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The concepts of triangle orthocenters in Minkowski planes

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Abstract. Let $(\mathbf{A^2}, \mathcal{C})$ be a Minkowski plane with a centrally symmetric, strictly convex C^1-curve $\mathcal{C}$ as the unit circle. Then $\mathcal{C}$ induces in $(\mathbf{A^2}, \mathcal{C})$ a *left-orthogonality structure* '$\dashv$' by setting tangents of $\mathcal{C}$ (and their parallels) left-orthogonal to the corresponding radii (and their paralles). If a line g is left-orthogonal to another one h, then h is *right-orthogonal* to g, $(h \vdash g)$. Based on those concepts of orthogonality in $(\mathbf{A^2}, \mathcal{C})$ *left-* and *right-altitudes* of a triangle are defined and one can discuss the existence of *left-* or *right-orthocentric triangles*. In general Minkowski planes these concepts of orthocenters are independent of a third type of a triangle-orthocenter, which is based on a circle-geometric definition due to Asplund and Grünbaum, c.f. [1].

Further results are the following: In every plane $(\mathbf{A^2}, \mathcal{C})$ there exist triplets of directions $\bar{g}_i$ such that the triangles $\mathcal{T}$ having sides g_i parallel to $\bar{g}_i$ are left-orthocentric. A plane $(\mathbf{A^2}, \mathcal{C})$ is euclidean, iff each triangle $\mathcal{T}$ is left-orthocentric. Constructing the altitudes of an altitude-triangle of a non (left- or right-)-orthocentric triangle $\mathcal{T}$ starts iteration processes with attractors (resp. repulsors) which can be called 'limit orthocenters' to the given triangle $\mathcal{T}$.

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Introduction and basic tools

a) While in the elementary euclidean plane every triangle has concurrent altitudes, this need not be true in other two-dimensional manifolds with respect to a given orthogonality structure. Nontrivial examples of such manifolds are for example Minkowski planes. The metric in such a real affine plane is based on a centric symmetric convex curve $\mathcal{C}$ as the unit circle of that Minkowski plane $(\mathbf{A^2}, \mathcal{C})$. Generalising euclidean orthogonality to such a plane there seem to be several possibilities to define a 'normal direction' to a given line g: We may start with the perpendicular bisector to a segment of g, which, in general, is curved (or even a domain) in $(\mathbf{A^2}, \mathcal{C})$. Segments in g differing in length lead to similar bisector curves b. Thus it would not make sense to use bisector curves b by themselves as 'normals' of g. Obviously, any two bisector curves b_1, b_2 intersect g in points M_1, M_2 such that their tangents t_1, t_2 in M_i are parallel. So we may use the direction of t_1 as the 'left-normal' direction to g. As t_1 automatically touches a 'Minkowski circle', (which is centered in a point of g), we will define 'left-orthogonality' in $(\mathbf{A^2}, \mathcal{C})$ as follows (see e.g. [10]): The tangent h of $\mathcal{C}$ is left-orthogonal to the corresponding radius of $\mathcal{C}$. Based on this orthogonality concept we will study triangle altitudes. In general the altitudes of a triangle in $(\mathbf{A^2}, \mathcal{C})$ will not be concurrent, such that the triangle is not orthocentric. So there arise the following questions:

> To a given arbitrary Minkowski plane find (all) triangles with concurrent altitudes.
>
> Characterize Minkowski planes in which every triangle has concurrent altitudes.
>
> If an arbitrary triangle in a Minkowski plane possesses a triangle of altitudes, one may construct the altitudes of that triangle again. When is this iterative process of constructing altitude-triangles of altitude-triangles convergent?

In the following we will consider Minkowski planes $(\mathbf{A^2}, \mathcal{C})$ with a centrally symmetric, *strictly* convex and *smooth* curve $\mathcal{C}$ as the unit circle in the (real) affine plane $\mathbf{A^2}$. Then the parallels h to a tangent $\overline{h}$ of $\mathcal{C}$ are called *left-orthogonal*[1] to the parallels g to the corresponding radius $\bar{g}$, $(h \dashv g)$, c.f [10, p. 77], and the set of directions $\{g\}$ is bijective to the set of corresponding left-normals $\{h\}$. If $h \dashv g$, then we call g *right orthogonal* to h, $(g \vdash h)$. From strict convexity of $\mathcal{C}$ follows uniqueness of the right-orthogonal direction $\|g$ to each direction $\|h$; smoothness of $\mathcal{C}$ causes uniqueness of the l-orthogonal direction $\|h$ to each direction $\|g$.

If (and only if) $\mathcal{C}$ is a Radon curve, (c.f. [9], [6]), then the l-orthogonality structure of $(\mathbf{A^2}, \mathcal{C})$ is symmetric. If (and only if) $\mathcal{C}$ is an ellipse, the Minkowski plane $(\mathbf{A^2}, \mathcal{C})$ is a euclidean plane and orthogonal lines are parallel to pairs of conjugate diameters of $\mathcal{C}$, (c.f. e.g. [8]).

b) The problem of characterizing Minkowski planes, where all triangles are orthocentric, can be solved mainly by geometric reasoning using well-known theorems of planar projective geometry. We will present these 'tools' in the following; (for references see e.g. [3], [5], [8]):

By adding an *ideal line u* to the (given) real affine plane $\mathbf{A^2}$ this plane becomes the 'projective extended affine plane' $\mathbf{P^2} := \mathbf{A^2} \cup u$, $(\mathbf{A^2} \cap u = \emptyset)$, which is a real projective plane.

A linear bijection π, (i.e. a *projectivity*), in the point set of a real projective line $g \in \mathbf{P^2}$ is an *'involutoric projectivity'* (shortly called an *'involution'*), $(\pi^2 = id_g)$, if there exists one pair of mutually assigned points.

If π is an involutoric projectivity on g, then every pair of corresponding points P, $P' := P^\pi$ is interchangable. Therewith an involution π is well defined already by two permissible pairs $(A, A' = A^\pi)$, $(B, B' = B^\pi)$.

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[1] We will further on abbreviate the prefixes 'left-' resp 'right-' by 'l-' resp. 'r-'.

Each pair of corresponding points (A, A') devides g into two projectively connected segments. The involutoric projectivity π is called an *elliptic involution*, if there exist two 'separating pairs' (A, A'), (B, B'); that means that each segment of g with respect to (A, A') contains one of the points B, B'. There are only separating pairs in an elliptic involution and no pair (F, F') with $F = F'$.

If the pairs of corresponding points do not separate each other, π is called a *hyperbolic involution* and there exist two fixed points $F_i = F_i'$.

c) We continue the listing of projective geometric facts with the following 'quadrangle properties'[2]:

> (QP1): Let $\mathcal{Q} \subset \mathbf{P}^2$ be a complete quadrangle consisting of four points $P_i \in \mathbf{P}^2$ in general position and their three pairs of connecting lines P_iP_j, P_kP_l, $(i \neq j \neq k \neq l)$. Every line g, which contains none of the points P_i, intersects the three pairs of sides of $\mathcal{Q}$ in pairs of an involutoric projectivity[3] δ; (c.f. Figure 1).

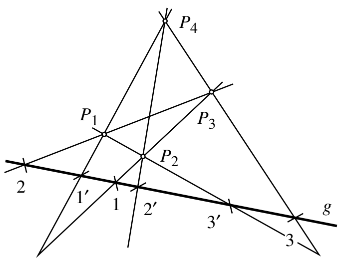

Figure 1

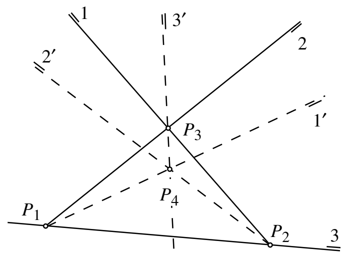

Figure 2

Another quadrangle property, which is somehow reverse to (QP1), is

> (QP2): Let δ be an involutoric projectivity on a line g and let $\{P_1, P_2, P_3\}$ be a triangle with $P_i \notin g$. Then $\{P_1, P_2, P_3\}$ can be completed by a uniquely determined fourth point P_4 to a quadrangle $\mathcal{Q}$ such that $\mathcal{Q}$ induces the given involution δ in g.

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[2]The following theorems are valid in projective planes over any commutative field K with $char\ K \neq 2$. With respect to the real geometry of Minkowski planes we restrict ourselves to real projective planes.

[3]The points $P_i \in \mathcal{Q}$ can be interpreted as the fundamental points of a pencil $\mathcal{P}_\mathcal{Q}$ of conics. Within this pencil the three pairs of sides of $\mathcal{Q}$ represent the singular curves of 2nd order. Let g be a line and δ the involutoric projectivity in g induced by $\mathcal{Q}$ according to (QP1); then corresponding points X, $X^\delta \in g$ belong to a conic of $\mathcal{P}_\mathcal{Q}$. Thus δ is induced by the pencil $\mathcal{P}_\mathcal{Q}$ of the conics through $\{P_1, \dots, P_4\}$ and is called " Desargues involution" in g with respect to $\mathcal{P}_\mathcal{Q}$. If δ is hyperbolic, there are two conics in $\mathcal{P}_\mathcal{Q}$ touching g in the fixed points of δ, c.f. [3].

REMARK. Putting $g = u$ the ideal line of the projectively extended affine plane $\mathbf{P^2}$ one deduces from (QP1) immediately that the ideal points $1, 2, 3$ of the sides of a triangle $\{P_1, P_2, P_3\}$ and those $1', 2', 3'$ of vertex transversals through a fourth point P_4 fit into one involutoric projectivity δ in u, c.f. [7] and Figure 2.

From (QP2) it follows that, to a given triangle $\{P_1, P_2, P_3\}$ and an involution δ in the (ideal) line u, the vertex transversals $P_i \vee i'$, $(i := P_j P_k \cap u,\ i' := i^\delta,\ i, j, k \in \{1, 2, 3\})$ are concurrent with a fourth point P_4. Obviously each triplet $\{P_i, P_j, P_k\}$ of $\{P_1, \ldots, P_4\}$, via δ, leads to the remaining fourth point P_l. This is, in the sense of Giering [7], the projective geometric explanation for the fact that, (in euclidean planes), a (general) triangle and its orthocenter represent four points, each being the orthocenter of the remaining three.

d) Let $\mathbf{P^2} := \mathbf{A^2} \cup u$ be a projective extended affine plane with a given elliptic involutoric projectivity δ in u. With respect to δ as the 'absolute involution', $\mathbf{A^2}$ becomes a euclidean plane. Orthogonal lines have δ-conjugate ideal points. Ellipses, whereby all pairs of conjugate diameters have δ-conjugate ideal points, are 'circles'. Transforming $\mathbf{A^2}$ by a certain affinity α will show such an ellipse $\mathcal{C}$ as an ordinary circle and $(\mathbf{A^2}, u, \delta' := \alpha^{-1}\delta\alpha)$ becomes the usual projective extended, elementary geometric plane. So it is possible to visualize an ellipse by an ordinary circle without loss of generality.

Minkowski planes containing only orthocentric triangles

We will show that if (and only if) every triangle of a Minkowski plane is l-orthocentric, then this plane must be euclidean:

THEOREM 1. *Assuming that* $(\mathbf{A^2}, \mathcal{C})$ *is a* Minkowski *plane with a strictly convex, centrally symmetric smooth gauge curve* $\mathcal{C}$ *and that every triangle in* $(\mathbf{A^2}, \mathcal{C})$ *has concurrent l-altitudes implies* $(\mathbf{A^2}, \mathcal{C})$ *to be a euclidean plane with an ellipse* $\mathcal{C}$ *as unit circle.*

Proof. a) As a first step we prove that there exist triangles $\mathcal{T} \subset (\mathbf{A^2}, \mathcal{C})$ such that they together with their l-orthocenter L form a quadrangle in $\mathbf{A^2}$.

Because of strict convexity of $\mathcal{C}$, a triangle $\mathcal{T}$ cannot have parallel l-altitudes, as there is a bijection of diameter directions of $\mathcal{C}$ and corresponding l-orthogonal directions. Thus the assumed l-orthocenter L neither can be an ideal point nor a point on the sides of $\mathcal{T}$ different from a vertex.

Furthermore, it cannot be true for all triangles $\mathcal{T}$ that the l-orthocenter L, (which we assumed to exist), coincides with a vertex. Triangles possessing that property, are called *l-right triangles*.

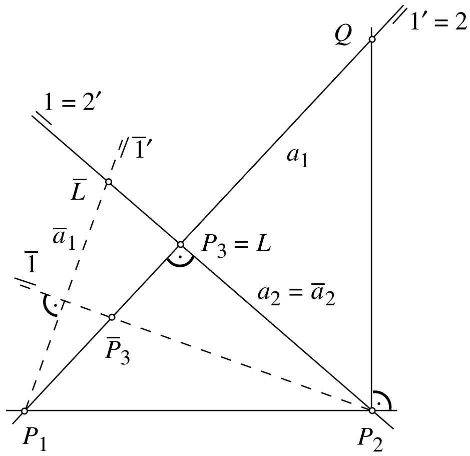

Figure 3

Let $\mathcal{T} = (P_1, P_2, P_3)$ be l-right with l-orthocenter $L = P_3$ and let us choose a point $\bar{P}_3 \in P_1P_3$ as vertex of a new triangle $\bar{\mathcal{T}} := (P_1, P_2, \bar{P}_3)$, only excluding P_1, P_3 and the intersection Q_2 of P_1P_3 with the l-normal to P_1P_2 through P_2; c.f. Figure 3. Therewith follows that the l-altitudes $a_1 \dashv P_2P_3$ and $\bar{a}_1 \dashv P_2\bar{P}_3$, $(\bar{a}_1 \neq P_1P_2)$, of $\mathcal{T}$ resp. $\bar{\mathcal{T}}$ must differ, while a_2 still equals $\bar{a}_2$. So the l-orthocenter $\bar{L}$ of $\bar{\mathcal{T}}$ will neither coincide with a vertex of $\bar{\mathcal{T}}$ nor with any other point of a side of $\bar{\mathcal{T}}$, which proves the existence of non l-right triangles.

b) Let $\mathcal{T} = (P_1, P_2, P_3)$ be a non l-right triangle with l-orthocenter L. Then, by (QP1), the quadrangle (P_1, P_2, P_3, L) induces a non-degenerated, involutoric projectivity δ in the ideal line u of the projective extension $\mathbf{P^2}$ of $(\mathbf{A^2}, \mathcal{C})$. As δ is already determined by two pairs of corresponding ideal points $i := P_jP_k \cap u$ and $i' = P_iL =: i^\delta$, $(i, j, k = 1, 2, 3)$, every l-orthocentric triangle $\bar{\mathcal{T}} = (\bar{P}_1, \bar{P}_2, \bar{P}_3)$ with two sides parallel to sides of $\mathcal{T}$ must lead to the same Desargues involution δ in u, c.f. Figure 4.

Thus δ is defined as well by the pairs $(2, 2')$, $(3, 3')$ as by $(2, 2')$, $(\bar{1}, \bar{1}')$. Let now $\tilde{\mathcal{T}}$ be another arbitrarily given (l-orthocentric) triangle; then, by (QP1) and by replacing each side of triangle $\mathcal{T}$ one by one with sides parallel to the sides of $\tilde{\mathcal{T}}$, it follows that every non l-right triangle $\tilde{\mathcal{T}}$, together with its l-orthocenter $\tilde{L}$ necessarily induces the same Desargues involution δ in u as $\mathcal{T} \cup \{L\}$ does. Obviously, if $\tilde{\mathcal{T}}$ is l-right, its l-altitudes and sides will suit to δ, too.

c) From b) follows that the l-orthogonality structure $\dashv$ in $(\mathbf{A^2}, \mathcal{C})$, defined by $\mathcal{C}$, is *symmetric* and it induces in u a regular involution δ, which is either elliptic or hyperbolic. Therewith

$\mathcal{C}$ is a Radon curve, (c.f. [9], [6]), which in addition integrates the vector field of l-normal vectors along the diameters of $\mathcal{C}$. Let $D(\delta)$ be a coordinate matrix of δ, then $\mathcal{C}$ solves the differential equation

$$\dot{\mathbf{x}} = D(\delta) \cdot \mathbf{x}.$$

The solutions of that differential equation are homothetic ellipses in a first case or homothetic (conjugate) hyperbolas in a second one, c.f. Figure 5. The latter case contradicts to the restriction that $\mathcal{C}$ is convex.

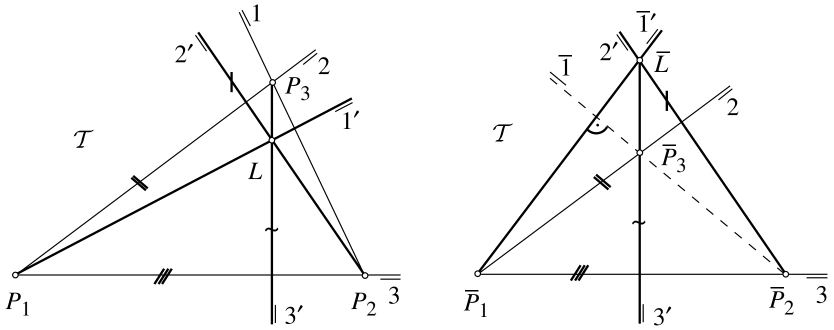

Figure 4

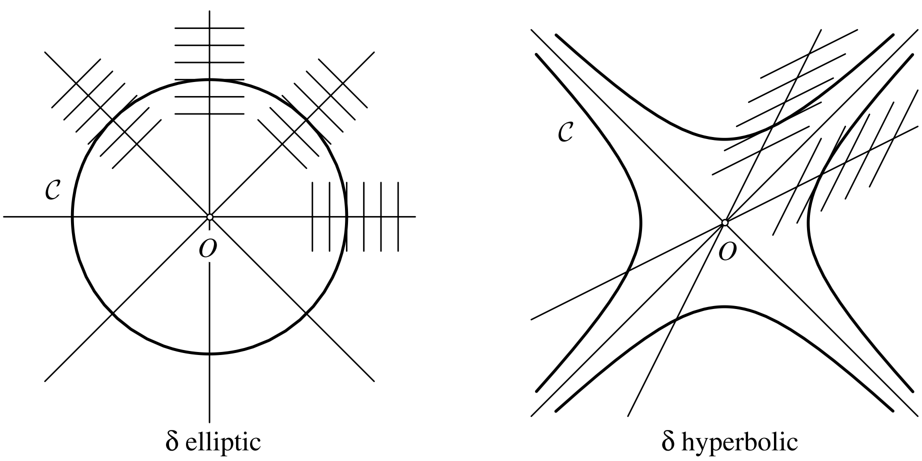

Figure 5

With respect to δ as the absolute involution in u the plane $(\mathbf{A}^2, \mathcal{C})$ is a euclidean plane with the unit circle $\mathcal{C}$. As in euclidean planes orthogonality is symmetric and every triangle is orthocentric, we conclude that euclidean planes are the only Minkowski planes, where all triangles are l-orthocentric. □

Orthocentric triangles in general Minkowski planes

Let $(\mathbf{A^2}, \mathcal{C})$ be a non-euclidean Minkowski plane, then (c.f. [10, p. 79]) $\mathcal{C}$ possesses at least one pair of mutually left and right-orthogonal radii $\mathbf{x}_1, \mathbf{x}_2$. By applying a certain affine transformation we can visualize $\{O, A_1 := \mathbf{x}_1, A_2 := \mathbf{x}_2\}$ as cartesian basis, such that $\mathcal{C}$ touches a (euclidean) circle $\mathcal{C}_e$ at $\pm\mathbf{x}_i$, c.f. Figure 6.

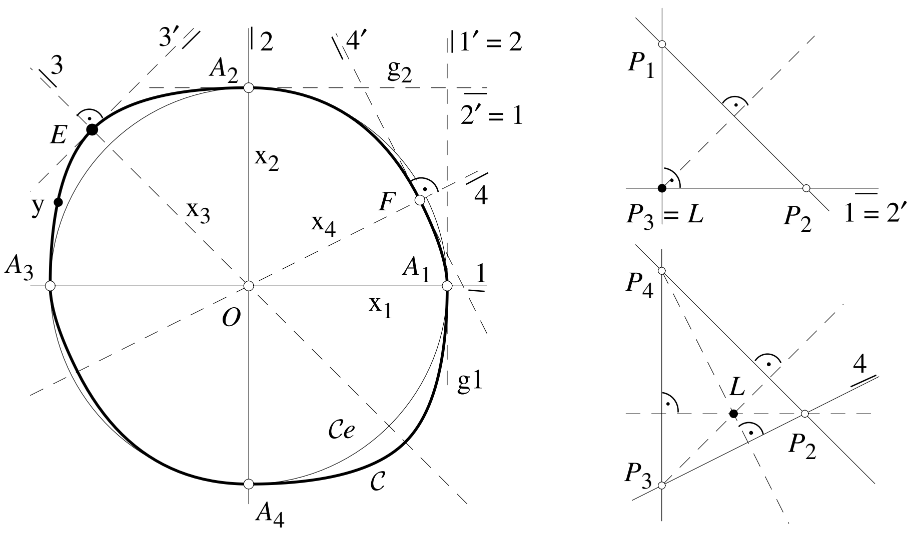

Figure 6

Because of $\mathcal{C} \neq \mathcal{C}_e$ there must exist a point $\mathbf{y} \in \mathcal{C}$ such that its euclidean norm is less or greater than 1. Applying the mean value theorem to that 'quarter arc' of $\mathcal{C}$ containing $\mathbf{y}$, it follows that there must exist a point $E := \mathbf{x}_3 \in \mathcal{C}$ with extremal euclidean norm. So E and the tangent vector $\mathbf{g}_3$ in E suit into the orthogonality structure of the euclidean plane $(\mathbf{A^2}, \mathcal{C}_e)$, too. Using the same argument for another arc of $\mathcal{C}$ containing neither $\mathbf{y}$ nor $-\mathbf{y}$ will lead to (at least) a fourth euclidean orthogonal pair $(\mathbf{x}_4, \mathbf{g}_4)$.

Now we have to distinguish two cases:

a) $\mathbf{x}_4$ equals the Minkowski unit vector $\mathbf{g}_3$ or $-\mathbf{g}_3$.
Then any triangle with sides parallel to three of the four directions $\mathbf{x}_i$ is a (euclidean) right triangle and thus l-orthocentric in a trivial manner.

b) $\mathbf{x}_4$ is not equal to the Minkowski unit vector $\mathbf{g}_3$ or $-\mathbf{g}_3$.
Then, besides l-right triangles with sides parallel to $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ or $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_4$, there exist l-orthocentric non-right triangles with sides parallel to $\mathbf{x}_1, \mathbf{x}_3, \mathbf{x}_4$ resp. $\mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$.

In the following we will show that in the 'neighbourhood' of each of the above mentioned l-orthocentric triangles there is a dense set of l-orthocentric triangles:

Let us choose two points $B_1 := \mathbf{y}_1,\ B_2 := \mathbf{y}_2 \in \mathcal{C}$ within an ε-neighbourhood of A_1 resp. A_2 such that the unit vectors $\mathbf{y}_i$ and their l-orthogonal unit vectors $\mathbf{g}_i$ form two different pairs $(\mathbf{y}_1, \mathbf{g}_1),\ (\mathbf{y}_2, \mathbf{g}_2)$ of a well defined involution $\pi(\mathbf{y}_1, \mathbf{y}_2) =: \pi_{12}$, c.f. Figure 7. Obviously π_{12} is always *elliptic*: From strict convexity of $\mathcal{C}$ it follows that the pairs $(\mathbf{y}_1, \mathbf{g}_1), (\mathbf{y}_2, \mathbf{g}_2)$ must separate each other. Using π_{12} as involution of conjugate diameters of a pencil of homothetic ellipses one can find two homothetic ellipses $\mathcal{C}_1, \mathcal{C}_2$ concentric with $\mathcal{C}$ and touching $\mathcal{C}$ in the points B_i. Thereby $\mathcal{C}$ will exceed the ringlike domain between $\mathcal{C}_1$ and $\mathcal{C}_2$, if we choose B_i suitably 'close' to A_i. So there exists at least one 'extremal' ellipse homothetic to $\mathcal{C}_1$, which touches $\mathcal{C}$ in an extremal point $E \neq B_1, B_2$ (c.f. Figure 7). As the pair of l-orthogonal directions $(OE, \mathbf{g}_E)$ suits into π_{12}, too, any triangle with sides parallel to $OB_1,\ OB_2,\ OE$ is l-orthocentric.

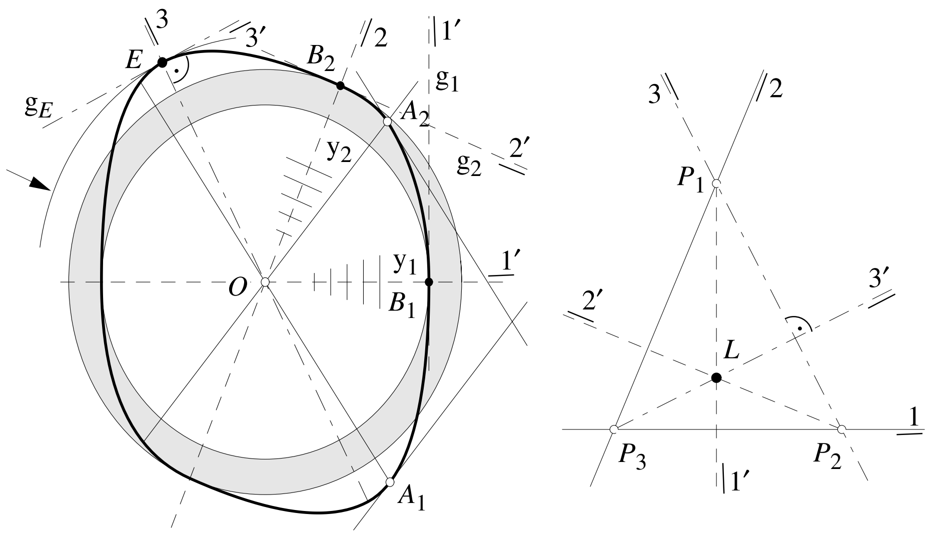

Figure 7

Obviously it will not happen for all possibly chosen points B_i that $OE \dashv\vdash OB_i$, $(i = 1$ or 2 and with $\dashv\vdash$ meaning '$\dashv$ or $\vdash$'): As B_i can be chosen within ε-neighbourhoods of A_i not containing E and because of $OA_1 \dashv\vdash OA_2$ it follows that OE not$\dashv\vdash$ OB_i, $(i = 1$ or $2)$. Therewith we have proved the following

THEOREM 2. *In any (non-euclidean) Minkowski plane with strictly convex smooth gauge circle $\mathcal{C}$ there exists a (dense) set of triplets of directions so that each triangle with sides parallel to such a triplet of directions is non-trivially l-orthocentric.*

Series of altitude triangles to a given triangle

Let T be a triangle which is not l-orthocentric. Then its l- resp. r-altitudes form a triangle T'. Let T'' be the triangle of l- resp. r-altitudes of T', and so on. So the question arises whether these iteration processes converge to *limit orthocenters* or not.

We may consider the series of purely l-altitude triangles or of purely r-altitude triangles as well as the series of mutually l- and r-altitudes, starting either with l-altitudes or with r-altitudes. In the following we will refer to (eventually existing) limit orthocenters as O_{ll}, O_{rr}, O_{lr} and O_{rl}. Figure 8 shows a triangle T with four limit orthocenters[4]. $O_{ll},\ O_{rr},\ O_{lr},\ O_{rl}$ and its 'beer mat point' B.

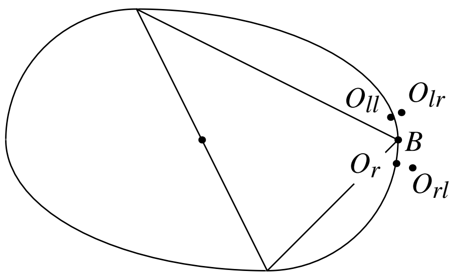

Figure 8

Obviously mutual application of l- and r-orthogonality leads to two sets of perspective similar altitude triangles. Thus O_{lr} and O_{rl} do exist in the projective extended Minkowski plane; they are the common centers of similarities (resp. translations), c.f. Figure 9, and they may occur as attractors as well as repulsors. In the following we will discuss the existence of the remaining limit orthocenters $O_{ll},\ O_{rr}$.

As a first step we ask for the limit of the direction of consecutive l- (r-) -orthogonals to an arbitrarily given direction OB_1. To avoid tedious formulations we will identify opposite

[4]The result is based on numerical treatment; the gauge circle $\mathcal{C}$ consists of a sequel of quarters of euclidean circles and ellipses. As T is inscribed to $\mathcal{C}$ such that one side is a diameter of $\mathcal{C}$, it follows that the 'beer mat point' B, (i.e. Asplund-Grünbaum's triangle orthocenter, c.f. [1]), coincides with the opposite vertex to that side. So in the sense of [1] T might be called Thales-*triangle*, as Figure 8 shows a generalized version of the elementary geometric theorem of Thales.

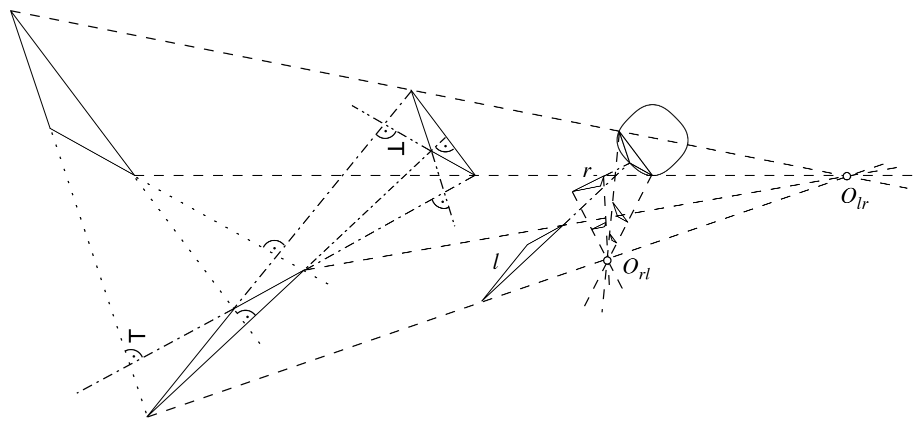

Figure 9

points of $\mathcal{C}$ and to ensure continuity in changes of directions we now have to assume that $\mathcal{C}$ is a C^1-curve. Then the following proposition holds:

PROPOSITION 1. *Let the gauge curve $\mathcal{C}$ of a Minkowski plane be a centrally symmetric, strictly convex C^1-curve, and let $\{B_i\}$ be a series of points of $\mathcal{C}$, starting with an arbitrarily given point B_1 and with B_{n+1} such that $OB_{n+1} \dashv (\vdash) OB_n$, $(n = 1, 2, \ldots)$. Then*

$$\lim_{n\to\infty}(B_n, B_{n+1}) = (A, \bar{A}) \text{ with } OA \dashv\vdash O\bar{A}.$$

The pair $(OA, O\bar{A})$ represents conjugate semidiameters *of $\mathcal{C}$.*

As $\mathcal{C}$ possesses at least one pair of conjugate semidiameters, c.f. [10, p.79], Proposition 1 is obvious.

REMARK 1. Recieving $(A, \bar{A})$ as the attractor pair of a limit process based on l-orthogonality implies that $(A, \bar{A})$ is repulsive with respect to r-orthogonality, c.f. Figure 10.

REMARK 2. Let $\mathcal{C}$ be a Radon curve, (c.f. [10, p.127] and [9]), what is equivalent to coinciding l- and r-orthogonality relations. In other words, to any point B_1 the above mentioned iteration process delivers (B_1, B_2) as fixed pair. If $\mathcal{C}$ is a Radon curve different from an ellipse, there exist triangles T, which are not orthocentric. The four limit orthocenters of T coincide, but they are different from the beer mat point B of T.

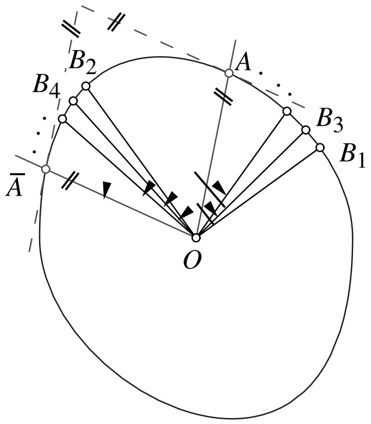

Figure 10

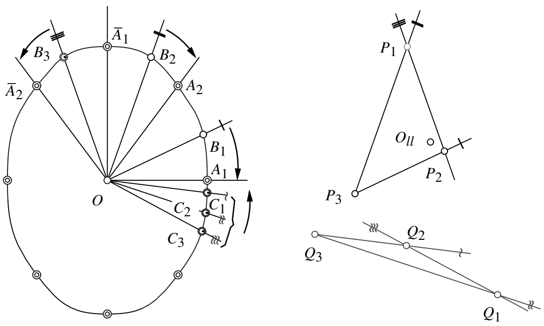

Figure 11

REMARK 3. Let $\mathcal{C}$ be a C^1 gauge curve free from corresponding pairs of Radon-arcs, so there exists no continuous set of directions with coinciding l- and r-orthogonal directions. Then, by Proposition 1, it is possible to construct *all* pairs of 'conjugate semidiameters' $(OA_i, O\bar{A}_i)$ of $\mathcal{C}$, $(i \in I \subset N)$.

Let T be a triangle with sides parallel to OB_j, $(j = 1, 2, 3)$ and let us consider the sequel of consecutive left-altitude triangles. If the limit pairs $(OA_j, O\bar{A}_j)$ to the three directions OB_j are different, then there will occur a limit orthocenter O_{ll}, c.f. Figure 11. If all three directions lead to the same limit pair $(OA, O\bar{A})$, T and its l-altitude triangle T' tend to triplets of parallel or coinciding lines, and so a limit orthocenter O_{ll} of T will not exist, c.f. Figure 11.

Final remark and acknowledgement

Theorem 1 gives one of many characterizations of euclidean planes (among Minkowski planes). It turns out that a proof of this theorem using facts from classical projective geometry can be carried out purely 'synthetically'. It seems promising to use projective geometric tools also to study higher dimensional analogues of Theorem 1.

The author wants to express sincere thanks to K. Nestler (Dresden) for stimulating and fruitful comments.

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