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A Multivariate and Asymmetric Generalization of Laplace Distribution

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Summary

Consider a sum of independent and identically distributed random vectors with finite second moments, where the number of terms has a geometric distribution independent of the summands. We show that the class of limiting distributions of such random sums, as the number of terms converges to infinity, consists of multivariate asymmetric distributions that are natural generalizations of univariate Laplace laws. We call these limits *multivariate asymmetric Laplace laws.* We give an explicit form of their multidimensional densities and show representations that effectively facilitate computer simulation of variates from this class. We also discuss the relation to other formerly considered classes of distributions containing Laplace laws.

Keywords: Bessel function; geometric compound; geometric stable law; heavy tailed modeling; elliptically contoured distribution; mixture; random summation; simulation.

1 Introduction

We discuss a class of multivariate and not necessary symmetric distributions, that naturally extend properties of and reduce to Laplace distribution in one dimension. Recall that a classical Laplace distribution with scale parameter σ can be defined either through its characteristic function (ch.f.),

$$\phi(t) = \frac{1}{1+\sigma^2 t^2}, \quad -\infty < t < \infty, \tag{1}$$

or through its probability density function (p.d.f.),

$$f(x) = \frac{1}{2\sigma} e^{-|x|/\sigma}, \quad -\infty < x < \infty. \tag{2}$$

Kozubowski and Podgórski (1998a) noticed that a class of distributions, introduced in Hinkley and Revankar (1977) and proposed by Madan et al. (1998) for modeling stock price data, admits the following characterizations, which parallel those of symmetric Laplace distribution given by (1),

- The characteristic function:

$$\phi(t) = \frac{1}{1+\sigma^2 t^2 - i\mu t}. \tag{3}$$

- The density function:

$$f(x) = \frac{1}{\sigma}\frac{\kappa}{1+\kappa^2} \begin{cases} \exp\left(-\frac{\kappa}{\sigma}x\right), & \text{if } x \geq 0 \\ \exp\left(\frac{1}{\sigma\kappa}x\right), & \text{if } x < 0, \end{cases} \tag{4}$$

  where $\kappa = 2\sigma/(\mu + \sqrt{4\sigma^2+\mu^2})$.

- Limiting distribution in the random summation scheme:

$$Y = \lim_{p\to 0} \sqrt{p} \sum_{i=1}^{N_p} (X_i - \mu(1-\sqrt{p})), \tag{5}$$

  where X_i are i.i.d. variables with mean μ and variance $2\sigma^2$, and independent of geometrically distributed N_p. Note that in the symmetric case, the centering $\mu(1-\sqrt{p})$ is not necessary in order to ensure existence of the distributional limit.

- Normal variance-mean mixture:

$$Y \stackrel{d}{=} \sigma\sqrt{2Z}X + \mu Z, \tag{6}$$

  where X and Z are independent with standard normal and standard exponential distributions, respectively. (Here and in the sequel we follow the standard convention that when expressing equalities in distribution, the random variables appearing on the same side of the equation are independent.)

- Exponential mixture:

$$Y \stackrel{d}{=} \sigma I Z, \tag{7}$$

where I takes on the values $\mp\kappa^{\pm 1}$ with probabilities $\frac{1}{2} \mp \frac{\mu}{2\sqrt{\mu^2+4\sigma^2}}$, while Z is standard exponential.

Since the above properties naturally extend those of symmetric Laplace laws (when $\mu = 0$), we shall refer to the distributions given by (3) as asymmetric Laplace (AL) laws.

A symmetric Laplace distribution has already been extended to the multivariate case: McGraw and Wagner (1968) listed a bivariate Laplace distribution as a special case of elliptically contoured law, while Johnson and Kotz (1970) provided its density function. The same distribution appeared in Anderson (1992) as a special case of multivariate Linnik law. In the following section, we extend the multivariate symmetric Laplace distributions as given in Anderson (1992) to asymmetric laws. We show that these distributions satisfy multivariate versions of the basic properties (3) - (7) of univariate Laplace laws, and thus we feel that the term *multivariate asymmetric Laplace (AL) laws* is a properly justified name for members of this class.

Ernst (1998) proposed another class of multivariate distributions based on elliptic contouring, and called them generalized Laplace distributions. However, his class does not coincide with those used before for multivariate generalizations of Laplace distributions. Further, unlike our AL laws, the generalizations of Laplace distributions of Ernst (1998) admit neither univariate nor multivariate symmetric Laplace marginals.

Johnson (1987) states that because density functions of multivariate Laplace distributions involve Bessel functions, they have limited practical use and their computer generation is troublesome as well. We believe that the first issue raised by Johnson (1987) is not a problem as Bessel functions are not hard to handle numerically and many modern packages have implemented them as standard functions (including Matlab, Maple, and Mathematica). We also address the second issue and present a simple and effective method of random variate generation of multivariate AL laws, extending the symmetric case of Anderson (1992). Thus, we argue that AL laws are ready for practical applications.

In this work we concentrate on density functions and methods of generating random variates from the AL family, referring to Kozubowski and Podgórski (1998b) for more in depth theoretical results.

2 Multivariate Asymmetric Laplace Distributions

We define multivariate AL distributions by an extension of the characteristic function (3), and list their properties which parallel the ones in the univariate case. In particular, we characterize Al densities and present a representation that leads to a simple method of computer simulation of multivariate AL random variables.

Definition 2.1 *A random vector* $\mathbf{Y}$ *in* $\mathcal{R}^d$ *is said to have multivariate asymmetric Laplace distribution (AL) if its ch.f. is given by*

$$\Psi(\mathbf{t}) = \left[1 + \frac{1}{2}\mathbf{t}'\boldsymbol{\Sigma}\mathbf{t} - i\mathbf{m}'\mathbf{t}\right]^{-1}, \tag{8}$$

where $\mathbf{m} \in \mathcal{R}^d$ *and* $\boldsymbol{\Sigma}$ *is a* $d \times d$ *non-negative definite symmetric matrix.*

Remark. Note that if $\mathbf{m} = \mathbf{0}$, the characteristic function depends on $\mathbf{t}$ only through $\mathbf{t}'\Sigma\mathbf{t}$ and thus the corresponding AL law belongs to the class of elliptically contoured multivariate distributions (see, e.g., Johnson (1987)).

As a direct consequences of the definition we obtain the next proposition, which shows that all marginal distributions of an AL r.v. $\mathbf{Y}$ are AL.

Proposition 2.1 *Let* $\mathbf{Y}' = (Y_1, \ldots, Y_d)$ *be AL with the parameters* $\mathbf{m}$ *and* $\boldsymbol{\Sigma}$, *and let* $\mathbf{A}$ *be an* $l \times d$ *real matrix. Then, the random vector* $\mathbf{Y}_\mathbf{A} = \mathbf{AY}$ *is AL with the parameters* $\mathbf{Am}$ *and* $\mathbf{A}\boldsymbol{\Sigma}\mathbf{A}'$.

Proof. The assertion follows from the relation

$$\boldsymbol{\Psi}_{\mathbf{AY}}(\mathbf{t}) = Ee^{i(\mathbf{AY})'\mathbf{t}} = Ee^{i\mathbf{Y}'\mathbf{A}'\mathbf{t}} = \boldsymbol{\Psi}_\mathbf{Y}(\mathbf{A}'\mathbf{t}) \tag{9}$$

and the fact that the matrix $\mathbf{A}\boldsymbol{\Sigma}\mathbf{A}'$ is non-negative definite whenever $\boldsymbol{\Sigma}$ is. □

2.1 Densities

The densities of multivariate AL laws can be written explicitly in terms of Bessel functions. Throughout this subsection, we assume that distributions are non-singular in $\mathcal{R}^d$, $d \geq 1$. Recall that the modified Bessel function of the third kind of order v has an integral representation

$$K_v(u) = \frac{1}{2}\left(\frac{u}{2}\right)^v \int_0^\infty z^{-v-1} \exp\left(-z - \frac{u^2}{4z}\right) dz, \tag{10}$$

valid for complex u with the non-negative real part of u^2. We have the following form of the AL densities (for $\mathbf{y} \neq \mathbf{0}$)

$$g(\mathbf{y}) = \frac{2e^{\mathbf{y}'\boldsymbol{\Sigma}^{-1}\mathbf{m}}}{(2\pi)^{d/2}|\boldsymbol{\Sigma}|^{1/2}} \left(\frac{\mathbf{y}'\boldsymbol{\Sigma}^{-1}\mathbf{y}}{2 + \mathbf{m}'\boldsymbol{\Sigma}^{-1}\mathbf{m}}\right)^{v/2} K_v\left(\sqrt{(2 + \mathbf{m}'\boldsymbol{\Sigma}^{-1}\mathbf{m})(\mathbf{y}'\boldsymbol{\Sigma}^{-1}\mathbf{y})}\right), \tag{11}$$

where $v = (2-d)/2$. This formula follows directly from the representation given in Subsection 2.3 (see Kozubowski and Podgórski (1998b) for the derivation). Let us discuss several special cases.
Case 1. In the elliptically contoured case ($\mathbf{m} = \mathbf{0}$), we obtain

$$g(\mathbf{y}) = \frac{2}{(2\pi)^{d/2}|\mathbf{\Sigma}|^{1/2}} \left(\mathbf{y}'\mathbf{\Sigma}^{-1}\mathbf{y}/2\right)^{v/2} K_v\left(\sqrt{2\mathbf{y}'\mathbf{\Sigma}^{-1}\mathbf{y}}\right), \quad \mathbf{y} \neq \mathbf{0},$$

which coincides with multivariate symmetric Laplace distributions of McGraw and Wagner (1968), Johnson and Kotz (1970), and Anderson (1992), but is different from the class of elliptically contoured distributions considered in Ernst (1998).
Case 2. In the one dimensional case we have $\mathbf{\Sigma} = \sigma_{11}$ and the ch.f. (8) simplifies to (3) with $\sigma^2 = \sigma_{11}/2$ and $\mu = \mathbf{m}$. Further, for $v = 1/2$, the Bessel function (10) simplifies to $K_{1/2}(u) = \sqrt{\pi/2}e^{-u}/u$, and the AL density (11) reduces to (4). In the symmetric case ($\mu = 0$) we obtain Laplace distribution with mean zero and variance $2\sigma^2$.
Case 3. In the bivariate case the formula simplifies to

$$g(\mathbf{y}) = \frac{e^{\mathbf{y}'\mathbf{\Sigma}^{-1}\mathbf{m}}}{\pi|\mathbf{\Sigma}|^{1/2}} K_0\left(\sqrt{(2+\mathbf{m}'\mathbf{\Sigma}^{-1}\mathbf{m})(\mathbf{y}'\mathbf{\Sigma}^{-1}\mathbf{y})}\right), \quad \mathbf{y} \neq \mathbf{0}.$$

Writing

$$\mathbf{\Sigma} = \sigma_1\sigma_2 \begin{bmatrix} \sigma_1/\sigma_2 & \rho \\ \rho & \sigma_2/\sigma_1 \end{bmatrix}$$

and letting $\mathbf{m}' = (m_1, m_2)$, we obtain the five parameter bivariate AL density,

$$g(x,y) = \frac{\exp\left[\left((m_1\sigma_2/\sigma_1 - m_2\rho)\,x + (m_2\sigma_1/\sigma_2 - m_1\rho)\,y\right)/(\sigma_1\sigma_2(1-\rho^2))\right]}{\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \times K_0\left(C(m_1,m_2,\sigma_1,\sigma_2,\rho)\sqrt{x^2\sigma_2/\sigma_1 - 2\rho xy + y^2\sigma_1/\sigma_2}\right),$$

where

$$C(m_1,m_2,\sigma_1,\sigma_2,\rho) = \frac{\sqrt{2\sigma_1\sigma_2(1-\rho^2) + m_1^2\sigma_2/\sigma_1 - 2m_1m_2\rho + m_2^2\sigma_1/\sigma_2}}{\sigma_1\sigma_2(1-\rho^2)}.$$

In Figure 1, we present several examples of bivariate AL densities. In the top-left figure, we have

$$g(x,y) = \frac{1}{\pi}K_0\left(\sqrt{2(x^2+y^2)}\right).$$

which might be called the standard bivariate Laplace density, as it corresponds to $m_1 = m_2 = 0$, $\rho = 0$, and $\sigma_1 = \sigma_2 = 1$. The top-right picture corresponds to

$$g(x,y) = \frac{5}{3\pi}K_0\left(\sqrt{2}\cdot 5/3 \cdot \sqrt{x^2 + 8/5\cdot xy + y^2}\right).$$

which arises when contouring is around the ellipsoid with the long axis along $y = -x$ with $\rho = -4/5$, while the other parameters are as in the previous case. The remaining pictures correspond to various cases when $m_1 \neq 0$ and $m_2 \neq 0$, and the distributions are no longer elliptically contoured. The middle-left figure is a graph of the density

$$g(x,y) = \frac{1}{\pi} e^{x+y} K_0\left(2\sqrt{x^2+y^2}\right),$$

while the middle-right one corresponds to

$$g(x,y) = \frac{5}{3\pi} e^{5(x+y)} K_0\left(\sqrt{3} \cdot 10/3 \cdot \sqrt{x^2 + 8/5 \cdot xy + y^2}\right).$$

At the bottom we present two cases with no "symmetries" between parameters.

Case 4. If d is odd, we obtain the density in the closed form. Indeed, suppose $d = 2r + 3$, where $r = 0, 1, 2, \ldots$, so that $v = (2-d)/2 = -r - 1/2$. Since $K_v(u) = K_{-v}(u)$ and the Bessel function K_v with $v = r + 1/2$ has the well known closed form,

$$K_{r+1/2}(u) = \sqrt{\frac{\pi}{2u}} e^{-u} \sum_{k=0}^{r} \frac{(r+k)!}{(r-k)!k!} (2u)^{-k},$$

the AL density (11) simplifies to

$$g(\mathbf{y}) = \frac{C^r e^{\mathbf{y}'\boldsymbol{\Sigma}^{-1}\mathbf{m} - C\sqrt{\mathbf{y}'\boldsymbol{\Sigma}^{-1}\mathbf{y}}}}{\left(2\pi\sqrt{\mathbf{y}'\boldsymbol{\Sigma}^{-1}\mathbf{y}}\right)^{r+1} |\boldsymbol{\Sigma}|^{1/2}} \sum_{k=0}^{r} \frac{(r+k)!}{(r-k)!k!} \left(2C\sqrt{\mathbf{y}'\boldsymbol{\Sigma}^{-1}\mathbf{y}}\right)^{-k}, \quad \mathbf{y} \neq \mathbf{0},$$

where $v = (2-d)/2$ and $C = \sqrt{2 + \mathbf{m}'\boldsymbol{\Sigma}^{-1}\mathbf{m}}$. In the three dimensional space, we have $r = 0$ and the density has a particularly explicit form,

$$g(\mathbf{y}) = \frac{e^{\mathbf{y}'\boldsymbol{\Sigma}^{-1}\mathbf{m} - C\sqrt{\mathbf{y}'\boldsymbol{\Sigma}^{-1}\mathbf{y}}}}{2\pi\sqrt{\mathbf{y}'\boldsymbol{\Sigma}^{-1}\mathbf{y}}|\boldsymbol{\Sigma}|^{1/2}}, \quad \mathbf{y} \neq \mathbf{0}.$$

2.2 Distributional Limits

We have a complete analogy with the one dimensional case. Namely, a random vector $\mathbf{Y}$ in $\mathcal{R}^d$ has an AL distribution with parameters $\mathbf{m}$ and $\boldsymbol{\Sigma}$, if and only if there exist a sequence $\mathbf{Y}_i$, $i \geq 1$, of i.i.d. r.v.'s in $\mathcal{R}^d$ with the mean vector $\mathbf{m}$ and the covariance matrix $\boldsymbol{\Sigma}$ such that

$$\lim_{p \to 0} \sqrt{p} \sum_{i=1}^{N_p} (\mathbf{Y}_i + (\sqrt{p} - 1)\mathbf{m}) \stackrel{d}{=} \mathbf{Y}, \tag{12}$$

where N_p is a geometrically distributed random variable independent of the sequence $\mathbf{Y}_i$. Relation (12) can be proved via Cramer-Wold device coupled with proposition 2.1, see Kozubowski and Podgórski (1998b) for details.

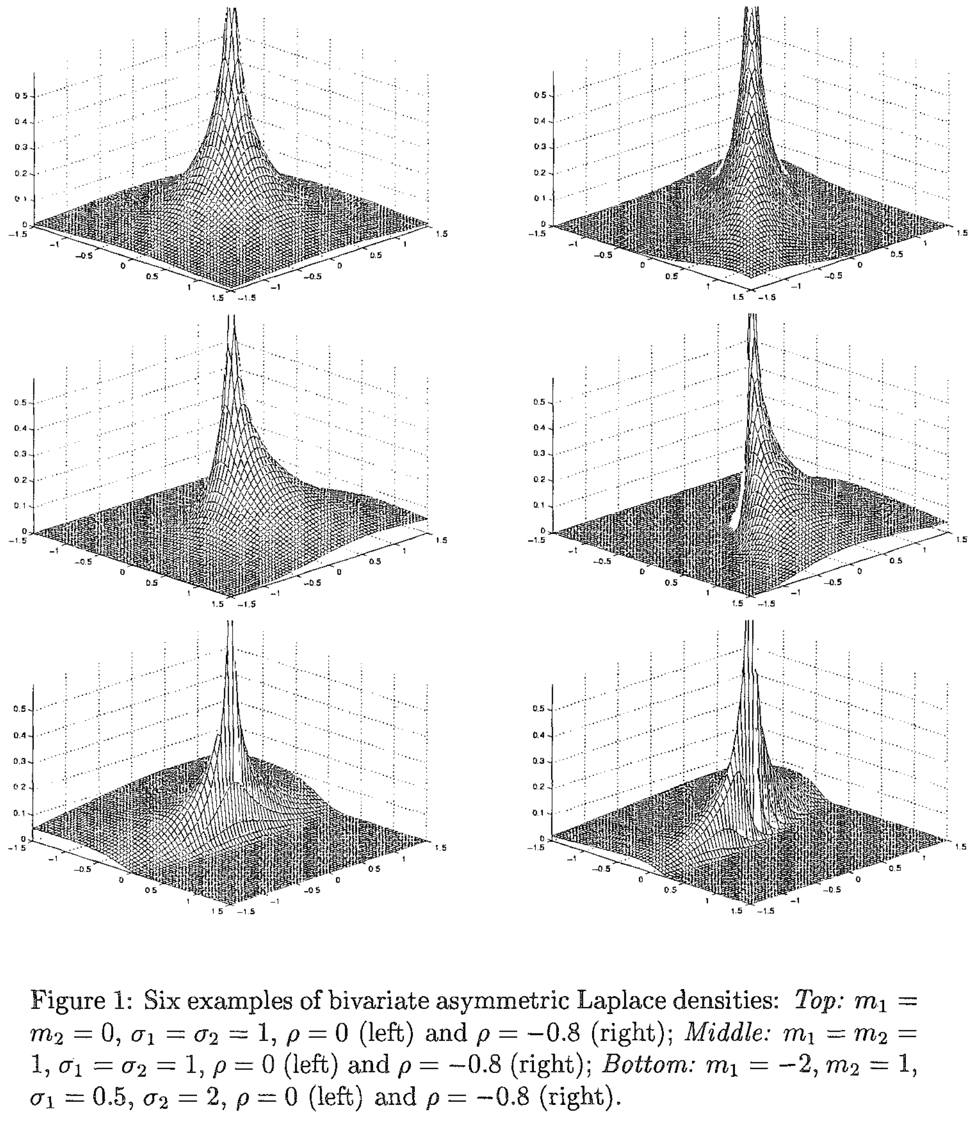

Figure 1: Six examples of bivariate asymmetric Laplace densities: *Top:* $m_1 = m_2 = 0$, $\sigma_1 = \sigma_2 = 1$, $\rho = 0$ (left) and $\rho = -0.8$ (right); *Middle:* $m_1 = m_2 = 1$, $\sigma_1 = \sigma_2 = 1$, $\rho = 0$ (left) and $\rho = -0.8$ (right); *Bottom:* $m_1 = -2, m_2 = 1$, $\sigma_1 = 0.5$, $\sigma_2 = 2$, $\rho = 0$ (left) and $\rho = -0.8$ (right).

2.3 Normal Distribution with Random Mean and Covariance

Let $\mathbf{Y}$ be an AL random vector with the parameters $\mathbf{m}$ and $\boldsymbol{\Sigma}$, let $\mathbf{X}$ be a normal random vector centered at zero and with the covariance matrix $\boldsymbol{\Sigma}$, and let Z be a standard exponential random variable independent of $\mathbf{X}$. Then the following representation holds

$$\mathbf{Y} \stackrel{d}{=} \mathbf{m}Z + Z^{1/2}\mathbf{X}. \tag{13}$$

To verify (13), write the right-hand side of (13) as

$$Ee^{i\mathbf{t}'(\mathbf{m}Z+\sqrt{Z}\mathbf{X})} = \int_0^\infty e^{i\mathbf{t}'\mathbf{m}Z}\mathbf{\Phi}(z^{1/2}\mathbf{t})e^{-z}dz, \tag{14}$$

conditioning on Z and denoting the ch.f. of X by $\mathbf{\Phi}$. Since $\mathbf{\Phi}(\mathbf{t}) = e^{-\mathbf{t}'\mathbf{\Sigma}\mathbf{t}/2}$, (14) produces (8).

Remark. It follows from the representation (13) that $E\mathbf{Y} = \mathbf{m}$ and $\text{Var}\mathbf{Y} = \mathbf{\Sigma} + \mathbf{mm}'$. Indeed, since $EX_iX_j = \sigma_{ij}$ and $EZ^2 = 2$, we have:

$$\begin{aligned} E(Y_iY_j) &= E[(m_iZ + Z^{1/2}X_i)(m_j + Z^{1/2}X_j)] \\ &= m_im_jEZ^2 + E(Z)E(X_iX_j) = 2m_im_j + \sigma_{ij}, \end{aligned}$$

so that

$$Cov(Y_i, Y_j) = E(Y_iY_j) - E(Y_i)E(Y_j) = 2m_im_j + \sigma_{ij} - m_im_j = m_im_j + \sigma_{ij}.$$

The above representation can be used to effectively simulate multivariate AL random vectors through generation of multivariate normal random vectors and standard exponential random variables. In Figure 2, we present results of computer generation of bivariate AL random vectors and compare the results with the corresponding bivariate normal samples. As in the univariate case, we see greater peakedness and longer tails of AL laws compared with the normal distribution. Note that for $\mathbf{m} = \mathbf{0}$ both random variables have the same theoretical means and covariances (see the Remark above), although from the figure it may appear that the AL r.v. has greater variability.

2.4 Polar Representation

The complete analog of the one-dimensional representation given by (7) is presently an open question. Below we discuss such a representation for the case $\mathbf{m} = \mathbf{0}$. Here, from (13), we obtain $\mathbf{Y} \stackrel{d}{=} \sqrt{Z}\tilde{R}\mathbf{U}$, where $\tilde{R} = \sqrt{\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{X}}$ and $\mathbf{U} = \sqrt{\mathbf{X}/(\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{X})}$. It is a well known that $\tilde{R}$ and $\mathbf{U}$ are independent, and $\tilde{R}^2$ has Chi-square distribution with d degrees of freedom, while $\mathbf{U}$ is distributed uniformly over the ellipsoid $\{\mathbf{y} : \mathbf{y}'\mathbf{\Sigma}^{-1}\mathbf{y} = 1\}$ (in the one dimensional case this independence simply means that for a normal random variable X centered at zero, $U = \text{sign}(X)$ and $\tilde{R} = |X|$ are independent). Further, set $R = \sqrt{Z}\tilde{R}$ to obtain the representation

$$\mathbf{Y} \stackrel{d}{=} R\mathbf{U}, \tag{15}$$

where R is a positive random variable, independent of $\mathbf{U}$, with density

$$f_R(x) = \frac{2x^{d/2}K_{d/2-1}(\sqrt{2}x)}{2^{d/4-1/2}\Gamma(d/2)}.$$

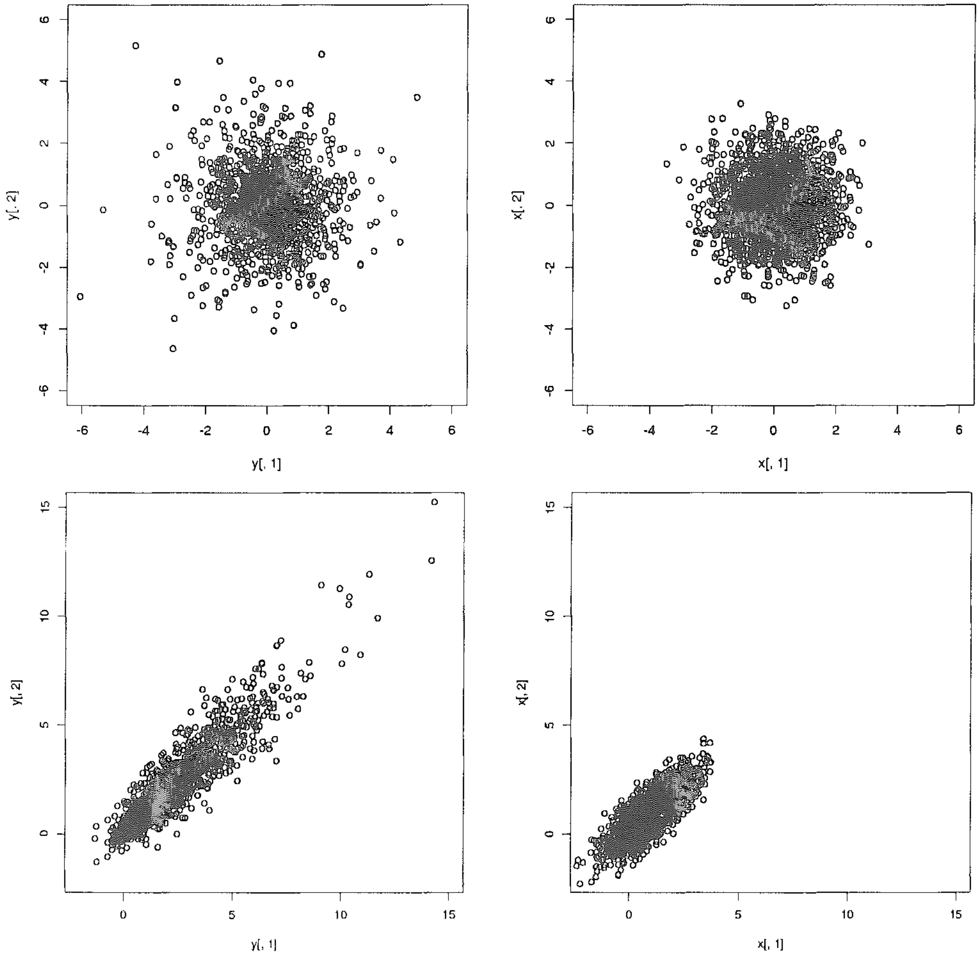

Figure 2: Monte Carlo simulations of bivariate asymmetric Laplace samples (left) vs. normal samples (right) (sample size equal to 2000). *Top:* $m_1 = m_2 = 0$, $\sigma_1 = \sigma_2 = 1$, $\rho = 0$. *Bottom:* $m_1 = m_2 = 1$, $\sigma_1 = \sigma_2 = 1$, $\rho = 4/5$.

Note that in the univariate case, where $d = 1$ and $\Sigma = \sigma_{11}$, we have $R \stackrel{d}{=} Z/\sqrt{2}$, where Z is standard exponential, while U is a random variable taking on values $\pm\sqrt{\sigma_{11}}$ with equal probabilities. Thus, the representation (15) coincides with (7) with $\sigma = \sqrt{\sigma_{11}/2}$.

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