&\phi^1 = \overline{\phi}^1 \quad \text{on} \quad S_\phi,\\
&K^1_{L\alpha} N_L = \overline{T}^1_\alpha \quad \text{on} \quad S_T,\\
&\mathcal{D}^1_K N_K = -\overline{\sigma}^1_e \quad \text{on} \quad S_D.
\end{aligned}
\qquad (6.2\text{-}20)
$$

Because of the dependence of $G_{K\alpha L\gamma}$, $R_{KL\gamma}$, and L_{KL} on the initial deformations and fields, (6.2-20) in general are equations with variable coefficients.

2.2 A Variational Principle

The symmetries shown in (6.2-17) imply that the differential operators in (6.2-20) are self-adjoint (see Section 6). It can be verified that the stationary condition of the following variational functional under the constraint of the boundary conditions on S_y and S_ϕ yields (6.2-18) and the boundary conditions on S_T and S_D:

$$
\begin{aligned}
\Pi(\mathbf{u},\phi^1) = &\int_{t_0}^{t} dt \int_V \Big(\frac{1}{2}\rho_0 \dot{u}_\alpha \dot{u}_\alpha - \frac{1}{2} G_{K\alpha L\gamma} u_{K,\alpha} u_{L,\gamma}\\
&- R_{KL\gamma}\phi^1_{,K} u_{L,\gamma} + \frac{1}{2} L_{KL}\phi^1_{,K}\phi^1_{,L} + \rho_0 f^1_\alpha u_\alpha - \rho^1_E \phi^1\Big) dV\\
&+ \int_{t_0}^{t} dt \int_{S_T} \overline{T}^1_\alpha u_\alpha dS - \int_{t_0}^{t} dt \int_{S_D} \overline{\sigma}^1_e \phi^1 dS.
\end{aligned}
\qquad (6.2\text{-}21)
$$

2.3 Linearization Using the Total Stress Formulation

With the total stress formulation in Section 7 of Chapter 1, the derivation for the equations of the incremental fields can be written in a more compact form as

$$\hat{T}_{KL} = \rho_0 \frac{\partial \hat{\psi}}{\partial S_{KL}} \cong \hat{T}^0_{KL} + \lambda \hat{T}^1_{KL},$$

$$\mathcal{D}_K = -\rho_0 \frac{\partial \hat{\psi}}{\partial \mathcal{E}_K} \cong \mathcal{D}^0_K + \lambda \mathcal{D}^1_K,$$

$$\hat{T}^1_{KL} = \rho_0 \frac{\partial^2 \hat{\psi}}{\partial S_{KL} \partial S_{MN}}\bigg|_{S^0_{KL}, \mathcal{E}^0_K} E^1_{MN} + \rho_0 \frac{\partial^2 \hat{\psi}}{\partial S_{KL} \partial \mathcal{E}_M}\bigg|_{S^0_{KL}, \mathcal{E}^0_K} \mathcal{E}^1_M,$$

$$\mathcal{D}^1_K = -\rho_0 \frac{\partial^2 \hat{\psi}}{\partial \mathcal{E}_K \partial S_{MN}}\bigg|_{S^0_{KL}, \mathcal{E}^0_K} E^1_{MN} - \rho_0 \frac{\partial^2 \hat{\psi}}{\partial \mathcal{E}_K \partial \mathcal{E}_M}\bigg|_{S^0_{KL}, \mathcal{E}^0_K} \mathcal{E}^1_M, \tag{6.2-22}$$

$$K_{Ki} = y_{i,L} \hat{T}_{KL} \cong \delta_{i\alpha} (K^0_{K\alpha} + \lambda K^1_{K\alpha}),$$

$$K^1_{K\alpha} = u_{\alpha,L} \hat{T}^0_{KL} + x_{\alpha,L} \hat{T}^1_{KL},$$

$$K^1_{L\gamma} = G_{L\gamma M\alpha} u_{\alpha,M} - R_{ML\gamma} \mathcal{E}^1_M,$$

$$\mathcal{D}^1_K = R_{KL\gamma} u_{\gamma,L} + L_{KL} \mathcal{E}^1_L,$$

where

$$G_{K\alpha L\gamma} = x_{\alpha,M}\, \rho_0 \frac{\partial^2 \hat{\psi}}{\partial S_{KM} \partial S_{LN}}\bigg|_{S^0_{KL}, \mathcal{E}^0_K} x_{\gamma,N} + \hat{T}^0_{KL} \delta_{\alpha\gamma} = G_{L\gamma K\alpha},$$

$$R_{KL\gamma} = -\rho_0 \frac{\partial^2 \hat{\psi}}{\partial \mathcal{E}_K \partial S_{ML}}\bigg|_{S^0_{KL}, \mathcal{E}^0_K} x_{\gamma,M}, \tag{6.2-23}$$

$$L_{KL} = -\rho_0 \frac{\partial^2 \hat{\psi}}{\partial \mathcal{E}_K \partial \mathcal{E}_L}\bigg|_{S^0_{KL}, \mathcal{E}^0_K} = L_{LK}.$$

Problem

6.2-1. Show (6.2-9).

3. VARIATIONAL APPROACH

The equations for the small incremental fields can also be obtained by making power series expansions in terms of the small incremental fields in the variational functional of nonlinear electroelasticity [36]. Consider the dynamic form of the total energy formulation in (1.8-5). Let

$$y_i = \delta_{i\alpha}(x_\alpha + \lambda u_\alpha)\,. \tag{6.3-1}$$

Other quantities of the present state can then be written as

$$\begin{aligned}
\phi &= \phi^0 + \lambda\phi^1 + \lambda^2\phi^2 \cdots,\\
f_i &= \delta_{i\alpha}(f_\alpha^0 + \lambda f_\alpha^1 + \lambda^2 f_\alpha^2 \cdots),\\
\rho_E &= \rho_E^0 + \lambda\rho_E^1 + \lambda^2\rho_E^2 \cdots,\\
\bar{y}_i &= \delta_{i\alpha}(\bar{\xi}_\alpha + \lambda\bar{u}_\alpha),\\
\bar{T}_i &= \delta_{i\alpha}(\bar{T}_\alpha^0 + \lambda\bar{T}_\alpha^1 + \lambda^2\bar{T}_\alpha^2 \cdots),\\
\bar{\phi} &= \bar{\phi}^0 + \lambda\bar{\phi}^1 + \lambda^2\bar{\phi}^2 \cdots,\\
\bar{\sigma}_e &= \bar{\sigma}_e^0 + \lambda\bar{\sigma}_e^1 + \lambda^2\bar{\sigma}_e^2 \cdots,
\end{aligned} \tag{6.3-2}$$

where, due to nonlinearity, higher powers of λ may arise. Note that in (6.3-2) the superscripts 0, 1, 2 are for orders of expansions, not for powers except in λ^2. We want to derive equations governing **u** and ϕ^1. From (6.3-1) and (6.3-2), we can further write

$$\begin{aligned}
S_{KL} &= S_{KL}^0 + \lambda S_{KL}^1 + \lambda^2 S_{KL}^2,\\
\mathcal{E}_K &= \mathcal{E}_K^0 + \lambda\mathcal{E}_K^1 + \lambda^2\mathcal{E}_K^2 \cdots,
\end{aligned} \tag{6.3-3}$$

where

$$\begin{aligned}
S_{KL}^0 &= (\xi_{\alpha,K}\xi_{\alpha,L} - \delta_{KL})/2,\\
S_{KL}^1 &= (\xi_{\alpha,K}u_{\alpha,L} + \xi_{\alpha,L}u_{\alpha,K})/2,\\
S_{KL}^2 &= u_{\alpha,K}u_{\alpha,L}/2,\\
\mathcal{E}_K^0 &= -\phi_{,K}^0, \quad \mathcal{E}_K^1 = -\phi_{,K}^1, \quad \mathcal{E}_K^2 = -\phi_{,K}^2.
\end{aligned} \tag{6.3-4}$$

Substituting (6.3-l)-(6.3-4) into the dynamic form of the Π in (1.8-5), we obtain

$$\Pi = \Pi^0 + \lambda\Pi^1 + \lambda^2\Pi^2 \cdots, \tag{6.3-5}$$

where

$$\begin{aligned}
\Pi^0 = &\int_{t_0}^{t} dt \int_V [\frac{1}{2}\rho_0\dot{x}_\alpha\dot{x}_\alpha - \rho_0\hat{\psi}(S_{KL}^0, \mathcal{E}_K^0)\\
&+ \rho_0 f_\alpha^0 x_\alpha - \rho_E^0\phi^0]dV\\
&+ \int_{t_0}^{t} dt \int_{S_T} \bar{T}_\alpha^0 x_\alpha dS - \int_{t_0}^{t} dt \int_{S_D} \bar{\sigma}_e^0\phi^0 dS,
\end{aligned} \tag{6.3-6}$$

$$
\begin{aligned}
\Pi^1 = &\int_{t_0}^{t} dt \int_V \; [\rho_0 \dot{x}_\alpha \dot{u}_\alpha - \rho_0 \frac{\partial \hat{\psi}}{\partial S_{KL}}\bigg|_{S_{KL}^0, \mathcal{E}_K^0} E_{KL}^1 - \rho_0 \frac{\partial \hat{\psi}}{\partial \mathcal{E}_K}\bigg|_{S_{KL}^0, \mathcal{E}_K^0} W_K^1 \\
&+ \rho_0 f_\alpha^0 u_\alpha + \rho_0 f_\alpha^1 x_\alpha - \rho_E^0 \phi^1 - \rho_E^1 \phi^0] dV \\
&+ \int_{t_0}^{t} dt \int_{S_T} (\overline{T}_\alpha^0 u_\alpha + \overline{T}_\alpha^1 x_\alpha) dS \\
&- \int_{t_0}^{t} dt \int_{S_D} \; (\overline{\sigma}_e^0 \phi^1 + \overline{\sigma}_e^1 \phi^0) dS,
\end{aligned}
\tag{6.3-7}
$$

and

$$
\begin{aligned}
\Pi^2 = &\int_{t_0}^{t} dt \int_V \; [\frac{1}{2} \rho_0 \dot{u}_\alpha \dot{u}_\alpha - \rho_0 \frac{\partial \hat{\psi}}{\partial S_{KL}}\bigg|_{S_{KL}^0, \mathcal{E}_K^0} S_{KL}^2 - \rho_0 \frac{\partial \hat{\psi}}{\partial \mathcal{E}_K}\bigg|_{S_{KL}^0, \mathcal{E}_K^0} \mathcal{E}_K^2 \\
&- \frac{1}{2} \rho_0 \frac{\partial^2 \hat{\psi}}{\partial S_{KL} \partial S_{MN}}\bigg|_{S_{KL}^0, \mathcal{E}_K^0} S_{KL}^1 S_{MN}^1 - \rho_0 \frac{\partial^2 \hat{\psi}}{\partial \mathcal{E}_M \partial S_{KL}}\bigg|_{S_{KL}^0, \mathcal{E}_K^0} \mathcal{E}_M^1 E_{KL}^1 \\
&- \frac{1}{2} \rho_0 \frac{\partial^2 \hat{\psi}}{\partial \mathcal{E}_M \partial \mathcal{E}_N}\bigg|_{S_{KL}^0, \mathcal{E}_K^0} \mathcal{E}_M^1 \mathcal{E}_N^1 \\
&+ \rho_0 f_\alpha^1 u_\alpha + \rho_0 f_\alpha^2 x_\alpha - \rho_E^0 \phi^2 - \rho_E^1 \phi^1 - \rho_E^2 \phi^0] dV \\
&+ \int_{t_0}^{t} dt \int_{S_T} (\overline{T}_\alpha^1 u_\alpha + \overline{T}_\alpha^2 x_\alpha) dS \\
&- \int_{t_0}^{t} dt \int_{S_D} \; (\overline{\sigma}_e^0 \phi^2 + \overline{\sigma}_e^1 \phi^1 + \overline{\sigma}_e^2 \phi^0) dS.
\end{aligned}
\tag{6.3-8}
$$

Comparing (6.3-6) to (1.7-1), we recognize (6.3-6) to be simply the variational functional for the initial deformation which may be dynamic. Since the initial deformation here satisfies the dynamic form of (6.2-1), Π^1 in (6.3-7) can be written into the following much simpler form:

$$
\begin{aligned}
\Pi^1 = &\int_{t_0}^{t} dt \int_V \; (\rho_0 f_\alpha^1 x_\alpha - \rho_E^1 \phi^0) dV \\
&+ \int_{t_0}^{t} dt \int_{S_T} \; \overline{T}_\alpha^1 x_\alpha dS - \int_{t_0}^{t} dt \int_{S_D} \; \overline{\sigma}_e^1 \phi^0 dS,
\end{aligned}
\tag{6.3-9}
$$

which does not depend on **u** and ϕ^1 anymore. If f_α^0, ρ_E^0, $\overline{T}_\alpha^0$, and $\overline{\sigma}_e^0$ are held constant, or, in other words, $f_\alpha^1 = \rho_E^1 = \overline{T}_\alpha^1 = \overline{\sigma}_e^1 = 0$, then $\Pi^1 = 0$ which simply shows that Π^0 is the variational functional for the initial deformation. Since we are interested in equations for the first-order incremental fields **u** and ϕ^1, we drop all second-order quantities involving ϕ^2, f_α^2, ρ_E^2, $\overline{T}_\alpha^2$ and $\overline{\sigma}_e^2$ in Π^2 and obtain

$$\Pi^2(\mathbf{u},\phi^1) = \int_{t_0}^{t} dt \int_V [\frac{1}{2}\rho_0 \dot{u}_\alpha \dot{u}_\alpha - \frac{1}{2}G_{K\alpha L\gamma} u_{K,\alpha} u_{L,\gamma} - R_{KL\gamma}\phi^1_{,K} u_{L,\gamma}$$

$$+\frac{1}{2}L_{KL}\phi^1_{,K}\phi^1_{,L} + \rho_0 f^1_\alpha u_\alpha - \rho^1_E \phi^1]dV \tag{6.3-10}$$

$$+\int_{t_0}^{t} dt \int_{S_T} \overline{T}^1_\alpha u_\alpha dS - \int_{t_0}^{t} dt \int_{S_D} \overline{\sigma}^1_e \phi^1 dS,$$

where

$$G_{K\alpha L\gamma} = x_{\alpha,M}\rho_0 \frac{\partial^2 \hat{\psi}}{\partial S_{KM}\partial S_{LN}}\bigg|_{S^0_{KL},\mathcal{E}^0_K} x_{\gamma,N} + \rho_0 \frac{\partial \hat{\psi}}{\partial S_{KL}}\bigg|_{S^0_{KL},\mathcal{E}^0_K} \delta_{\alpha\gamma} = G_{L\gamma K\alpha},$$

$$R_{KL\gamma} = -\rho_0 \frac{\partial^2 \hat{\psi}}{\partial \mathcal{E}_K \partial S_{ML}}\bigg|_{S^0_{KL},\mathcal{E}^0_K} x_{\gamma,M}, \tag{6.3-11}$$

$$L_{KL} = -\rho_0 \frac{\partial^2 \hat{\psi}}{\partial \mathcal{E}_K \partial \mathcal{E}_L}\bigg|_{S^0_{KL},\mathcal{E}^0_K} = L_{LK}.$$

Equation (6.3-11) are the same as (6.2-23). When (1.8-1) is introduced into (6.3-11), with the use of (1.8-2) and (1.8-6)-(1.8-8), (6.2-17) will result.

4. SMALL BIASING FIELDS

In some applications, the biasing deformations and fields are also infinitesimal. In this case, usually only their first-order effects on the incremental fields need to be considered. Then the following energy density of a cubic polynomial is sufficient:

$$\rho_0\psi(E_{KL},\mathcal{E}_K) = \frac{1}{2}c_{ABCD}S_{AB}S_{CD} - e_{ABC}\mathcal{E}_A S_{BC} - \frac{1}{2}\chi_{AB}\mathcal{E}_A\mathcal{E}_B$$

$$+\frac{1}{6}c_{ABCDEF}S_{AB}S_{CD}S_{EF} + \frac{1}{2}k_{ABCDE}\mathcal{E}_A S_{BC}S_{DE} \tag{6.4-1}$$

$$-\frac{1}{2}b_{ABCD}\mathcal{E}_A\mathcal{E}_B S_{CD} - \frac{1}{6}\chi_{ABC}\mathcal{E}_A\mathcal{E}_B\mathcal{E}_C,$$

where the subscripts indicating the orders of the material constants have been dropped. For small biasing fields it is convenient to introduce the small displacement vector $\boldsymbol{w}$ of the initial deformation (see Figure 6.2-1), given as

$$x_\alpha = \delta_{\alpha K} X_K + w_\alpha . \qquad (6.4\text{-}2)$$

Then, neglecting terms quadratic in the gradients of $\boldsymbol{w}$ and ϕ^0, the effective material constants take the following form [35]:

$$G_{K\alpha L\gamma} = c_{K\alpha L\gamma} + \hat{c}_{K\alpha L\gamma}, \quad R_{KL\gamma} = e_{KL\gamma} + \hat{e}_{KL\gamma}, \quad L_{KL} = \varepsilon_{KL} + \hat{\varepsilon}_{KL}, \qquad (6.4\text{-}3)$$

where

$$\begin{aligned}
\hat{c}_{K\alpha L\gamma} &= T^0_{KL}\delta_{\alpha\gamma} + c_{K\alpha LN} w_{\gamma,N} + c_{KNL\gamma} w_{\alpha,N} \\
&\quad + c_{K\alpha L\gamma AB} S^0_{AB} + k_{AK\alpha L\gamma} \mathcal{E}^0_A, \\
\hat{e}_{KL\gamma} &= e_{KLM} w_{\gamma,M} - k_{KL\gamma AB} S^0_{AB} + b_{AKL\gamma} \mathcal{E}^0_A \\
&\quad + \varepsilon_0 (\mathcal{E}^0_K \delta_{L\gamma} - \mathcal{E}^0_L \delta_{K\gamma} - \mathcal{E}^0_M \delta_{M\gamma} \delta_{KL}), \qquad (6.4\text{-}4) \\
\hat{\varepsilon}_{KL} &= b_{KLAB} S^0_{AB} + \chi_{KLA} \mathcal{E}^0_A + \varepsilon_0 (S^0_{MM} \delta_{KL} - 2S^0_{KL}), \\
S^0_{AB} &\cong (w_{A,B} + w_{B,A})/2, \\
\mathcal{E}^0_K &= -\phi^0_{,K}.
\end{aligned}$$

It is important to note that the third-order material constants are necessary for a complete description of the lowest order effects of the biasing fields.

5. THEORY OF INITIAL STRESS

In certain applications, e.g., buckling of thin structures, consideration of initial stresses without initial deformations is sufficient. Such a theory is called the initial stress theory in elasticity. It can be obtained from the theory for incremental fields derived in Section 2. We set $\mathbf{x} = \mathbf{X}$ in the equations for small fields superposed on finite biasing fields. Furthermore, for buckling analysis, a quadratic expression of ψ with second-order material constants only and the corresponding linear constitutive relations are sufficient. The biasing fields can be treated as infinitesimal fields. Then the effective material constants sufficient for describing the buckling phenomenon take the following simple form:

$$\begin{aligned}
G_{K\alpha L\gamma} &= c_{K\alpha L\gamma} + T^0_{KL}\delta_{\alpha\gamma}, \\
R_{KL\gamma} &= e_{KL\gamma} + \varepsilon_0 (\mathcal{E}^0_K \delta_{L\gamma} - \mathcal{E}^0_L \delta_{K\gamma} - \mathcal{E}^0_M \delta_{M\gamma} \delta_{KL}), \qquad (6.5\text{-}1) \\
L_{KL} &= \varepsilon_{KL},
\end{aligned}$$

where T^0_{KL} is the initial stress and $\mathcal{E}^0_K$ is the initial electric field.

Results obtained in buckling analyses of a few thin piezoelectric beams, plates, and shells show that the buckling load of a piezoelectric structure is often related to the corresponding elastic buckling load obtained from an analysis neglecting the piezoelectric coupling in the following manner

$$P_{\text{cr}}^{\text{Piezoelectric}} = (1+\lambda k^2)P_{\text{cr}}^{\text{Elastic}}, \tag{6.5-2}$$

where λ is a small, positive number. λ may depend on the material and geometry of the structure. $k^2 = e^2/c\varepsilon > 0$ is an electromechanical coupling factor. When (6.5-2) is true the electromechanical coupling tends to increase the buckling load. In such a case an elastic analysis ignoring the piezoelectric coupling yields a conservative estimate of the buckling load. This is not surprising in view of the piezoelectric stiffening effect. Specific results on buckling of thin piezoelectric structures can be found in the references in a review article [37].

6. FREQUENCY PERTURBATION

Many piezoelectric devices are resonant devices for which frequency consideration is of fundamental importance in design. Analysis based on linear piezoelectricity can provide understanding of the operating principles and basic design tools. This type of analysis is represented by Mindlin's early work on the eigenvalue problem of Section 6 of Chapter 4 [38]. However, devices designed based on linear piezoelectricity are deficient in certain applications. Knowledge of the frequency stability due to environmental effects (e.g., temperature change, force, and acceleration) which cause biasing deformations and frequency shifts is often required for a successful design. For the lowest order effect of the biasing fields, we need to study the eigenvalue problem of an electroelastic body vibrating with the presence of a small bias. From (6.4-4) we have

$$\begin{aligned}
&-[(c_{L\gamma M\alpha}+\varepsilon\hat{c}_{L\gamma M\alpha})u_{\alpha,M}+(e_{ML\gamma}+\varepsilon\hat{e}_{ML\gamma})\phi^1_{,M}]_{,L}=\rho_0\lambda u_\gamma, \quad \text{in} \quad V,\\
&[-(e_{KL\gamma}+\varepsilon\hat{e}_{KL\gamma})u_{\gamma,L}+(\varepsilon_{KL}+\varepsilon\hat{\varepsilon}_{KL})\phi^1_{,L}]_{,K}=0, \quad \text{in} \quad V,\\
&u_\alpha=0, \quad \text{on} \quad S_u,\\
&K^1_{L\gamma}N_L=[c_{L\gamma M\alpha}+\varepsilon\hat{c}_{L\gamma M\alpha})u_{\alpha,M}+(e_{ML\gamma}+\varepsilon\hat{e}_{ML\gamma})\phi^1_{,M}]N_L=0, \quad \text{on} \quad S_T,\\
&\phi^1=0, \quad \text{on} \quad S_\phi,\\
&\mathcal{D}^1_K N_K=[(e_{KL\gamma}+\varepsilon\hat{e}_{KL\gamma})u_{\gamma,L}-(\varepsilon_{KL}+\varepsilon\hat{\varepsilon}_{KL})\phi^1_{,L}]N_K=0, \quad \text{on} \quad S_D,
\end{aligned} \tag{6.6-1}$$

where $\lambda = \omega^2$, ω and $\{\mathbf{u}, \phi^1\} = \mathbf{U}$ are the resonance frequency and the corresponding mode, respectively, when the biasing fields are present and may be called a perturbed frequency and mode. ε is an artificially introduced dimensionless number to show the smallness of the biasing fields. In terms of the abstract notation in Section 6 of Chapter 4, Equation (6.6-1) can be written as [39]

$$\begin{aligned}
&(\mathbf{A} + \varepsilon\hat{\mathbf{A}})\mathbf{U} = \lambda\mathbf{B}\mathbf{U}, \quad \text{in} \quad V, \\
&u_\alpha = 0, \quad \text{on} \quad S_u, \\
&[(\mathbf{K} + \varepsilon\hat{\mathbf{K}})\mathbf{U}]_{L\gamma} N_L = 0, \quad \text{on} \quad S_T, \\
&\phi^1 = 0, \quad \text{on} \quad S_\phi, \\
&[(\mathbf{D} + \varepsilon\hat{\mathbf{D}})\mathbf{U}]_K N_K = 0, \quad \text{on} \quad S_D,
\end{aligned} \tag{6.6-2}$$

where

$$\begin{aligned}
&\mathbf{AU} = \{-(c_{L\gamma M\alpha} u_{\alpha,M} + e_{ML\gamma}\phi^1_{,M})_{,L}, (-e_{KL\gamma} u_{\gamma,L} + \varepsilon_{KL}\phi^1_{,L})_{,K}\}, \\
&\hat{\mathbf{A}}\mathbf{U} = \{-(\hat{c}_{L\gamma M\alpha} u_{\alpha,M} + \hat{e}_{ML\gamma}\phi^1_{,M})_{,L}, (-\hat{e}_{KL\gamma} u_{\gamma,L} + \hat{\varepsilon}_{KL}\phi^1_{,L})_{,K}\}, \\
&\mathbf{BU} = \{\rho_0 u_\gamma, 0\}, \\
&(\mathbf{KU})_{L\gamma} = c_{L\gamma M\alpha} u_{\alpha,M} + e_{ML\gamma}\phi^1_{,M}, \\
&(\hat{\mathbf{K}}\mathbf{U})_{L\gamma} = \hat{c}_{L\gamma M\alpha} u_{\alpha,M} + \hat{e}_{ML\gamma}\phi^1_{,M}, \\
&(\mathbf{DU})_K = e_{KL\gamma} u_{\gamma,L} - \varepsilon_{KL}\phi^1_{,L}, \\
&(\hat{\mathbf{D}}\mathbf{U})_K = \hat{e}_{KL\gamma} u_{\gamma,L} - \hat{\varepsilon}_{KL}\phi^1_{,L}.
\end{aligned} \tag{6.6-3}$$

We make the following expansions:

$$\begin{aligned}
&\lambda \cong \lambda^{(0)} + \varepsilon\lambda^{(1)}, \\
&\mathbf{U} = \begin{Bmatrix} u_\alpha \\ \phi^1 \end{Bmatrix} \cong \begin{Bmatrix} u^{(0)}_\alpha \\ \phi^{(0)} \end{Bmatrix} + \varepsilon \begin{Bmatrix} u^{(1)}_\alpha \\ \phi^{(1)} \end{Bmatrix} = \mathbf{U}^{(0)} + \varepsilon\mathbf{U}^{(1)}.
\end{aligned} \tag{6.6-4}$$

Substituting (6.6-4) into (6.6-2), collecting terms of equal powers of ε, the following perturbation problems of successive orders can be obtained. Zero-order:

$$\begin{aligned}
&\mathbf{AU}^{(0)}=\lambda^{(0)}\mathbf{BU}^{(0)}, \quad \text{in} \quad V,\\
&u_\alpha^{(0)}=0, \quad \text{on} \quad S_u,\\
&(\mathbf{KU}^{(0)})_{L\gamma}N_L=0, \quad \text{on} \quad S_T,\\
&\phi^{(0)}=0, \quad \text{on} \quad S_\phi,\\
&(\mathbf{DU}^{(0)})_K N_K=0, \quad \text{on} \quad S_D,
\end{aligned} \tag{6.6-5}$$

which we recognize to be the eigenvalue problem for vibrations of a linear piezoelectric body without biasing fields, treated in Section 6 of Chapter 4. The solution to the zero-order problem, $\lambda^{(0)}$ and $\mathbf{U}^{(0)}$, is assumed known and the first-order problem below is to be solved:

$$\begin{aligned}
&\mathbf{AU}^{(1)}+\hat{\mathbf{A}}\mathbf{U}^{(0)}=\lambda^{(0)}\mathbf{BU}^{(1)}+\lambda^{(1)}\mathbf{BU}^{(0)}, \quad \text{in} \quad V,\\
&u_\alpha^{(1)}=0, \quad \text{on} \quad S_u,\\
&(\mathbf{KU}^{(1)}+\hat{\mathbf{K}}\mathbf{U}^{(0)})_{L\gamma}N_L=0, \quad \text{on} \quad S_T,\\
&\phi^{(1)}=0, \quad \text{on} \quad S_\phi,\\
&(\mathbf{DU}^{(1)}+\hat{\mathbf{D}}\mathbf{U}^{(0)})_K N_K=0, \quad \text{on} \quad S_D.
\end{aligned} \tag{6.6-6}$$

The equations for the first-order problem can be written as

$$\mathbf{AU}^{(1)}=\lambda^{(0)}\mathbf{BU}^{(1)}+\lambda^{(1)}\mathbf{BU}^{(0)}-\hat{\mathbf{A}}\mathbf{U}^{(0)}. \tag{6.6-7}$$

Multiply both sides of (6.6-7) by $\mathbf{U}^{(0)}$

$$\begin{aligned}
&<\mathbf{AU}^{(1)};\mathbf{U}^{(0)}>\\
&\quad=\lambda^{(0)}<\mathbf{BU}^{(1)};\mathbf{U}^{(0)}>+\lambda^{(1)}<\mathbf{BU}^{(0)};\mathbf{U}^{(0)}>\\
&\quad-<\hat{\mathbf{A}}\mathbf{U}^{(0)};\mathbf{U}^{(0)}>.
\end{aligned} \tag{6.6-8}$$

Similar to (4.6-7), it can be shown that

$$\begin{aligned}
&<\mathbf{AU}^{(0)};\mathbf{U}^{(1)}>\\
&\quad=-\int_S [(\mathbf{KU}^{(0)})_{L\gamma}N_L u_\gamma^{(1)}+(\mathbf{DU}^{(0)})_K N_K\phi^{(1)}]dS\\
&\quad+\int_S [(\mathbf{KU}^{(1)})_{M\alpha}N_M u_\alpha^{(0)}+(\mathbf{DU}^{(1)})_L N_L\phi^{(0)}]dS\\
&\quad+<\mathbf{U}^{(0)};\mathbf{AU}^{(1)}>.
\end{aligned} \tag{6.6-9}$$

With (6.6-5) and (6.6-6), Equation (6.6-9) becomes

$$\begin{aligned}
&<\mathbf{AU}^{(0)};\mathbf{U}^{(1)}>=-\int_{S_T} (\hat{\mathbf{K}}\mathbf{U}^{(0)})_{M\alpha}N_M u_\alpha^{(0)}dS\\
&\quad-\int_{S_D} (\hat{\mathbf{D}}\mathbf{U}^{(0)})_L N_L\phi^{(0)}dS+<\mathbf{U}^{(0)};\mathbf{AU}^{(1)}>.
\end{aligned} \tag{6.6-10}$$

Substitute (6.6-10) into (6.6-8):

$$\begin{aligned}&\int_{S_T}(\hat{\mathbf{K}}\mathbf{U}^{(0)})_{M\alpha}N_M u_\alpha^{(0)}dS\\&+\int_{S_D}(\hat{\mathbf{D}}\mathbf{U}^{(0)})_L N_L\phi^{(0)}dS+<\mathbf{A}\mathbf{U}^{(0)};\mathbf{U}^{(1)}>\\&=\lambda^{(0)}<\mathbf{U}^{(1)};\mathbf{B}\mathbf{U}^{(0)}>+\lambda^{(1)}<\mathbf{B}\mathbf{U}^{(0)};\mathbf{U}^{(0)}>\\&-<\hat{\mathbf{A}}\mathbf{U}^{(0)};\mathbf{U}^{(0)}>,\end{aligned} \tag{6.6-11}$$

which can be further written as

$$\begin{aligned}&<\mathbf{A}\mathbf{U}^{(0)}-\lambda^{(0)}\mathbf{B}\mathbf{U}^{(0)};\mathbf{U}^{(1)}>\\&+\int_{S_T}(\hat{\mathbf{K}}\mathbf{U}^{(0)})_{M\alpha}N_M u_\alpha^{(0)}dS+\int_{S_D}(\hat{\mathbf{D}}\mathbf{U}^{(0)})_L N_L\phi^{(0)}dS\\&=\lambda^{(1)}<\mathbf{B}\mathbf{U}^{(0)};\mathbf{U}^{(0)}>-<\hat{\mathbf{A}}\mathbf{U}^{(0)};\mathbf{U}^{(0)}>.\end{aligned} \tag{6.6-12}$$

With Equation (6.6-5)$_1$, from (6.6-12)

$$\begin{aligned}\lambda^{(1)}&=\frac{1}{<\mathbf{B}\mathbf{U}^{(0)};\mathbf{U}^{(0)}>}\\&\times\Big[<\hat{\mathbf{A}}\mathbf{U}^{(0)};\mathbf{U}^{(0)}>+\int_{S_T}(\hat{\mathbf{K}}\mathbf{U}^{(0)})_{M\alpha}N_M u_\alpha^{(0)}dS\\&+\int_{S_D}(\hat{\mathbf{D}}\mathbf{U}^{(0)})_L N_L\phi^{(0)}dS\Big].\end{aligned} \tag{6.6-13}$$

The above expressions are for the eigenvalue $\lambda=\omega^2$. For ω we make the following expansion:

$$\omega\cong\omega^{(0)}+\varepsilon\omega^{(1)}. \tag{6.6-14}$$

Then

$$\begin{aligned}\lambda=\omega^2&\cong(\omega^{(0)}+\varepsilon\omega^{(1)})^2\\&\cong(\omega^{(0)})^{(2)}+2\varepsilon\omega^{(0)}\omega^{(1)}\cong\lambda^{(0)}+\varepsilon\lambda^{(1)}.\end{aligned} \tag{6.6-15}$$

Hence

$$\begin{aligned}\frac{\varepsilon\omega^{(1)}}{\omega^{(0)}}&\cong\frac{1}{2(\omega^{(0)})^2}\varepsilon\lambda^{(1)}\\&=\frac{1}{2(\omega^{(0)})^2}\frac{1}{<\mathbf{B}\mathbf{U}^{(0)};\mathbf{U}^{(0)}>}\\&\times\Big[<\varepsilon\hat{\mathbf{A}}\mathbf{U}^{(0)};\mathbf{U}^{(0)}>+\int_{S_T}(\varepsilon\hat{\mathbf{K}}\mathbf{U}^{(0)})_{M\alpha}N_M u_\alpha^{(0)}dS\\&+\int_{S_D}(\varepsilon\hat{\mathbf{D}}\mathbf{U}^{(0)})_L N_L\phi^{(0)}dS\Big],\end{aligned} \tag{6.6-16}$$

or

$$\frac{\omega-\omega^{(0)}}{\omega^{(0)}} \cong \frac{1}{2(\omega^{(0)})^2} \frac{1}{\int_V \rho_0 u_\alpha^{(0)} u_\alpha^{(0)} dV}$$
$$\times\Big\{\int_V \Big[-(\varepsilon \hat{c}_{L\gamma M\alpha} u_{\alpha,M}^{(0)} + \varepsilon \hat{e}_{ML\gamma} \phi_{,M}^{(0)})_{,L} u_\gamma^{(0)}$$
$$+(-\varepsilon \hat{e}_{KL\gamma} u_{\gamma,L}^{(0)} + \varepsilon \hat{\varepsilon}_{KL} \phi_{,L}^{(0)})_{,K} \phi^{(0)}\Big] dV \tag{6.6-17}$$
$$+\int_{S_T} (\varepsilon \hat{c}_{L\gamma M\alpha} u_{\alpha,M}^{(0)} + \varepsilon \hat{e}_{ML\gamma} \phi_{,M}^{(0)}) N_L u_\gamma^{(0)} dS$$
$$+\int_{S_D} (\varepsilon \hat{e}_{KL\gamma} u_{\gamma,L}^{(0)} - \varepsilon \hat{\varepsilon}_{KL} \phi_{,L}^{(0)}) N_K \phi^{(0)} dS\Big]\Big\}.$$

With integration by parts, we can write (6.6-17) further as

$$\frac{\omega-\omega^{(0)}}{\omega^{(0)}} \cong \frac{1}{2(\omega^{(0)})^2} \frac{1}{\int_V \rho_0 u_\alpha^{(0)} u_\alpha^{(0)} dV} \tag{6.6-18}$$
$$\times \int_V (\varepsilon \hat{c}_{L\gamma M\alpha} u_{\alpha,M}^{(0)} u_{\gamma,L}^{(0)} + 2\varepsilon \hat{e}_{ML\gamma} \phi_{,M}^{(0)} u_{\gamma,L}^{(0)} - \varepsilon \hat{\varepsilon}_{KL} \phi_{,L}^{(0)} \phi_{,K}^{(0)}) dV.$$

When ε is set to 1, (6.6-18) becomes the well-known first-order perturbation integral for frequency shifts [40].

7. ELECTROSTRICTIVE CERAMICS

As the linear coupling between mechanical and electric fields, piezoelectricity cannot exist in isotropic materials. Mathematically this is the consequence of the fact that a third-rank isotropic tensor with a pair of symmetric indices has to vanish. Electrostriction is a nonlinear electroelastic coupling effect that exists in all dielectrics, isotropic or anisotropic. In the simplest description, electrostriction can be described by the term $b_{ABCD}\mathcal{E}_A\mathcal{E}_B S_{CD}$ in the energy density (6.4-1).

7.1 Nonlinear Theory

Electrostrictive ceramics are macroscopically isotropic due to their polycrystalline structure. For isotropic materials, there are not many independent components of the material tensors, linear or nonlinear. Instead of (6.4-1), it is more convenient to use representations based on tensor invariants of the strain tensor S with components S_{KL} and the material electric field vector $\mathcal{E}$ with components $\mathcal{E}_K$. The invariant representation automatically yields three-dimensional constitutive relations with a few independent material parameters. With the integrity bases for isotropic functions of a symmetric tensor and a vector, it can be determined that for isotropic electroelastic ceramics the energy density function ψ can be written as

$$\psi = \psi(I_1, I_2, I_3, I_4, I_5, I_6), \tag{6.7-1}$$

where the six invariants I_1 through I_6 are given by [1]

$$\begin{aligned} &I_1 = \mathrm{tr}(\mathcal{S}), \quad I_2 = \mathrm{tr}(\mathcal{S}^2), \quad I_3 = \mathrm{tr}(\mathcal{S}^3), \\ &I_4 = \mathcal{E}\cdot\mathcal{E}, \quad I_5 = \mathcal{E}\cdot\mathcal{S}\cdot\mathcal{E}, \quad I_6 = \mathcal{E}\cdot\mathcal{S}^2\cdot\mathcal{E}. \end{aligned} \tag{6.7-2}$$

In (6.7-2), $\mathcal{S}^2$ stands for $\mathcal{S}\cdot\mathcal{S}$. Equations (6.7-1) and (6.7-2) imply the following constitutive relations for the symmetric stress tensor and the polarization vector:

$$\begin{aligned} \mathbf{T}^S &= \frac{\partial\Sigma}{\partial I_1}\mathbf{1} + 2\frac{\partial\Sigma}{\partial I_2}\mathcal{S} + 3\frac{\partial\Sigma}{\partial I_3}\mathcal{S}^2 + \frac{\partial\Sigma}{\partial I_5}\mathcal{E}\otimes\mathcal{E} \\ &+ \frac{\partial\Sigma}{\partial I_6}[\mathcal{E}\otimes(\mathcal{S}\cdot\mathcal{E}) + (\mathcal{S}\cdot\mathcal{E})\otimes\mathcal{E}], \end{aligned} \tag{6.7-3}$$

$$\mathcal{P} = -2\frac{\partial\Sigma}{\partial I_4}\mathcal{E} - 2\frac{\partial\Sigma}{\partial I_5}\mathcal{S}\cdot\mathcal{E} - 2\frac{\partial\Sigma}{\partial I_6}\mathcal{S}^2\cdot\mathcal{E}, \tag{6.7-4}$$

where $\mathbf{1}$ is the unit tensor of rank two, and $\otimes$ represents tensor or dyadic product. Equation (6.7-3) and (6.7-4) are the most general constitutive relations of isotropic, nonlinear electroelastic materials. Although seemingly simple, they can be complicated functions of $\mathcal{S}_{KL}$ and $\mathcal{E}_K$. Under the inversion of $\mathcal{E} \to -\mathcal{E}$, we have $\mathcal{P} \to -\mathcal{P}$ and $\mathbf{T}^S \to \mathbf{T}^S$, indicating that $\mathcal{P}$ is odd and $\mathbf{T}^S$ is even in $\mathcal{E}$. Therefore linear dependence of $\mathbf{T}^S$ on $\mathcal{E}$ (piezoelectricity) is not allowed, but higher order couplings are possible. In particular, electrostrictive effect can be seen from, e.g., the fourth term on the right-hand side of (6.7-3), which is due to I_5.

7.2 Effects of a Small, Electrical Bias

Electrostrictive ceramics operate under a biasing electric field. If a small biasing electric field $\mathcal{E}^0$ is applied, the small biasing fields are purely electrical because there is no linear electromechanical coupling in the material. In such a case, the effective material constants under the electrical bias are

$$G_{K\alpha L\gamma} = c_{K\alpha LN},$$
$$R_{KL\gamma} = b_{AKL\gamma}\mathcal{E}_A^0 + \varepsilon_0(\mathcal{E}_K^0\delta_{L\gamma} - \mathcal{E}_L^0\delta_{K\gamma} - \mathcal{E}_M^0\delta_{M\gamma}\delta_{KL}), \tag{6.7-5}$$
$$L_{KL} = \varepsilon_{KL}.$$

Thus electrostrictive ceramics appear to be piezoelectric under a biasing electric field, and the effective piezoelectric constants $R_{KL\gamma}$ are tunable by the biasing electric field $\mathcal{E}^0$.

For the simplest model of electrostrictive ceramics, consider the case of infinitesimal deformation. We construct an energy density function as follows:

$$\rho_0\psi = c_1 I_1^2 + c_2 I_2 - \varepsilon_0\chi I_4/2 + b_1 I_1 I_4 + b_2 I_5, \tag{6.7-6}$$

where c_1 and c_2 are elastic constants, χ is the relative dielectric permittivity, b_1 and b_2 are electrostrictive constants. The constitutive relations generated by (6.7-4) are

$$T_{KL}^S = (2c_1 I_1 + b_1 I_4)\delta_{KL} + 2c_2 S_{KL} + b_2\mathcal{E}_K\mathcal{E}_L,$$
$$\mathcal{P}_K = (\varepsilon_0\chi - 2b_1 I_1)\mathcal{E}_K - 2b_2 S_{KL}\mathcal{E}_L. \tag{6.7-7}$$

Under a biasing electric field $\mathcal{E}^0$ in the x_3 direction, from (6.7-5), the effective piezoelectric constants can be obtained as

$$R_{113} = R_{131} = R_{223} = R_{232} = -\mathcal{E}_3^0 b_2 - \varepsilon_0\mathcal{E}_3^0,$$
$$R_{311} = R_{322} = -2\mathcal{E}_3^0 b_1 + \varepsilon_0\mathcal{E}_3^0,$$
$$R_{333} = -2\mathcal{E}_3^0(b_1 + b_2) - \varepsilon_0\mathcal{E}_3^0, \tag{6.7-8}$$
$$\text{All other } R_{KL\gamma} = 0.$$

Note that since there are only two electrostrictive material constants, the following relation exists

$$2R_{113} + R_{311} = R_{333}. \tag{6.7-9}$$

The nonzero tensor components of the electrostrictive constants are related to the material constants in (6.7-6) by

$$b_{3113} = b_{3131} = b_{3223} = b_{3232} = -b_2,$$
$$b_{3311} = b_{3322} = -2b_1, \tag{6.7-10}$$
$$b_{3333} = -2(b_1 + b_2).$$

Chapter 7

CUBIC AND OTHER EFFECTS

In this chapter we derive equations for cubic nonlinear effects. Some other effects not included in the general framework of Chapter 1 are also discussed.

1. CUBIC THEORY

1.1 Cubic Effects

By cubic theory we mean that effects of all terms up to the third power of the displacement and potential gradients or their products are included [6]. Cubic theory is an approximate theory for relatively weak nonlinearities, and can be obtained by expansions and truncations from the nonlinear theory in Chapter 1. From

$$X_M = \delta_{Mj} y_j - u_M , \tag{7.1-1}$$

by repeated use of the chain rule of differentiation, we obtain, to the second order in products of the derivative of u_M,

$$\begin{aligned} X_{M,i} &= \delta_{Mi} - u_{M,L} X_{L,i} = \delta_{Mi} - u_{M,L}(\delta_{Li} - u_{L,K} X_{K,i}) \\ &= \delta_{Mi} - u_{M,L}\delta_{Li} + u_{M,L}u_{L,K}X_{K,i} \\ &\cong \delta_{Mi} - u_{M,L}\delta_{Li} + u_{M,L}u_{L,K}\delta_{Ki} . \end{aligned} \tag{7.1-2}$$

From (1.1-16), retaining terms up to the second order in the derivative of u_M, we find

$$J \cong 1 + u_{K,K} + \frac{1}{2}(u_{K,K})^2 - \frac{1}{2}u_{K,L}u_{L,K} . \tag{7.1-3}$$

From (7.1-2) and (7.1-3)

$$\begin{aligned} JX_{L,i} \cong \ & \delta_{Li} - u_{L,R}\delta_{Ri} + \delta_{Li}u_{R,R} + u_{L,K}u_{K,R}\delta_{Ri} \\ & - u_{R,R}u_{L,K}\delta_{Ki} + \frac{1}{2}\delta_{Li}u_{K,K}u_{R,R} - \frac{1}{2}\delta_{Li}u_{K,R}u_{R,K} . \end{aligned} \tag{7.1-4}$$

From $(1.5\text{-}5)_2$, $(1.5\text{-}3)_2$, (1.5-11), and (7.1-4), retaining terms up to cubic in the small field variables, we obtain

$$F_{Lj} \cong \delta_{jM}\Big[\underset{2}{c}_{LMAB}\, u_{A,B} + e_{ALM}\phi_{,A} + \frac{1}{2}\underset{2}{c}_{LMAB}\, u_{K,A}u_{K,B}$$

$$+ \underset{2}{c}_{LKAB}\, u_{M,K}u_{A,B} + \frac{1}{2}\underset{3}{c}_{LMABCD}\, u_{A,B}u_{CD}$$

$$+ e_{ALK}u_{M,K}\phi_{,A} - \underset{1}{d}_{ABCLM}\, u_{B,C}\phi_{,A} - \frac{1}{2}b_{ABLM}\phi_{,A}\phi_{,B}$$

$$+ \frac{1}{2}\underset{2}{c}_{LRAB}\, u_{M,R}u_{K,A}u_{K,B} + \frac{1}{2}\underset{3}{c}_{LKABCD}\, u_{M,K}u_{A,B}u_{CD}$$

$$+ \frac{1}{2}\underset{3}{c}_{LMABcD}\, u_{A,B}u_{K,C}u_{K,D} + \frac{1}{6}\underset{4}{c}_{LMABCDEF}\, u_{A,B}u_{CD}u_{E,F}$$

$$- \underset{1}{d}_{ABCLK}\, u_{B,C}u_{M,K}\phi_{,A} - \frac{1}{2}\underset{1}{d}_{ABCLM}\, u_{K,B}u_{K,C}\phi_{,A}$$

$$- \frac{1}{2}\underset{2}{d}_{ABCDELM}\, u_{B,C}u_{D,E}\phi_{,A} - \frac{1}{2}b_{ABLK}u_{M,K}\phi_{,A}\phi_{,B}$$

$$+ \frac{1}{2}\underset{1}{a}_{ABCDLM}\, u_{c,D}\phi_{,A}\phi_{,B} + \frac{1}{6}\underset{3}{d}_{ABCLM}\, \phi_{,A}\phi_{,B}\phi_{,C} \Big],$$

(7.1-5)

and

$$\mathcal{P}_L \cong e_{LBC}u_{B,C} - \underset{2}{\chi}_{AL}\, \phi_{,A} + \frac{1}{2}e_{LBC}u_{K,B}u_{K,C}$$

$$- \frac{1}{2}\underset{1}{d}_{LBCDE}\, u_{B,C}u_{D,E} - b_{ALCD}u_{C,D}\phi_{,A}$$

$$+ \frac{1}{2}\underset{3}{\chi}_{ABL}\, \phi_{,A}\phi_{,B} - \frac{1}{2}\underset{1}{d}_{LBCDE}\, u_{B,C}u_{K,D}u_{K,E}$$

$$- \frac{1}{6}\underset{2}{d}_{LBCDEFG}\, u_{B,C}u_{D,E}u_{F,G} - \frac{1}{2}b_{ALCD}u_{K,C}u_{K,D}\phi_{,A}$$

$$+ \frac{1}{2}\underset{1}{a}_{ALCDEF}\, u_{C,D}u_{E,F}\phi_{,A} + \frac{1}{2}\underset{3}{d}_{ABLDE}\, u_{D,E}\phi_{,A}\phi_{,B} - \frac{1}{6}\underset{4}{\chi}_{ABCL}\, \phi_{,A}\phi_{,B}\phi_{,C}.$$

(7.1-6)

From (1.5-5)$_3$,(1.5-10), and (7.1-4):

$$M_{Lj} \cong \varepsilon_0 \delta_{jM} \Big[\phi_{,L}\phi_{,M} - \frac{1}{2}\phi_{,K}\phi_{,K}\delta_{LM} - \phi_{,K}\phi_{,M}u_{K,L}$$
$$- \phi_{,K}\phi_{,M}u_{L,K} + \phi_{,L}\phi_{,M}u_{K,K} - \phi_{,L}\phi_{,K}u_{K,M} \tag{7.1-7}$$
$$+ \phi_{,K}\phi_{,R}u_{R,K}\delta_{LM} + \frac{1}{2}\phi_{,K}\phi_{,K}u_{L,M} - \frac{1}{2}\phi_{,R}\phi_{,R}u_{K,K}\delta_{LM} \Big],$$

and

$$\varepsilon_0 J C_{KL}^{-1} \mathcal{E}_K \cong \varepsilon_0 \Big[-\phi_{,L} + \phi_{,K}u_{L,K} - \phi_{,L}u_{K,K} + \phi_{,K}u_{K,L}$$
$$- \phi_{,M}u_{L,K}u_{K,M} + \phi_{,K}u_{M,M}u_{L,K} - \frac{1}{2}\phi_{,L}u_{K,K}u_{M,M} \tag{7.1-8}$$
$$+ \frac{1}{2}\phi_{,L}u_{K,M}u_{M,K} - \phi_{,M}u_{L,K}u_{M,K} + \phi_{,M}u_{M,L}u_{K,K} - \phi_{,M}u_{M,K}u_{K,L} \Big].$$

Note that the fourth-order material constants are needed for a complete description of the cubic effects.

1.2 Quadratic Effects

If we keep terms up to the second order of the gradients only, we obtain the quadratic or second-order theory below:

$$F_{Lj} \cong \delta_{jM} \Big[\underset{2}{c}_{LMAB} u_{A,B} + e_{ALM}\phi_{,A} + \frac{1}{2} \underset{2}{c}_{LMAB} u_{K,A}u_{K,B}$$
$$+ \underset{2}{c}_{LKAB} u_{M,K}u_{A,B} + \frac{1}{2} \underset{3}{c}_{LMABCD} u_{A,B}u_{CD} \tag{7.1-9}$$
$$+ e_{ALK}u_{M,K}\phi_{,A} - \underset{1}{d}_{ABCLM} u_{B,C}\phi_{,A} - \frac{1}{2} b_{ABLM}\phi_{,A}\phi_{,B} \Big],$$

$$\mathcal{P}_L \cong e_{LBC}u_{B,C} - \underset{2}{\chi}_{AL} \phi_{,A} + \frac{1}{2} e_{LBC}u_{K,B}u_{K,C}$$
$$- \frac{1}{2} \underset{1}{d}_{LBCDE} u_{B,C}u_{D,E} - b_{ALCD}u_{C,D}\phi_{,A} + \frac{1}{2} \underset{3}{\chi}_{ABL} \phi_{,A}\phi_{,B}, \tag{7.1-10}$$

$$M_{Lj} \cong \varepsilon_0 \delta_{jM} [\phi_{,L}\phi_{,M} - \frac{1}{2}\phi_{,K}\phi_{,K}\delta_{LM}$$
$$- \phi_{,K}\phi_{,M}u_{K,L} - \phi_{,K}\phi_{,M}u_{L,K}], \tag{7.1-11}$$

$$\varepsilon_0 J C_{KL}^{-1} \mathcal{E}_K \cong \varepsilon_0 \Big[-\phi_{,L} + \phi_{,K}u_{L,K} - \phi_{,L}u_{K,K} + \phi_{,K}u_{K,L} \Big], \tag{7.1-12}$$

where the third-order material constants are needed.

2. NONLOCAL EFFECTS

2.1 Nonlocal Theory

Nonlocality comes from the consideration of long-range interactions. In one-dimensional lattice dynamics it has been shown that nonlocal theory includes, besides the interactions between neighboring atoms, interactions among non-neighboring atoms as well [41]. Nonlocality in constitutive relations is needed in modeling certain phenomena. Consider an electroelastic body V. Within the linear theory of piezoelectricity the nonlocal constitutive relations are given by [42]

$$\begin{aligned} T_{ij}(\mathbf{x}) &= \int_V [c_{ijkl}(\mathbf{x},\mathbf{x}')S_{kl}(\mathbf{x}') - e_{kij}(\mathbf{x},\mathbf{x}')E_k(\mathbf{x}')]dV(\mathbf{x}'), \\ D_i(\mathbf{x}) &= \int_V [e_{ikl}(\mathbf{x},\mathbf{x}')S_{kl}(\mathbf{x}') + \varepsilon_{ik}(\mathbf{x},\mathbf{x}')E_k(\mathbf{x}')]dV(\mathbf{x}'). \end{aligned} \tag{7.2-1}$$

As a special case, when the nonlocal material moduli are Dirac delta functions, Equations (7.2-1) reduce to the classical constitutive relations in (2.1-11). Substitution of (7.2-1) into the equation of motion and the charge equation results in integral-differential equations which are usually difficult to solve.

2.2 Thin Film Capacitance

In the following we give an example of what is probably the simplest nonlocal problem [43]. Consider an unbounded dielectric plate as shown in Figure 7.2-1. The plate is electroded and a voltage is applied. We want to obtain its capacitance from the nonlocal theory.

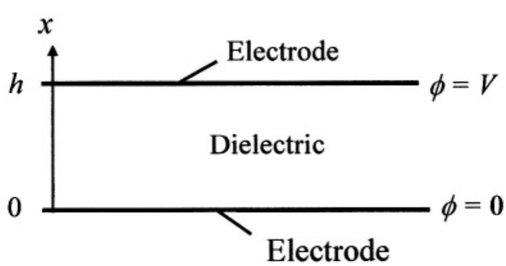

Figure 7.2-1. A thin dielectric plate.

The problem is one-dimensional. The boundary-value problem is

$$\frac{dD}{dx}=0, \quad 0<x<h,$$
$$D=\varepsilon_0 E+P, \quad 0<x<h,$$
$$P=\varepsilon_0\chi\int_0^h K(x',x)E(x')dx', \quad 0<x<h, \tag{7.2-2}$$
$$E=-\frac{d\phi}{dx}, \quad 0<x<h,$$
$$\phi(0)=0, \quad \phi(h)=V.$$

When the kernel function $K(x',x)$ has the following special form

$$K(x',x)=\delta(x'-x), \tag{7.2-3}$$

Equation (7.2-2) reduces to the usual classical form. χ is the dimensionless relative electric susceptibility which differs from the one in (1.5-11) by a factor of ε_0. The dielectric material of the capacitor is assumed to be homogeneous and isotropic. Hence $K(x',x)$ must be invariant under translation and inversion. We have

$$K(x',x)=K(x'-x)=K(x-x'). \tag{7.2-4}$$

$K(x',x)$ should have a localized behavior, large near $x'=x$ and decaying away from there. We chose the following kernel function

$$K(x'-x)=\frac{1}{2\alpha}e^{-\frac{|x'-x|}{\alpha}}, \quad \alpha>0, \tag{7.2-5}$$

where α is a microscopic parameter with the dimension of a length. It is a characteristic length of microscopic interactions. It is easy to verify that $K(x',x)$ has the following properties:

$$\lim_{\substack{\alpha\to 0^+\\ x\neq x'}} K=0, \quad \int_{-\infty}^{+\infty} Kdx=1. \tag{7.2-6}$$

Hence

$$\lim_{\alpha\to 0^+} K=\delta(x'-x), \tag{7.2-7}$$

which shows that $K(x',x)$ does include the local form as a limit case. We also note that the above $K(x',x)$ is the fundamental solution of the following differential operator

$$-\alpha^2\frac{\partial^2 K}{\partial x^2}+K=\delta(x'-x). \tag{7.2-8}$$

Integrating $(7.2\text{-}2)_1$ once, with $(7.2\text{-}2)_{2,3}$ we obtain

$$D = \varepsilon_0 E(x) + \varepsilon_0 \chi \int_0^h K(x',x)E(x')dx' = -\sigma_e, \tag{7.2-9}$$

where σ_e is an integration constant which physically represents the surface free charge density on the electrode at $x = h$. Equation (7.2-9) can be written as

$$E(x) = -\chi \int_0^h K(x',x)E(x')dx' - \frac{\sigma}{\varepsilon_0}, \tag{7.2-10}$$

which is a Fredholm integral equation of the second kind for the electric field E. Instead of solving (7.2-10) directly, we proceed as follows. With (7.2-8), we differentiate (7.2-10) with respect to x twice and obtain

$$\begin{aligned}
\frac{d^2E(x)}{dx^2} &= -\chi \int_0^h \frac{\partial^2 K(x',x)}{\partial x^2} E(x')dx' \\
&= -\chi \int_0^h \frac{1}{\alpha^2}[K(x',x) - \delta(x'-x)]E(x')dx' \\
&= \frac{1}{\alpha^2}[-\chi \int_0^h K(x',x)E(x')dx' - \frac{\sigma_e}{\varepsilon_0}] \\
&\quad + \frac{1}{\alpha^2}\chi \int_0^h \delta(x'-x)E(x')dx' + \frac{1}{\alpha^2}\frac{\sigma_e}{\varepsilon_0} \\
&= \frac{1}{\alpha^2}[E(x) + \chi E(x) + \frac{\sigma_e}{\varepsilon_0}].
\end{aligned} \tag{7.2-11}$$

Hence a solution E of the integral equation (7.2-10) also satisfies the following differential equation

$$\alpha^2 \frac{d^2E}{dx^2} - (1+\chi)E = \frac{\sigma_e}{\varepsilon_0}. \tag{7.2-12}$$

The general solution to (7.2-12) can be obtained easily. It has two exponential terms from the corresponding homogeneous equation, and a constant term which is the particular solution. The general solution contains two new integration constants. These two integration constants result from the differentiation in obtaining the differential equation (7.2-12) from the original integral equation (7.2-10). Hence the solution to (7.2-12) may not satisfy (7.2-10). Therefore we substitute the general solution to (7.2-12) back into (7.2-10), which determines the two new integration constants. Then, with the boundary conditions $(7.2\text{-}2)_{5,6}$, we can determine σ_e and another integration constant resulting from integrating E for ϕ, and thus obtain the nonlocal electric potential distribution ϕ

$$\phi = [\frac{x}{h} + \frac{\chi}{kh} \frac{\sinh k(x - \frac{h}{2}) + \sinh \frac{kh}{2}}{\cosh \frac{kh}{2} + k\alpha \sinh \frac{kh}{2}}]$$

$$\times (1 + \frac{2\chi}{kh} \frac{\tanh \frac{kh}{2}}{1 + k\alpha \tanh \frac{kh}{2}})^{-1} V, \qquad (7.2\text{-}13)$$

$$\phi_0 = \frac{x}{h} V,$$

where ϕ_0 is the classical local solution, and

$$k = \frac{\sqrt{1+\chi}}{\alpha}. \qquad (7.2\text{-}14)$$

The nonlocal electric field distribution E and the local solution E_0 are

$$E = [1 + \chi \frac{\cosh k(x - \frac{h}{2})}{\cosh \frac{kh}{2} + k\alpha \sinh \frac{kh}{2}}](1 + \frac{2\chi}{kh} \frac{\tanh \frac{kh}{2}}{1 + k\alpha \tanh \frac{kh}{2}})^{-1} E_0, \qquad (7.2\text{-}15)$$

$$E_0 = -\frac{V}{h}.$$

Denoting the capacitance per unit electrode area from the local theory by C_0 and the one from the nonlocal theory by C, we have

$$C = (1 + \frac{2\chi}{kh} \frac{\tanh \frac{kh}{2}}{1 + k\alpha \tanh \frac{kh}{2}})^{-1} C_0, \quad C_0 = \frac{\varepsilon_0 (1+\chi)}{h}. \qquad (7.2\text{-}16)$$

With the expression of k in (7.2-14), we write $(7.16\text{-}1)_1$ in the following form:

$$\frac{C}{C_0} = (1 + \frac{\chi}{\sqrt{1+\chi}\frac{h}{2\alpha}} \frac{\tanh(\sqrt{1+\chi}\frac{h}{2\alpha})}{1 + \sqrt{1+\chi}\tanh(\sqrt{1+\chi}\frac{h}{2\alpha})})^{-1}. \qquad (7.2\text{-}17)$$

It is seen that the thin film capacitance from the nonlocal theory differs from the result of the local theory. The nonlocal solution depends on the ratio $h/2\alpha$ of the film thickness to the microscopic characteristic length. From (7.2-17) we immediately have

$$C/C_0 < 1, \tag{7.2-18}$$

which shows that the nonlocal result is smaller than the local result. From (7.2-17) we also have the following limit behavior

$$\lim_{\frac{h}{\alpha}\to\infty} \frac{C}{C_0} = 1, \tag{7.2-19}$$

which shows that when the film thickness is large compared to the microscopic characteristic length, the nonlocal solution approaches the local solution. We also have the limit

$$\lim_{\frac{h}{\alpha}\to 0} \frac{C}{C_0} = \frac{1}{1+\chi} < 1\,, \tag{7.2-20}$$

which shows that the nonlocal and local solutions differ more for materials with larger χ. We plot C/C_0 from (7.2-17) as a function of $h/2\alpha$ for values of $\chi = 1$, 10, and 100 in Figure 7.2-2. It is seen that for a film with a moderate value of $\chi = 100$, when the thickness $h/2\alpha \approx 10$, there is a deviation of about 10% from the local theory which has a fixed value of 1. The figure shows that $C/C_0 < 1$ and the deviation from 1 becomes larger as h becomes smaller and disappears when h is large:

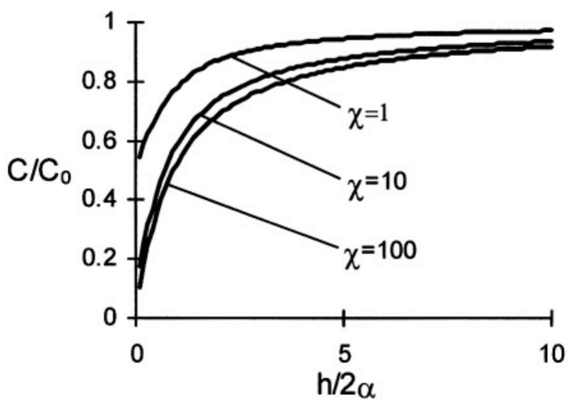

Figure 7.2-2. Capacitance for $\chi = 1$, 10, and 100.

The spatial distribution of the electric field for $\chi = 10$ and for two values of $h/2\alpha = 1$ and 5, respectively, is shown in Figure 7.2-3. It is interesting to see that the field is large near the electrodes compared to the local solution with the fixed value of 1. The curve with $h/2\alpha = 5$ has a larger electric field near the electrodes than the curve with $h/2\alpha = 1$. This is a boundary effect exhibited by the nonlocal theory. Even for a thick capacitor, (7.2-15) still yields

$$\lim_{\frac{h}{\alpha}\to\infty}\frac{E(h)}{E_0}=1+\frac{\chi}{1+\sqrt{1+\chi}}>1\,. \tag{7.2-21}$$

For our case, with $\chi = 10$, Equation (7.2-21) yields a limit value of 3.32. For materials with a large χ the value of (7.2-21) can be large. Since E is larger near the electrodes and D is a constant, P must be smaller near the electrodes than near the center of the plate.

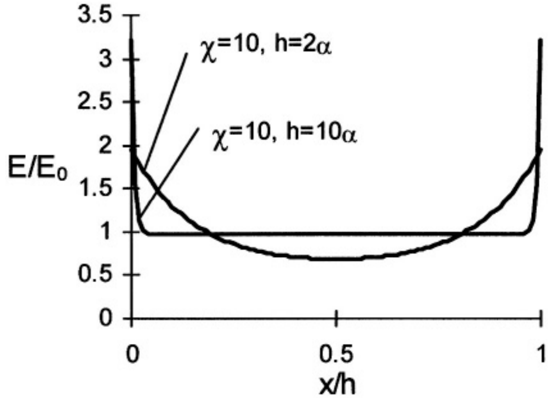

Figure 7.2-3. Electric field distribution for $\chi = 10$, $h/2\alpha = 1$ and 5.

The spatial distribution of the normalized deviation of the electric potential from the local solution for $\chi = 10$ and for two values of $h/2\alpha = 1$ and 5, respectively, are shown in Figure 7.2-4. The curve with $h/2\alpha = 5$ shows a smaller deviation.

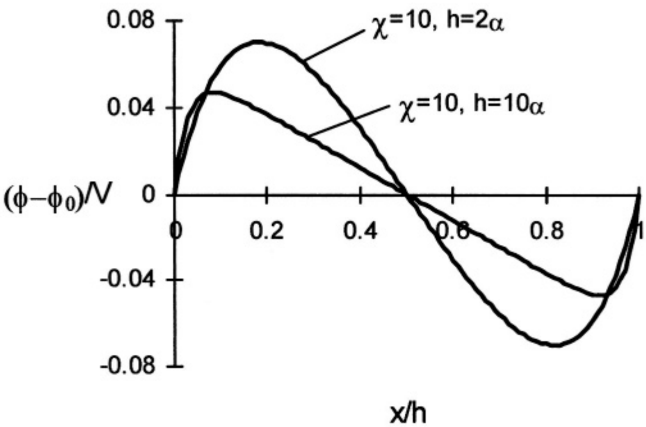

Figure 7.2-4. Electric potential deviation for $\chi = 10$, $h/2\alpha = 1$ and 5.

Finally, we note that in (7.2-12) the small parameter α appears as the coefficient of the term with the highest derivative. Hence when α tends to zero we have a singular perturbation problem of boundary layer type of a differential equation. For this type of problem, when the small parameter is set to zero, certain boundary conditions have to be dropped because the order of the differential equation is lowered. Equation (7.2-12) is a consequence of an integral-differential equation of ϕ defined by (7.2-2), which only needs two boundary conditions. In the solution procedure, two of the integration constants in the general solution to (7.2-12) were determined by the integral equation (7.2-10). However, if we take (7.2-12) as our starting point, we need two more boundary conditions. This is because (7.2-12) is a fourth-order differential equation for ϕ (considering it has already been integrated once with an integration constant σ_e). Then when α is set to zero, two boundary conditions have to be dropped.

3. GRADIENT EFFECTS

3.1 Gradient Effect as a Weak Nonlocal Effect

Gradient effects in constitutive relations can be shown to be related to weak nonlocal effects. For example, consider a one-dimensional nonlocal constitutive relation between Y and X in a homogeneous, unbounded medium. We have

$$\begin{aligned}
Y(x) &= \int_{-\infty}^{+\infty} K(x'-x)X(x')dx' \\
&= \int_{-\infty}^{+\infty} K(x'-x)X[x+(x'-x)]dx' \\
&= \int_{-\infty}^{+\infty} K(x'-x)[X(x)+X'(x)(x'-x)+\cdots]dx' \\
&\cong \int_{-\infty}^{+\infty} K(x'-x)[X(x)+X'(x)(x'-x)]d(x'-x) \\
&= \int_{-\infty}^{+\infty} K(x'-x)X(x)d(x'-x)+\int_{-\infty}^{+\infty} K(x'-x)X'(x)(x'-x)d(x'-x) \\
&= X(x)\int_{-\infty}^{+\infty} K(x'-x)d(x'-x)+X'(x)\int_{-\infty}^{+\infty} K(x'-x)(x'-x)d(x'-x) \\
&= aX(x)+bX'(x),
\end{aligned} \tag{7.3-1}$$

where

$$a = \int_{-\infty}^{+\infty} K(x'-x)d(x'-x),$$
$$b = \int_{-\infty}^{+\infty} K(x'-x)(x'-x)d(x'-x). \qquad (7.3\text{-}2)$$

Therefore, to the lowest order of approximation, the nonlocal relation reduces to a local one, and to the next order a gradient term arises.

3.2 Gradient Effect and Lattice Dynamics

Gradient terms can also be introduced in the following procedure. Consider the extensional motion of a one-dimensional spring-mass system (see Figure 7.3-1).

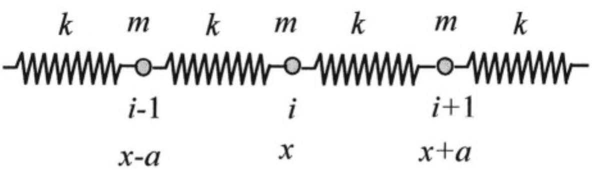

Figure 7.3-1. A spring-mass system.

The motion of the i-th particle is governed by the finite difference equation

$$m\ddot{u}(i) = k[u(i+1) - u(i)] - k[u(i) - u(i-1)], \qquad (7.3\text{-}3)$$

or, with the introduction of x

$$\begin{aligned} m\ddot{u}(x) &= k[u(x+a) + u(x-a) - 2u(x)] \\ &= k\Big[u(x) + u'(x)a + \frac{1}{2}u''(x)a^2 + \frac{1}{6}u'''(x)a^3 + \frac{1}{24}u''''(x)a^4 + \cdots \\ &\quad + u(x) - u'(x)a + \frac{1}{2}u''(x)a^2 - \frac{1}{6}u'''(x)a^3 \\ &\quad + \frac{1}{24}u''''(x)a^4 + \cdots - 2u(x)\Big] \\ &\cong k\left[u''(x)a^2 + \frac{1}{12}u''''(x)a^4\right] = T'(x), \end{aligned} \qquad (7.3\text{-}4)$$

where the extensional force T is given by the following constitutive relation

$$T = ka^2 u'(x) + \frac{ka^4}{12} u'''(x), \tag{7.3-5}$$

which depends on the strain u' and its second gradient. It should be noted that, according to Mindlin [44], a continuum theory with the first strain gradient is fundamentally flawed in that it is qualitatively inconsistent with lattice dynamics and the second strain gradient needs to be included to correct the inconsistency.

3.3 Polarization Gradient

Mindlin [45] generalized the theory of piezoelectricity by allowing the stored energy density to depend on the polarization gradient $P_{j,i}$

$$\Pi(u_i, P_i, \phi) = \int_V \left[W(S_{ij}, P_i, P_{j,i}) - \frac{1}{2}\varepsilon_0 \phi_{,i}\phi_{,i} + \phi_{,i} P_i \right] dV, \tag{7.3-6}$$

where boundary terms are dropped for simplicity. The stationary conditions of the above functional for independent variations of u_i, ϕ and P_i are

$$\begin{aligned} &\left(\frac{\partial W}{\partial S_{ij}}\right)_{,i} = 0, \\ &-\varepsilon_0 \phi_{,ii} + P_{i,i} = 0, \\ &-\frac{\partial W}{\partial P_i} + \left(\frac{\partial W}{\partial P_{j,i}}\right)_{,j} - \phi_{,i} = 0. \end{aligned} \tag{7.3-7}$$

Equations (7.3-7) represent seven equations for u_i, P_i and ϕ. If the dependence of W on the polarization gradient is dropped, Equations (7.3-7) reduce to the theory of linear piezoelectricity. The inclusion of polarization gradient is supported by lattice dynamics [46,47]. The polarization gradient theory and lattice dynamics both predict the thin film capacitance to be smaller than the classical result [47], as shown in Figure 7.2-2.

3.4 Electric Field Gradient and Electric Quadrupole

3.4.1 Governing Equations

Electric field gradient can also be included in constitutive relations [48]. Electric field gradient theory is equivalent to the theory of dielectrics with electric quadrupoles [1], because electric quadrupole is the thermodynamic conjugate of the electric field gradient. Consider the following functional [49]

$$\Pi(u_i,\phi) = \int_V \left[W(S_{ij}, E_i, E_{i,j}) - \frac{1}{2}\varepsilon_0 E_i E_i - f_i u_i + \rho_e \phi \right] dV \\ - \int_S \left(\bar{t}_i u_i + \bar{d}\phi + \bar{\pi}\frac{\partial \phi}{\partial \mathbf{n}} \right) dS, \tag{7.3-8}$$

where $\bar{d}$ is related to surface free charge. The presence of the $\bar{\pi}$ term is variationally consistent. We choose

$$W(S_{ij}, E_i, E_{i,j}) - \frac{1}{2}\varepsilon_0 E_i E_i \\ = H(S_{ij}, E_i) - \varepsilon_0 \gamma_{ijk} E_i E_{j,k} - \frac{1}{2}\varepsilon_0 \alpha_{ijkl} E_{i,j} E_{k,l}, \tag{7.3-9}$$

where H is the usual electric enthalpy function of piezoelectric materials given in (2.1-9), which is repeated below:

$$H(S_{ij}, E_i) = \frac{1}{2} c_{ijkl} S_{ij} S_{kl} - e_{ikl} E_i S_{kl} - \frac{1}{2}\varepsilon_{ij} E_i E_j. \tag{7.3-10}$$

γ_{ijk} and α_{ijkl} are new material constants due to the introduction of the electric field gradient into the energy density function. γ_{ijk} has the dimension of length. α_{ijkl} has the dimension of (length)^2. Physically they may be related to characteristic lengths of microstructural interactions of the material. Since $E_{i,j} = E_{j,i}$, α_{ijkl} has the same structure as c_{ijkl} as required by crystal symmetry, and γ_{ijk} has the same structure as e_{ijk}. For W to be negative definite in the case of pure electric phenomena without mechanical fields, we require α_{ijkl} to be positive definite like ε_{ij}.

With the following constraints

$$S_{ij} = (u_{i,j} + u_{j,i})/2, \quad E_i = -\phi_{,i}, \tag{7.3-11}$$

from the variational functional in (7.3-8), for independent variations of u_i and ϕ in V, we have

$$T_{ji,j} + f_i = 0, \\ D_{i,i} = \rho_e, \tag{7.3-12}$$

where we have denoted

$$
\begin{aligned}
T_{ij} &= \frac{\partial W}{\partial S_{ij}} = c_{ijkl}S_{kl} - e_{kij}E_k, \\
D_i &= \varepsilon_0 E_i + P_i = \varepsilon_{ij}E_j + e_{ikl}S_{kl} \\
&\quad - \varepsilon_0(\gamma_{kij} - \gamma_{ijk})E_{j,k} - \varepsilon_0\alpha_{ijkl}E_{k,lj}, \\
P_i &= \Pi_i - \Pi_{ij,j} = \varepsilon_0\chi_{ij}E_j + e_{ikl}S_{kl} \\
&\quad - \varepsilon_0(\gamma_{kij} - \gamma_{ijk})E_{j,k} - \varepsilon_0\alpha_{ijkl}E_{k,lj}, \\
\Pi_i &= -\frac{\partial W}{\partial E_i} = e_{ikl}S_{kl} + \varepsilon_0\chi_{ij}E_j + \varepsilon_0\gamma_{ijk}E_{j,k}, \\
\Pi_{ij} &= -\frac{\partial W}{\partial E_{i,j}} = \varepsilon_0\gamma_{kij}E_k + \varepsilon_0\alpha_{ijkl}E_{k,l},
\end{aligned}
\tag{7.3-13}
$$

and $\varepsilon_{ij} = \varepsilon_0(\delta_{ij}+\chi_{ij})$. χ_{ij} is the relative electric susceptibility. When the energy density does not depend on the electric field gradient, the equations reduce to the linear theory of piezoelectricity. The first variation of the functional in (7.3-8) also implies the following as possible forms of boundary conditions on S

$$
\begin{aligned}
&T_{ji}n_j = \bar{t}_i \quad \text{or} \quad \delta u_i = 0, \\
&\int_S \left[(D_i n_i - \bar{d})\delta\phi + \Pi_{ij}n_j(\nabla_s\delta\phi)_i\right]dS = 0, \\
&\Pi_{ij}n_j n_i = \bar{\pi} \quad \text{or} \quad \delta\left(\frac{\partial\phi}{\partial \mathbf{n}}\right) = 0,
\end{aligned}
\tag{7.3-14}
$$

where ∇_s is the surface gradient operator. One obvious possibility of Equation **(7.3-14)**$_2$ is $\delta\phi = 0$ on S. With substitutions from (7.3-13) and (7.3-11), Equation (7.3-12) can be written as four equations for u_i and ϕ:

$$
\begin{aligned}
c_{ijkl}u_{k,lj} + e_{kij}\phi_{,kj} + f_i &= \rho\ddot{u}_i, \\
e_{ikl}u_{k,li} - \varepsilon_{ij}\phi_{,ij} + \varepsilon_0\alpha_{ijkl}\phi_{,ijkl} &= \rho_e,
\end{aligned}
\tag{7.3-15}
$$

where we have added the acceleration term.

3.4.2 Anti-Plane Problems of Ceramics

For anti-plane motions of polarized ceramics, Equations (7.3-15) reduce to a much simpler form. Consider

$$u_1 = u_2 = 0, \quad u_3 = u_3(x_1, x_2, t),$$
$$\phi = \phi(x_1, x_2, t). \tag{7.3-16}$$

The non-vanishing strain and electric field components are

$$\begin{Bmatrix} S_5 \\ S_4 \end{Bmatrix} = \nabla u, \quad \begin{Bmatrix} E_1 \\ E_2 \end{Bmatrix} = -\nabla \phi. \tag{7.3-17}$$

For ceramics poled in the x_3 direction, the nontrivial components of T_{ij} and D_i are

$$\begin{Bmatrix} T_5 \\ T_4 \end{Bmatrix} = c\nabla u + e\nabla \phi,$$
$$\begin{Bmatrix} D_1 \\ D_2 \end{Bmatrix} = e\nabla u - \varepsilon \nabla \phi + \varepsilon_0 \alpha \nabla(\nabla^2 \phi), \tag{7.3-18}$$
$$D_3 = -\varepsilon_0(\gamma_{31} - \gamma_{15})\nabla^2 \phi,$$

where ∇^2 is the two-dimensional Laplacian, $c = c_{44}$, $e = e_{15}$, $\varepsilon = \varepsilon_{11}$, and $\alpha = \alpha_{11}$. The nontrivial ones of (7.3-15) take the form

$$c\nabla^2 u + e\nabla^2 \phi + f = \rho \ddot{u},$$
$$e\nabla^2 u - \varepsilon \nabla^2 \phi + \varepsilon_0 \alpha \nabla^2 \nabla^2 \phi = \rho_e, \tag{7.3-19}$$

where $f = f_3$.

3.4.3 Thin Film Capacitance

To see the most basic effects of the electric field gradient, consider the infinite plate capacitor shown in Figure 7.3-1.

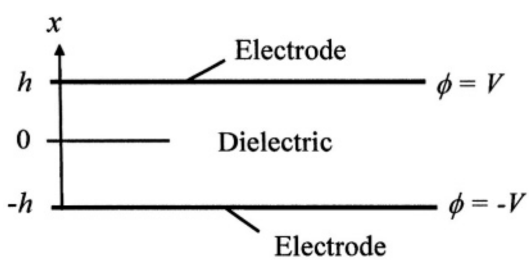

Figure 7.3-1. A thin dielectric plate.

The problem is one-dimensional. We assume that the material is isotropic so that there is no piezoelectric coupling. The equations and boundary conditions from the electric field gradient theory are

$$\begin{aligned}
&\frac{dD}{dx} = 0, \quad -h < x < h, \\
&D = \varepsilon_0 E + P, \quad -h < x < h, \\
&P = \varepsilon_0 \chi E - \varepsilon_0 \alpha \frac{d^2 E}{dx^2}, \quad -h < x < h, \\
&E = -\frac{d\phi}{dx}, \quad -h < x < h, \\
&\phi(-h) = -V, \quad \phi(h) = V.
\end{aligned} \tag{7.3-20}$$

From (7.3-20) the following equation for ϕ can be obtained:

$$\frac{d^4\phi}{dx^4} - k^2 \frac{d^2\phi}{dx^2} = 0, \tag{7.3-21}$$

where

$$k^2 = \frac{1+\chi}{\alpha}, \tag{7.3-22}$$

The general solution to (7.3-22) can be obtained in a straightforward manner. The anti-symmetric solution for ϕ is

$$\phi = C_1 x + C_2 \sinh kx, \tag{7.3-23}$$

where C_1 and C_2 are integration constants. Due to the introduction of the electric field gradient, the order of the equation for ϕ is now higher than the Laplace equation in the classical theory. Therefore more boundary conditions than in the classical theory are needed. Following Mindlin [47], we prescribe

$$P(x = h) = -\lambda \varepsilon_0 \chi \frac{V}{h}, \tag{7.3-24}$$

where $0 \le \lambda \le 1$ is a parameter. $\lambda = 1$ represents the classical solution. Equation (7.3-24) is for Mindlin's polarization theory. When it is directly introduced here for the electric field gradient theory, it is not variationally consistent. This can be resolved by translating it into a different form mathematically while still keeping its physical interpretation, which is left as an exercise. With the solution in (7.3-23), the boundary conditions in (7.3-20) and (7.3-24), and the identification of the relation between an integration constant and the surface charge on the electrode at $x = h$, we obtain the capacitance C per unit area, the potential ϕ and the electric field E as

$$\frac{C}{C_0}=\frac{1+\lambda\chi\dfrac{\tanh kh}{kh}}{1+\chi\dfrac{\tanh kh}{kh}}\,, \tag{7.3-25}$$

$$\frac{\phi}{V}=\frac{1+\lambda\chi\dfrac{\tanh kh}{kh}}{1+\chi\dfrac{\tanh kh}{kh}}\frac{x}{h}+\left(1-\frac{1+\lambda\chi\dfrac{\tanh kh}{kh}}{1+\chi\dfrac{\tanh kh}{kh}}\right)\frac{\sinh kx}{\sinh kh}, \tag{7.3-26}$$

$$\frac{E}{E_0}=\frac{1+\lambda\chi\dfrac{\tanh kh}{kh}}{1+\chi\dfrac{\tanh kh}{kh}}+\left(1-\frac{1+\lambda\chi\dfrac{\tanh kh}{kh}}{1+\chi\dfrac{\tanh kh}{kh}}\right)\frac{\cosh kx}{\sinh kh}kh, \tag{7.3-27}$$

where

$$C_0=\frac{\varepsilon}{2h},\quad E_0=-\frac{V}{h}\,, \tag{7.3-28}$$

are the capacitance and electric field from the classical theory, and $\varepsilon = \varepsilon_0(1+\chi)$ is the electric permittivity. Equation (7.3-25) is exactly the same as the result of the polarization gradient theory [47], and its behavior is qualitatively the same as what is shown in Figure 7.2-2.

3.4.4 A Line Source

Consider the potential field of a line charge Q_e at the origin [50]. We need to solve Equation (7.3-19) with a concentrated electric source. Eliminating u we obtain

$$\begin{gathered}-\bar{\varepsilon}\nabla^2\phi+\varepsilon_0\alpha\nabla^2\nabla^2\phi=Q_e\delta(x_1,x_2),\\ \bar{\varepsilon}=\varepsilon(1+k^2),\quad k^2=e^2/(\varepsilon c).\end{gathered} \tag{7.3-29}$$

Equation (7.3-29) can be rewritten as

$$(-\bar{\varepsilon}+\varepsilon_0\alpha\nabla^2)\nabla^2\phi=Q_e\delta(x_1,x_2)\,. \tag{7.3-30}$$

Therefore $\nabla^2\phi$ is the fundamental solution of the differential operator in Equation (7.3-30), which is known. Hence

$$\nabla^2\phi=\frac{1}{r}\frac{d}{dr}(r\frac{d\phi}{dr})=-\frac{Q_e}{2\pi\varepsilon_0\alpha}K_0(\beta r), \tag{7.3-31}$$

where K_0 is the zero order modified Bessel function of the second-kind. Since

$$xK_0(x) = -\frac{d}{dx}[xK_1(x)], \quad K_1(x) = -\frac{d}{dx}[K_0(x)], \tag{7.3-32}$$

integrating Equation (7.3-31) twice we obtain

$$\phi = -\frac{Q_e}{2\pi\bar{\varepsilon}}[\ln r + K_0(\beta r)],$$
$$\beta^2 = \bar{\varepsilon}/(\varepsilon_0\alpha), \tag{7.3-33}$$

where the ln$r$ term is the classical solution. Since

$$K_0(x) \to -\ln x, \quad x \to 0,$$
$$K_0(x) \to \left(\frac{\pi}{2x}\right)^{1/2} e^{-x}, \quad x \to \infty, \tag{7.3-34}$$

we have

$$\phi \to \frac{Q_e}{2\pi\bar{\varepsilon}} \ln\beta = \frac{Q_e}{4\pi\bar{\varepsilon}} \ln\frac{\bar{\varepsilon}}{\varepsilon_0\alpha}, \quad r \to 0,$$
$$\phi \to -\frac{Q_e}{2\pi\bar{\varepsilon}} \ln r, \quad r \to \infty. \tag{7.3-35}$$

The potential field is plotted in Figure 7.3-2.

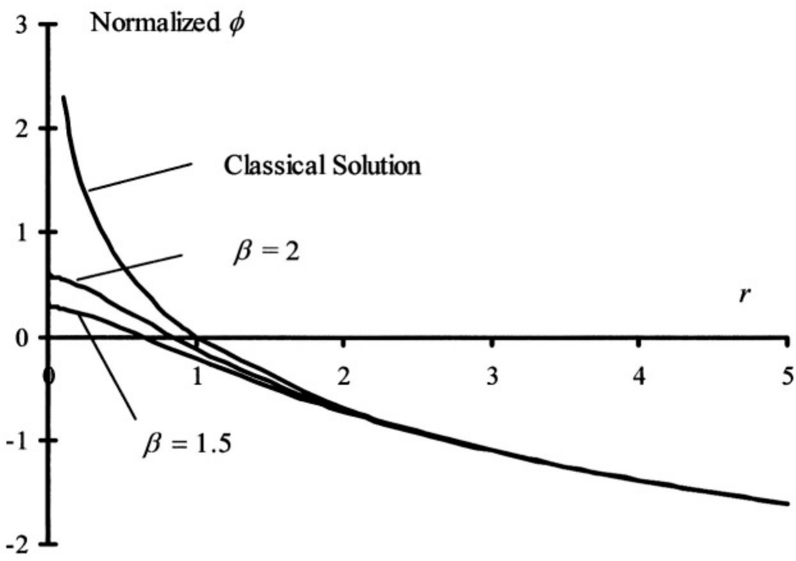

Figure 7.3-2. Normalized potential field ($-2\pi\bar{\varepsilon}\phi/Q_e$) of a line source.

For far field ϕ approaches the classical solution. At the source point ϕ is not singular. This is fundamentally different from the classical solution. When α approaches zero, Equation (7.3-33) reduces to the classical result. The curve with the larger value of β is closer to the classical solution. These qualitative behaviors are as expected.

3.4.5 Dispersion of Plane waves

In the source-free case, eliminating ϕ from (7.3-19) we obtain

$$\bar{c}\nabla^2 u + \frac{\varepsilon_0}{\varepsilon}\alpha\nabla^2(\rho\ddot{u} - c\nabla^2 u) = \rho\ddot{u},$$
$$\bar{c} = c(1+k^2). \tag{7.3-36}$$

Consider the propagation of the following plane wave

$$u = \exp[i(\xi x_1 - \omega t)]. \tag{7.3-37}$$

Substitution of Equation (7.3-37) into the homogeneous form of Equation (7.3-36) yields the following dispersion relation [50]

$$\omega^2 = \frac{c}{\rho}\xi^2 \frac{1 + k^2 + \frac{\varepsilon_0}{\varepsilon}\alpha\xi^2}{1 + \frac{\varepsilon_0}{\varepsilon}\alpha\xi^2}. \tag{7.3-38}$$

Different from the plane waves in linear piezoelectricity, Equation (7.3-38) shows that the waves are dispersive, and the dispersion is caused by the electric field gradient through electromechanical coupling. The dispersion disappears when $k = 0$, or when there is no electromechanical coupling. We note that the dispersion is more pronounced when $\xi\sqrt{\alpha}$ is not small, or when the wavelength $2\pi/\xi$ is not large when compared to the microscopic characteristic length $\sqrt{\alpha}$. When $\xi\sqrt{\alpha}$ just begins to show its effect, Equation (7.3-38) can be approximated by

$$\omega^2 \cong \frac{\bar{c}}{\rho}\xi^2\left[1 - \frac{k^2}{1+k^2}\frac{\varepsilon_0}{\varepsilon}\alpha\xi^2\right]. \tag{7.3-39}$$

As a numerical example we consider polarized ceramics PZT-7A. For polarized ferroelectric ceramics the grain size, which may be taken as the microscopic characteristic length $\sqrt{\alpha}$, is at sub-micron range. We plot

Equation (7.3-39) in Figure 7.3-3 for different values of $\sqrt{\alpha}$. It can be seen that larger values of $\sqrt{\alpha}$ yields more dispersion, as expected.

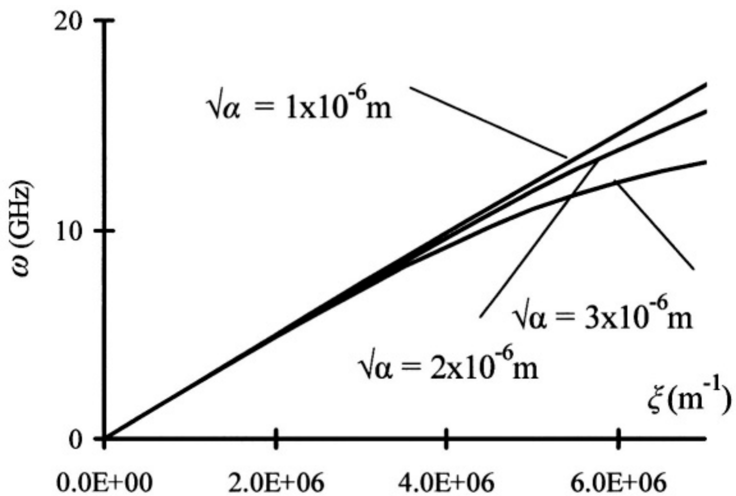

Figure 7.3-3. Dispersion curves of plane waves.

Problem

7.3-1. Study the capacitance of the dielectric plate in Figure 7.3-1 using the electric field gradient theory with the following additional boundary condition instead of (7.3-24)

$$E(x = h) = -\lambda \frac{V}{h}, \tag{7.3-40}$$

where $1 \leq \lambda$ is aparameter. $\lambda = 1$ represents the classical solution.

4. THERMAL AND VISCOUS EFFECTS

4.1 Equations in Spatial Form

Thermal and viscous effects often appear together and are treated in this section. The energy equation in the global balance laws in (1.2-3) needs to be extended to include thermal effects, and the second law of thermodynamics needs to be added as follows:

$$\frac{D}{Dt}\int_v \rho(\frac{1}{2}\mathbf{v}\cdot\mathbf{v}+e)dv$$
$$=\int_v [(\rho\mathbf{f}+\mathbf{F}^E)\cdot\mathbf{v}+w^E+\rho\gamma]dv+\int_s (\mathbf{t}\cdot\mathbf{v}-\mathbf{n}\cdot\mathbf{q})ds, \qquad (7.4\text{-}1)$$
$$\frac{D}{Dt}\int_v \rho\eta dv \geq \int_v \frac{\rho\gamma}{\theta}dv - \int_s \frac{\mathbf{q}\cdot\mathbf{n}}{\theta}ds,$$

where $\mathbf{q}$ is the heat flux vector, η is the entropy per unit mass, γ is the body heat source per unit mass, and θ is the absolute temperature. The above integral balance laws can be localized to yield

$$\rho\dot{e}=\tau_{ij}v_{j,i}+\rho\gamma-q_{i,i}+w^E,$$
$$\rho\dot{\eta}\geq\frac{\rho\gamma}{\theta}-(\frac{q_i}{\theta})_{,i}. \qquad (7.4\text{-}2)$$

Eliminating γ in (7.4-2), we obtain the Clausius-Duhem inequality as

$$\rho(\theta\dot{\eta}-\dot{e})+\tau_{ij}v_{j,i}+\rho E_i\dot{\pi}_i-\frac{q_i\theta_{,i}}{\theta}\geq 0. \qquad (7.4\text{-}3)$$

The free energy ψ can be introduced through the following Legendre transform:

$$\psi=e-\theta\eta-E_i\pi_i, \qquad (7.4\text{-}4)$$

then the energy equation $(7.4\text{-}2)_1$ and the C–D inequality (7.4-3) become

$$\rho(\dot{\psi}+\eta\dot{\theta}+\dot{\eta}\theta)=\tau_{ij}v_{j,i}-P_i\dot{E}_i+\rho\gamma-q_{i,i}, \qquad (7.4\text{-}5)$$

and

$$-\rho(\dot{\psi}+\eta\dot{\theta})+\tau_{ij}v_{j,i}-P_i\dot{E}_i-\frac{q_i\theta_{,i}}{\theta}\geq 0. \qquad (7.4\text{-}6)$$

7.2 Equations in Material Form

Introducing the material heat flux and temperature gradient

$$Q_K=JX_{K,k}q_k, \quad \Theta_K=\theta_{,K}=\theta_{,k}y_{k,K}, \qquad (7.4\text{-}7)$$

the energy equation and the C–D inequality can be written as

$$
\begin{aligned}
&\rho_0(\dot{\psi}+\eta\dot{\theta}+\dot{\eta}\theta)=T^S_{KL}E\dot{S}_{KL}-\mathcal{P}_K\dot{\mathcal{E}}_K+\rho_0\gamma-Q_{K,K},\\
&-\rho_0(\dot{\psi}+\eta\dot{\theta})+T^S_{KL}\dot{S}_{KL}-\mathcal{P}_K\dot{\mathcal{E}}_K-\frac{Q_K\Theta_K}{\theta}\geq 0.
\end{aligned}
\tag{7.4-8}
$$

7.3 Constitutive Relations

For constitutive relations we start with the following:

$$
\begin{aligned}
&\psi=\psi(S_{KL},\mathcal{E}_K,\theta,\Theta_K),\\
&T^S_{KL}=T^S_{KL}(S_{KL},\mathcal{E}_K,\theta,\Theta_K,\dot{S}_{KL},\dot{\mathcal{E}}_K),\\
&\mathcal{P}_K=\mathcal{P}_K(S_{KL},\mathcal{E}_K,\theta,\Theta_K,\dot{S}_{KL},\dot{\mathcal{E}}_K),\\
&Q_K=Q_K(S_{KL},\mathcal{E}_K,\theta,\Theta_K,\dot{S}_{KL},\dot{\mathcal{E}}_K).
\end{aligned}
\tag{7.4-9}
$$

Substitution of (7.4-9) into the C–D inequality $(7.4\text{-}8)_2$ yields

$$
\begin{aligned}
&-\rho_0\frac{\partial\psi}{\partial\Theta_K}\dot{\Theta}_K-\rho_0(\eta+\frac{\partial\psi}{\partial\theta})\dot{\theta}\\
&+(T^S_{KL}-\rho_0\frac{\partial\psi}{\partial S_{KL}})S_{KL}-(\mathcal{P}_K+\rho_0\frac{\partial\psi}{\partial\mathcal{E}_K})\dot{\mathcal{E}}_K-\frac{1}{\theta}Q_K\Theta_K\geq 0.
\end{aligned}
\tag{7.4-10}
$$

Since (7.4-10) is linear in $\dot{\Theta}_K$ and $\dot{\theta}$, for the inequality to hold ψ cannot depend on Θ_K, and η is related to ψ by

$$
\eta=-\frac{\partial\psi}{\partial\theta}.
\tag{7.4-11}
$$

We break T^S_{KL} and $\mathcal{P}_K$ into reversible and dissipative parts as follows:

$$
\begin{aligned}
&T^S_{KL}=T^R_{KL}+T^D_{KL},\quad \mathcal{P}_K=\mathcal{P}^R_K+\mathcal{P}^D_K,\\
&T^R_{KL}=\rho_0\frac{\partial\psi}{\partial S_{KL}},\quad \mathcal{P}^R_K=-\rho_0\frac{\partial\psi}{\partial\mathcal{E}_K},\\
&T^D_{KL}=T^D_{KL}(S_{KL},\mathcal{E}_K,\theta,\Theta_K,\dot{S}_{KL},\dot{\mathcal{E}}_K),\\
&\mathcal{P}^D_K=\mathcal{P}^D_K(S_{KL},\mathcal{E}_K,\theta,\Theta_K,\dot{S}_{KL},\dot{\mathcal{E}}_K).
\end{aligned}
\tag{7.4-12}
$$

Then what is left for the C–D inequality (7.4-10) is

$$
T^D_{KL}\dot{S}_{KL}-\mathcal{P}^D_K\dot{\mathcal{E}}_K-\frac{1}{\theta}Q_K\Theta_K\geq 0\,.
\tag{7.4-13}
$$

From $(7.4\text{-}8)_1$ and (7.4-11) we obtain the heat equation

$$
\rho_0\theta\dot{\eta}=T^D_{KL}\dot{S}_{KL}-\mathcal{P}^D_K\dot{\mathcal{E}}_K+\rho_0\gamma-Q_{K,K}\,.
\tag{7.4-14}
$$

4.4 Boundary-Value Problem

In summary, the nonlinear equations for thermoviscoelectroelasticity are

$$\begin{aligned}
&\rho_0 = \rho J, \\
&K_{Lk,L} + \rho_0 f_k = \rho_0 \ddot{y}_k, \\
&\mathcal{D}_{K,K} = \rho_E, \\
&\rho_0 \theta \dot{\eta} = T^D_{KL} \dot{S}_{KL} - \mathcal{P}^D_K \dot{\mathcal{E}}_K + \rho_0 \gamma - Q_{K,K},
\end{aligned} \tag{7.4-15}$$

with constitutive relations

$$\begin{aligned}
&\psi = \psi(S_{KL}, \mathcal{E}_K, \theta), \quad \eta = -\frac{\partial \psi}{\partial \theta}, \\
&T^S_{KL} = T^R_{KL} + T^D_{KL}, \quad \mathcal{P}_K = \mathcal{P}^R_K + \mathcal{P}^D_K, \\
&T^R_{KL} = \rho_0 \frac{\partial \psi}{\partial S_{KL}}, \quad \mathcal{P}^R_K = -\rho_0 \frac{\partial \psi}{\partial \mathcal{E}_K}, \\
&T^D_{KL} = T^D_{KL}(S_{KL}, \mathcal{E}_K, \theta, \Theta_K, \dot{S}_{KL}, \dot{\mathcal{E}}_K), \\
&\mathcal{P}^D_K = \mathcal{P}^D_K(S_{KL}, \mathcal{E}_K, \theta, \Theta_K, \dot{S}_{KL}, \dot{\mathcal{E}}_K). \\
&Q_K = Q_K(S_{KL}, \mathcal{E}_K, \theta, \Theta_K, \dot{S}_{KL}, \dot{\mathcal{E}}_K),
\end{aligned} \tag{7.4-16}$$

which are restricted by

$$T^D_{KL} \dot{S}_{KL} - \mathcal{P}^D_K \dot{\mathcal{E}}_K - \frac{1}{\theta} Q_K \Theta_K \geq 0. \tag{7.4-17}$$

The equation for the conservation of mass in $(7.4\text{-}15)_1$ can be used to determine ρ separately from the other equations in (7.4-15). Equations $(7.4\text{-}15)_{2,3,4}$ can be written as five equations for $y_i(X_L, t)$, $\phi(X_L, t)$ and $\theta(X_L, t)$. On the boundary surface *S*, the thermal boundary conditions may be either prescribed temperature or heat flux

$$N_L Q_L = \overline{Q}. \tag{7.4-18}$$

4.5 Linear Equations

For small deformations and weak electric fields

$$\begin{aligned}
&D_{i,i} = \rho_e, \\
&T_{ji,j} + \rho_0 f_i = \rho_0 \ddot{u}_i, \\
&\rho_0 \theta \dot{\eta} = T^D_{ij} \dot{S}_{ij} - P^D_i \dot{E}_i - q_{i,i}.
\end{aligned} \tag{7.4-19}$$

The reversible part of the constitutive equations for small deformations and weak electric fields are determined by $\psi = \psi(S_{ij}, E_i, \theta)$ and

$$\eta = -\frac{\partial\psi}{\partial\theta}, \quad T_{ij}^R = \rho_0 \frac{\partial\psi}{\partial S_{ij}}, \quad P_i^R = -\rho_0 \frac{\partial\psi}{\partial E_i}. \tag{7.4-20}$$

In order to linearize the constitutive relations we expand ψ into a power series about $\theta = T_0$, $S_{ij} = 0$, and $E_i = 0$, where T_0 is a reference temperature. Denoting $T = \theta - T_0$, assuming $|T/T_0| << 1$, and keeping quadratic terms only, we can write

$$\begin{aligned} \rho_0\psi = {} & \frac{1}{2} c_{ijkl} S_{ij} S_{kl} - e_{kij} E_k S_{ij} - \frac{1}{2}\chi_{ij} E_i E_j \\ & - \frac{1}{2}\frac{\alpha}{T_0} T^2 - \beta_{kl} S_{kl} T - \lambda_k E_k T, \end{aligned} \tag{7.4-21}$$

where β_{ij} are the thermoelastic constants, λ_k are the pyroelectric constants and α is related to the specific heat. Equations (7.4-20) and (7.4-21) yield

$$\begin{aligned} T_{ij}^R &= c_{ijkl} S_{kl} - e_{kij} E_k - \beta_{ij} T, \\ D_i^R &= \varepsilon_0 E_i + P_i^R = e_{ijk} S_{jk} + \varepsilon_{ij} E_j + \lambda_k T, \\ \rho_0 \eta &= \frac{\alpha}{T_0} T + \beta_{kl} S_{kl} + \lambda_k E_k, \end{aligned} \tag{7.4-22}$$

which are the equations for linear thermopiezoelectricity given by Mindlin [51]. For the dissipative part of the constitutive relations we choose the linear relations

$$\begin{aligned} T_{ij}^D &= \mu_{ijkl} \dot{S}_{kl} - \alpha_{kij} \dot{E}_k, \\ D_i^D &= P_i^D = \beta_{ijk} \dot{S}_{jk} + \zeta_{ij} \dot{E}_j, \\ q_k &= -\kappa_{kl} \theta_{,l}. \end{aligned} \tag{7.4-23}$$

In the following we will assume $\beta_{ijk} = \alpha_{ijk}$. Equations (7.4-23) are restricted by

$$T_{ij}^D \dot{S}_{ij} - P_i^D \dot{E}_i - \frac{q_i \theta_{,i}}{\theta} \geq 0. \tag{7.4-24}$$

A dissipation function can be introduced as follows:

$$\chi(\dot{S}_{kl}, \dot{E}_j) = \frac{1}{2}\mu_{ijkl} \dot{S}_{kl} \dot{S}_{ij} - \alpha_{kij} \dot{E}_k \dot{S}_{ij} - \frac{1}{2}\zeta_{ij} \dot{E}_i \dot{E}_j, \tag{7.4-25}$$

whereby Equations $(7.4\text{-}19)_3$, $(7.4\text{-}23)_{1,2}$ and (7.4-24) can be written as

$$\rho_0 T_0 \dot{\eta} = 2\chi(\dot{S}_{ij}, \dot{E}_i) - q_{i,i},$$

$$T_{ij}^D = \frac{\partial\chi}{\partial\dot{S}_{ij}}, \quad P_i^D = -\frac{\partial\chi}{\partial\dot{E}_i}, \tag{7.4-26}$$

$$2\chi(\dot{S}_{ij}, \dot{E}_i) + \frac{\kappa_{ij}\theta_{,i}\theta_{,j}}{\theta} \geq 0.$$

Equation **(7.4-26)**$_4$ implies that

$$\chi(\dot{S}_{kl}, \dot{E}_j) \geq 0, \quad \kappa_{ij}\theta_{,i}\theta_{,j} \geq 0, \tag{7.4-27}$$

which further implies that μ_{ijkl}, $-\zeta_{ij}$, and κ_{ij} are positive definite. The formal similarity between (7.4-25) and the first three terms on the right-hand side of (7.4-21) suggests that the structures of μ_{ijkl}, ζ_{ij}, and α_{ijk} are the same as those of c_{ijkl}, χ_{ij}, and e_{ijk}, which are known for various crystal classes.

When the thermoelastic and pyroelectric effects are small, they can be neglected. Then the above equations for the linear theory reduce to two one-way coupled systems of equations, where one represents the problem of viscopiezoelectricity with the following constitutive relations

$$\begin{aligned} T_{ij}^R &= c_{ijkl}S_{kl} - e_{kij}E_k, \quad D_i^R = e_{ijk}S_{jk} + \varepsilon_{ij}E_j, \\ T_{ij}^D &= \mu_{ijkl}\dot{S}_{kl} - \alpha_{kij}\dot{E}_k, \quad D_i^D = \alpha_{ijk}\dot{S}_{jk} + \zeta_{ij}\dot{E}_j, \end{aligned} \tag{7.4-28}$$

and the other governs the temperature field

$$\begin{aligned} \rho_0 T_0 \dot{\eta} &= T_{ij}^D \dot{S}_{ij} - D_i^D \dot{E}_i - q_{i,i}, \\ \rho_0 \eta &= \frac{\alpha}{T_0} T. \end{aligned} \tag{7.4-29}$$

Equations (7.4-28) can be substituted into **(7.4-19)**$_{1,2}$ for four equations for y_i and ϕ. Once the mechanical and electric fields are found, they can be substituted into (7.4-29) to solve for the temperature field T.

Under harmonic excitation with an $e^{i\omega t}$ factor, the linear constitutive relations in (7.4-28) can be written as

$$\begin{aligned} T_{ij} &= c_{ijkl}S_{kl} + \mu_{ijkl}\dot{S}_{kl} - e_{kuj}E_k - \alpha_{kij}\dot{E}_k \\ &= (c_{ijkl} + i\omega\mu_{ijkl})S_{kl} - (e_{kuj} + i\omega\alpha_{kij})E_k, \\ D_i &= e_{ijk}S_{jk} + \alpha_{ijk}\dot{S}_{jk} + \varepsilon_{ij}E_j + \zeta_{ij}\dot{E}_j \\ &= (e_{ijk} + i\omega\alpha_{ijk})S_{jk} + (\varepsilon_{ij} + i\omega\zeta_{ij})E_j, \end{aligned} \tag{7.4-30}$$

Formally, the material constants become complex and frequency-dependent.

5. SEMICONDUCTION

Piezoelectric materials are either dielectrics or semiconductors. Mechanical fields and mobile charges in piezoelectric semiconductors can interact, and this is called the acoustoelectric effect. An acoustic wave traveling in a piezoelectric semiconductor can be amplified by application of a dc electric field. The acoustoelectric effect and the acoustoelectric amplification of acoustic waves have led to piezoelectric semiconductor devices. The basic behavior of piezoelectric semiconductors can be described by a simple extension of the theory of piezoelectricity.

5.1 Governing Equations

Consider a homogeneous, one-carrier piezoelectric semiconductor under a uniform dc electric field $\overline{E}_j$. The steady state current is $\overline{J}_i = q\overline{n}\mu_{ij}\overline{E}_j$, where q is the carrier charge which may be the electronic charge or its opposite, $\overline{n}$ is the steady state carrier density which produces electrical neutrality,and μ_{ij} is the carrier mobility. When an acoustic wave propagates through the material, perturbations of the electric field, the carrier density and the current are denoted by E_j, n and J_i. The linear theory for small signals consists of the equations of motion, Gauss's law, and conservation of charge [52]

$$\begin{aligned} T_{ji,j} &= \rho\ddot{u}_i, \\ D_{i,i} &= qn, \\ q\dot{n} + J_{i,i} &= 0. \end{aligned} \tag{7.5-1}$$

The above equations are accompanied by the following constitutive relations:

$$\begin{aligned} T_{ij} &= c_{ijkl}S_{kl} - e_{kij}E_k, \\ D_i &= e_{ijk}S_{jk} + \varepsilon_{ij}E_j, \\ J_i &= q\overline{n}\mu_{ij}E_j + qn\mu_{ij}\overline{E}_j - qd_{ij}n_{,j}, \end{aligned} \tag{7.5-2}$$

where d_{ij} are the carrier diffusion constants. Equations (7.5-1) can be written as five equations for $\mathbf{u}$, ϕ and n

$$\begin{aligned} c_{ijkl}u_{k,lj} + e_{kij}\phi_{,kj} + f_i &= \rho\ddot{u}_i, \\ e_{ikl}u_{k,li} - \varepsilon_{ij}\phi_{,ij} &= qn, \\ \dot{n} - \overline{n}\mu_{ij}\phi_{,ij} + \mu_{ij}\overline{E}_j n_{,i} - d_{ij}n_{,ij} &= 0. \end{aligned} \tag{7.5-3}$$

On the boundary of a finite body with a unit outward normal n_i, the mechanical displacement u_i or the traction vector $T_{ij}n_i$, the electric potential ϕ or the normal component of the electric displacement vector $D_i n_i$, and the carrier density n or the normal current $J_i n_i$ may be prescribed.

The Acoustoelectric effect and amplification of acoustic waves can also be achieved through composite structures of piezoelectric dielectrics and nonpiezoelectric semiconductors. In these composites the acoustoelectric effect is due to the combination of the piezoelectric effect and semiconduction in each component phase.

5.2 Surface Waves

As an example, consider the propagation of anti-plane surface waves in a piezoelectric dielectric half-space carrying a thin, nonpiezoelectric semiconductor film of silicon (see Figure 6.5-1) [53].

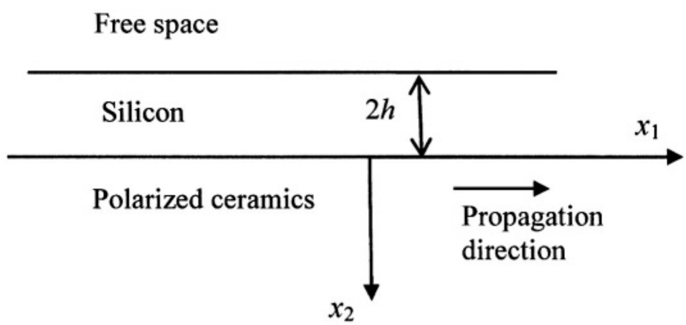

Figure 6.5-1. A ceramic half-space with a silicon film.

5.2.1 Equations for a Thin Film

The film is assumed to be very thin in the sense that its thickness is much smaller than the wavelength of the waves we are interested in. For thin films the following stress components can be approximately taken to vanish

$$T_{2j} = 0, \quad j = 1,2,3. \tag{7.5-4}$$

According to the compact matrix notation, with the range of p, q as 1,2, ... and 6, Equation (7.5-4) can be written as

$$T_q = 0, \quad q = 2,4,6. \tag{7.5-5}$$

For convenience we introduce a convention that subscripts *u, v, w* take the values 2, 4, 6 while subscripts *r, s, t* take the remaining values 1, 3, 5. Then Equation $(7.5\text{-}2)_{1,2}$ can be written as

$$\begin{aligned} T_r &= c_{rs}S_s + c_{ru}S_u - e_{kr}E_k, \\ T_v &= c_{vs}S_s + c_{vw}S_w - e_{kv}E_k = 0, \\ D_i &= e_{is}S_s + e_{iu}S_u + \varepsilon_{ij}E_j, \end{aligned} \qquad (7.5\text{-}6)$$

where (7.5-5) has been used. From $(7.5\text{-}6)_2$ we have

$$S_u = -c_{uv}^{-1}c_{vs}S_s + c_{uv}^{-1}e_{kv}E_k. \qquad (7.5\text{-}7)$$

Substitution of (7.5-7) into $(7.5\text{-}6)_{1,3}$ gives the constitutive relations for the film

$$\begin{aligned} T_r &= c_{rs}^p S_s - e_{kr}^p E_k, \\ D_i &= e_{is}^p S_s + \varepsilon_{ij}^p E_j, \end{aligned} \qquad (7.5\text{-}8)$$

where the film material constants are

$$\begin{aligned} c_{rs}^p &= c_{rs} - c_{rv}c_{vw}^{-1}c_{ws}, \quad e_{ks}^p = e_{ks} - e_{kw}c_{wv}^{-1}c_{vs}, \\ \varepsilon_{kj}^p &= \varepsilon_{kj} + e_{kv}c_{vw}^{-1}e_{jw}. \end{aligned} \qquad (7.5\text{-}9)$$

We now introduce another convention that subscripts *a, b, c* and *d* assume 1 and 3 but not 2. Then Equation (9.5-8) can be written as

$$\begin{aligned} T_{ab} &= c_{abcd}^p S_{cd} - e_{kab}^p E_k, \\ D_i &= e_{iab}^b S_{ab} + \varepsilon_{ij}^p E_j. \end{aligned} \qquad (7.5\text{-}10)$$

Integrating the equations in $(7.5\text{-}1)_1$ for $i = 1,3$ and $(7.5\text{-}1)_{2,3}$ with respect to x_2 through the film thickness, we obtain the following two-dimensional equations of motion, Gauss's law and conservation of charge:

$$\begin{aligned} T_{ab,a} + \frac{1}{2h}[T_{2b}(x_2 = h) - T_{2b}(x_2 = -h)] &= \rho\ddot{u}_b, \\ D_{a,a} + \frac{1}{2h}[D_2(x_2 = h) - D_2(x_2 = -h)] &= qn, \\ q\dot{n} + J_{a,a} + \frac{1}{2h}[J_2(x_2 = h) - J_2(x_2 = -h)] &= 0, \end{aligned} \qquad (7.5\text{-}11)$$

where u_a, T_{ab}, D_a, J_a and n are averages of the corresponding quantities along the film thickness.

5.2.2 Fields in the Ceramic Half-Space

From the equations in Section 6 of Chapter 3, the equations for the ceramic half-space are

$$\bar{c}_{44}\nabla^2 u_3 = \rho\ddot{u}_3,$$
$$\nabla^2\psi = 0, \tag{7.5-12}$$

$$\psi = \phi - \frac{e_{15}}{\varepsilon_{11}} u_3, \tag{7.5-13}$$

and

$$\begin{aligned} T_{23} &= \bar{c}_{44} u_{3,2} + e_{15}\psi_{,2}, \\ T_{31} &= \bar{c}_{44} u_{3,1} + e_{15}\psi_{,1}, \\ D_1 &= -\varepsilon_{11}\psi_{,1}, \\ D_2 &= -\varepsilon_{11}\psi_{,2}, \end{aligned} \tag{7.5-14}$$

where

$$\bar{c}_{44} = c_{44} + \frac{e_{15}^2}{\varepsilon_{11}} = c_{44}(1 + k_{15}^2), \quad k_{15}^2 = \frac{e_{15}^2}{\varepsilon_{11} c_{44}}. \tag{7.5-15}$$

For a surface wave solution we must have

$$u_3, \phi \to 0, \quad x_2 \to +\infty. \tag{7.5-16}$$

Consider the possibility of solutions in the following form:

$$\begin{aligned} u_3 &= A\exp(-\xi_2 x_2)\exp[i(\xi_1 x_1 - \omega t)], \\ \psi &= B\exp(-\xi_1 x)\exp[i(\xi_1 x_1 - \omega t)], \end{aligned} \tag{7.5-17}$$

where A and B are undetermined constants, and ξ_2 should be positive for decaying behavior away from the surface. Equation $(7.5\text{-}17)_2$ already satisfies $(7.5\text{-}12)_2$. For $(7.5\text{-}17)_1$ to satisfy $(7.5\text{-}12)_1$ we must have

$$\bar{c}_{44}(\xi_1^2 - \xi_2^2) = \rho\omega^2, \tag{7.5-18}$$

which leads to the following expression for ξ_2

$$\xi_2^2 = \xi_1^2 - \frac{\rho\omega^2}{\bar{c}_{44}} = \xi_1^2(1 - \frac{v^2}{v_T^2}) > 0, \tag{7.5-19}$$

where

$$v^2 = \frac{\omega^2}{\xi_1^2}, \quad v_T^2 = \frac{\bar{c}_{44}}{\rho}. \tag{7.5-20}$$

The following are needed for prescribing boundary and continuity conditions:

$$\phi = [B\exp(-\xi_1 x_2) + \frac{e_{15}}{\varepsilon_{11}} A\exp(-\xi_2 x_2)]\exp[i(\xi_1 x_1 - \omega t)],$$

$$T_{23} = -[A\bar{c}_{44}\xi_2 \exp(-\xi_2 x_2)$$
$$+ e_{15} B\xi_1 \exp(-\xi_1 x_2)]\exp[i(\xi_1 x_1 - \omega t)], \qquad (7.5\text{-}21)$$
$$D_2 = \varepsilon_{11} B\xi_1 \exp(-\xi_1 x_2)\exp[i(\xi_1 x_1 - \omega t)].$$

5.2.3 Fields in the Free Space

Electric fields can also exist in the free space of $x_2 < 0$, which is governed by

$$\nabla^2\phi = 0, \quad x_2 < 0,$$
$$\phi \to 0, \quad x_2 \to -\infty. \qquad (7.5\text{-}22)$$

A surface wave solution to (7.5-22) is

$$\phi = C\exp(\xi_1 x_2)\exp[i(\xi_1 x_1 - \omega t)], \qquad (7.5\text{-}23)$$

where C is an undetermined constant. From (7.5-23), in the free space

$$D_2 = -\varepsilon_0 \xi_1 C\exp(\xi_1 x_2)\exp[i(\xi_1 x_1 - \omega t)]. \qquad (7.5\text{-}24)$$

5.2.4 Fields in the Semiconductor Film

The semiconductor film is one-dimensional with $n = n(x_1, t)$. Consider the case when the dc biasing electric field is in the x_1 direction. Let

$$u_3 = A\exp[i(\xi_1 x_1 - \omega t)], \quad \phi = C\exp[i(\xi_1 x_1 - \omega t)],$$
$$n = N\exp[i(\xi_1 x_1 - \omega t)], \qquad (7.5\text{-}25)$$

where N is an undetermined constant. Equation (7.5-25) already satisfies the continuity of displacement between the film and the ceramic half-space, and the continuity of electric potential between the film and the free space. We use a prime to indicate the elastic and dielectric constants as well as the mass density of the film. Silicon is a cubic crystal with m3m symmetry. The elastic and dielectric constants are given by

$$\begin{pmatrix} c'_{11} & c'_{12} & c'_{12} & 0 & 0 & 0 \\ c'_{12} & c'_{11} & c_{12} & 0 & 0 & 0 \\ c'_{12} & c'_{12} & c'_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & c'_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c'_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & c'_{44} \end{pmatrix}, \begin{pmatrix} \varepsilon'_{11} & 0 & 0 \\ 0 & \varepsilon'_{11} & 0 \\ 0 & 0 & \varepsilon'_{11} \end{pmatrix}. \qquad (7.5\text{-}26)$$

From (7.5-10) and (7.5-2)$_3$ we obtain:

$$
\begin{aligned}
T_{13} &= c_{55}^p S_{13} = c_{44}^p u_{3,1} = c_{44}^p i\xi_1 A \exp[i(\xi_1 x_1 - \omega t)], \\
D_1 &= \varepsilon_{11}^p E_1 = -\varepsilon_{11}^p \phi_{,1} = -\varepsilon_{11}^p i\xi_1 C \exp[i(\xi_1 x_1 - \omega t)], \\
J_1 &= -q\bar{n}\mu_{11}\phi_{,1} + qn\mu_{11}\overline{E}_1 - qd_{11}n_{,1} \\
&= (-q\bar{n}\mu_{11} i\xi_1 C + qN\mu_{11}\overline{E}_1 - qd_{11} i\xi_1 N)\exp[i(\xi_1 x_1 - \omega t)].
\end{aligned}
\tag{7.5-27}
$$

5.2.5 Continuity Conditions and Dispersion Relation

Substitution of (7.5-21), (7.5-23), (7.5-24), (7.5-25) and (7.5-27) into the continuity condition of the electric potential between the ceramic half-space and the film, (7.5-11)$_1$ for $b = 3$, and (7.5-11)$_{2,3}$ yields

$$
\begin{aligned}
& B + \frac{e_{15}}{\varepsilon_{11}} A = C, \\
& -c_{44}^p \xi_1^2 A - \frac{1}{2h}(A\bar{c}_{44}\xi_2 + e_{15} B \xi_1) = -\rho' \omega^2 A, \\
& \varepsilon_{11}^p \xi_1^2 C + \frac{1}{2h}(\varepsilon_{11} B \xi_1 + \varepsilon_0 \xi_1 C) = qN, \\
& -qi\omega N + i\xi_1(-q\bar{n}\mu_{11} i\xi_1 C + qN\mu_{11}\overline{E}_1 - qd_{11} i\xi_1 N) = 0,
\end{aligned}
\tag{7.5-28}
$$

which is a system of linear, homogeneous equations for A, B, C and N. For nontrivial solutions the determinant of the coefficient matrix has to vanish

$$
\begin{vmatrix}
\dfrac{e_{15}}{\varepsilon_{11}} & 1 & -1 & 0 \\
\rho'\omega^2 - c_{44}^p \xi_1^2 - \dfrac{\bar{c}_{44}\xi_2}{2h} & -\dfrac{e_{15}\xi_1}{2h} & 0 & 0 \\
0 & \dfrac{\varepsilon_{11}\xi_1}{2h} & \dfrac{\varepsilon_0 \xi_1}{2h} + \varepsilon_{11}^p \xi_1^2 & -q \\
0 & 0 & q\bar{n}\mu_{11}\xi_1^2 & \begin{matrix} -qi\omega + i\xi_1 q\mu_{11}\overline{E}_1 \\ + qd_{11}\xi_1^2 \end{matrix}
\end{vmatrix} = 0,
\tag{7.5-29}
$$

which determines the dispersion relation, a relation between ω and ξ_1, of the surface wave. In terms of the surface wave speed $v = \omega / \xi_1$, Equation (7.5-29) can be written in the following form:

$$\left(\frac{v^2}{v_T'^2}-1\right)\frac{c_{44}^p}{\bar{c}_{44}}2h\xi_1-\sqrt{1-\frac{v^2}{v_T^2}}+\bar{k}_{15}^2$$
$$=\frac{\bar{k}_{15}^2}{1+\frac{\varepsilon_0}{\varepsilon_{11}}+\frac{\varepsilon_{11}^p}{\varepsilon_{11}}\xi_1\,2h+\frac{q\bar{n}\mu_{11}2h}{\varepsilon_{11}[d_{11}\xi_1\;+i(\mu_{11}\bar{E}_1-v)]}}, \tag{7.5-30}$$

where (7.5-19) has been used, and

$$v_T'^2=\frac{c_{44}^p}{\rho'},\quad \bar{k}_{15}^2=\frac{e_{15}^2}{\varepsilon_{11}\bar{c}_{44}}. \tag{7.5-31}$$

When $h = 0$, i.e., the semiconductor film does not exist, (7.5-30) reduces to

$$v^2=v_T^2\left[1-\frac{\bar{k}_{15}^4}{(1+\varepsilon_{11}/\varepsilon_0)^2}\right]=v_{B-G}^2, \tag{7.5-32}$$

which is the speed of the Bleustein-Gulyaev wave in (5.3-20).

When $\bar{k}_{15}^2=0$, i.e., the half-space is non-piezoelectric, electromechanical coupling disappears and the wave is purely elastic. In this case Equation (7.5-30) reduces to

$$\left(\frac{v^2}{v_T'^2}-1\right)\frac{c_{44}^p}{\bar{c}_{44}}2h\xi_1-\sqrt{1-\frac{v^2}{v_T^2}}=0, \tag{7.5-33}$$

which is the equation that determines the speed of Love wave (an anti-plane surface wave in an elastic half-space carrying an elastic layer) in the limit when the film is very thin compared to the wavelength ($\xi_1 h \ll 1$). Love waves are known to exist when the elastic stiffness of the layer is smaller that that of the half-space.

The denominator of the right hand side of (7.5-30) indicates that a complex wave speed may be expected and the imaginary part of the complex wave speed may change its sign (transition from a damped wave to a growing wave) when $\mu_{11}\bar{E}_1\xi_1-\omega$ changes sign or

$$v=\frac{\omega}{\xi_1}=\mu_{11}\bar{E}_1, \tag{7.5-34}$$

i.e., the acoustic wave speed is equal to the carrier drift speed [52].

When semiconduction is small, Equation (7.5-30) can be solved by an iteration or perturbation procedure. As the lowest (zero) order of approximation, we neglect the small semiconduction and denote the zero-order solution by $v_{(0)}$. Then from (7.5-30),

$$\left(\frac{v_{(0)}^2}{v_T'^2}-1\right)\frac{c_{44}^p}{\bar{c}_{44}}2h\xi_1-\sqrt{1-\frac{v_{(0)}^2}{v_T^2}+\bar{k}_{15}^2}$$
$$=\frac{\bar{k}_{15}^2}{1+\frac{\varepsilon_0}{\varepsilon_{11}}+\frac{\varepsilon_{11}^p}{\varepsilon_{11}}\xi_1\,2h},\tag{7.5-35}$$

which is dispersive. For the next order, we substitute $v_{(0)}$ into the right-hand side of (7.5-30) and obtain the following equation for $v_{(1)}$

$$\left(\frac{v_{(1)}^2}{v_T'^2}-1\right)\frac{c_{44}^p}{\bar{c}_{44}}2h\xi_1-\sqrt{1-\frac{v_{(1)}^2}{v_T^2}+\bar{k}_{15}^2}$$
$$=\frac{\bar{k}_{15}^2}{1+\frac{\varepsilon_0}{\varepsilon_{11}}+\frac{\varepsilon_{11}^p}{\varepsilon_{11}}\xi_1\,2h+\frac{q\bar{n}\mu_{11}2h}{\varepsilon_{11}[d_{11}\xi_1+i(\mu_{11}\bar{E}_1-v_{(0)})]}},\tag{7.5-36}$$

which suggests a wave that is both dispersive and dissipative.

For numerical results consider PZT-5H. Since $c'_{44}>c_{44}$, the counterpart of the elastic Love wave does not exist, but a modified Bleustein-Gulyaev wave is expected. We plot the real parts of $v_{(0)}$ and $v_{(1)}$ versus ξ_1 in Figure 7.5-2. The dimensionless wave number X and the dimensionless wave speed Y of different orders are defined by

$$\begin{aligned}X&=\xi_1\Big/\frac{\pi}{2h},\\ Y_{(0)}&=v_{(0)}/v_{B-G},\\ Y_{(1)}&=\mathrm{Re}\{v_{(1)}\}/v_{B-G}.\end{aligned}\tag{7.5-37}$$

γ is a dimensionless number given by

$$\gamma=\mu_{11}\bar{E}_1/v_{B-G},\tag{7.5-38}$$

which may be considered as a normalized electric field. It represents the ratio of the carrier drift velocity and the speed of the Bleustein-Gulyave wave. Because of the use of thin film equations for the semiconductor film, the solution is valid only when the wavelength is much larger than the film thickness ($X << 1$). It can be seen that semiconduction causes additional dispersion. This conduction induced dispersion varies according to the dc biasing electric field.

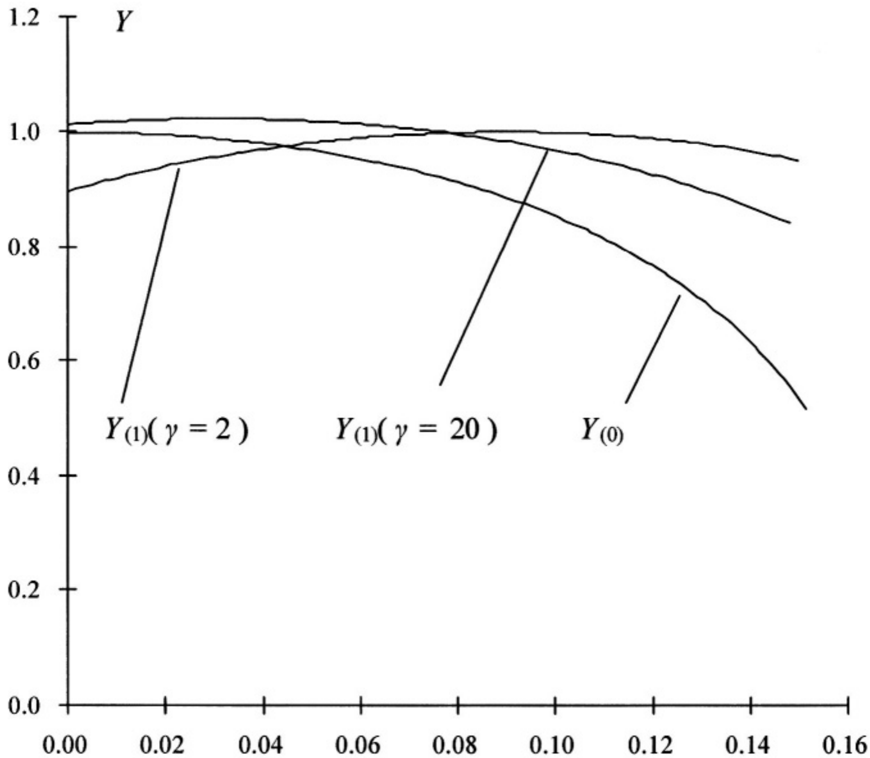

Figure 7.5-2. Dispersion relations.

Figure 7.5-3 shows the imaginary part of $v_{(1)}$ versus γ. The dimensionless number describing the decaying behavior of the waves is defined by

$$Y = \mathrm{Im}\{v_{(1)}\}/v_{B-G} \,. \tag{7.5-39}$$

When the dc bias is large enough (approximately $\gamma > 1$) the decay constant becomes negative indicating wave amplification. The transition from damped waves to growing waves indeed occurs when (7.5-34) is true for $v_{(0)}$.

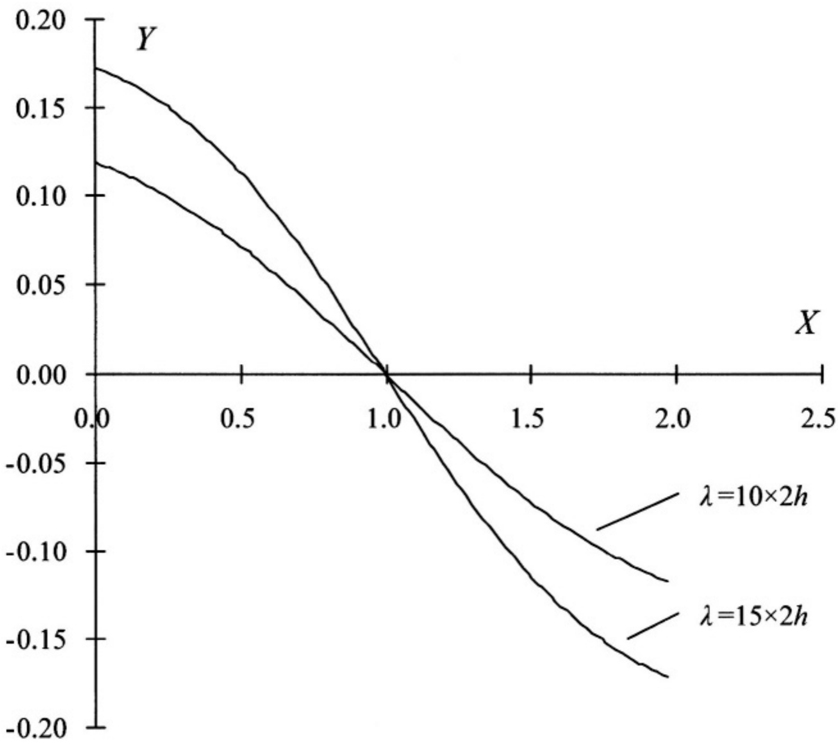

Figure 7.5-3. Dissipation as a function of the dc bias.

6. DYNAMIC THEORY

The theory of linear piezoelectricity is based on the quasistatic approximation. In piezoelectricity theory, the mechanical equations are dynamic but the electromagnetic equations appear to be static. The electric field and the magnetic field are not directly coupled in Maxwell's equations. When the complete set of Maxwell's equations is included, the fully dynamic theory is called piezoelectromagnetism [54].

6.1 Governing Equations

For a piezoelectric but nonmagnetizable dielectric body, the three-dimensional equations of linear piezoelectromagnetism consist of the equations of motion and Maxwell's equations, as shown by

$$
\begin{aligned}
&T_{ji,j} + \rho f_i = \rho \ddot{u}_i, \\
&\varepsilon_{ijk} E_{k,j} = -\dot{B}_i, \quad \varepsilon_{ijk} H_{k,j} = \dot{D}_i, \\
&B_{i,i} = 0, \quad D_{i,i} = 0,
\end{aligned}
\tag{7.6-1}
$$

as well as the following constitutive relations

$$
\begin{aligned}
&T_{ij} = c_{ijkl} S_{kl} - e_{kij} E_k, \\
&D_i = e_{ijk} S_{jk} + \varepsilon_{ij} E_j, \\
&B_i = \mu_0 H_i,
\end{aligned}
\tag{7.6-2}
$$

where B_i is the magnetic induction, H_i is the magnetic field, and μ_0 is the magnetic permeability of free space. With Equation (7.6-2), Equation (7.6-1) becomes

$$
\begin{aligned}
&c_{ijkl} u_{k,li} - e_{kij} E_{k,i} = \rho \ddot{u}_j, \\
&\varepsilon_{ijk} E_{k,j} = -\dot{B}_i, \\
&\frac{1}{\mu_0} \varepsilon_{ijk} B_{k,j} = e_{ikl} \dot{u}_{k,l} + \varepsilon_{ik} \dot{E}_k.
\end{aligned}
\tag{7.6-3}
$$

6.2 Quasistatic Approximation

The quasistatic approximation made in Section 2 of Chapter 1 can be considered as the lowest order approximation of the dynamic theory given by (7.6-3) through the following perturbation procedure [5]. Consider an acoustic wave with frequency ω in a piezoelectric crystal of size L. We scale the various independent and dependent variables with respect to characteristic quantities

$$
\begin{aligned}
&\xi_i = \frac{x_i}{L}, \quad \tau = \omega t, \\
&U_i = \frac{u_i}{L}, \quad b_i = c B_i,
\end{aligned}
\tag{7.6-4}
$$

where

$$
c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}},
\tag{7.6-5}
$$

is the speed of light in free space and the scaling yields a **b** in the same units as **E**. Then Equation (7.6-3) takes the following form:

$$\frac{1}{L} c_{ijkl} \frac{\partial^2 U_k}{\partial \xi_l \partial \xi_i} - \frac{1}{L} e_{kij} \frac{\partial E_k}{\partial \xi_i} = \rho \omega^2 L \frac{\partial^2 U_j}{\partial \tau^2},$$

$$\frac{1}{L} \varepsilon_{ijk} \frac{\partial E_k}{\partial \xi_j} = -\frac{\omega}{c} \frac{\partial b_i}{\partial \tau}, \tag{7.6-6}$$

$$\frac{1}{cL\mu_0} \varepsilon_{ijk} \frac{\partial b_k}{\partial \xi_j} = \omega \varepsilon_0 \frac{e_{ikl}}{\varepsilon_0} \frac{\partial^2 U_k}{\partial \xi_l \partial \tau} + \omega \varepsilon_0 \frac{\varepsilon_{ik}}{\varepsilon_0} \frac{\partial E_k}{\partial \tau},$$

or

$$c_{ijkl} \frac{\partial^2 U_k}{\partial \xi_l \partial \xi_i} - e_{kij} \frac{\partial E_k}{\partial \xi_i} = \rho \omega^2 L^2 \frac{\partial^2 U_j}{\partial \tau^2},$$

$$\varepsilon_{ijk} \frac{\partial E_k}{\partial \xi_j} = -\eta \frac{\partial b_i}{\partial \tau}, \tag{7.6-7}$$

$$\varepsilon_{ijk} \frac{\partial b_k}{\partial \xi_j} = \eta \left(\frac{e_{ikl}}{\varepsilon_0} \frac{\partial^2 U_k}{\partial \xi_l \partial \tau} + \frac{\varepsilon_{ik}}{\varepsilon_0} \frac{\partial E_k}{\partial \tau} \right),$$

where

$$\eta = \frac{\omega L}{c} << 1. \tag{7.6-8}$$

To the lowest order

$$c_{ijkl} \frac{\partial^2 U_k}{\partial \xi_l \partial \xi_i} - e_{kij} \frac{\partial E_k}{\partial \xi_i} = \rho \omega^2 L^2 \frac{\partial^2 U_j}{\partial \tau^2},$$

$$\varepsilon_{ijk} \frac{\partial E_k}{\partial \xi_j} = 0, \tag{7.6-9}$$

$$\varepsilon_{ijk} \frac{\partial b_k}{\partial \xi_j} = 0,$$

or

$$c_{ijkl} u_{k,li} - e_{kij} E_{k,i} = \rho \ddot{u}_j,$$

$$\varepsilon_{ijk} E_{k,j} = 0, \tag{7.6-10}$$

$$\varepsilon_{ijk} H_{k,j} = 0.$$

6.3 Anti-Plane Problems of Ceramics

For anti-plane motions of polarized ceramics we have [55]

$$
\begin{aligned}
&u_1 = u_2 = 0, \quad u_3 = u_3(x_1, x_2, t), \\
&E_1 = E_1(x_1, x_2, t), \quad E_2 = E_2(x_1, x_2, t), \quad E_3 = 0, \\
&H_1 = H_2 = 0, \quad H_3 = H_3(x_1, x_2, t).
\end{aligned} \tag{7.6-11}
$$

The non-vanishing components of S_{ij}, T_{ij}, D_i and B_i are

$$
\begin{aligned}
&S_4 = u_{3,2}, \quad S_5 = u_{3,1}, \\
&T_4 = c_{44} u_{3,2} - e_{15} E_2, \quad T_5 = c_{44} u_{3,1} - e_{15} E_1, \\
&D_1 = e_{15} u_{3,1} + \varepsilon_{11} E_1, \quad D_2 = e_{15} u_{3,2} + \varepsilon_{11} E_2, \\
&B_3 = \mu_0 H_3.
\end{aligned} \tag{7.6-12}
$$

The nontrivial ones of the equations of motion and Maxwell's equations in (7.6-1) take the following form:

$$
\begin{aligned}
&c_{44}(u_{3,11} + u_{3,22}) - e_{15}(E_{1,1} + E_{2,2}) = \rho \ddot{u}_3, \\
&e_{15}(u_{3,11} + u_{3,22}) + \varepsilon_{11}(E_{1,1} + E_{2,2}) = 0, \\
&E_{2,1} - E_{1,2} = -\mu_0 \dot{H}_3, \\
&H_{3,2} = e_{15} \dot{u}_{3,1} + \varepsilon_{11} \dot{E}_1, \quad -H_{3,1} = e_{15} \dot{u}_{3,2} + \varepsilon_{11} \dot{E}_2.
\end{aligned} \tag{7.6-13}
$$

Eliminating the electric field components from $(7.6\text{-}13)_{1,2}$,

$$
\bar{c}_{44}(u_{3,11} + u_{3,22}) = \rho \ddot{u}_3, \tag{7.6-14}
$$

where $\bar{c}_{44} = c_{44} + e_{15}^2 / \varepsilon_{11}$. Differentiating $(7.6\text{-}13)_3$ with respect to time once and substituting from $(7.6\text{-}13)_{4,5}$, we have

$$
H_{3,11} + H_{3,22} = \varepsilon_{11} \mu_0 \ddot{H}_3. \tag{7.6-15}
$$

The above equations can be written in coordinate independent forms as

$$
\begin{aligned}
&\bar{c}_{44} \nabla^2 u_3 = \rho \ddot{u}_3, \quad \nabla^2 H_3 = \varepsilon_0 \mu_0 \ddot{H}_3, \\
&\dot{\mathbf{D}} = -\mathbf{i}_3 \times \nabla H_3,
\end{aligned} \tag{7.6-16}
$$

where ∇ and ∇^2 are the two-dimensional gradient operator and Laplacian, respectively. $\mathbf{D}$ is the electric displacement in the (x_1,x_2) plane. $\mathbf{i}_3$ is the unit vector in the x_3 direction. Equations $(7.6\text{-}16)_{1,2}$ govern the displacement and magnetic fields. Once u_3 and H_3 are determined, D_1 and D_2 can be obtained from Equation $(7.6\text{-}16)_3$. Then the electric field and the stress components can be obtained from constitutive relations. From Equations $(7.6\text{-}16)_3$ and

$(3.6\text{-}9)_{3,4}$, it can be seen that physically the ψ introduced by Bleustein [18] is related to H_3.

6.4 Surface Waves

To see the dynamic effects more specifically, we study the propagation of surfaces waves in a ceramic half-space [55]. The corresponding quasistatic problem was analyzed in Section 3 of Chapter 5. Consider a ceramic half-space poled in the x_3 direction (see Figure 7.6-1).

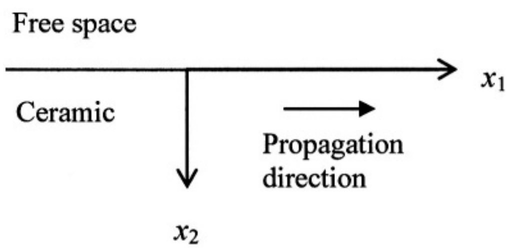

Figure 7.6-1. A ceramic half-space.

Consider surface waves propagating in the x_1 direction with

$$\begin{aligned} u_3 &= U\exp(-\xi_2 x_2)\cos(\xi_1 x_1 - \omega t), \\ H_3 &= H\exp(-\eta_2 x_2)\cos(\xi_1 x_1 - \omega t), \end{aligned} \tag{7.6-17}$$

where U, H, ξ_1, ξ_2, η_2 and ω are undetermined constants. Substitution of Equations (7.6-17) into (7.6-14) and (7.6-15) results in

$$\begin{aligned} \xi_2^2 &= \xi_1^2 - \rho\omega^2/\bar{c}_{44} > 0, \\ \eta_2^2 &= \xi_1^2 - \varepsilon_{11}\mu_0\omega^2 > 0, \end{aligned} \tag{7.6-18}$$

where the inequalities are for decaying behavior from the surface. From Equations $(7.6\text{-}13)_{4,5}$ and (7.6-17) we obtain

$$\begin{aligned} E_1 &= \frac{1}{\varepsilon_{11}\omega}[e_{15}\omega\xi_1 U\exp(-\xi_2 x_2) \\ &\quad + \eta_2 H\exp(-\eta_2 x_2)]\sin(\xi_1 x_1 - \omega t), \\ E_2 &= \frac{1}{\varepsilon_{11}\omega}[e_{15}\omega\xi_2 U\exp(-\xi_2 x_2) \\ &\quad + \xi_1 H\exp(-\eta_2 x_2)]\cos(\xi_1 x_1 - \omega t). \end{aligned} \tag{7.6-19}$$

6.4.1 A Half-Space with an Electroded Surface

First consider the case when the surface at $x_2 = 0$ is electroded with a perfect conductor for which we have $E_1 = 0$. The electrode is assumed to be very thin with negligible mass. Hence we have the traction-free condition $T_4 = 0$ on the surface. Then from (7.6-12) and $(7.6\text{-}19)_1$ we can write

$$\begin{aligned} &e_{15}\omega\xi_1 U + \eta_2 H = 0, \\ &\varepsilon_{11}\bar{c}_{44}\omega\xi_2 U + e_{15}\xi_1 H = 0. \end{aligned} \tag{7.6-20}$$

For nontrivial solutions of U and H, the determinant of the coefficient matrix has to vanish which, with (7.6-18), leads to

$$\sqrt{1-\frac{v^2}{v_T^2}}\sqrt{1-\alpha n^2\frac{v^2}{v_T^2}} = \bar{k}_{15}^2, \tag{7.6-21}$$

where

$$\begin{aligned} &v^2 = \frac{\omega^2}{\xi_1^2}, \quad v_T^2 = \frac{\bar{c}_{44}}{\rho}, \quad \alpha = \frac{v_T^2}{c^2}, \\ &c^2 = \frac{1}{\varepsilon_0\mu_0}, \quad n^2 = \frac{\varepsilon_{11}}{\varepsilon_0}, \quad \bar{k}_{15}^2 = \frac{e_{15}^2}{\varepsilon_{11}\bar{c}_{44}}. \end{aligned} \tag{7.6-22}$$

In Equations (7.6-22), v is the surface wave speed, v_T is the speed of plane shear waves propagating in the x_1 direction, α is the ratio of acoustic and light wave speeds which is normally a very small number, c is the speed of light in a vacuum, and n is the refractive index in the x_1 direction. Equation (7.6-21) is an equation for the surface wave speed v. Waves with speed determined by (7.6-21) are clearly nondispersive. Since α is very small, it is simpler and more revealing to examine the following perturbation solution of (7.6-21) for small α

$$v^2 \cong v_T^2(1-k_{15}^4)(1-\alpha n^2\bar{k}_{15}^4). \tag{7.6-23}$$

It is seen that the effect of electromagnetic coupling on the wave speed of Bleustein-Gulyaev waves is of the order of $\alpha n^2\bar{k}_{15}^4$. As a numerical example we consider PZT-7A. Calculation shows that

$$\begin{aligned} &\bar{k}_{15} = 0.671, \quad n^2 = 460, \\ &\alpha = 6.85\times 10^{-9}, \quad \alpha n^2\bar{k}_{15}^4 = 6.38\times 10^{-7}. \end{aligned} \tag{7.6-24}$$

Hence the modification on the wave speed is very small and is negligible in most applications. When α is set to zero, or when the speed of light approaches infinity, Equation (7.6-23) reduces to the speed of the Bleustein-Gulyaev waves in Section 3 of Chapter 5. The above solution serves as a good example for illustrating the quasistatic approximation, which can only be done from the dynamic theory.

6.4.2 A Half-Space with an Unelectroded Surface

When the surface of the half-space at $x_2 = 0$ is unelectroded, electromagnetic waves also exist in the free space of $x_2 < 0$. The solution for the free space $x_2 < 0$ can be written as:

$$H_3 = \overline{H} \exp(\overline{\eta}_2 x_2) \cos(\xi_1 x_1 - \omega t), \tag{7.6-25}$$

where $\overline{H}$ and $\overline{\eta}_2$ are undetermined constants. Substitution of (7.6-25) into (7.6-15) with ε_{11} replaced by ε_0 for free space, we obtain

$$\overline{\eta}_2^2 = \xi_1^2 - \varepsilon_0 \mu_0 \omega^2 > 0\,. \tag{7.6-26}$$

The electric field generated by H_3 in (7.6-25) through (7.6-13) with u_3 dropped and ε_{11} replaced by ε_0 for fee space, is given by

$$\begin{aligned} E_1 &= -\frac{1}{\varepsilon_0 \omega} \overline{\eta}_2 \overline{H} \exp(\overline{\eta}_2 x_2) \sin(\xi_1 x_1 - \omega t), \\ E_2 &= \frac{1}{\varepsilon_0 \omega} \xi_1 \overline{H} \exp(\overline{\eta}_2 x_2) \cos(\xi_1 x_1 - \omega t). \end{aligned} \tag{7.6-27}$$

We require the continuity of E_1 and H_3 at $x_2 = 0$ as well as the vanishing of shear stress T_4. This implies that

$$\begin{aligned} &\frac{1}{\varepsilon_{11}\omega}(e_{15}\omega\xi_1 U + \eta_2 H) + \frac{1}{\varepsilon_0 \omega}\overline{\eta}_2 \overline{H} = 0, \\ &H - \overline{H} = 0, \\ &\varepsilon_{11}\overline{c}_{44}\omega\xi_2 U + e_{15}\xi_1 H = 0. \end{aligned} \tag{7.6-28}$$

Vanishing of the determinant of the coefficient matrix leads to

$$\sqrt{1 - \frac{v^2}{v_T^2}} \left(\sqrt{1 - \alpha n^2 \frac{v^2}{v_T^2}} + n^2 \sqrt{1 - \alpha \frac{v^2}{v_T^2}} \right) = k_{15}^2\,, \tag{7.6-29}$$

which is an equation for v. Again, the waves are nondispersive. When α is set to zero the result of Section 3 of Chapter 5 will be obtained. A perturbation solution of (7.6-29) to the first order in α is

$$v^2 \cong v_T^2\left[1-\frac{k_{15}^4}{(1+n^2)^2}\right]\left[1-\alpha 2n^2\frac{k_{15}^4}{(1+n^2)^3}\right], \quad (7.6\text{-}30)$$

and calculation shows that, for PZT-7A,

$$\alpha 2n^2\frac{k_{15}^4}{(1+n^2)^3}=1.30\times 10^{-16}. \quad (7.6\text{-}31)$$

6.5 Electromagnetic Radiation

Next we consider electromagnetic radiation from a vibrating circular cylinder of ceramics poled in the x_3 direction as shown in Figure 7.6-2 [56].

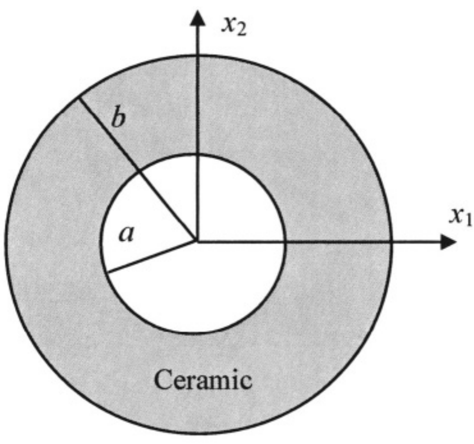

Figure 7.6-2. A Circular cylinder of ceramics poled in the x_3 direction.

The cylinder is mechanically driven at $r = b$. The surface at $r = b$ is unelectroded. Electromagnetic waves propagate away from the cylinder (radiation).

6.5.1 Boundary-Value Problem

For the special case of a solid cylinder ($a = 0$), from Equation (7.6-16) the boundary-value problem is:

$$\begin{aligned}
&v_T^2\nabla^2 u_3 = \ddot{u}_3, \quad c^2\nabla^2 H_3 = \ddot{H}_3, \quad r < a, \\
&c_0^2\nabla^2 H_3 = \ddot{H}_3, \quad r > a, \\
&u_3, H_3 \quad \text{finite}, \quad r = 0, \\
&H_3 \quad \text{outgoing}, \quad r \to \infty, \\
&T_{r3} = \tau \sin \nu\theta \exp(-i\omega t), \quad r = b, \\
&H_3, E_\theta \quad \text{continuous}, \quad r = b.
\end{aligned} \tag{7.6-32}$$

6.5.2 Interior Fields

For fields inside the cylinder, in polar coordinates, from Equation (7.6-16) we have

$$\begin{aligned}
&v_T^2\left(\frac{\partial^2 u_3}{\partial r^2} + \frac{1}{r}\frac{\partial u_3}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u_3}{\partial \theta^2}\right) = \ddot{u}_3, \\
&c^2\left(\frac{\partial^2 H_3}{\partial r^2} + \frac{1}{r}\frac{\partial H_3}{\partial r} + \frac{1}{r^2}\frac{\partial^2 H_3}{\partial \theta^2}\right) = \ddot{H}_3,
\end{aligned} \tag{7.6-33}$$

and

$$\begin{aligned}
\varepsilon_{11}\dot{E}_r &= \frac{1}{r}H_{3,\theta} - e_{15}\dot{u}_{3,r}, \\
\varepsilon_{11}\dot{E}_\theta &= -H_{3,r} - e_{15}\frac{1}{r}\dot{u}_{3,\theta}.
\end{aligned} \tag{7.6-34}$$

Consider the possibility of

$$\begin{aligned}
u_3(r,\theta,t) &= u(r)\sin \nu\theta \exp(-i\omega t), \\
H_3(r,\theta,t) &= H(r)\cos \nu\theta \exp(-i\omega t),
\end{aligned} \tag{7.6-35}$$

where ν is allowed to assume any real, positive value for the moment (for solutions periodic in θ, ν has to be an integer). Other values of ν may also be physically meaningful. For example, $\nu = 1/2$ with $-\pi \le \theta \le \pi$ represents a crack at $\theta = \pi$. Substitution of (7.6-35) into (7.6-33)results in

$$\begin{aligned}
&\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \left(\alpha^2 - \frac{\nu^2}{r^2}\right)u = 0, \\
&\frac{\partial^2 H}{\partial r^2} + \frac{1}{r}\frac{\partial H}{\partial r} + \left(\beta^2 - \frac{\nu^2}{r^2}\right)H = 0,
\end{aligned} \tag{7.6-36}$$

where we have denoted

$$\alpha = \frac{\omega}{v_T}, \quad \beta = \frac{\omega}{c}. \tag{7.6-37}$$

Equation (7.6-36) can be written as Bessel's equations of order ν. Then general solutions for u_3 and H_3 can be written as

$$\begin{aligned} u_3 &= [C_1 J_\nu(\alpha r) + C_2 Y_\nu(\alpha r)] \sin \nu\theta \exp(-i\omega t), \\ H_3 &= [C_3 J_\nu(\beta r) + C_4 Y_\nu(\beta r)] \cos \nu\theta \exp(-i\omega t), \end{aligned} \tag{7.6-38}$$

where J_ν and Y_ν are the ν-*th* order Bessel functions of the first and second kind. $C_1 - C_4$ are undetermined constants. From (7.6-38) we obtain the following expressions that are useful for boundary and/or continuity conditions:

$$\begin{aligned} D_r &= \frac{\nu}{i\omega r}[C_3 J_\nu(\beta r) + C_4 Y_\nu(\beta r)] \sin \nu\theta \exp(-i\omega t), \\ D_\theta &= \frac{\beta}{i\omega}[C_3 J'_\nu(\beta r) + C_4 Y'_\nu(\beta r)] \cos \nu\theta \exp(-i\omega t), \end{aligned} \tag{7.6-39}$$

$$\begin{aligned} E_r &= \{\frac{\nu}{\varepsilon_{11}\omega r}[C_3 J_\nu(\beta r) + C_4 Y_\nu(\beta r)] \\ &\quad - \frac{e_{15}\alpha}{\varepsilon_{11}}[C_1 J'_\nu(\alpha r) + C_2 Y'_\nu(\alpha r)]\} \sin \nu\theta \exp(-i\omega t), \\ E_\theta &= \{\frac{\beta}{\varepsilon_{11} i\omega}[C_3 J'_\nu(\beta r) + C_4 Y'_\nu(\beta r)] \\ &\quad - \frac{e_{15}\nu}{\varepsilon_{11} r}[C_1 J_\nu(\alpha r) + C_2 Y_\nu(\alpha r)]\} \cos \nu\theta \exp(-i\omega t), \end{aligned} \tag{7.6-40}$$

$$\begin{aligned} T_{rz} &= \{\bar{c}_{44}\alpha[C_1 J'_\nu(\alpha r) + C_2 Y'_\nu(\alpha r)] \\ &\quad - \frac{e_{15}\nu}{\varepsilon_{11} i\omega r}[C_3 J_\nu(\beta r) + C_4 Y_\nu(\beta r)]\} \sin \nu\theta \exp(-i\omega t), \\ T_{\theta z} &= \{\frac{\bar{c}_{44}\nu}{r}[C_1 J_\nu(\alpha r) + C_2 Y_\nu(\alpha r)] \\ &\quad - \frac{e_{15}}{i\omega\varepsilon_{11}}[C_3 J'_\nu(\beta r) + C_4 Y'_\nu(\beta r)]\} \cos \nu\theta \exp(-i\omega t), \end{aligned} \tag{7.6-41}$$

where a superimposed prime indicates differentiation with respect to the whole argument of a function.

6.5.3 Exterior Fields

In the free space of $r > b$, the electromagnetic fields are given by

$$
\begin{aligned}
H_3 &= [C_5 H_\nu^{(1)}(\gamma r) + C_6 H_\nu^{(2)}(\gamma r)] \cos\nu\theta \exp(-i\omega t), \\
D_r &= \frac{\nu}{i\omega r}[C_5 H_\nu^{(1)}(\gamma r) + C_6 H_\nu^{(2)}(\gamma r)] \sin\nu\theta \exp(-i\omega t), \\
D_\theta &= \frac{\gamma}{i\omega}[C_5 H_\nu^{(1)\prime}(\gamma r) + C_6 H_\nu^{(2)\prime}(\gamma r)] \cos\nu\theta \exp(-i\omega t), \qquad (7.6\text{-}42) \\
E_r &= \frac{\nu}{i\omega r\varepsilon_0}[C_5 H_\nu^{(1)}(\gamma r) + C_6 H_\nu^{(2)}(\gamma r)] \sin\nu\theta \exp(-i\omega t), \\
E_\theta &= \frac{\gamma}{i\omega\varepsilon_0}[C_5 H_\nu^{(1)\prime}(\gamma r) + C_6 H_\nu^{(2)\prime}(\gamma r)] \cos\nu\theta \exp(-i\omega t),
\end{aligned}
$$

where $H_\nu^{(1)}$ and $H_\nu^{(2)}$ are the v-th order Hankel function of the first and second kind, and

$$\gamma = \frac{\omega}{c_0}. \qquad (7.6\text{-}43)$$

6.5.4 Boundary and Continuity Conditions

Since Y_ν is singular at the origin, terms associated with C_2 and C_4 have to be dropped. To satisfy the radiation condition at $r \to \infty$ we must have $C_6 = 0$. What need to be satisfied at $r = b$ are

$$
\begin{aligned}
T_{rz}(b) &= \bar{c}_{44}\alpha C_1 J_\nu'(\alpha b) - \frac{e_{15}\nu}{\varepsilon_{11} i\omega b} C_3 J_\nu(\beta b) = \tau, \\
H_3(b^-) &= C_3 J_\nu(\beta b) = C_5 H_\nu^{(1)}(\gamma b) = H_3(b^+), \\
E_\theta(b^-) &= \frac{\beta}{\varepsilon_{11} i\omega} C_3 J_\nu'(\beta b) - \frac{e_{15}\nu}{\varepsilon_{11} b} C_1 J_\nu(\alpha b) \qquad (7.6\text{-}44) \\
&= \frac{\gamma}{i\omega\varepsilon_0} C_5 H_\nu^{(1)\prime}(\gamma b) = E_\theta(b^+).
\end{aligned}
$$

Note that when $\nu = 0$ (axi-symmetric), Equation $(7.6\text{-}44)_1$ becomes uncoupled to $(7.6\text{-}44)_{2,3}$. In this case H_3 cannot be excited by τ. Hence there is no radiation. In the following we consider the case of $\nu \neq 0$. From Equation (7.6-44)

$$C_1 = b\frac{\beta b J_\nu'(\beta b) H_\nu^{(1)}(\gamma b) - \frac{\varepsilon_{11}}{\varepsilon_0}\gamma b J_\nu(\beta b) H_\nu^{(1)'}(\gamma b)}{\Delta}\frac{\tau}{\bar{c}_{44}},$$

$$C_3 = i\omega e_{15} b\frac{\nu J_\nu(\alpha b) H_\nu^{(1)}(\gamma b)}{\Delta}\frac{\tau}{\bar{c}_{44}}, \qquad (7.6\text{-}45)$$

$$C_5 = i\omega e_{15} b\frac{\nu J_\nu(\alpha b) J_\nu(\beta b)}{\Delta}\frac{\tau}{\bar{c}_{44}},$$

where

$$\begin{aligned}\Delta = {} & \alpha b \beta b J_\nu'(\alpha b) J_\nu'(\beta b) H_\nu^{(1)}(\gamma b) \\ & - \bar{k}_{15}^2 \nu^2 J_\nu(\alpha b) J_\nu(\beta b) H_\nu^{(1)}(\gamma b) \\ & - \frac{\varepsilon_{11}}{\varepsilon_0}\alpha b \gamma b J_\nu'(\alpha b) J_\nu(\beta b) H_\nu^{(1)'}(\gamma b).\end{aligned} \qquad (7.6\text{-}46)$$

$\Delta = 0$ yields a frequency equation. The corresponding modes are coupled acousto-electromagnetic modes.

6.5.5 Electromagnetic Radiation

We calculate the radiation at far fields with large r using the following asymptotic expressions of Bessel functions with large arguments

$$\begin{aligned}H_\nu^{(1)}(x) &\cong \sqrt{\frac{2}{\pi x}}\exp i\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right), \\ H_\nu^{(1)'}(x) &\cong i\sqrt{\frac{2}{\pi x}}\exp i\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right).\end{aligned} \qquad (7.6\text{-}47)$$

Then

$$\begin{aligned}H_3 &\cong C_5\sqrt{\frac{2}{\pi\gamma r}}\exp i\left(\gamma r - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)\cos\nu\theta\exp(-i\omega t), \\ E_\theta &\cong \frac{\gamma}{\omega\varepsilon_0}C_5\sqrt{\frac{2}{\pi\gamma r}}\exp i\left(\gamma r - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)\cos\nu\theta\exp(-i\omega t),\end{aligned} \qquad (7.6\text{-}48)$$

which are clearly outgoing. To calculate radiated power we need the radial component of the Poynting vector which, when averaged over a period of time, with the complex notation, is given by

$$S_r = \frac{1}{2}(\mathbf{E}^* \times \mathbf{H})_r = \frac{1}{2}\mathrm{Re}\{E_\theta^* H_3\} = \frac{C_5 C_5^*}{\pi\omega\varepsilon_0 r}\cos^2 \nu\theta\,, \qquad (7.6\text{-}49)$$

where an asterisk represents complex conjugate. Equation (7.6-49) shows that the energy flux is inversely proportional to r. It also shows the angular distribution of the power radiation. The radiated power per unit length of the cylinder is

$$S = \int_0^{2\pi} S_r r d\theta = \frac{C_5 C_5^*}{2\pi\omega\varepsilon_0}(2\pi + \frac{1}{2\nu}\sin 4\nu\pi)\,. \qquad (7.6\text{-}50)$$

We are interested in the frequency range of acoustic waves. Therefore αb is finite, $\beta b << 1$, and $\gamma b <<1$. For small arguments we have

$$J_\nu(x) \cong \frac{x^\nu}{2^\nu \Gamma(1+\nu)}, \quad H_\nu^{(1)}(x) \cong -i\frac{2^\nu \Gamma(\nu)}{\pi x^\nu},$$
$$\frac{xJ_\nu'(x)}{J_\nu(x)} \cong \nu, \quad \frac{xH_\nu^{(1)\prime}(x)}{H_\nu^{(1)}(x)} \cong -\nu. \qquad (7.6\text{-}51)$$

Then, approximately,

$$C_5 = \frac{i\omega e_{15} b J_\nu(\alpha b)}{(1+\frac{\varepsilon_{11}}{\varepsilon_0})\alpha b J_\nu'(\alpha b) - \bar{k}_{15}^2 \nu J_\nu(\alpha b)} \frac{\tau}{\bar{c}_{44}} \frac{1}{H_\nu^{(1)}(\gamma b)}\,. \qquad (7.6\text{-}52)$$

In this approximate form, the denominator of the first factor of (7.6-52) represents the frequency equation for quasistatic electromechanical resonances in piezoelectricity. With Equation (7.6-52), the radiated power can be written as

$$S = \frac{\omega e_{15}^2 b^2}{2\pi\varepsilon_0}\left|\frac{J_\nu(\alpha b)}{(1+\frac{\varepsilon_{11}}{\varepsilon_0})\alpha b J_\nu'(\alpha b) - \bar{k}_{15}^2 \nu J_\nu(\alpha b)}\frac{\tau}{\bar{c}_{44}}\right|^2 \frac{2\pi + \dfrac{\sin 4\nu\pi}{2\nu}}{H_\nu^{(1)}(\gamma b)[H_\nu^{(1)}(\gamma b)]^*}$$

(7.6-53)

From Equation (7.6-53) we make the following observations:

(i) S is large near resonance frequencies. It is singular at these frequencies unless some damping is present.

(ii) In the limit of $\omega \to 0$, α, β, and γ all $\to 0$. In this case $S \to 0$ as expected.

(iii) S is proportional to the square of a piezoelectric constant. For materials with strong piezoelectric coupling, the radiated power is much more than that of a material with weak coupling.

Problems

7.6-1. Study piezoelectromagnetic SH waves in a ceramic plate [57].

7.6-2. Study piezoelectromagnetic SH surface waves in a ceramic half-space carrying a thin layer of isotropic conductor or dielectric [32].

7.6-3. Study piezoelectromagnetic SH gap waves between two ceramic half-spaces [58].

Chapter 8

PIEZOELECTRIC DEVICES

This chapter presents analyses of piezoelectric devices that use the equations developed in previous chapters. Sections 1 and 2 are based on the linear theory of piezoelectricity. Sections 3 to 5 use the theory for small fields superposed on biasing fields. Device problems are usually complicated mathematical problems. Exactly solutions cannot be obtained. Structural theories or numerical methods are necessary. All problems treated in this chapter have approximations in order to make them simple. Since interest in devices varies, the chapter is written in such a way that sections can be read independently.

1. GYROSCOPES

Gyroscopes have important applications in automobiles, video cameras, smart weapon systems, machine control, robotics, and navigation. Traditional mechanical gyroscopes are based on the inertia of a rotating rigid body. New types of gyroscopes have also been developed, e.g., vibratory gyroscopes and optical gyroscopes. These gyroscopes are based on different physical principles and they differ greatly in size, weight, accuracy, and cost. Each type of gyroscope has its own advantages and disadvantages, and each is used in different applications.

For vibratory gyroscopes, the excitation and detection of vibrations can be achieved electrostatically or piezoelectrically, or in other ways. Piezoelectric gyroscopes make use of two vibration modes of a piezoelectric body. The two modes have material particles moving in perpendicular directions so that they are coupled by Coriolis force when the gyroscope is rotating. Furthermore, the two modes must have the same frequency ω_0 so that the gyroscope operates at the so-called double resonance condition with high sensitivity. When a gyroscope is excited into mechanical vibration by an applied alternating electric voltage in one of the two modes (the primary mode) and is attached to a body rotating with an angular rate $\mathbf{\Omega}$, Coriolis force excites the other mode (the secondary mode) through which $\mathbf{\Omega}$ can be detected from electrical signals (voltage or current) accompanying the secondary mode.

1.1 Governing Equations

A Piezoelectric gyroscope is in small amplitude vibration in a reference frame that rotates with it. The equilibrium state in the rotating reference frame has initial deformation and stress due to centrifugal force. Therefore an exact description of the motion of a piezoelectric gyroscope requires the equations for small, dynamic fields superposed on static initial fields due to centrifugal force. The governing equations in the rotating frame can be written as

$$\begin{aligned} &T_{ji,j} - 2\rho\varepsilon_{ijk}\Omega_j\dot{u}_k - \rho(\Omega_i\Omega_j u_j - \Omega_j\Omega_j u_i) + O(\Omega^2) = \rho\ddot{u}_i, \\ &D_{i,i} = 0, \\ &T_{ij} = c_{ijkl}S_{kl} - e_{kij}E_k, \quad D_i = e_{ijk}S_{jk} + \varepsilon_{ij}E_j, \\ &S_{ij} = (u_{i,j} + u_{j,i})/2, \quad E_i = -\phi_{,i}, \end{aligned} \tag{8.1-1}$$

where $O(\Omega^2)$ represents terms due to the initial fields. Since piezoelectric gyroscopes are very small (on the order of 10 mm), their operating frequency ω_0 is very high, on the order of hundreds of kHz or higher. Piezoelectric gyroscopes are used to measure an angular rate $\mathbf{\Omega}$ much smaller than ω_0. In such a case, the centrifugal force due to rotation and the initial fields which are proportional to $\mathbf{\Omega}^2$ are much smaller compared to the Coriolis force that is proportional to $\omega_0\mathbf{\Omega}$. Therefore, the effect of rotation on the motion of piezoelectric gyroscopes is dominated by the Coriolis force. This is fundamentally different from the relatively well-studied subject of vibrations of a rotating elastic body for machinery application. In machinery dynamics large bodies with low resonance frequencies are often in relatively fast rotations, and the centrifugal force is one of the dominant forces. The high frequency vibration of a small rotating piezoelectric body is characterized by that the Coriolis force is much larger than the centrifugal force. This presents a new class of mechanics problems.

1.2 An Example

The operating principle and the basic behaviors of piezoelectric gyroscopes can be best explained by the simple example below [59]. Consider a concentrated mass *M* connected to two thin rods of polarized ceramics as shown in Figure 8.1-1.

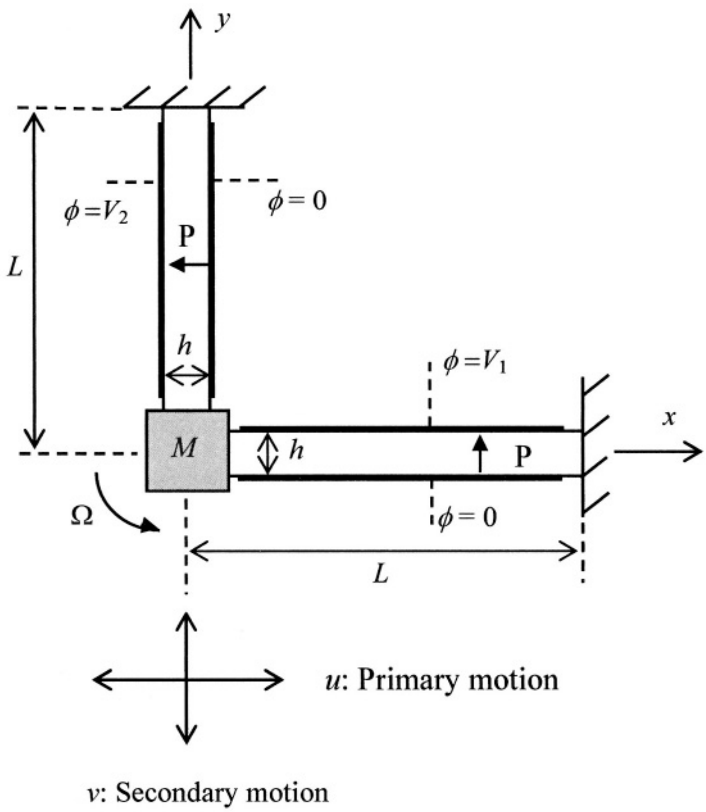

Figure 8.1-1. A simple piezoelectric gyroscope.

The two rods are electroded at the side surfaces, with electrodes shown by the thick lines. Under a time-harmonic driving voltage $V_1(t)$, the rod along the x direction is driven into extensional vibrations. If the entire system is rotating about the normal of the (x,y) plane at an angular rate $\mathbf{\Omega}$, it results in a voltage output $V_2(t)$ across the width of the rod along the y direction. $V_2(t)$ is proportional to $\mathbf{\Omega}$ when $\mathbf{\Omega}$ is small, which can be used to detect $\mathbf{\Omega}$.

1.2.1 A Zero-Dimensional Model

For long and thin rods $(L >> h)$ the flexural rigidity is very small. The rods do not resist bending but can still provide extensional forces. When the mass of M is much larger than that of the rods, the inertial effect of the rods can be discounted. Then the mechanical behavior of the rods is effectively like two elastic springs with the addition of piezoelectric couplings. Let the displacements of M in the x and y directions be $u(t)$ and $v(t)$. We consider small amplitude vibrations of M in the co-rotating (x,y) coordinate system. For each rod we also associate a local coordinate system with the x_1 axis along the axis of the rod and the x_3 axis along the poling direction.

Consider the rod along the x direction first. Neglecting the dynamical effect in the rod due to inertia, the axial strain in the rod can be written as

$$S_1 = -u / L . \tag{8.1-2}$$

With respect to the local coordinate system, the electric field corresponding to the configuration of the driving electrodes can be written as

$$E_1 = E_2 = 0, \quad E_3 = -V_1 / h, \tag{8.1-3}$$

where the driving voltage V_1 is considered given and is time-harmonic. For thin rods in extension, the dominating stress component is the axial stress component T_1. All other stress components can be treated as zero (see Chapter 4, Section 7). Under the above stress and electric field conditions, the constitutive relations take the form

$$\begin{aligned} S_1 &= s_{11}T_1 + d_{31}E_3, \\ D_3 &= d_{31}T_1 + \varepsilon_{33}E_3, \end{aligned} \tag{8.1-4}$$

where D_3 is the component of the electric displacement vector in the local coordinate system, and s_{11}, d_{31}, and ε_{33} are the relevant elastic, piezoelectric, and dielectric constants. From Equation (8.1-4) we can solve for T_1 and D_3 in terms of S_1 and E_3, with the result

$$\begin{aligned} T_1 &= \frac{1}{s_{11}} S_1 - \frac{d_{31}}{s_{11}} E_3 = -\frac{1}{s_{11}} \frac{u}{L} + \frac{d_{31}}{s_{11}} \frac{V_1}{h}, \\ D_3 &= \frac{d_{31}}{s_{11}} S_1 + \bar{\varepsilon}_{33} E_3 = -\frac{d_{31}}{s_{11}} \frac{u}{L} - \bar{\varepsilon}_{33} \frac{V_1}{h}, \end{aligned} \tag{8.1-5}$$

where Equations (8.1-2) and (8.1-3) have been used, and

$$\bar{\varepsilon}_{33} = \varepsilon_{33}(1 - k_{31}^2), \quad k_{31}^2 = d_{31}^2 / (\varepsilon_{33} s_{11}) . \tag{8.1-6}$$

The axial force in the rod and the electric charge on the electrode at the upper surface of the rod are given by

$$F_1 = T_1 h = -Ku + \frac{d_{31}}{s_{11}} V_1,$$
$$Q_1 = -D_3 L = \frac{d_{31}}{s_{11}} u + C_0 V_1, \tag{8.1-7}$$

where

$$K = \frac{h}{s_{11} L}, \quad C_0 = \frac{\bar{\varepsilon}_{33} L}{h}, \tag{8.1-8}$$

represent the elastic stiffness and the static capacitance of the rod. The electric current on the electrode is related to the charge by

$$I_1 = -\dot{Q}_1 = -\frac{d_{31}}{s_{11}} \dot{u} - C_0 \dot{V}_1. \tag{8.1-9}$$

Similarly, for the rod along the y direction, the axial force and electric current are given by

$$F_2 = -Kv + \frac{d_{31}}{s_{11}} V_2,$$
$$I_2 = -\frac{d_{31}}{s_{11}} \dot{v} - C_0 \dot{V}_2. \tag{8.1-10}$$

In gyroscope applications, neither V_2 nor I_2 is directly known. The output receiving or sensing electrodes across the rod along the y direction are connected by an electric circuit. For time-harmonic cases, we have the following circuit condition

$$I_2 = V_2 / Z, \tag{8.1-11}$$

where Z is the impedance of the output circuit (also called the load circuit in this book), which depends on the structure of the circuit and, in general, is also a function of the frequency of the time-harmonic motion. In the special cases when $Z = 0$ or ∞, we have shorted or open output circuit conditions with $V_2 = 0$ or $I_2 = 0$.

The equations of motion are

$$F_1 = M(\ddot{u} - 2\Omega\dot{v} - \Omega^2 u),$$
$$F_2 = M(\ddot{v} + 2\Omega\dot{u} - \Omega^2 v), \tag{8.1-12}$$

where the Coriolis and centrifugal accelerations are included.

1.2.2 An Analytical Solution

For time-harmonic motions we use the complex notation

$$(u, v, V_1, V_2, I_1, I_2) = (\bar{u}, \bar{v}, \bar{V}_1, \bar{V}_2, \bar{I}_1, \bar{I}_2)e^{i\omega t}, \tag{8.1-13}$$

where I_1 and I_2 are the input and output currents. The vibration of the system is governed by the following three linear equations for $\bar{u}$, $\bar{v}$, and $\bar{V}_2$, with $\bar{V}_1$ as the driving term:

$$\begin{aligned}
&[M(\omega^2 + \Omega^2) - K]\bar{u} + 2i\omega\Omega M\bar{v} = -\frac{d_{31}}{s_{11}}\bar{V}_1, \\
&-2i\omega\Omega M\bar{u} + [M(\omega^2 + \Omega^2) - K]\bar{v} + \frac{d_{31}}{s_{11}}\bar{V}_2 = 0, \\
&\frac{d_{31}}{s_{11}}\bar{v} + C_0(1 + \frac{Z_0}{Z})\bar{V}_2 = 0.
\end{aligned} \tag{8.1-14}$$

For forced vibration, some damping is introduced into the system by the complex elastic constant $s_{11}(1 - iQ^{-1})$, where Q, the quality factor, is a large number. The output voltage and current, as well as the driving current, are determined as

$$\begin{aligned}
&\frac{\bar{V}_2}{\bar{V}_1} = \frac{1}{\Delta} 2i\omega\Omega\omega_0^2 \frac{k_{31}^2}{1 - k_{31}^2} \frac{Z}{Z + Z_0}, \\
&\frac{-\bar{I}_1}{\bar{V}_1 / Z_0} = 1 - \frac{1}{\Delta} \frac{k_{31}^2}{1 - k_{31}^2} \omega_0^2 \left[\omega^2 + \Omega^2 - \omega_0^2 \left(1 + \frac{k_{31}^2}{1 - k_{31}^2} \frac{Z}{Z + Z_0}\right)\right], \\
&\frac{\bar{I}_2}{\bar{V}_1 / Z_0} = \frac{1}{\Delta} 2i\omega\Omega\omega_0^2 \frac{k_{31}^2}{1 - k_{31}^2} \frac{Z_0}{Z + Z_0}.
\end{aligned} \tag{8.1-15}$$

1.2.3 Numerical Results

Output voltage as a function of the driving frequency ω is plotted in Figure 8.1-2 for the case of open sensing electrodes ($Z = \infty$ for large sensing voltage) and two values of Ω. There are two resonance frequencies with values near 1. Near the two resonance frequencies, the voltage sensitivity assumes maximal values. If smaller values of Q are used, the peaks will become narrower and higher. Although higher peaks suggest higher voltage sensitivity, narrower peaks require better control in tuning the sensor into what is called the double-resonance condition, with the driving frequency and the resonance frequencies of the primary and the secondary modes very close. We note that when Ω is doubled, the output voltage is essentially doubled as well, thus suggesting a linear response to Ω.

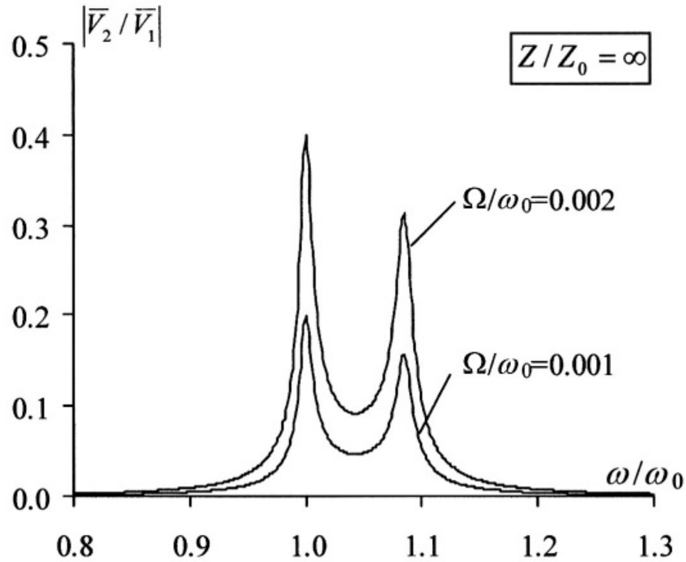

Figure 8.1-2. Voltage sensitivity versus the driving frequency ω.

The dependence of the normalized maximum output voltage (the value of one of the two peaks shown in Figure 8.1-2) on the rotation rate Ω is shown in Figure 8.1-3 for two values of the load Z. When Ω is much smaller than ω_0, the relation between the output voltage and Ω is essentially linear, as shown in $(8.1\text{-}15)_1$. Therefore these gyroscopes are convenient for detecting a rotation rate that is relatively slow compared to the operating frequency. Since piezoelectric gyroscopes can be made very small with high resonance frequencies, the relatively slow rotation rate that the gyroscopes can detect linearly can still cover a variety of applications. When Ω is not small, the quadratic effect of Ω in the denominator of $(8.15\text{-}1)_1$ begins to show its effect, which determines the range of the sensor for a linear response. Since the response is linear in small Ω, in the analysis of piezoelectric gyroscopes the centrifugal force (which represents higher order effects of Ω) can often be neglected. The contribution to sensitivity is from the Coriolis force which is linear in Ω. The above behaviors are also observed in many other gyroscopes.

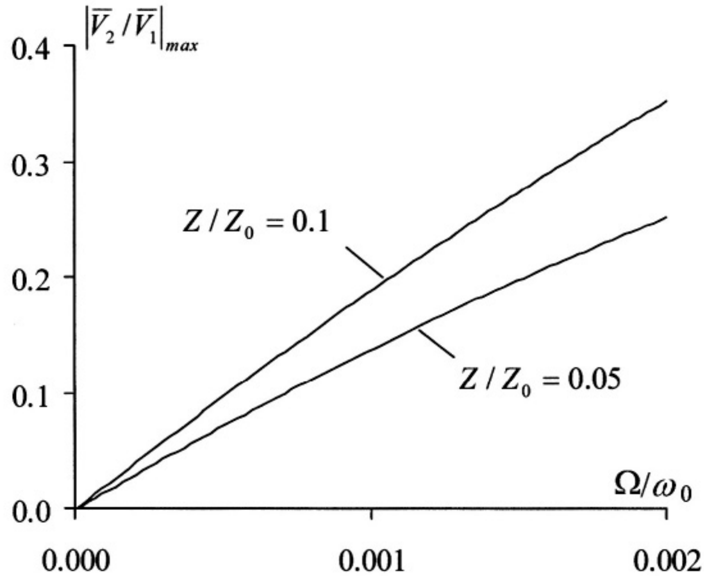

Figure 8.1-3. Voltage sensitivity versus the rotation rate **Ω.**

2. TRANSFORMERS

Piezoelectric materials can be used to make transformers for raising or lowering electric voltages. A piezoelectric transformer is widely used in several types of electronic equipment. A piezoelectric transformer is a resonant device operating at a particular resonance frequency of a vibrating piezoelectric body. The basic behavior of a piezoelectric transformer is governed by the linear theory of piezoelectricity.

In this section, we perform an analytical study [60] on Rosen transformers operating with extensional modes of rods. An approximate one-dimensional model similar to Section 7 of Chapter 4 will be developed. The one-dimensional model is simple enough to allow analytical studies. At the same time, the model provides useful information for understanding the operating principle of the transformer and its design.

2.1 A One-Dimensional Model

Consider a ceramic Rosen transformer of length $a+b$, width w and thickness h as shown in Figure 8.2-1.

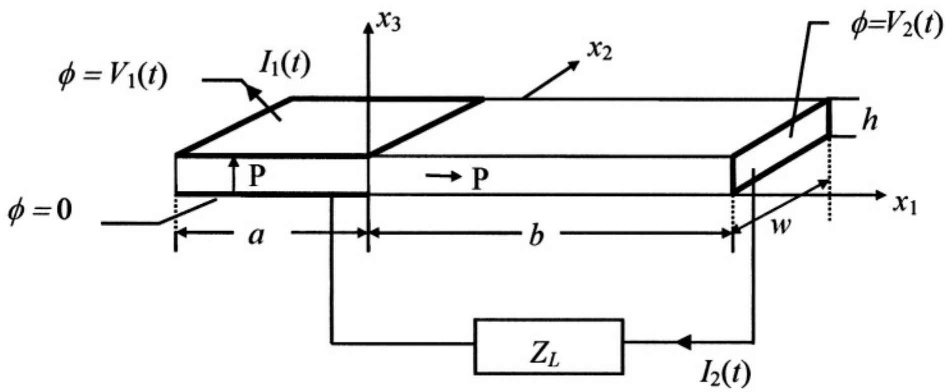

Figure 8.2-1. A Rosen ceramic piezoelectric transformer.

We assume that the transformer has a slender shape with a, $b >> w >> h$. The driving portion $-a < x_1 < 0$ is poled in the x_3 direction and is electroded at $x_3 = 0$ and $x_3 = h$, with electrodes in the areas bounded by the thick lines shown in the figure. In the receiving portion $0 < x_1 < b$, the ceramic is poled in the x_1 direction with one output electrode at the end $x_1 = b$. The other output electrode is shared with the driving portion (where $\phi = 0$). Under a time-harmonic driving voltage $V_1(t)$ with a proper frequency, the transformer can be driven into extensional vibration and produce an output voltage $V_2(t)$. The output electrodes are usually connected by a load circuit whose impedance is denoted by Z_L. Because the transformer is non-uniform, with materials in the two portions oriented differently, we analyze each portion separately below.

2.1.1 Driving Portion

Since the rod is slender and we are considering extensional motions only, for the driving portion $-a < x_1 < 0$, we make the usual assumption of a unidirectional state of stress:

$$T_2 = T_3 = T_4 = T_5 = T_6 = 0. \tag{8.2-1}$$

For the electric field, based on the electrode configuration, we approximately have

$$\phi = \frac{x_3}{h} V_1, \tag{8.2-2}$$

which implies that

$$E_1 = E_2 = 0, \quad E_3 = -\frac{1}{h} V_1. \tag{8.2-3}$$

The relevant equation of motion and constitutive relations are

$$T_{1,1} = \rho \ddot{u}_1, \qquad (8.2\text{-}4)$$
$$S_1 = s_{11}T_1 + d_{31}E_3, \quad D_3 = d_{31}T_1 + \varepsilon_{33}E_3.$$

From Equations (8.2-1) to (8.2-4) we obtain the equation for u_1 and the expressions for T_1 and D_3 as

$$\frac{1}{s_{11}}u_{1,11} = \rho \ddot{u}_1, \qquad (8.2\text{-}5)$$
$$T_1 = \frac{1}{s_{11}}(u_{1,1} + \frac{d_{31}}{h}V_1), \quad D_3 = \frac{d_{31}}{s_{11}}u_{1,1} - \bar{\varepsilon}_{33}\frac{V_1}{h},$$

where

$$\bar{\varepsilon}_{33} = \varepsilon_{33}(1-k_{31}^2), \quad k_{31}^2 = \frac{d_{31}^2}{\varepsilon_{33}s_{11}}. \qquad (8.2\text{-}6)$$

At the left end, we have the following boundary condition

$$T_1 = \frac{1}{s_{11}}(u_{1,1} + \frac{d_{31}}{h}V_1) = 0, \quad x_1 = -a. \qquad (8.2\text{-}7)$$

The current flowing out of the driving electrode at $x_3 = h$ is given by

$$I_1 = -\dot{Q}_1, \quad Q_1 = -w\int_{-a}^{0} D_3 dx_1, \qquad (8.2\text{-}8)$$

where Q_1 is the charge on the driving electrode at $x_3 = h$.

2.1.2 Receiving Portion

For the receiving portion $0 < x_1 < b$, the stress assumption (8.2-1) is still valid. For the electric field, since the portion is not electroded on its lateral surfaces, we approximately have

$$D_2 = D_3 = 0, \qquad (8.2\text{-}9)$$

which implies, from the constitutive equations, that

$$E_2 = E_3 = 0. \qquad (8.2\text{-}10)$$

Hence in the receiving portion, the dominating electric field component is

$$E_1 = -\phi_{,1}, \quad \phi = \phi(x_1, t). \qquad (8.2\text{-}11)$$

The relevant equation of motion, the charge equation, and constitutive relations are

$$T_{1,1} = \rho \ddot{u}_1, \quad D_{1,1} = 0, \qquad (8.2\text{-}12)$$
$$S_1 = s_{33}T_1 + d_{33}E_1, \quad D_1 = d_{33}T_1 + \varepsilon_{33}E_1.$$

Then the equation for u_1 and the expressions for T_1, D_1 and ϕ are

$$\frac{1}{\bar{s}_{33}} u_{1,11} = \rho \ddot{u}_1, \quad T_1 = \frac{1}{\bar{s}_{33}}(u_{1,1} - k_{33}^2 c_1),$$
$$D_1 = \frac{d_{33}}{s_{33}} c_1, \quad \phi = \frac{1}{\bar{d}_{33}}(c_1 x_1 - u_1) + c_2, \tag{8.2-13}$$

where

$$\bar{s}_{33} = s_{33}(1 - k_{33}^2), \quad \bar{d}_{33} = d_{33}(1 - \frac{1}{k_{33}^2}), \quad k_{33}^2 = \frac{d_{33}^2}{s_{33}\varepsilon_{33}}, \tag{8.2-14}$$

and $c_1(t)$ and $c_2(t)$ are two integration constants which may still be functions of time. The following boundary conditions need to be satisfied at the right end:

$$T_1 = \frac{1}{\bar{s}_{33}}(u_{1,1} - k_{33}^2 c_1) = 0, \quad \phi = V_2(t), \quad x_1 = b. \tag{8.2-15}$$

Physically, c_1 is related to the electric charge Q_2 and hence the current I_2 on the receiving electrode at $x_1 = b$

$$D_1 = \frac{d_{33}}{s_{33}} c_1 = -\frac{Q_2}{wh}, \quad I_2 = -\dot{Q}_2. \tag{8.2-16}$$

2.1.3 Continuity Conditions

At the junction of the two portions $x_1 = 0$, the following continuity and boundary conditions need to be prescribed:

$$u_1(0^-) = u_1(0^+),$$
$$T_1(0^-) = \frac{1}{s_{11}}(u_{1,1} + \frac{d_{31}}{h} V_1)|_{0^-} = \frac{1}{\bar{s}_{33}}(u_{1,1} - k_{33}^2 c_1)|_{0^+} = T_1(0^+), \tag{8.2-17}$$
$$\phi(0^+) = \frac{1}{2} V_1.$$

We need to solve the two second-order equations $(8.2\text{-}5)_1$ and $(8.2\text{-}13)_1$ for u_1 in two regions. Since piezoelectric transformers operate under a time-harmonic driving voltage, we employ the complex notation and write

$$\{u_1, \phi, V_1, V_2 I_1, I_2, c_1, c_2\} = \mathrm{Re}\{\{U, \Phi, \bar{V}_1, \bar{V}_2 \bar{I}_1, \bar{I}_{2,} C_1, C_2\} e^{i\omega t}\}. \tag{8.2-18}$$

For the output electrodes, under the complex notation, we have the following circuit condition

$$\bar{I}_2 = \bar{V}_2 / Z_L. \tag{8.2-19}$$

When $Z_L = 0$ or ∞, we have shorted or open output electrodes.

2.2 Free Vibration Analysis

The basic mechanism of a piezoelectric transformer can be shown by its vibration modes from a free vibration analysis. For free vibrations we set $V_1 = 0$ and $c_1 = 0$. Physically this means that the driving electrodes are shorted and the receiving electrodes are open. Mathematically the equations and boundary conditions all become homogeneous. From Equations $(8.2\text{-}5)_1$, $(8.2\text{-}13)_1$, (8.2-7), (8.2-15) and (8.2-17), the equations and boundary conditions we need to solve reduce to

$$
\begin{gathered}
-\frac{1}{s_{11}}U_{,11} = \rho\omega^2 U\ , \quad -a < x_1 < 0\,, \\
-\frac{1}{\bar{s}_{33}}U_{,11} = \rho\omega^2 U\,, \quad 0 < x_1 < b\,, \\
\frac{1}{s_{11}}U_{,1}(-a) = 0\,, \quad \frac{1}{\bar{s}_{33}}U_{,1}(b) = 0\,, \\
U(0^-) = U(0^+)\,, \quad \frac{1}{s_{11}}U_{,1}(0^-) = \frac{1}{\bar{s}_{33}}U_{,1}(0^+)\,,
\end{gathered}
\tag{8.2-20}
$$

where ω is an unknown resonance frequency at which nontrivial solutions of U to the above equations exist. Mathematically, Equation (8.2-20) is an eigenvalue problem. The solution with $\omega = 0$ is a rigid body displacement with U being a constant, which is not of interest here. The solution to (8.2-20) is

$$
\omega = \omega_{(n)}, \quad n = 1,2,3,\cdots, \tag{8.2-21}
$$

$$
U^{(n)} = \begin{cases} \left(\dfrac{s_{11}}{\bar{s}_{33}}\right)^{\frac{1}{2}} \sin k_{(n)}x_1 + \dfrac{1}{\tan \bar{k}_{(n)}b}\cos k_{(n)}x_1, & -a < x_1 < 0, \\ \sin \bar{k}_{(n)}x_1 + \dfrac{1}{\tan \bar{k}_{(n)}b}\cos \bar{k}_{(n)}x_1, & 0 < x_1 < b, \end{cases} \tag{8.2-22}
$$

where

$$
k_{(n)}^2 = \rho s_{11}\omega_{(n)}^2, \quad \bar{k}_{(n)}^2 = \rho \bar{s}_{33}\omega_{(n)}^2, \tag{8.2-23}
$$

and $\omega_{(n)}$ is the n-th root of the following frequency equation

$$
\frac{\tan ka}{\tan \bar{k}b} = -\left(\frac{s_{11}}{\bar{s}_{33}}\right)^{\frac{1}{2}}. \tag{8.2-24}
$$

We note that the width w and thickness h do not appear in the frequency equation. This is as expected from a one-dimensional model. Once U is known, Φ in the receiving portion can be found from $(8.2\text{-}13)_4$. We have

$$\Phi^{(n)} = \begin{cases} 0, & -a < x_1 < 0, \\ -\dfrac{1}{\bar{d}_{33}}[\sin \bar{k}_{(n)}x_1 + \dfrac{1}{\tan \bar{k}_{(n)}b}(\cos \bar{k}_{(n)}x_1 - 1)], & 0 < x_1 < b. \end{cases} \quad (8.2\text{-}25)$$

2.3 Forced Vibration Analysis

For forced vibration driven by $V_1(t) = \bar{V}_1 e^{i\omega t}$, from $(8.2\text{-}5)_1$, $(8.2\text{-}13)_1$, (8.2-7), (8.2-15) and (8.2-17), we need to solve the following non-homogeneous problem for U:

$$\begin{gathered} -\frac{1}{s_{11}}U_{,11} = \rho\omega^2 U, \quad -a < x_1 < 0, \\ -\frac{1}{\bar{s}_{33}}U_{,11} = \rho\omega^2 U, \quad 0 < x_1 < b, \\ \frac{1}{s_{11}}(U_{,1} + \frac{d_{31}}{h}\bar{V}_1) = 0, \quad x_1 = -a, \\ \frac{1}{\bar{s}_{33}}(U_{,1} - k_{33}^2 C_1) = 0, \quad x_1 = b, \\ U(0^-) = U(0^+), \quad \frac{1}{s_{11}}[U_{,1}(0^-) + \frac{d_{31}}{h}\bar{V}_1] = \frac{1}{\bar{s}_{33}}[U_{,1}(0^+) - k_{33}^2 C_1]. \end{gathered} \quad (8.2\text{-}26)$$

The solution to (8.2-26) allows an arbitrary constant which is chosen to be 0. Then the solution to (8.2-26) can be written as

$$U = \begin{cases} (\alpha_{11}\dfrac{d_{31}\bar{V}_1}{h} + \beta_{11}k_{33}^2 C_1)\sin kx_1 \\ \quad + (\alpha_{12}\dfrac{d_{31}\bar{V}_1}{h} + \beta_{12}k_{33}^2 C_1)\cos kx_1, \quad -a < x_1 < 0, \\ (\alpha_{22}\dfrac{d_{31}\bar{V}_1}{h} + \beta_{22}k_{33}^2 C_1)\sin \bar{k}x_1 \\ \quad + (\alpha_{12}\dfrac{d_{31}\bar{V}_1}{h} + \beta_{12}k_{33}^2 C_1)\cos \bar{k}x_1, \quad 0 < x_1 < b, \end{cases} \quad (8.2\text{-}27)$$

where

$$k=\omega\sqrt{\rho s_{11}}, \quad \bar{k}=\omega\sqrt{\rho\bar{s}_{33}},$$

$$\begin{aligned}
\alpha_{11} &= -\frac{1}{\Delta\cos ka}\bar{s}_{33}\cos\bar{k}b\sin ka(\cos ka-1)-\frac{1}{k\cos ka},\\
\beta_{11} &= \frac{1}{\Delta}s_{11}\sin ka(1-\cos\bar{k}b), \quad \alpha_{12}=\frac{1}{\Delta}\bar{s}_{33}\cos\bar{k}b(\cos ka-1),\\
\beta_{12} &= -\frac{1}{\Delta}s_{11}\cos ka(1-\cos\bar{k}b), \quad \alpha_{22}=\frac{1}{\Delta}\bar{s}_{33}\sin\bar{k}b(\cos ka-1),\\
\beta_{22} &= -\frac{1}{\Delta\cos\bar{k}b}s_{11}\sin\bar{k}b\cos ka(1-\cos\bar{k}b)+\frac{1}{\bar{k}\cos\bar{k}b},\\
\Delta &= s_{11}\bar{k}\sin\bar{k}b\cos ka+\bar{s}_{33}k\sin ka\cos\bar{k}b.
\end{aligned} \tag{8.2-28}$$

Note that $\Delta = 0$ implies the frequency equation (8.2-24). With Equations (8.2-27), from $(8.2\text{-}13)_4$ and $(8.2\text{-}17)_3$, we obtain the voltage distribution in the receiving portion as

$$\begin{aligned}
\Phi &= \frac{1}{2}\bar{V}_1+\frac{1}{\bar{d}_{33}}[C_1x_1-(\alpha_{22}\frac{d_{31}\bar{V}_1}{h}+\beta_{22}k_{33}^2C_1)\sin\bar{k}x_1\\
&\quad +(\alpha_{12}\frac{d_{31}\bar{V}_1}{h}+\beta_{12}k_{33}^2C_1)(1-\cos\bar{k}x_1)], \quad 0<x_1<b.
\end{aligned} \tag{8.2-29}$$

Although $\bar{V}_1$ is considered given, Equations (8.2-27) and (8.2-29) still have an unknown constant C_1. From Equations (8.2-29) and $(8.2\text{-}15)_2$ the output voltage is

$$\bar{V}_2=\Phi(b)=\Gamma_1\bar{V}_1-Z_2\bar{I}_2, \tag{8.2-30}$$

where

$$\begin{aligned}
\Gamma_1 &= \frac{1}{2}+\frac{d_{31}}{\bar{d}_{33}}\frac{1}{h}[\alpha_{12}(1-\cos\bar{k}b)-\alpha_{22}\sin\bar{k}b],\\
Z_2 &= \frac{1}{1-k_{33}^2}\frac{1}{b}[b-\beta_{22}k_{33}^2\sin\bar{k}b+\beta_{12}k_{33}^2(1-\cos\bar{k}b)]\frac{b}{i\omega\varepsilon_{33}wh},
\end{aligned} \tag{8.2-31}$$

and Equation (8.2-16) has been used to replace C_1 by $\bar{I}_2$. From Equations (8.2-8), $(8.2\text{-}5)_3$, and (8.2-27), the driving current is

$$\bar{I}_1=-\bar{V}_1/Z_1+\Gamma_2\bar{I}_2, \tag{8.2-32}$$

where

$$1/Z_1 = \frac{1}{a}[a - \frac{k_{31}^2}{1-k_{31}^2}(\alpha_{12} + \alpha_{11}\sin ka - \alpha_{12}\cos ka)]\frac{i\omega\bar{\varepsilon}_{33}wa}{h},$$
$$\Gamma_2 = k_{33}^2 \frac{s_{33}d_{31}}{s_{11}d_{33}}\frac{1}{h}(\beta_{12} + \beta_{11}\sin ka - \beta_{12}\cos ka). \tag{8.2-33}$$

Equations (8.2-30) and (8.2-32) are in fact the general form of the input-output relation for a linear system. Equations (8.2-31) and (8.2-33) are specific expressions of the coefficients in the linear relations, which depend on the specific system which, in this case, is the Rosen transformer in Figure 8.2-1. Discussions in the rest of this section will depend on the general forms of (8.2-30) and (8.2-32), but not on the specific expressions (8.2-31) and (8.2-33). Since $\bar{V}_1$ is given, solving (8.2-30), (8.2-32), and the circuit condition (8.2-19), we obtain the transforming ratio, the output current, and the input admittance as

$$\frac{\bar{V}_2}{\bar{V}_1} = \frac{\Gamma_1 Z_L}{Z_L + Z_2}, \quad \bar{I}_2 = \frac{\Gamma_1}{Z_L + Z_2}\bar{V}_1,$$
$$-\frac{\bar{I}_1}{\bar{V}_1} = \frac{1}{Z_1} - \frac{\Gamma_1\Gamma_2}{Z_L + Z_2}. \tag{8.2-34}$$

The dependence of the transforming ratio and the input admittance on the load Z_L is of great interest in transformer design. This is explicitly shown in (8.2-34). It can be seen that for small loads the transforming ratio is linear in Z_L and the input admittance approaches a finite limit. For large Z_L the transforming ratio approaches Γ_1 (saturation) and the input admittance approaches $1/Z_1$. The input and output powers are given by

$$P_1 = \frac{1}{4}(\bar{I}_1\bar{V}_1^* + \bar{I}_1^*\bar{V}_1), \quad P_2 = \frac{1}{4}(\bar{I}_2\bar{V}_2^* + \bar{I}_2^*\bar{V}_2), \tag{8.2-35}$$

where an asterisk means complex conjugate. Then the efficiency of the transformer can be written as

$$\eta = \frac{P_2}{P_1}$$
$$= \frac{Z_1 Z_1^* \Gamma_1 \Gamma_1^* (Z_L + Z_L^*)}{Z_1 Z_1^* [\Gamma_1^* \Gamma_2^* (Z_L + Z_2) + \Gamma_1 \Gamma_2 (Z_L^* + Z_2^*)] - (Z_L + Z_2)(Z_L^* + Z_2^*)(Z_1 + Z_1^*)}. \tag{8.2-36}$$

Equation (8.2-36) shows that for small loads η depends on Z_L linearly, and for large loads η decreases to 0. When the load is a pure resistor, Z_L is real. In this case, Equation (8.2-36) can be written in a simpler form as

$$\eta = \frac{\lambda Z_L}{1 + \mu Z_L + \nu Z_L^2}, \tag{8.2-37}$$

where λ, μ, and ν are real functions of Γ_1, Γ_1, Z_1 and Z_2. Equation (8.2-37) has only three real parameters which can be easily determined from experiments. Equation (8.2-37) implies that the efficiency as a function of a resistor load has a maximum. Equations (8.2-34), (8.2-36), and (8.2-37) are valid for piezoelectric transformers in general.

2.4 Numerical Results

As an example, consider a transformer made from polarized ceramics PZT–5H. For forced vibration analysis, some damping in the system is necessary. This is achieved by allowing the elastic material constants s_{11} and s_{33} to assume complex values, which can represent viscous type damping in the material. In our calculations s_{11} and s_{33} are replaced by $s_{11}(1-iQ^{-1})$ and $s_{33}(1-iQ^{-1})$, where Q is a real number. For polarized ceramics the value of Q is in the order of 10^2 to 10^3. For free vibration we consider the case of $Q = 0$. In the calculation for forced vibration we fix $Q = 1000$. We consider a transformer with $a = b = 22$ mm, $w = 10$ mm, and $h = 2$ mm.

With the above data, the first root of the frequency equation (8.2-24), that is, the resonant frequency of the operating mode, is found to be

$$f = \omega_{(1)} / 2\pi = 36.35 \text{ kHz}. \tag{8.2-38}$$

The mode shape of the operating mode $U^{(1)}$ is shown in Figure 8.2-2, and is normalized by its maximum. The location of the nodal point is not simply in the middle of the transformer, although $a = b$ in this example. This is because the material is not uniform. The location of the nodal point depends on the compliances s_{11} and $\bar{s}_{33}$. In this example we have $\bar{s}_{33} < s_{11}$. Hence the nodal point appears in the left half of the bar, where the material is less rigid in the x_1 direction than the material of the right half.

In Figure 8.2-2, $U^{(1)}$ is continuous at $x_1 = 0$ but $U_{,1}^{(1)}$ is not, as dictated by $(8.2\text{-}20)_{5,6}$. Accurate prediction of vibration modes and their nodal points is particularly important for Rosen piezoelectric transformers. Theoretically, Rosen transformers can be mounted at their nodal points. Then mounting will not affect the vibration and the performance of these transformers.

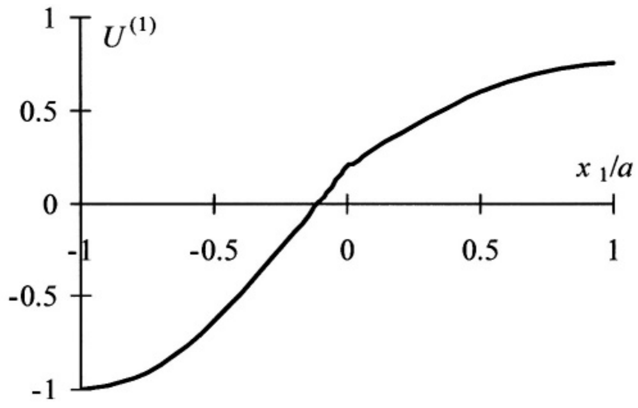

Figure 8.2-2. Normalized mechanical displacement $U^{(1)}$ of the first extensional mode.

Normalized $\Phi^{(1)}$ as a function of x_1 is shown in Figure 8.2-3. It is seen that $\Phi^{(1)}$ rises in the receiving portion [0, b]. This is why the present non-uniform ceramic rod can work as a transformer. We note that the rate of change of $\Phi^{(1)}$ is large near $x_1 = 0$ where the extensional stress is large. On the other hand, the rate of change is small near $x_1 = b$, which is a free end with vanishing extensional stress.

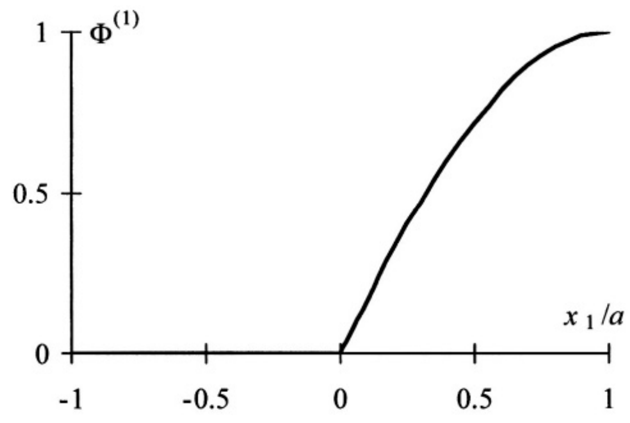

Figure 8.2-3. Normalized electric potential $\Phi^{(1)}$ of the first extensional mode.

Figure 8.2-4 shows the transforming ratio as a function of the driving frequency for two values of the load Z_L. The transforming ratio assumes its maximum as expected near the first resonance frequency (f = 36.35 kHz). This shows that the transformer is a resonant device operating at a particular frequency. The output voltage can be tens of times as high as the input voltage. When the load is larger, the output voltage is larger. At the same time, the output current is usually smaller.

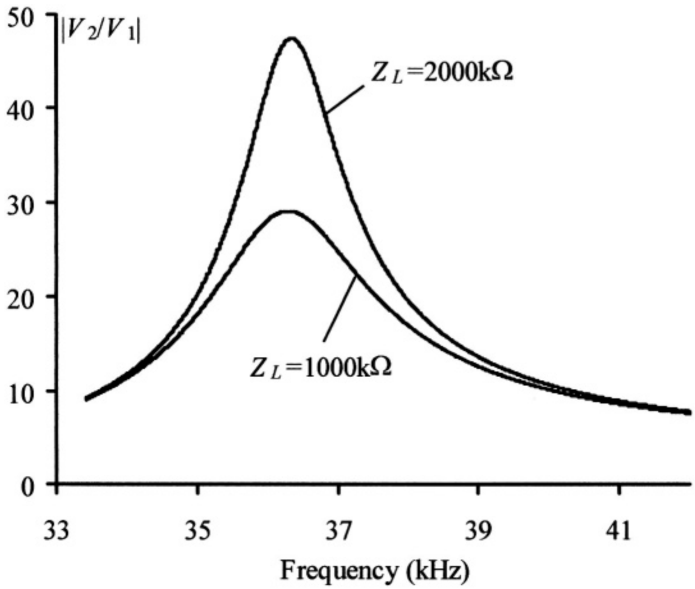

Figure 8.2-4. Transforming ratio versus driving frequency.

Figure 8.2-5 shows the transforming ratio as a function of the load for two values of the driving frequency that are close to the first resonance frequency. As the load increases from 0, the transforming ratio increases from 0 almost linearly. As the load increases further, there is a slower rate of increase in the transforming ratio. For very large loads, the transforming ratio is almost a constant, exhibiting a saturation type of behavior. Physically, for very large values of the load, the output electrodes are essentially open. In this case, the output voltage is saturated and the output current essentially vanishes.

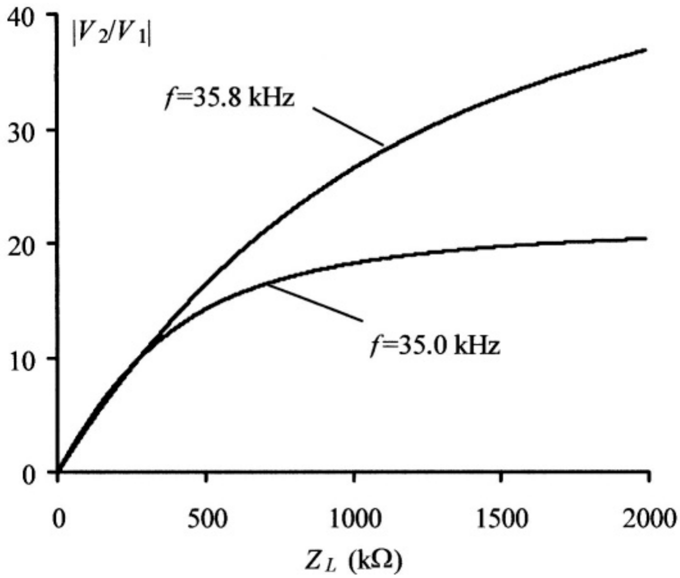

Figure 8.2-5. Transforming ratio versus load.

In Figure 8.2-6 the transforming ratio versus the aspect ratio *b/h* is plotted for different values of the load and driving frequency. The transforming ratio increases essentially linearly with *b/h*.

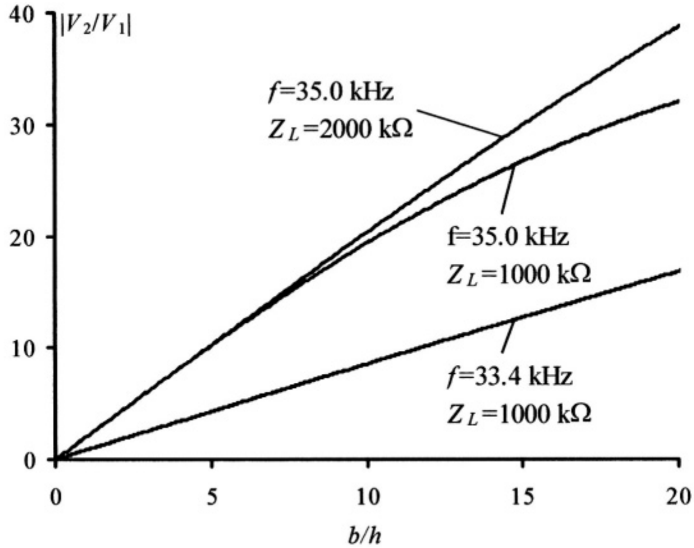

Figure 8.2-6. Transforming ratio versus *b/h*.

Figure 8.2-7 shows the efficiency as a function of the load. The figure shows two values of the driving frequency, one close to and the other far away from the first resonant frequency. As the load increases from 0, the efficiency first increases from 0 linearly and then reaches a maximum. After the maximum, the efficiency decreases monotonically. The efficiency-load curve is not sensitive to the driving frequency.

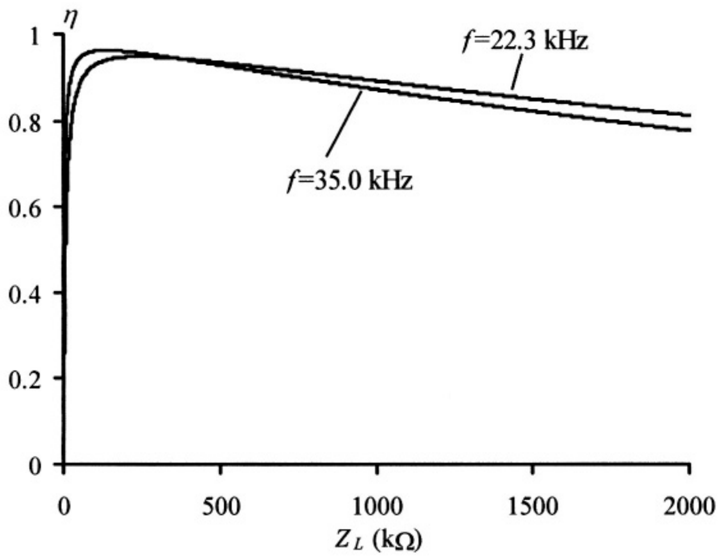

Figure 8.2-7. Efficiency versus load.

3. PRESSURE SENSORS

Piezoelectric crystals are often used to make pressure sensors. Piezoelectric pressure sensors may detect pressure either from pressure-induced voltage (or charge) or from frequency shifts. The thickness-stretch deformation of a ceramic plate analyzed in Section 2 of Chapter 3 shows how a pressure sensor may work. The voltage or charge developed in the plate is proportional to the normal surface traction, and can be used to detect the traction.

In this section, we analyze a pressure sensor that measures pressure by frequency shifts induced by the biasing fields due to the pressure. Consider the reference state of the structure shown in Figure 8.3-1 [61].

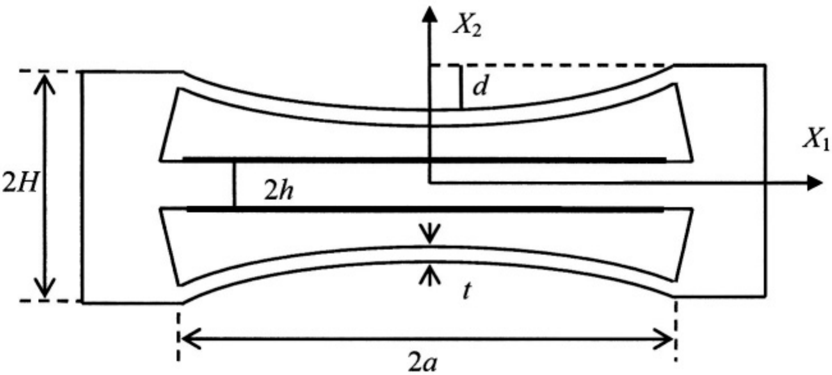

Figure 8.3-1. A pressure sensor.

The X_3 axis is normal to the paper determined by the right-hand rule. The shell is a cylindrical structure which is relatively long in the X_3 direction. Figure 8.3-1 shows a cross-section with a unit length in the X_3 direction. A quartz thickness-shear resonator of thickness $2h$ between two electrodes represented by the thick lines is sealed in the shell structure. External pressure on the shell is transmitted to the plate, causing biasing fields and hence frequency shifts in the resonator, which can then be used to measure the pressure. The shallow shell is part of a circular cylindrical shell. The shell is shallow in the sense that $a >> d$. The shell is also assumed to be very thin with $t << a$. The radius of the corresponding circular cylindrical shell (see Figure 8.3-2) can be found as

$$R = \frac{a^2 + d^2}{2d} >> a. \tag{8.3-1}$$

When the structure in Figure 8.3-1 is subject to a surrounding pressure p, extensional stresses develop in the shell. As an approximation we neglect bending moments and shear forces everywhere in the shell including the edges, and treat it as a membrane that does resist bending. The free body diagram of the lower piece of the shell when under pressure p is shown in Figure 8.3-2. The problem is statically determinate. The membrane extensional stress Q/t induced by p is simply the hoop stress for a circular cylindrical shell under uniform pressure

$$\frac{Q}{t} = \frac{pR}{t},$$
$$Q = pR = p\frac{a^2 + d^2}{2d} \cong \frac{pa^2}{2d}, \tag{8.3-2}$$

which is a large force when $a >> d$. The membrane stress in (8.3-2) compresses the piezoelectric plate axially and generates a pressure-induced initial stress in the resonator. Under the axial stress induced by p the resonator vibrates in a thickness-shear mode at a resonance frequency that is slightly perturbed from that of a stress-free resonator. In order to calculate this frequency shift and hence exhibit the mechanism of the pressure sensor, the theory in Chapter 6 for incremental vibrations superposed on biasing deformations is needed.

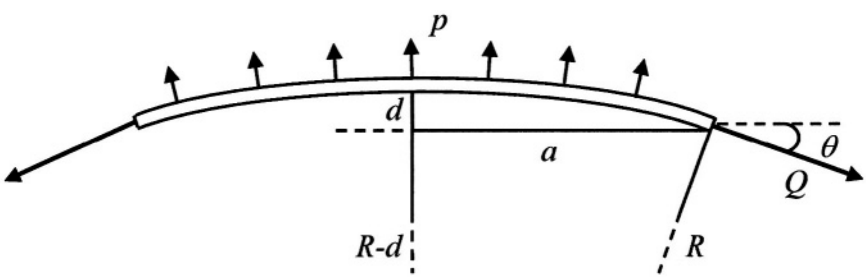

Figure 8.3-2. Free body diagram of the shell.

Consider the quartz plate alone. The stress-free reference configuration of the plate before the surrounding pressure p is applied is shown in Figure 8.3-3(a). The static biasing deformation is due to the surrounding pressure p on the shell or the related axial force N, and is shown in Figure 8.3-3(b). The incremental thickness-shear vibration is shown in Figure 8.3-3(c). We study motions independent of X_3, with $\partial/\partial X_3 = 0$ and $u_3 = 0$. Quartz is a material with very weak piezoelectric coupling. For a frequency analysis this coupling is neglected and an elastic analysis will be performed.

The biasing fields in the crystal plate caused by p through N are assumed to be small and are governed by the linear theory of elasticity. The two major surfaces of the plate are traction-free. The plate is under the axial compressional force N which depends on the pressure p as follows

$$N = 2Q\cos\theta + p2H \cong 2(Q + pH), \tag{8.3-3}$$

where the approximation is valid for a shallow shell with a small θ. The two terms on the right-hand side of (8.3-3) are due to the pressure p on the shells and the pressure that acts on the side walls of the structure with height $2H$. They are transferred to the crystal plate as N.

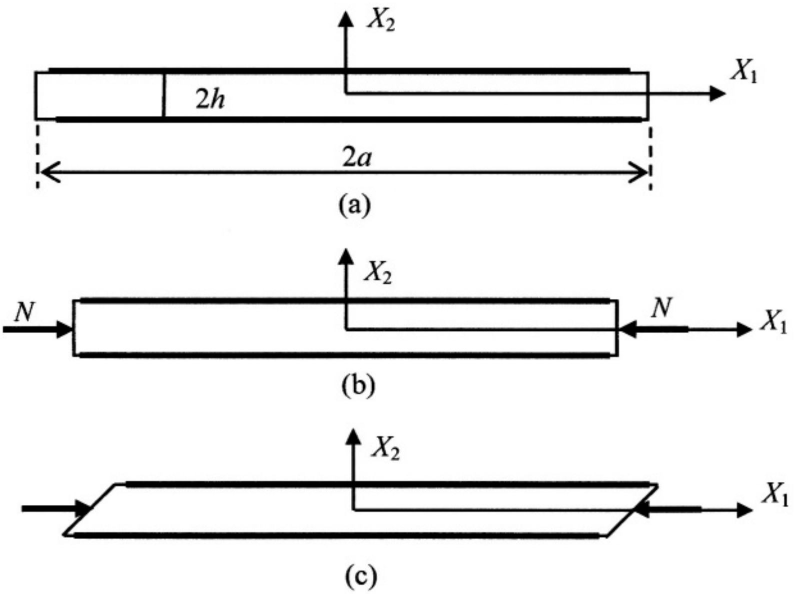

Figure 8.3-3. Reference (a), initial (b), and present (c) configurations of the quartz plate.

Consider a rotated Y-cut quartz plate which is of monoclinic symmetry. The deformations due to compression by N are governed by

$$\bar{c}_{11} w_{1,11} = 0, \quad -a < X_1 < a, \tag{8.3-4}$$

with boundary conditions

$$\bar{c}_{11} w_{1,1} = -N/2h, \quad X_1 = \pm a, \tag{8.3-5}$$

where $w_1 = w_1(X_1)$ is the displacement in the X_1 direction and $\bar{c}_{11} = c_{11} - c_{12}^2 / c_{22}$. From Equations (8.3-4) and (8.3-5) the initial strain in the X_1 direction is found to be

$$S_1^0 = w_{1,1} = -\frac{N}{2h\bar{c}_{11}}. \tag{8.3-6}$$

The related thickness expansion due to Poisson's effect can be obtained from the stress relaxation condition in the plate's normal direction as

$$S_2^0 = w_{2,2} = -\frac{c_{21}}{c_{22}} w_{1,1} = \frac{c_{21}}{c_{22}} \frac{N}{2h\bar{c}_{11}}. \tag{8.3-7}$$

Since the biasing deformations are uniform and the plate is thin with $a >> h$, for the incremental motion we neglect edge effects and consider motions independent of X_1. We study thickness-shear vibration in the X_1 direction with

$$u_1 = u_1(X_2, t), \quad u_2 = u_3 = 0\,. \tag{8.3-8}$$

The relevant incremental stress components are found to be

$$\begin{aligned} K^1_{21} &= c_{66}(1 + cS^0_1)u_{1,2}, \\ K^1_{22} &= (c_{261}S^0_1 + c_{262}S^0_2)u_{1,2}, \\ K^1_{23} &= (c_{461}S^0_1 + c_{462}S^0_2)u_{1,2}, \end{aligned} \tag{8.3-9}$$

where, under the compact matrix notation, c_{261}, c_{262}, c_{461}, and c_{462} are the third-order elastic constants and

$$c = 2 + \frac{c_{661}}{c_{66}} - \frac{c_{662}c_{21}}{c_{66}c_{22}}\,. \tag{8.3-10}$$

Since the third-order elastic constants are usually larger in value than the second-order elastic constants, K^1_{22} and K^1_{23} are not necessarily small compared to K^1_{21}. For rotated Y-cut quartz:

$$c_{261} = c_{262} = c_{461} = c_{462} = 0\,, \tag{8.3-11}$$

and hence $K^1_{22} = K^1_{23} = 0$. The equations of motion are

$$K^1_{21,2} = \rho_0 \ddot{u}_1\,, \quad K_{22,2} = 0, \quad K_{23,2} = 0\,. \tag{8.3-12}$$

Equations $(8.3\text{-}12)_{2,3}$ are trivially satisfied. Equation $(8.3\text{-}12)_1$ take the following form

$$c_{66}(1 + cS^0_1)u_{1,22} = \rho_0 \ddot{u}_1\,. \tag{8.3-13}$$

Consider odd thickness-shear modes which are excitable by a thickness electric field. Let

$$u_1(X_2, t) = u(t)\sin\frac{n\pi}{2h}X_2, \quad n = 1,3,5\cdots, \tag{8.3-14}$$

which satisfies the boundary conditions

$$K^1_{21}(X_2 = \pm h) = 0\,. \tag{8.3-15}$$

Substitution of (8.3-14) into (8.3-13) results in the following ordinary differential equation for $u(t)$,

$$\ddot{u} + \omega_0^2(1 + cS^0_1)u = 0\,, \tag{8.3-16}$$

where

$$\omega_0^2 = \frac{n^2\pi^2 c_{66}}{4h^2\rho_0}\,. \tag{8.3-17}$$

ω_0 is the resonance frequency of the n-th odd thickness-shear mode of the plate when the biasing fields are not present. Letting

$$u(t) = e^{i\omega t} \tag{8.3-18}$$

in Equation (8.3-16), we obtain the frequency

$$\omega = \omega_0 \sqrt{1 + cS_1^0} \cong \omega_0 (1 + \frac{cS_1^0}{2}), \tag{8.3-19}$$

which implies the following frequency shift due to pressure

$$\frac{\omega - \omega_0}{\omega_0} = \frac{cS_1^0}{2} = -\frac{c}{2}\frac{p}{\bar{c}_{11}}(\frac{a^2}{2dh} + \frac{H}{h}). \tag{8.3-20}$$

A few observations can be made from Equation (8.3-20). The frequency shift varies according to pressure and therefore can be used to measure the pressure. For sensor applications, the frequency shift should ideally be proportional to pressure (for small pressure). The frequency shift depends on a^2/dh, which is a large number. Thus a shallow shell magnifies the pressure and increases sensitivity. A thicker resonator or a more rigid resonator with a large h or a large c_{11} results in lower sensitivity. Under the assumption of the membrane state of thin shells, shell thickness does not have an effect on sensitivity.

Consider the fundamental thickness-shear mode corresponding to $n = 1$ in (8.3-14). Relative frequency shift versus pressure is shown in Figure 8.3-4 for different values of *a/d,* while $a/h = 20$ and $H/h = 5$ *are* fixed.

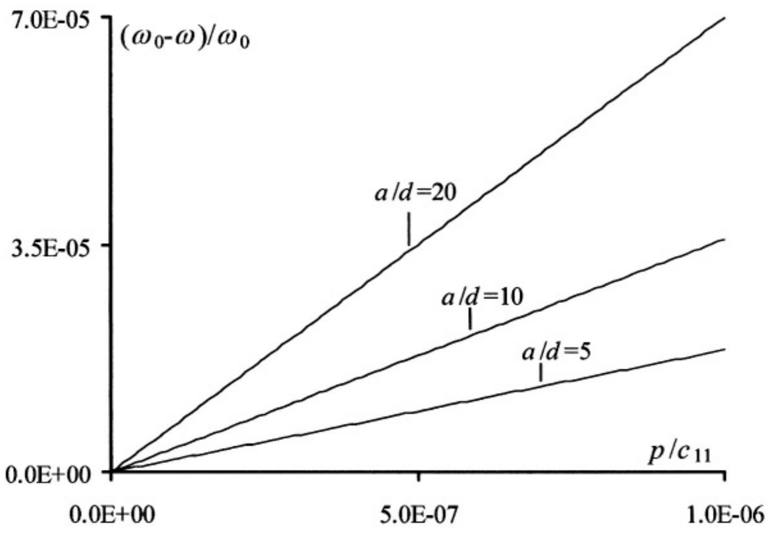

Figure 8.3-4. Frequency shift versus pressure for different values of *a/d.*

Figure 8.3-4 shows that for small pressure the frequency shift is proportional to pressure. The slope of the straight lines is related to sensitivity. Larger slopes represent higher sensitivity. Clearly, for larger values of *a/d,* i.e., shallower shells, the sensitivity is higher. The above analysis and the linear results in Figure 8.3-4 are for low pressure. For high pressure, the response is nonlinear.

The above analysis is based on the membrane theory of shells. When the shells become thicker, the effect of bending moment and transverse shear force cannot be ignored, especially near the edges of the shells.

4. TEMPERATURE SENSORS

A temperature change causes changes in the geometry of a piezoelectric body. The material constants are also temperature-dependent. Hence a temperature change causes shifts in resonance frequencies in a piezoelectric body. This effect can be used to make a thermometer. Consider a crystal plate connected to two elastic layers by rigid end walls as shown in Figure 8.4-1 [62].

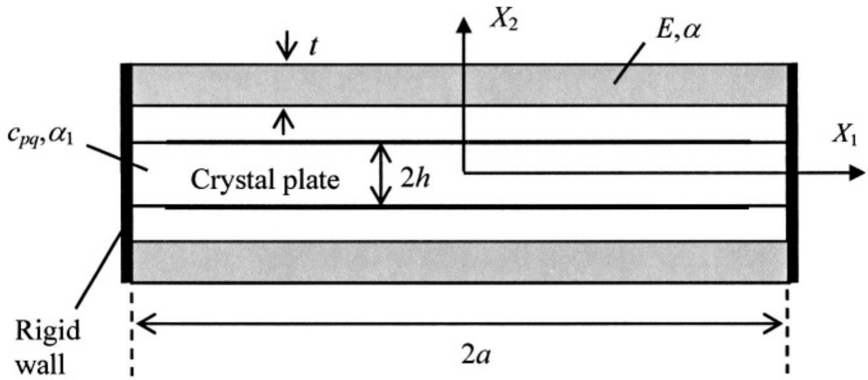

Figure 8.4-1. A crystal plate as a temperature sensor.

The structure is long in the X_3 direction, and Figure 8.4-1 shows a cross-section. The elastic layers are isotropic with Young's modulus E and thermal expansion coefficient α. The crystal plate is piezoelectric with two electrodes for vibration excitation represented by the thick lines at the top and bottom of the plate. The thermal expansion coefficients of the crystal plate and the elastic layers are different. Under a temperature change they expand differently if allowed to do so freely. Due to the rigid end connections, they have to be of the same length for all temperatures. Thus, under a temperature change, initial strains and stresses are developed in both the crystal plate and the elastic layers.

The operating mode of the crystal plate is the thickness-shear mode, with only one displacement component in the X_1 direction. The frequency is determined by the plate thickness. We need to study the thickness-shear vibration of a crystal plate with initial stresses and strains due to a temperature change and the constraint. The theory for small fields superposed on finite biasing fields in a thermoelectroelastic body is needed. Such a theory can be obtained by linearizing the nonlinear electroelastic equation in Section 4 of Chapter 7 in the manner of Chapter 6 about a thermoelectroelastic bias [63].

Consider a quartz plate of rotated Y-cut. Quartz has very weak piezoelectric coupling. The coupling is necessary for electrically forced vibrations. For free vibration frequency analysis, the weak piezoelectric coupling can be neglected and an elastic analysis is sufficient. The equations for small fields superposed on a thermomechanical bias in a thermoelastic body are summarized below.

4.1 Equations for Small Fields Superposed on a Thermal Bias

Consider the following three configurations of a thermoelastic body as shown in Figure 8.4-2.

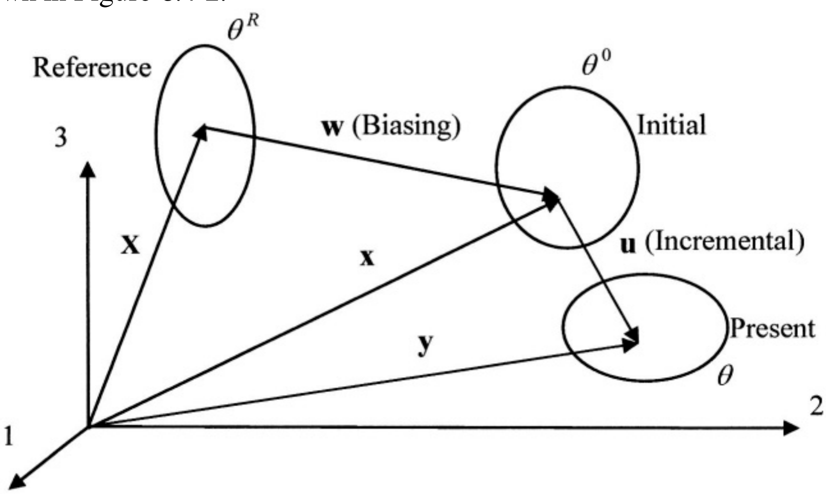

Figure 8.4-2. Reference, initial and present configurations of a thermoelastic body.

(i) The reference configuration: At this state the body is undeformed with a reference temperature of θ^R. A generic point is denoted by **X** with rectangular coordinates X_K. The mass density is denoted by ρ_0.

(ii) The initial configuration: In this state the body is deformed finitely and statically, with an initial temperature of θ^0. The position of the material point associated with **X** is given by $x_\alpha = x_\alpha(\mathbf{X})$. The initial displacement is given by $\mathbf{w} = \mathbf{x} - \mathbf{X}$. The initial deformations satisfy the following equations of static nonlinear thermoelasticity

$$\begin{aligned} &K^0_{LK,L} + \rho_0 f^0_k = 0, \\ &0 = -Q^0_{K,K} + \rho_0 \gamma^0. \end{aligned} \tag{8.4-1}$$

The above equations are adjoined by constitutive relations defined by the specification of the free energy ψ and the heat flux vector Q_K

$$\psi = \psi(S_{KL}, \theta), \quad Q_K = Q_K(S_{KL}, \theta, \Theta_K), \tag{8.4-2}$$

where Θ_K is the material temperature gradient. For the initial state

$$\begin{aligned} &K^0_{Lk} = x_{k,K}\rho_0 \frac{\partial \psi}{\partial S_{KL}}\bigg|_{S^0_{KL},\theta^0}, \quad Q^0_K = Q_K(S^0_{KL}, \theta^0, \Theta^0_K), \\ &S^0_{KL} = (x_{\alpha,K}x_{\alpha,L} - \delta_{KL})/2, \quad \Theta^0_K = \theta^0_{,K}. \end{aligned} \tag{8.4-3}$$

(iii) The present configuration: Time-dependent, small deformations are applied to the initial configuration of the deformed body. The final position of the material point associated with **X** is given by $y_i = y_i(\mathbf{X},t)$. The small, incremental displacement vector is denoted by **u**. The equations for the incremental fields are

$$\begin{aligned} &K^1_{K\alpha,K} + \rho_0 f^1_\alpha = \rho_0 \ddot{u}_\alpha, \\ &\rho_0(\theta^0 \dot{\eta}^1 + \theta^1 \dot{\eta}^0) = -Q^1_{K,K} + \rho_0 \gamma^1, \end{aligned} \tag{8.4-4}$$

where the effective linear constitutive relations for the small incremental stress tensor, entropy, and heat flux are

$$\begin{aligned} &K^1_{K\alpha} = G_{K\alpha L\gamma} u_{\gamma,L} - \rho_0 \Lambda_{K\alpha} \theta^1, \\ &\eta^1 = \Lambda_{L\gamma} u_{\gamma,L} + \alpha \theta^1, \\ &Q^1_K = A_{KM\alpha} u_{\alpha,M} + C_K \theta^1 + F_{KL} \theta^1_{,L}. \end{aligned} \tag{8.4-5}$$

In Equation (8.4-5), the effective material constants are defined by

$$G_{K\alpha L\gamma} = x_{\alpha,M}\,\rho_0 \left.\frac{\partial^2 \psi}{\partial S_{KM}\partial S_{LN}}\right|_{S_{KL}^0,\theta^0} x_{\gamma,N}$$

$$+ \rho_0 \left.\frac{\partial \psi}{\partial S_{KL}}\right|_{S_{KL}^0,\theta^0} \delta_{\alpha\gamma} = G_{L\gamma K\alpha},$$

$$\Lambda_{L\alpha} = -\left.\frac{\partial^2 \psi}{\partial S_{KL}\partial \theta}\right|_{S_{KL}^0,\theta^0} x_{\alpha,K}, \quad \alpha = -\left.\frac{\partial^2 \psi}{\partial \theta^2}\right|_{S_{KL}^0,\theta^0}, \tag{8.4-6}$$

$$A_{KM\alpha} = \left.\frac{\partial Q_K}{\partial S_{LM}}\right|_{S_{KL}^0,\theta^0,\Theta_K^0} x_{\alpha,L}, \quad C_K = \left.\frac{\partial Q_K}{\partial \theta}\right|_{S_{KL}^0,\theta^0,\Theta_K^0},$$

$$F_{KL} = \left.\frac{\partial Q_K}{\partial \Theta_L}\right|_{S_{KL}^0,\theta^0,\Theta_K^0}.$$

High frequency incremental motions can be assumed to be isentropic for which $\eta^1 = 0$. Then from $(8.4\text{-}5)_2$ the incremental temperature field θ^1 can be obtained and substituted into $(8.4\text{-}5)_1$, resulting in

$$K_{K\alpha}^1 = \overline{G}_{K\alpha L\gamma} u_{\gamma,L}, \tag{8.4-7}$$

where the isentropic effective constants are

$$\overline{G}_{K\alpha L\gamma} = G_{K\alpha L\gamma} + \rho_0 \frac{\Lambda_{K\alpha}\Lambda_{L\gamma}}{\alpha} = \overline{G}_{L\gamma K\alpha}. \tag{8.4-8}$$

In many applications the biasing deformations are small. In this case, only their linear effects on the incremental fields need to be considered. Then the following free energy density is sufficient:

$$\rho_0\psi(S_{KL},\theta) = -a(\theta) - f_{AB}(\theta)S_{AB}$$
$$+ \frac{1}{2}c_{ABCD}(\theta)S_{AB}S_{CD} + \frac{1}{6}c_{ABCDEF}(\theta)S_{AB}S_{CD}S_{EF}, \tag{8.4-9}$$

where a and f_{AB} are related to the specific heat and the thermoelastic constants. Then

$$G_{K\alpha L\gamma}(\theta^0) = c_{K\alpha L\gamma}(\theta^0) + \hat{c}_{K\alpha L\gamma}(\theta^0),$$

$$\hat{c}_{K\alpha L\gamma}(\theta^0) = c_{K\alpha LN}(\theta^0)w_{\gamma,N} + c_{KNL\gamma}(\theta^0)w_{\alpha,N}$$
$$+ c_{K\alpha L\gamma AB}(\theta^0)S_{AB}^0 + T_{KL}^0\delta_{\alpha\gamma}, \tag{8.4-10}$$

$$T_{KL}^0 = c_{KLMN}(\theta^0)S_{MN}^0 - f_{KL}(\theta^0), \quad S_{AB}^0 = (w_{A,B} + w_{B,A})/2.$$

For sensor applications the linear effects of the initial temperature variation θ^0 - θ^R are of main interest. Then, Equation (8.4-10) can be approximated to the first order for all of the mechanical and thermal biasing fields by

$$G_{K\alpha L\gamma}(\theta^0) \cong c_{K\alpha L\gamma}(\theta^R) + \left.\frac{\partial c_{K\alpha L\gamma}}{\partial \theta}\right|_{\theta^R} (\theta^0 - \theta^R)$$
$$+ c_{K\alpha LN}(\theta^R) w_{\gamma,N} + c_{KNL\gamma}(\theta^R) w_{\alpha,N} + c_{K\alpha L\gamma AB}(\theta^R) S^0_{AB} \tag{8.4-11}$$
$$+ [c_{KLMN}(\theta^R) S^0_{MN} - \lambda_{KL}(\theta^R)(\theta^0 - \theta^R)]\delta_{\alpha\gamma},$$

where

$$\lambda_{KL}(\theta) = \frac{df_{KL}}{d\theta} \tag{8.4-12}$$

are the thermoelastic constants.

4.2 Analysis of a Temperature Sensor

We now use the above equations for incremental vibrations superposed on a thermomechanical bias to analyze [62] the temperature sensor in Figure 8.4-1. The reference configuration of the crystal plate in Figure 8.4-1 is redrawn in Figure 8.4-3(a). The biasing deformation is due to a constrained thermal expansion which is shown in Figure 8.4-3(b). The incremental motion is the thickness-shear vibration shown in Figure 8.4-3(c).

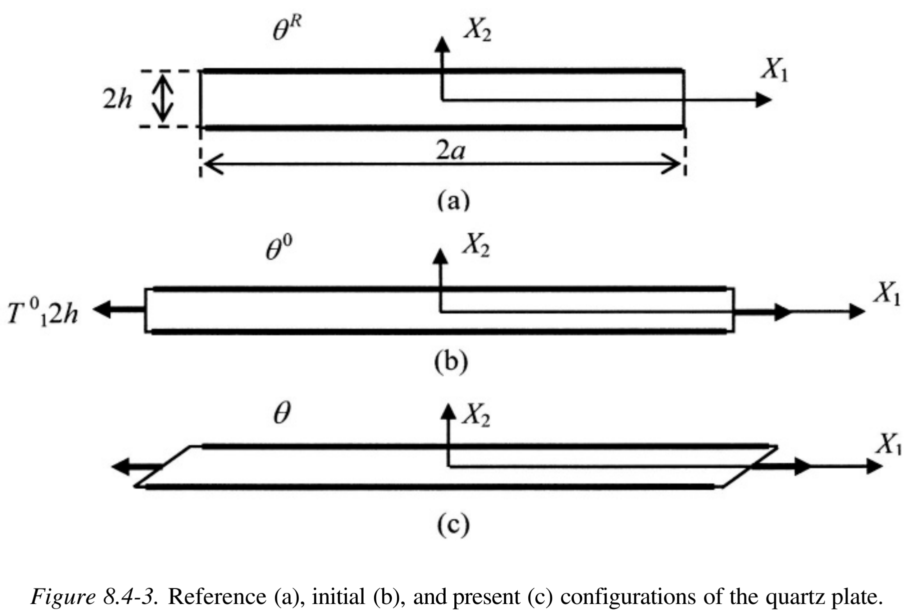

Figure 8.4-3. Reference (a), initial (b), and present (c) configurations of the quartz plate.

Consider plane-strain motions independent of X_3, with $\partial/\partial X_3 = 0$, $w_3 = 0$, and $u_3 = 0$. The biasing fields are assumed to be small and are governed by the linear theory of thermoelasticity. The relevant constitutive relations for plane-strain deformations are

$$\begin{aligned} T_1^0 &= c_{11}S_1^0 + c_{12}S_2^0 - \lambda_1 \Delta\theta, \\ T_2^0 &= c_{21}S_1^0 + c_{22}S_2^0 - \lambda_1 \Delta\theta, \end{aligned} \qquad (8.4\text{-}13)$$

where, under the compact matrix notation, T_1^0 and T_2^0 are the initial extensional stresses in the X_1 and X_2 directions, and S_1^0 and S_2^0 are the corresponding initial strains. In Equation (8.4-13) we have denoted $\Delta\theta = \theta^0 - \theta^R$. The plate is assumed to be thin with $T_2^0 = 0$. We can then solve for S_2^0 from $(8.4\text{-}13)_2$ and substitute the resulting expression into $(8.4\text{-}13)_1$ to obtain the following one-dimensional constitutive relation

$$\begin{aligned} S_1^0 &= \frac{1}{\bar{c}_{11}} T_1^0 + \alpha_1 \Delta\theta, \\ \bar{c}_{11} &= c_{11} - c_{12}^2 / c_{22}, \\ \bar{\lambda}_1 &= \lambda_1 (1 - c_{12}/c_{22}), \quad \alpha_1 = \bar{\lambda}_1 / \bar{c}_{11}. \end{aligned} \qquad (8.4\text{-}14)$$

The relevant constitutive relation for the elastic layers is

$$S_1^0 = \frac{1}{E} T_1^0 + \alpha \Delta\theta. \qquad (8.4\text{-}15)$$

The solution to the constrained thermal expansion problem of the structure in Figure 8.4-1 is

$$\begin{aligned} T_1^0 &= \frac{Et\bar{c}_{11}}{Et + \bar{c}_{11}h}(\alpha - \alpha_1)\Delta\theta, \\ w_{1,1} &= S_1^0 = \frac{\alpha Et + \alpha_1 \bar{c}_{11} h}{Et + \bar{c}_{11}h}\Delta\theta, \\ w_{2,2} &= S_2^0 = -\frac{c_{21}}{c_{22}} E_1^0 + \frac{\lambda_1}{c_{22}}\Delta\theta. \end{aligned} \qquad (8.4\text{-}16)$$

Since the biasing deformations are uniform and the plate is long and thin, for the incremental motion we neglect edge effects and consider motions independent of X_1. We study thickness-shear vibration in the X_1 direction with

$$u_1 = u_1(X_2,t), \quad u_2 = u_3 = 0. \tag{8.4-17}$$

The isentropic modification of the effective elastic constants given by (8.4-8) is a very small modification for quartz. Therefore in the following example we will neglect the modification. The relevant stress components for the incremental fields are found to be

$$\begin{aligned} K_{21}^1 &= \{c_{66}(\theta^0) + [2c_{66}(\theta^R) + c_{661}(\theta^R)]S_1^0 + c_{662}(\theta^R)S_2^0\}u_{1,2} \\ &= c_{66}(\theta^R)(1 + 2\overline{\alpha}\Delta\theta)u_{1,2}, \\ K_{22}^1 &= [c_{261}(\theta^R)S_1^0 + c_{262}(\theta^R)S_2^0]u_{1,2}, \\ K_{23}^1 &= [c_{461}(\theta^R)S_1^0 + c_{462}(\theta^R)S_2^0]u_{1,2}, \end{aligned} \tag{8.4-18}$$

where, under the compact matrix notation, c_{261}, c_{262}, c_{461}, and c_{462} are the third-order elastic constants and

$$\begin{aligned} 2\overline{\alpha} &= \frac{1}{c_{66}(\theta^R)} \frac{dc_{66}}{d\theta}\bigg|_{\theta^R} \\ &+ \frac{2c_{66}(\theta^R) + c_{661}(\theta^R)}{c_{66}(\theta^R)} \frac{\alpha(\theta^R)E(\theta^R)t + \alpha_1(\theta^R)\overline{c}_{11}(\theta^R)h}{E(\theta^R)t + \overline{c}_{11}(\theta^R)h} \\ &+ \frac{c_{662}(\theta^R)}{c_{66}(\theta^R)}\left[-\frac{c_{21}(\theta^R)}{c_{22}(\theta^R)} \frac{\alpha(\theta^R)E(\theta^R)t + \alpha_1(\theta^R)\overline{c}_{11}(\theta^R)h}{E(\theta^R)t + \overline{c}_{11}(\theta^R)h} + \frac{\lambda_1(\theta^R)}{c_{22}(\theta^R)}\right]. \end{aligned} \tag{8.4-19}$$

As a special case, note that for an unconstrained plate we have, by setting $E = 0$ in (8.4-19),

$$\begin{aligned} 2\overline{\alpha} &= \frac{1}{c_{66}(\theta^R)} \frac{dc_{66}}{d\theta}\bigg|_{\theta^R} + \frac{2c_{66}(\theta^R) + c_{661}(\theta^R)}{c_{66}(\theta^R)} + \alpha_1(\theta^R) \\ &+ \frac{c_{662}(\theta^R)}{c_{66}(\theta^R)}\left[-\frac{c_{21}(\theta^R)}{c_{22}(\theta^R)}\alpha_1(\theta^R) + \frac{\lambda_1(\theta^R)}{c_{22}(\theta^R)}\right]. \end{aligned} \tag{8.4-20}$$

Since the third-order elastic constants are usually larger in value than the second-order elastic constants, K_{22}^1 and K_{23}^1 are not necessarily small compared to K_{21}^1. For rotated Y-cut quartz,

$$c_{261} = c_{262} = c_{461} = c_{462} = 0, \tag{8.4-21}$$

and hence $K^1_{22} = K^1_{23} = 0$. The equations of motion are

$$K^1_{21,2} = \rho_0 \ddot{u}_1, \quad K^1_{22,2} = 0, \quad K^1_{23,2} = 0. \tag{8.4-22}$$

Equations $(8.4\text{-}22)_{2,3}$ are trivially satisfied. Equation $(8.4\text{-}22)_1$ takes the following form

$$c_{66}(\theta^R)(1 + 2\bar{\alpha}\Delta\theta)u_{1,22} = \rho_0 \ddot{u}_1. \tag{8.4-23}$$

Consider the following odd thickness-shear modes which are excitable by a thickness electric field

$$u_1(X_2, t) = u(t)\sin\frac{n\pi}{2b}X_2, \quad n = 1,3,5\cdots, \tag{8.4-24}$$

which satisfy the boundary conditions

$$K^1_{21}(X_2 = \pm b) = 0. \tag{8.4-25}$$

Substitution of (8.4-24) into (8.4-23) yields the following ordinary differential equation for *u(t)*

$$\ddot{u} + \omega_0^2(1 + 2\bar{\alpha}\Delta\theta)u = 0, \tag{8.4-26}$$

where

$$\omega_0^2 = \frac{n^2\pi^2 c_{66}(\theta^R)}{4b^2\rho_0}. \tag{8.4-27}$$

ω_0 is the resonance frequency of the *n*-th odd thickness-shear mode of the crystal plate when the biasing fields are not present. Letting

$$u(t) = Ue^{i\omega t}, \tag{8.4-28}$$

in Equation (8.4-26), where U is a constant, we obtain the resonance frequency as

$$\omega = \omega_0\sqrt{1 + 2\bar{\alpha}\Delta\theta} \cong \omega_0(1 + \bar{\alpha}\Delta\theta), \tag{8.4-29}$$

which implies the following frequency shift due to a temperature change

$$\frac{\omega - \omega_0}{\omega_0} = \bar{\alpha}\Delta\theta. \tag{8.4-30}$$

For a numerical example, consider an AT-cut quartz plate. The temperature derivative of the relevant material constant c_{66} at 25°C is

$$\frac{1}{c_{66}}\frac{dc_{66}}{d\theta} = 126.7\times10^{-6}\ /^{\circ}\mathrm{C}. \tag{8.4-31}$$

The thermal expansion coefficient of quartz is

$$\alpha_1 = 13.71\times10^{-6}\ /^{\circ}\mathrm{C}. \tag{8.4-32}$$

For the plate thickness we choose h = 0.9696 mm. For thickness-shear vibration without biasing deformations, the fundamental thickness-shear frequency (n =1 in (8.4-27)) is $\omega_0/2\pi = 10^6$ Hz. We want to study effects of the constraints on the thermally induced frequency shift.

Figure 8.4-4 shows the frequency shift versus temperature change for different values of α/α_1, ratio of the thermal expansion coefficients of the crystal plate and the elastic layers. The frequency shift is linear in temperature variation. This is ideal for a temperature sensor. This linearity assumes a small temperature variation. The slopes of the straight lines in the figure are related to the sensitivity of the sensor. It can be seen that the sensitivity is not sensitive to α/α_1.

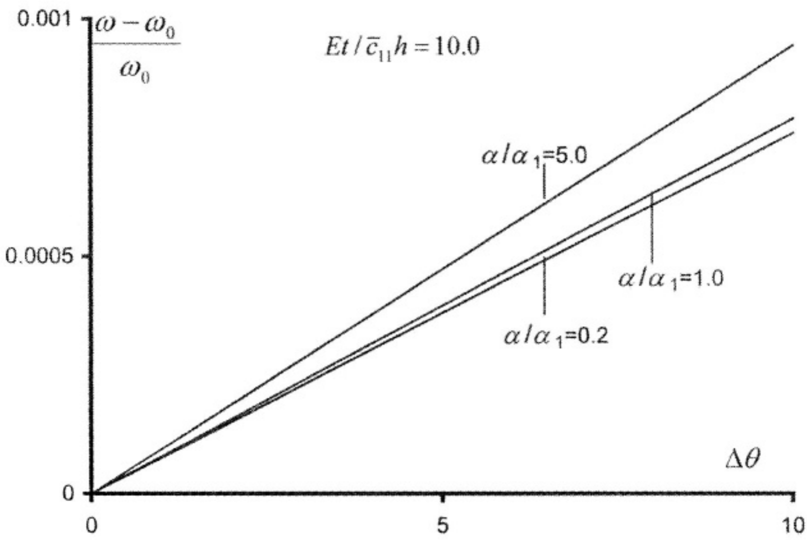

Figure 8.4-4. Frequency shift versus temperature change for different α/α_1.

Figure 8.4-5 shows the frequency shift versus temperature change for different values of the ratio of extensional stiffness of the crystal plate and the elastic layers, $Et/\bar{c}_{11}h$. Again, the frequency shift is insensitive to the stiffness ratio.

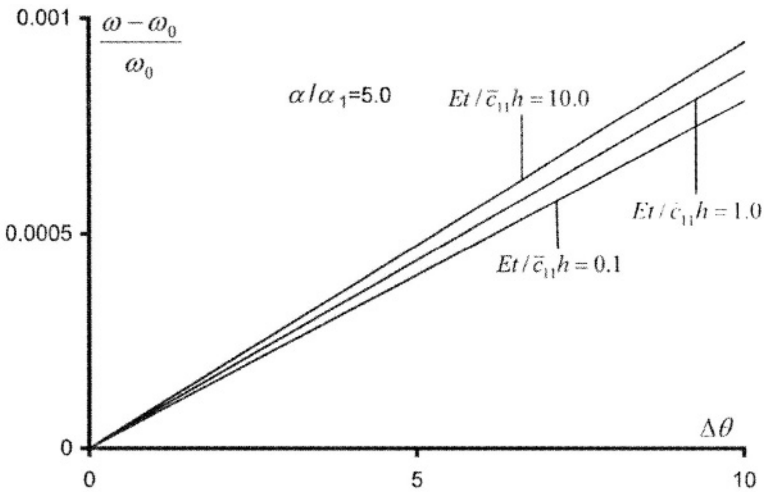

Figure 8.4-5. Frequency shift versus temperature change for different $Et/\bar{c}_{11}h$.

A close examination of the numerical procedure shows that the first term on the right-hand side of (8.4-19), i.e., the dependence of the elastic constant c_{66} on temperature, is dominant. This term does not vary with the constraint. Therefore the sensitivity of this particular sensor does not vary much when a plate is in constraint or free thermal expansion.

5. VIBRATION SENSITIVITY OF A RESONATOR

Piezoelectric crystal resonators are key components of telecommunication, timekeeping, and navigation equipment. A resonator is a resonant device operating at a desired frequency. When a resonator is mounted on objects under constant acceleration (missiles, satellites, etc.), acceleration-induced inertial forces cause biasing deformations and hence frequency shifts in the resonator. The acceleration sensitivity of a resonator is defined as a vector $\vec{\Gamma}$. When the resonator is under a constant acceleration $\vec{a}$, the sensitivity defines the acceleration perturbed frequency by $\omega(\vec{a})=\omega_0(1+\vec{\Gamma}\cdot\vec{a})$, where ω_0 is the unperturbed frequency. Calculations of acceleration sensitivity and its minimization have been among the main issues in resonator design for a long time. Acceleration sensitivity can be calculated by the theory of small fields superposed on finite biasing fields in an electroelastic body and the corresponding perturbation theory. The research on acceleration sensitivity is very active, with military requirements driving the need to advance from the 10^{-10}/g production technology available today to the 10^{-12}/g or better in the near future.

In certain applications, the acceleration is also time-dependent (vibration). In the case of a time-harmonic acceleration with frequency ω_v, the acceleration can be written as $\vec{a}\cos\omega_v t$, where $\vec{a}$ is a constant vector. In this case, it is more appropriate to consider the vibration sensitivity. When the frequency of the vibration is much lower than the operating frequency of the resonator, that is, $\omega_v << \omega_0$, it has been assumed that the acceleration sensitivity $\vec{\Gamma}$ calculated from a constant acceleration analysis is still applicable. The frequency of a resonator under a tune-dependent acceleration is then approximately given by $\omega(\vec{a}) = \omega_0(1 + \vec{\Gamma}\cdot\vec{a}\cos\omega_v t)$. This approach may be called a quasistatic analysis. Mathematically, since the vibration induced biasing fields are time-dependent, the equations for the incremental motion have time-dependent coefficients. To predict the behavior of the solution of an equation with time-dependent coefficients, we need a dynamic approach. In this case, the results can be more complicated than with a quasistatic approach.

In this section, the problem of resonator vibration sensitivity is analyzed directly from the equations for small fields superposed on time-dependent biasing fields [64]. Consider a thin crystal plate as shown in Figure 8.5-1 (a), with $a >> b$.

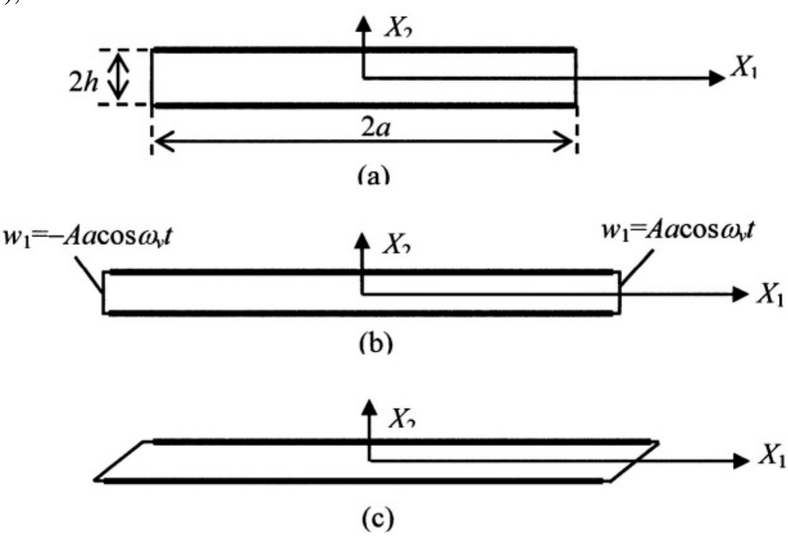

Figure 8.5-1. The reference configuration (a), the biasing extensional deformation (b), and the incremental thickness-shear deformation (c) of a thin crystal plate.

For plane-stain motions, the fields are independent of X_3 and $w_3 = u_3 = 0$. The two major surfaces of the plate are traction-free. The two ends of the plate at $X_1 = \pm a$ move with prescribed extensional displacements $\pm Aa\cos\omega_v t$, where A is a dimensionless number and ω_v is the vibration frequency. We assume that the biasing fields are infinitesimal: hence A is a very small number ($A << 1$). The plate is driven into extensional motions under the prescribed end displacements (see Figure 8.5-1(b)). Since the plate is thin, $T^0_2 = 0$ is taken to be approximately true throughout. The equation for extensional motions takes the following form:

$$\bar{c}_{11} w_{1,11} = \rho_0 \ddot{w}_1, \quad -a < X_1 < a, \tag{8.5-1}$$

with boundary conditions

$$w_1(X_1 = \pm a) = \pm Aa\cos\omega_v t, \tag{8.5-2}$$

where $w_1 = w_1(X_1,t)$ and $\bar{c}_{11} = c_{11} - c_{12}^2 / c_{22}$. Once the extensional displacement is found, the related thickness expansion or contraction due to Poisson's effect can be obtained from the plate stress relaxation condition as

$$w_{2,2} = -\frac{c_{21}}{c_{22}} w_{1,1}. \tag{8.5-3}$$

Consider low frequency forcing with ω_v much lower than the resonance frequency of the first extensional mode of the plate, i.e., $\omega_v << \omega_{(1)} = \frac{\pi}{2a}\sqrt{\frac{\bar{c}_{11}}{\rho_0}}$. Then the inertial term $\rho_0 \ddot{w}_1$ in (8.5-1) may be neglected. The solution to (8.5-1) and (8.5-2) is

$$w_1 = AX_1 \cos\omega_v t, \quad w_2 = -\frac{c_{21}}{c_{22}} AX_2 \cos\omega_v t, \tag{8.5-4}$$

which produces the following spatially homogeneous, time-dependent biasing strains:

$$S_1^0 = A\cos\omega_v t, \quad S_2^0 = -\frac{c_{21}}{c_{22}} A\cos\omega_v t. \tag{8.5-5}$$

Since the biasing deformations are uniform and the plate is assumed to be thin, for the incremental motion the edge effects are neglected and motions are assumed to be independent of X_1. For thickness-shear vibration in the X_1 direction (see Figure 8.5-1(c)),

$$u_1 = u_1(X_2,t), \quad u_2 = u_3 = 0. \tag{8.5-6}$$

The relevant stress components are found to be

$$K_{21}^{0}=c_{66}(1+\varepsilon\cos\omega_v t)u_{1,2},$$
$$K_{22}^{0}=(c_{261}S_1^0+c_{262}S_2^0)u_{1,2},\qquad(8.5\text{-}7)$$
$$K_{23}^{0}=(c_{461}S_1^0+c_{462}S_2^0)u_{1,2},$$

where, under the compact matrix notation, c_{261}, c_{262}, c_{461}, and c_{462} are the third-order elastic constants. ε is a small ($\varepsilon \ll 1$), dimensionless number given by

$$\varepsilon=(2+\frac{c_{661}}{c_{66}}-\frac{c_{662}c_{21}}{c_{66}c_{22}})A.\qquad(8.5\text{-}8)$$

Since the third-order elastic constants are usually larger in value than the second-order elastic constants, $K^1{}_{22}$ and $K^1{}_{23}$ are not necessarily small compared to $K^1{}_{21}$. For rotated Y-cut quartz,

$$c_{261}=c_{262}=c_{461}=c_{462}=0,\qquad(8.5\text{-}9)$$

and hence $K^1{}_{22}=K^1{}_{23}=0$. The equations of motion are

$$K_{21,2}^{1}=\rho_0\ddot{u}_1,\quad K_{22,2}^{1}=0,\quad K_{23,2}^{1}=0.\qquad(8.5\text{-}10)$$

Equations $(8.5\text{-}10)_{2,3}$ are trivially satisfied. Equation $(8.5\text{-}10)_1$ takes the following form

$$c_{66}(1+\varepsilon\cos\omega_v t)u_{1.22}=\rho_0\ddot{u}_1.\qquad(8.5\text{-}11)$$

For odd thickness-shear modes that are excitable by a thickness electric field, let

$$u_1(X_2,t)=u(t)\sin\frac{n\pi}{2b}X_2,\quad n=1,3,5\cdots,\qquad(8.5\text{-}12)$$

which satisfies the boundary conditions

$$K_{21}^{1}(X_2=\pm b)=0.\qquad(8.5\text{-}13)$$

Substituting (8.5-12) into (8.5-11) gives the following ordinary differential equation for $u(t)$:

$$\ddot{u}+\omega_0^2(1+\varepsilon\cos\omega_v t)u=0,\qquad(8.5\text{-}14)$$

where

$$\omega_0^2=\frac{n^2\pi^2c_{66}}{4b^2\rho_0}.\qquad(8.5\text{-}15)$$

ω_0 is the resonant frequency of the n-th odd thickness-shear mode of the plate when the biasing fields are not present. Equation (8.5-14) is the well-known Mathieu equation.

Consider the case when the frequency of the biasing fields is much lower than the operating frequency of the resonator, i.e., $\omega_v << \omega_0$. We seek solutions to (8.5-14) in the following form

$$u(t) = \alpha(t)\cos[\omega_0 t + \beta(t)], \tag{8.5-16}$$

where $\alpha(t)$ and $\beta(t)$ are dimensionless functions. They are slowly varying in the following sense

$$\dot{\alpha} << \omega_0, \quad \dot{\beta} << \omega_0. \tag{8.5-17}$$

Differentiating (8.5-16) with respect to time twice, substituting the expressions for u and $\ddot{u}$ into (8.5-14), setting the sum of the coefficients of the terms with $\sin(\omega_0 t+\beta)$ and the sum of the coefficients of the terms with $\cos(\omega_0 t+\beta)$ to zero separately, we obtain the following two equations for $\alpha(t)$ and $\beta(t)$:

$$\begin{aligned} &\alpha\ddot{\beta} + 2\dot{\alpha}\dot{\beta} + 2\dot{\alpha}\omega_0 = 0, \\ &\ddot{\alpha} - 2\omega_0\alpha\dot{\beta} - \alpha\dot{\beta}\dot{\beta} + \varepsilon\alpha\omega_0^2 \cos\omega_v t = 0. \end{aligned} \tag{8.5-18}$$

Under Equation (8.5-17), we neglect terms associated with the second derivatives of $\alpha(t)$ and $\beta(t)$ or the product of their first derivatives in (8.5-18) and obtain the following approximate system of first-order equations:

$$\begin{aligned} &2\dot{\alpha}\omega_0 = 0, \\ &-2\omega_0\alpha\dot{\beta} + \varepsilon\alpha\omega_0^2 \cos\omega_v t = 0. \end{aligned} \tag{8.5-19}$$

Equation (8.5-19) shows that to the first order, α is a constant and the corresponding β is found to be

$$\beta(t) = \frac{\omega_0}{2\omega_v}\varepsilon\sin\omega_v t + \gamma, \tag{8.5-20}$$

where γ is an integration constant. Equation (8.5-20) shows that $\beta(t)$ is indeed slowly varying in the sense of (8.5-17) due to the smallness of ε. Within the first-order approximation

$$u(t) = \alpha\cos(\omega_0 t + \frac{\omega_0}{2\omega_v}\varepsilon\sin\omega_v t + \gamma), \tag{8.5-21}$$

which is a frequency modulated signal. The modulated frequency may be defined as

$$\omega(t) = \frac{d}{dt}[\omega_0 t + \beta(t)] = \omega_0(1 + \frac{\varepsilon}{2}\cos\omega_v t). \tag{8.5-22}$$

For a more refined solution that shows the time dependence of α and a more refined expression for β, some second-order terms in (8.5-18) need to be kept. The first-order solution shows that although both $\alpha(t)$ and $\beta(t)$ are slowly varying, $\alpha(t)$ is even slower than $\beta(t)$. Therefore we drop $\ddot{\alpha}$ but keep $\dot{\beta}\dot{\beta}$ (which was dropped in the first-order approximation) in $(8.5\text{-}18)_2$ and obtain

$$-2\omega_0\dot{\beta} - \dot{\beta}\dot{\beta} + \varepsilon\omega_0^2 \cos\omega_v t = 0 . \tag{8.5-23}$$

From (8.5-23) we have

$$\dot{\beta} = -\omega_0 + \omega_0\sqrt{1+\varepsilon\cos\omega_v t} . \tag{8.5-24}$$

Equation (8.5-24) implies the following approximate expression of the frequency

$$\begin{aligned} \omega &= \omega_0 + \dot{\beta} = \omega_0\sqrt{1+\varepsilon\cos\omega_v t} \\ &= \omega_0(1+\frac{\varepsilon}{2}\cos\omega_v t - \frac{\varepsilon^2}{8}\cos^2\omega_v t + \cdots), \end{aligned} \tag{8.5-25}$$

which is more refined than (8.5-22). Multiplying $(8.5\text{-}18)_1$ by $\alpha(t)$ gives

$$\alpha^2\ddot{\beta} + 2\alpha\dot{\alpha}\dot{\beta} + 2\alpha\dot{\alpha}\omega_0 = 0 , \tag{8.5-26}$$

which can be written as

$$\frac{d}{dt}(\alpha^2\dot{\beta} + \alpha^2\omega_0) = 0 . \tag{8.5-27}$$

Integration of (8.5-27) yields

$$\alpha^2\dot{\beta} + \alpha^2\omega_0 = \lambda^2\omega_0 , \tag{8.5-28}$$

where λ is an integration constant. Solving (8.5-28) for $\alpha(t)$, we obtain

$$\alpha^2 = \frac{\lambda^2}{1+\dot{\beta}/\omega_0} . \tag{8.5-29}$$

To the lowest order, we have, approximately

$$\alpha = \lambda(1 - \frac{\dot{\beta}}{2\omega_0}) = \frac{\lambda}{2}(3 - \sqrt{1+\varepsilon\cos\omega_v t}) , \tag{8.5-30}$$

which is indeed much slower than $\beta(t)$. Equations (8.5-25) and (8.5-30) show that both the frequency and the amplitude of $u(t)$ are modulated.

As a numerical example, consider a Y-cut quartz resonator. For geometric parameters we choose a = 14.25 mm and b = 0.9696 mm. The biasing deformation is specified by A = 1. Then ε = 0.4221. The above parameters imply that for the extensional biasing deformation, $\omega_{(1)}/2\pi = 10^5$ Hz. For the thickness-shear motion without biasing deformation, the fundamental thickness-shear frequency (n = 1 in (8.5-15)) is $\omega_0/2\pi = 10^6$ Hz. We choose $\omega_v/2\pi = 10^5$ Hz. We are considering a very small effect. The above parameters are chosen to greatly exaggerate this small effect. Therefore some of the parameters do not quite satisfy the assumptions that $A \ll 1$, $\varepsilon \ll 1$, $\omega_v \ll \omega_{(1)}$ and $\omega_v \ll \omega_0$. The zero-, first-, and approximate second-order solutions are summarized below:

$$u(t) = \cos \omega_0 t,$$

$$u(t) = \cos(\omega_0 t + \frac{\omega_0}{2\omega_v} \varepsilon \sin \omega_v t), \tag{8.5-31}$$

$$u(t) = \frac{1}{2}(3 - \sqrt{1 + \varepsilon \cos \omega_v t}) \cos(\omega_0 t + \frac{\omega_0}{2\omega_v} \varepsilon \sin \omega_v t).$$

where we have set λ = 1 and γ = 0. Equation $(8.5\text{-}31)_1$ is the zero-order solution when the biasing field is not present, which is a pure sinusoidal signal. Equation $(8.5\text{-}31)_2$ shows the first-order effect of the biasing field. This is a frequency modulated signal as shown in Figure 8.5-2. This is the same as the result from the quasistatic analysis.

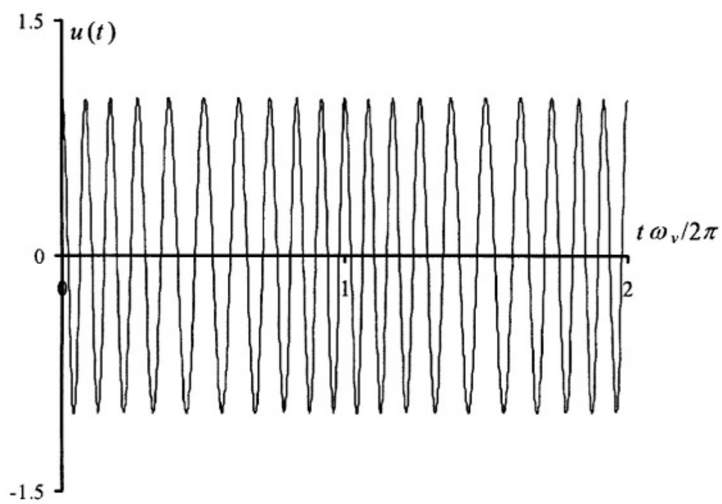

Figure 8.5-2. First-order solution with frequency modulation.

Equation (8.5-31)$_3$ includes some second-order effects on the amplitude modulation and is shown in Figure 8.5-3.

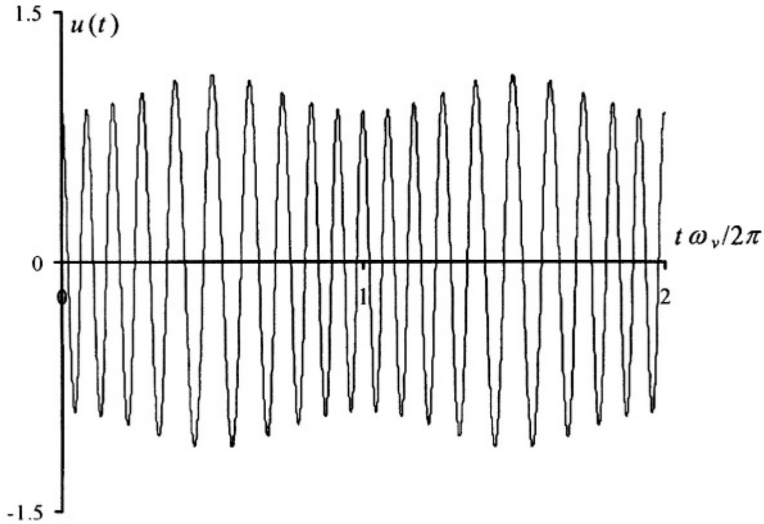

Figure 8.5-3. Second-order solution with frequency and amplitude modulation.

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Appendix 1

List of Notation

At present the IEEE Standard on Piezoelectricity [11] is concerned with the linear theory of piezoelectricity only. For nonlinear electroelasticity notation and terminology vary among researchers. The following are used in this book:

δ_{ij}, δ_{KL} – Kronecker delta
δ_{iK}, δ_{Ki} – Shifter
ε_{ijk}, ε_{IJK} – Permutation tensor
X_K – Reference position of a material point
y_i – Present position of a material point
u_K – Mechanical displacement vector
J – Jacobian
C_{KL} – Deformation tensor
S_{KL} – Finite strain tensor
S_{kl} – Linear strain tensor
v_i – Velocity vector
d_{ij} – Deformation rate tensor
ω_{ij} – Spin tensor
D/Dt – Material time derivative
ρ_0 – Reference mass density
ρ – Present mass density
Q_e – Free charge
ρ_e – Free charge density per unit present volume
ρ_E – Free charge density per unit reference volume
σ_e – Surface free charge per unit present area
σ_E – Surface free charge per unit reference area
ε_0 – Permittivity of free space
ϕ – Electrostatic potential
E_i – Electric field
P_i – Electric polarization per unit present volume

π_i – Electric polarization per unit mass
D_i – Electric displacement vector
$\mathcal{E}_K$ – Reference electric field vector
$\mathcal{P}_K$ – Reference electric polarization vector
$\mathcal{D}_K$ – Reference electric displacement vector
F_j^E – Electric body force per unit present volume
C_j^E – Electric body couple per unit present volume
w^E – Electric body power per unit present volume
f_j – Body force per unit mass
σ_{ij} – Cauchy stress tensor
σ_{ij}^E – Electrostatic stress tensor
$\sigma_{ij}^S, F_{Lj}, T_{KL}^S$ – Symmetric stress tensor in spatial, two-point, and material form
$\sigma_{ij}^M, M_{Lj}, T_{KL}^M$ – Symmetric Maxwell stress tensor in spatial, two-point, and material form
$\tau_{ij}, K_{Lj}, \hat{T}_{KL}$ – Total stress tensor in spatial, two-point, and material form
T_{kl} – Linear stress tensor
T_k – Mechanical surface traction per unit reference area
t_k – Mechanical surface traction per unit present area
e – Internal energy per unit mass
ψ – Free energy per unit mass
$\hat{\psi}$ – Total free energy per unit mass
θ – Absolute temperature
η – Entropy per unit mass
γ – Body heat source per unit mass
q_k – Present heat flux vector
Q_K – Reference heat flux vector

Appendix 2

Electroelastic Material Constants

Material constants for a few common piezoelectrics are summarized below. Numerical results given in this book are calculated from these constants. It is convenient to have these constants all in one place.

Permittivity of free space $\varepsilon_0 = 8.854 \times 10^{-12}$ Faraday/m.

Permeability of free space $\mu_0 = 12.57 \times 10^{-7}$ Henry/m.

Boltzmann constant $k = 1.38 \times 10^{-23}$ J/K.

Electronic charge $q_e = 1.602 \times 10^{-19}$ Coulomb.

Polarized ceramics

The material matrices for PZT-5H are [65]

$$\rho = 7500\ kg/m^3,$$

$$[c_{pq}] = \begin{pmatrix} 12.6 & 7.95 & 8.41 & 0 & 0 & 0 \\ 7.95 & 12.6 & 8.41 & 0 & 0 & 0 \\ 8.41 & 8.41 & 11.7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2.3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2.3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2.325 \end{pmatrix} \times 10^{10}\ \mathrm{N/m^2},$$

$$[e_{ip}] = \begin{pmatrix} 0 & 0 & 0 & 0 & 17 & 0 \\ 0 & 0 & 0 & 17 & 0 & 0 \\ -6.5 & -6.5 & 23.3 & 0 & 0 & 0 \end{pmatrix} \mathrm{Coulomb/m^2},$$

$$[\varepsilon_{ij}]=\begin{pmatrix}1700\varepsilon_0 & 0 & 0\\ 0 & 1700\varepsilon_0 & 0\\ 0 & 0 & 1470\varepsilon_0\end{pmatrix}=$$

$$\begin{pmatrix}1.505 & 0 & 0\\ 0 & 1.505 & 0\\ 0 & 0 & 1.302\end{pmatrix}\times 10^{-8}\,\mathrm{C/(Volt\text{-}m)},$$

where the superscript T in the piezoelectric matrix indicates transpose. For PZT-5H, an equivalent set of material constants are [65]

$$s_{11}=16.5,\quad s_{33}=20.7,\quad s_{44}=43.5,$$

$$s_{12}=-4.78,\quad s_{13}=-8.45\times 10^{-12}\,\mathrm{m^2/N},$$

$$d_{31}=-274,\quad d_{15}=741,\quad d_{33}=593\times 10^{-12}\,\mathrm{C/N},$$

$$\varepsilon_{11}=3130\varepsilon_0,\quad \varepsilon_{33}=3400\varepsilon_0.$$

When poling is along other directions, the material matrices can be obtained by tensor transformations. For PZT-5H, when poling is along the x_1 axis, we have

$$[c_{pq}]=\begin{pmatrix}11.7 & 8.41 & 8.41 & 0 & 0 & 0\\ 8.41 & 12.6 & 7.95 & 0 & 0 & 0\\ 8.41 & 7.95 & 12.6 & 0 & 0 & 0\\ 0 & 0 & 0 & 2.325 & 0 & 0\\ 0 & 0 & 0 & 0 & 2.3 & 0\\ 0 & 0 & 0 & 0 & 0 & 2.3\end{pmatrix}\times 10^{10}\,\mathrm{N/m^2},$$

$$[e_{ip}]=\begin{pmatrix}23.3 & -6.5 & -6.5 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 17\\ 0 & 0 & 0 & 0 & 17 & 0\end{pmatrix}\mathrm{C/m^2},$$

$$[\varepsilon_{ij}]=\begin{pmatrix}1.302 & 0 & 0\\ 0 & 1.505 & 0\\ 0 & 0 & 1.505\end{pmatrix}\times 10^{-8}\,\mathrm{C/Vm}.$$

When poling is along the x_2 axis

$$[c_{pq}]=\begin{pmatrix} 12.6 & 8.41 & 7.95 & 0 & 0 & 0 \\ 8.41 & 11.7 & 8.41 & 0 & 0 & 0 \\ 7.95 & 8.41 & 12.6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2.3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2.325 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2.3 \end{pmatrix}\times 10^{10}\,\mathrm{N/m^2},$$

$$[e_{ip}]=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 17 \\ -6.5 & 23.3 & -6.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 17 & 0 & 0 \end{pmatrix}\mathrm{C/m^2},$$

$$[\varepsilon_{ij}]=\begin{pmatrix} 1.505 & 0 & 0 \\ 0 & 1.302 & 0 \\ 0 & 0 & 1.505 \end{pmatrix}\times 10^{-8}\,\mathrm{C/Vm}.$$

Material constants of a few other polarized ceramics are given in the following tables [66]:

Material	c_{11}	c_{12}	c_{13}	c_{33}	c_{44}	c_{66}
PZT-4	13.9	7.78	7.40	11.5	2.56	3.06
PZT-5A	12.1	7.59	7.54	11.1	2.11	2.26
PZT-6B	16.8	8.47	8.42	16.3	3.55	4.17
PZT-5H	12.6	7.91	8.39	11.7	2.30	2.35
PZT-7A	14.8	7.61	8.13	13.1	2.53	3.60
PZT-8	13.7	6.99	7.11	12.3	3.13	3.36
$BaTiO_3$	15.0	6.53	6.62	14.6	4.39	4.24
	$\times 10^{10}\,\mathrm{N/m^2}$					

Material	e_{31}	e_{33}	e_{15}	ε_{11}	ε_{33}
PZT-4	-5.2	15.1	12.7	0.646	0.562
PZT-5A	-5.4	15.8	12.3	0.811	0.735
PZT-6B	-0.9	7.1	4.6	0.360	0.342
PZT-5H	-6.5	23.3	17.0	1.505	1.302

PZT-7A	-2.1	9.5	9.2	0.407	0.208
PZT-8	-4.0	13.2	10.4	0.797	0.514
$BaTiO_3$	-4.3	17.5	11.4	0.987	1.116
	C/m^2			$\times 10^{-8}$ C/Vm	

Density	PZT-5H	PZT-5A	PZT-6B	PZT-4
Kg/m^3	7500	7750	7550	7500

Density	PZT-7A	PZT-8	$BaTiO_3$
Kg/m^3	7600	7600	5700

Quartz

When referred to the crystal axes, the second-order material constants for left-hand quartz have the following values [67]

$$\rho = 2649 \text{ kg/m}^3,$$

$$[c_{pq}] = \begin{pmatrix} 86.74 & 6.99 & 11.91 & -17.91 & 0 & 0 \\ 6.99 & 86.74 & 11.91 & 17.91 & 0 & 0 \\ 11.91 & 11.91 & 107.2 & 0 & 0 & 0 \\ -17.91 & 17.91 & 0 & 57.94 & 0 & 0 \\ 0 & 0 & 0 & 0 & 57.94 & -17.91 \\ 0 & 0 & 0 & 0 & -17.91 & 39.88 \end{pmatrix}$$

$\times 10^9$ N/m2,

$$[e_{ip}] = \begin{pmatrix} 0.171 & -0.171 & 0 & -0.0406 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.0406 & -0.171 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{C/m}^2,$$

$$[\varepsilon_{ij}] = \begin{pmatrix} 39.21 & 0 & 0 \\ 0 & 39.21 & 0 \\ 0 & 0 & 41.03 \end{pmatrix} \times 10^{-12} \text{C/Vm}.$$

Temperature derivatives of the elastic constants of quartz at 25 °C are [68]

pq	11	33	12	13
$(1/c_{pq})(dc_{pq}/dT)$ $(10^{-6}/°C)$	18.16	-66.60	-1222	-178.6

pq	44	66	14
$(1/c_{pq})(dc_{pq}/dT)$ $(10^{-6}/°C)$	-89.72	126.7	-49.21

For quartz there are 31 nonzero third-order elastic constants. 14 are given in the following table. These values, at 25 °C, and based on a least-squares fit, are all in 10^{11} N/m^2 [69]

Constant	Value	Standard error
c_{111}	-2.10	0.07
c_{112}	-3.45	0.06
c_{113}	+0.12	0.06
c_{114}	-1.63	0.05
c_{123}	-2.94	0.05
c_{124}	-0.15	0.04
c_{133}	-3.12	0.07
c_{134}	+0.02	0.04
c_{144}	-1.34	0.07
c_{155}	-2.00	0.08
c_{222}	-3.32	0.08
c_{333}	-8.15	0.18
c_{344}	-1.10	0.07
c_{444}	-2.76	0.17

In addition, there are 17 relations among the third-order elastic constants of quartz [70]

$$c_{122} = c_{111} + c_{112} - c_{222}, \quad c_{156} = \frac{1}{2}(c_{114} + 3c_{124}),$$

$$c_{166} = \frac{1}{4}(-2c_{111} - c_{112} + 3c_{222}),$$

$$c_{224} = -c_{114} - 2c_{124}, \quad c_{256} = \frac{1}{2}(c_{114} - c_{124}),$$

$$c_{266} = \frac{1}{4}(2c_{111} - c_{112} - c_{222}),$$

$$c_{366} = \frac{1}{2}(c_{113} - c_{123}), \quad c_{456} = \frac{1}{2}(-c_{144} + c_{155}),$$

$$c_{223} = c_{113}, \quad c_{233} = c_{133}, \quad c_{234} = -c_{134}, \quad c_{244} = c_{155}, \quad c_{255} = c_{144},$$

$$c_{355} = c_{344}, \quad c_{356} = c_{134}, \quad c_{455} = -c_{444}, \quad c_{466} = c_{124}.$$

For the fourth-order elastic constants there are 69 nonzero ones of which 23 are independent [71]

$$c_{1111}, \quad c_{3333}, \quad c_{4444}, \quad c_{6666}, \quad c_{1112}, \quad c_{1113}, \quad c_{1123}, \quad c_{2214}, \quad c_{3331},$$

$$c_{4456}, \quad c_{5524}, \quad c_{4443}, \quad c_{1133}, \quad c_{3344}, \quad c_{1456}, \quad c_{1155}, \quad c_{1134}, \quad c_{2356},$$

$$c_{4423}, \quad c_{4413}, \quad c_{3314}, \quad c_{6614}, \quad c_{6624}.$$

There are 46 relations [71]

$$c_{2222} = c_{1111}, \quad c_{2266} = \frac{1}{6}(c_{1111} - c_{1112}), \quad c_{2223} = c_{1113},$$

$$c_{2221} = c_{1112}, \quad c_{6612} = \frac{1}{6}(c_{1111} - 4c_{6666} - c_{1112}), \quad c_{2213} = c_{1123},$$

$$c_{1166} = c_{2266}, \quad c_{1122} = \frac{1}{3}(-c_{1111} + 4c_{1112} + 8c_{6666}),$$

$$c_{6613} = \frac{1}{4}(c_{1113} - c_{1123}),$$

$$c_{5555} = c_{4444}, \quad c_{4455} = \frac{1}{3}c_{4444}, \quad c_{6623} = c_{6613},$$

$$c_{1124} = -c_{2214} + c_{6614} + c_{6624},$$

$$c_{3312} = -c_{1133}, \quad c_{1114} = 3(-c_{2214} + 2c_{6614} - 2c_{6624}), \quad c_{2233} = c_{1133},$$

$$c_{2256} = \frac{1}{2}(-2c_{2214} + 3c_{6614} - 5c_{6624}), \quad c_{6633} = c_{1133},$$

$$c_{2224} = 3(c_{2214} - 3c_{6614} + c_{6624}),$$

$$c_{3355} = c_{3344}, \quad c_{1156} = \frac{1}{2}(-2c_{2214} + 7c_{6614} - c_{6624}), \quad c_{3332} = c_{3331},$$

$$c_{1256} = \frac{1}{2}(-2c_{2214} + 3c_{6614} - c_{6624}), \quad c_{5534} = -c_{4443},$$

$$c_{6665} = \frac{3}{2}(c_{6614} - c_{6624}),$$

$$c_{4442} = -4c_{4456} - c_{5524}, \quad c_{1234} = c_{1134} - 2c_{2356}, \quad c_{2255} = c_{4412},$$

$$c_{5514} = 2c_{4456} + c_{5524}, \quad c_{1356} = 2c_{1134} - 3c_{2356}, \quad c_{5566} = c_{1456},$$

$$c_{5556} = 3c_{4456}, \quad c_{2234} = 4c_{2356} - 3c_{1134}, \quad c_{3324} = -c_{3314},$$

$$c_{4441} = 2c_{4456} - c_{5524}, \quad c_{6634} = c_{1234}, \quad c_{3356} = c_{3314}, \quad c_{5512} = c_{4412},$$

$$c_{1144} = c_{4412}, \quad c_{5523} = c_{4413}, \quad c_{2456} = c_{1456}, \quad c_{2244} = c_{1155}, \quad c_{5513} = c_{4423},$$

$$c_{4466} = c_{1456}, \quad c_{4412} = c_{1155} - 4c_{1456}, \quad c_{3456} = \frac{1}{2}(c_{4423} - c_{4413}).$$

The fourth-order elastic constants are usually unknown. Some scattered results are [71]

$$c_{1111} = 1.59 \times 10^{13}\,\mathrm{N/m^2} \pm 20\%,$$

$$c_{3333} = 1.84 \times 10^{13}\,\mathrm{N/m^2} \pm 20\%,$$

and [72]

$$c^{E}_{6666} = 77 \times 10^{11}\,\mathrm{N/m^2}.$$

AT-cut quartz is a special case of rotated Y-cut quartz (θ = 35.25°) whose material constants are [4]

$$[c_{pq}] = \begin{pmatrix} 86.74 & -8.25 & 27.15 & -3.66 & 0 & 0 \\ -8.25 & 129.77 & -7.42 & 5.7 & 0 & 0 \\ 27.15 & -7.42 & 102.83 & 9.92 & 0 & 0 \\ -3.66 & 5.7 & 9.92 & 38.61 & 0 & 0 \\ 0 & 0 & 0 & 0 & 68.81 & 2.53 \\ 0 & 0 & 0 & 0 & 2.53 & 29.01 \end{pmatrix} \times 10^9\,\mathrm{N/m^2},$$

$$[e_{ip}] = \begin{pmatrix} 0.171 & -0.152 & -0.0187 & 0.067 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.108 & -0.095 \\ 0 & 0 & 0 & 0 & -0.0761 & 0.067 \end{pmatrix} \text{C/m}^2,$$

$$[\varepsilon_{ij}] = \begin{pmatrix} 39.21 & 0 & 0 \\ 0 & 39.82 & 0.86 \\ 0 & 0.86 & 40.42 \end{pmatrix} \times 10^{-12}\,\text{C/Vm}.$$

Langasite

The second-order material constants of $La_3Ga_5SiO_{14}$ are [73]

$$\rho = 5743\ \text{kg/m}^3,$$

$$[c_{pq}] = \begin{pmatrix} 18.875 & 10.475 & 9.589 & -1.412 & 0 & 0 \\ 10.475 & 18.875 & 9.589 & 1.412 & 0 & 0 \\ 9.589 & 9.589 & 26.14 & 0 & 0 & 0 \\ -1.412 & 1.412 & 0 & 5.35 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5.35 & -1.412 \\ 0 & 0 & 0 & 0 & -1.412 & 4.2 \end{pmatrix}$$

$$\times 10^{10}\,\text{N/m2},$$

$$[e_{ip}] = \begin{pmatrix} -0.44 & 0.44 & 0 & -0.08 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.08 & 0.44 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{C/m}^2,$$

$$[\varepsilon_{ij}] = \begin{pmatrix} 18.92\varepsilon_0 & 0 & 0 \\ 0 & 18.92\varepsilon_0 & 0 \\ 0 & 0 & 50.7\varepsilon_0 \end{pmatrix}$$

$$= \begin{pmatrix} 167.5 & 0 & 0 \\ 0 & 167.5 & 0 \\ 0 & 0 & 448.9 \end{pmatrix} \times 10^{-12}\,\text{C/Vm}.$$

The third-order material constants of $La_3Ga_5SiO_{14}$ at 20°C are given in [73]. The third-order elastic constants c_{pqr} (in 10^{10} N/m^2) are

c_{111}	-97.2	c_{134}	-4.1
c_{112}	0.7	c_{144}	-4.0
c_{113}	-11.6	c_{155}	-19.8
c_{114}	-2.2	c_{222}	-96.5
c_{123}	0.9	c_{333}	-183.4
c_{124}	-2.8	c_{344}	-38.9
c_{133}	-72.1	c_{444}	20.2

The third-order piezoelectric effect constants e_{ipq} (in C/m^2) are

e_{111}	9.3	e_{124}	-4.8
e_{113}	-3.5	e_{134}	6.9
e_{114}	1.0	e_{144}	-1.7
e_{122}	0.7	e_{315}	-4

The third-order electrostriction constants H_{pq} (in $10^{-9}N/V^2$) are

H_{11}	-26	H_{31}	-24
H_{12}	65	H_{33}	-40
H_{13}	20	H_{41}	-170
H_{14}	-43	H_{44}	-44

The third-order dielectric permeability ε_{111} (in 10^{-20} F/V) are

ε_{111}	-0.5

Lithium Niobate

The second-order material constants for lithium niobate are [74]

$$\rho = 4700 \text{ kg/m}^3,$$

$$[c_{pq}] = \begin{pmatrix} 2.03 & 0.53 & 0.75 & 0.09 & 0 & 0 \\ 0.53 & 2.03 & 0.75 & -0.09 & 0 & 0 \\ 0.75 & 0.75 & 2.45 & 0 & 0 & 0 \\ 0.09 & -0.09 & 0 & 0.60 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.60 & 0.09 \\ 0 & 0 & 0 & 0 & 0.09 & 0.75 \end{pmatrix} \times 10^{11}\,\mathrm{N/m^2},$$

$$[e_{ip}] = \begin{pmatrix} 0 & 0 & 0 & 0 & 3.70 & -2.50 \\ -2.50 & 2.50 & 0 & 3.70 & 0 & 0 \\ 0.20 & 0.20 & 1.30 & 0 & 0 & 0 \end{pmatrix} \mathrm{C/m^2},$$

$$[\varepsilon_{ij}] = \begin{pmatrix} 38.9 & 0 & 0 \\ 0 & 38.9 & 0 \\ 0 & 0 & 25.7 \end{pmatrix} \times 10^{-11}\,\mathrm{C/Vm}.$$

The third-order material constants of lithium niobate are given in [75]. The third-order elastic constants c_{pqr} (in 10^{11} N/m^2) are

Constant	Value	Standard error
c_{111}	-21.2	4.0
c_{112}	-5.3	1.2
c_{113}	-5.7	1.5
c_{114}	2.0	0.8
c_{123}	-2.5	1.0
c_{124}	0.4	0.3
c_{133}	-7.8	1.9
c_{134}	1.5	0.3
c_{144}	-3.0	0.2
c_{155}	-6.7	0.3
c_{222}	-23.3	3.4
c_{333}	-29.6	7.2

c_{344}	-6.8	0.7
c_{444}	-0.3	0.4

The third-order piezoelectric constants e_{ipq} $(= -k_{1\,ipq})$ are

Constant	Value	Standard error
e_{115}	17.1	6.6
e_{116}	-4.7	6.4
e_{125}	19.9	2.1
e_{126}	-15.9	5.3
e_{135}	19.6	2.7
e_{136}	-0.9	2.7
e_{145}	20.3	5.7
e_{311}	14.7	6.0
e_{312}	13.0	11.4
e_{313}	-10.0	8.7
e_{314}	11.0	4.6
e_{333}	-17.3	5.9
e_{344}	-10.2	5.6
	C/m^2	

The third-order electrostirctive constants l_{pq} (compressed from b_{ijkl} + $\varepsilon_0\delta_{ij}\delta_{kl}$ - $\varepsilon_0\delta_{ik}\delta_{jl}$ - $\varepsilon_0\delta_{il}\delta_{kl}$) (in $10^{-9}F/m^2$) are

Constant	Value	Standard error
l_{11}	1.11	0.39
l_{12}	2.19	0.56
l_{13}	2.32	0.67

l_{31}	0.19	0.61
l_{33}	-2.76	0.41
l_{14}	1.51	0.17
l_{41}	1.85	0.17
l_{44}	-1.83	0.11

The third-order dielectric constants ε_{ip} (in 10^{-19} F/V) are

Constant	Value	Standard error
ε_{31}	-2.81	0.06
ε_{22}	-2.40	0.09
ε_{33}	-2.91	0.06

Lithium Tantalate

The second-order material constants for lithium niobate are [74]

$$\rho = 7450 \text{ kg/m}^3,$$

$$[c_{pq}] = \begin{pmatrix} 2.33 & 0.47 & 0.80 & -0.11 & 0 & 0 \\ 0.47 & 2.33 & 0.80 & 0.11 & 0 & 0 \\ 0.80 & 0.80 & 2.45 & 0 & 0 & 0 \\ -0.11 & -0.11 & 0 & 0.94 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.94 & -0.11 \\ 0 & 0 & 0 & 0 & -0.11 & 0.93 \end{pmatrix} \times 10^{11} \text{ N/m}^2,$$

$$[e_{ip}] = \begin{pmatrix} 0 & 0 & 0 & 0 & 2.6 & -1.6 \\ -1.6 & 1.6 & 0 & 2.6 & 0 & 0 \\ 0 & 0 & 1.9 & 0 & 0 & 0 \end{pmatrix} \text{C/m}^2,$$

$$[\varepsilon_{ij}] = \begin{pmatrix} 36.3 & 0 & 0 \\ 0 & 36.3 & 0 \\ 0 & 0 & 38.2 \end{pmatrix} \times 10^{-11} \text{ C/Vm}.$$

Silicon

Silicon is of cubic symmetry m3m and is nonpiezoelectric. For silicon we have [76]

$$\rho = 2332 \text{ kg/m}^3,$$

$$c_{11} = 16.57, \quad c_{44} = 7.956, \quad c_{12} = 6.39 \times 10^{10} \text{ N/m}^2,$$

$$\varepsilon_{11} = 11.8\varepsilon_0, \quad \varepsilon_0 = 8.854 \times 10^{-12} \text{ Farads/m.}$$

The mobility of electrons and holes in silicon are

$$\mu_n = 1500, \quad \mu_p = 480 \text{ cm}^2/\text{V} - \text{sec}.$$

The diffusion constants can be determined from the Einstein relation

$$D = \frac{k\theta}{q_e}\mu,$$

where θ is the absolute temperature. For the third-order elastic constants there are twenty nonzero ones among which six are independent

$$c_{111} = -825, \quad c_{112} = -451, \quad c_{123} = -64,$$

$$c_{144} = 12, \quad c_{155} = -310, \quad c_{456} = -64 \text{ GPa.}$$

and the other fourteen are determined from the following relations:

$$c_{113} = c_{112}, \quad c_{122} = c_{112}, \quad c_{133} = c_{112}, \quad c_{166} = c_{155},$$

$$c_{222} = c_{111}, \quad c_{223} = c_{112}, \quad c_{233} = c_{112}, \quad c_{244} = c_{155},$$

$$c_{255} = c_{144}, \quad c_{266} = c_{155}, \quad c_{333} = c_{111}, \quad c_{344} = c_{155},$$

$$c_{355} = c_{155}, \quad c_{366} = c_{144}.$$