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Marko Loparic* · Yves Pochet · Laurence A. Wolsey

The uncapacitated lot-sizing problem with sales and safety stocks**

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Abstract. We examine a variant of the uncapacitated lot-sizing model of Wagner-Whitin involving sales instead of fixed demands, and lower bounds on stocks. Two extended formulations are presented, as well as a dynamic programming algorithm and a complete description of the convex hull of solutions. When the lower bounds on stocks are non-decreasing over time, it is possible to describe an extended formulation for the problem and a combinatorial separation algorithm for the convex hull of solutions. Finally when the lower bounds on stocks are constant, a simpler polyhedral description is obtained for the case of Wagner-Whitin costs.

Key words. lot-sizing – production planning – mixed integer programming – integral polyhedra – extended formulations

1. Introduction

The original uncapacitated single item lot-sizing model of Wagner-Whitin [15] has been extended in many directions, to include among others backlogging [16], capacities [3], [10], start-ups [5] and production in series [9]. There is a vast literature both on these problems and on more general multi-item problems containing the single-item problem as subproblem [7], [12]. In all these models demand is assumed to be known exactly, and the usual objective is to minimize total cost. Recently we have encountered several models constructed by an industrial partner in which demand is not pre-specified, but bounds on potential sales are presented, and the objective is profit maximization [6]. In an attempt to improve the formulation of these models, we have been led to consider a single item lot-sizing problem with sales, and in addition, for reason of practical applicability, we have also incorporated lower bounds on stocks (safety stocks) in the model.

Below we first specify the ULS^3 problem (Uncapacitated Lot-sizing Problem with Sales and Safety Stocks), and then formulate it as a mixed integer program. We then derive equivalent formulations in which there is a fixed demand (positive or negative) as well as potential sales. Next we analyse the structure of the optimal solutions which allows us to conclude in standard fashion that dynamic programming provides a poly-

M. Loparic: CORE, Université Catholique de Louvain. e-mail: loparic@core.ucl.ac.be

Y. Pochet: CORE and IAG, Université Catholique de Louvain. e-mail: pochet@core.ucl.ac.be

L.A. Wolsey: CORE and INMA, Université Catholique de Louvain. e-mail: wolsey@core.ucl.ac.be

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nomial algorithm for ULS^3. We then terminate the introduction with an overview of later sections.

Problem ULS^3 is specified by a time horizon n, an initial stock $L_0 \geq 0$, and for each period $t = 1, \dots, n$, upper bounds $u_t \geq 0$ on sales, lower bounds on stocks $L_t \geq 0$, and objective coefficients consisting of unit selling prices p_t, unit production and storage costs c_t and h_t, and fixed set-up costs of production f_t.

Introducing variables

x_t: production in period t,
s_t: stock at the end of period t (s_0: initial stock),
v_t: sales in period t,
$y_t \in \{0, 1\}$, a set-up variable with $y_t = 1$ if $x_t > 0$,

we obtain the profit maximization formulation

$$\max \sum_{t=1}^{n} p_t v_t - \sum_{t=1}^{n} c_t x_t - \sum_{t=1}^{n} f_t y_t - \sum_{t=1}^{n} h_t s_t,$$

$$(F1) \qquad \begin{aligned} s_{t-1} + x_t &= v_t + s_t, && \text{for } t = 1, \dots n, && (1)\\ 0 \leq v_t &\leq u_t, && \text{for } t = 1, \dots n, && (2)\\ x_t &\leq M y_t, && \text{for } t = 1, \dots n, && (3)\\ s_t &\geq L_t, && \text{for } t = 1, \dots n, && (4)\\ s_0 &= L_0, && && (5)\\ x_t \geq 0, 0 \leq y_t &\leq 1, && \text{for } t = 1, \dots n, && (6)\\ y_t &\text{ integral}, && \text{for } t = 1, \dots n, && (7) \end{aligned}$$

where M is a large positive constant. Constraint (3) forces y_t to one when x_t is positive, but there is always an optimal solution with $x_t < M$ unless $c_t + \sum_{i=t}^{n} h_i < 0$. Any value of M greater than $\sum_{t=1}^{n} u_t + \max_{t=1,\dots,n} L_t$ is sufficient. Note that it is possible to eliminate the variables x_t or s_t from the objective function, and so one can assume for convenience either that $c_t = 0$ for all t, or that $h_t = 0$ for all t.

This model could also be interpreted as a (deterministic) lost sales model related to classical lost sales model in (stochastic) inventory theory. In this case, the upper bound on sales would be the demand, and the lost sales would be $(u_t - v_t)$, for all t.

We now present an equivalent problem. The difference is the introduction of (possibly negative) demands d_t and the new stock variables σ_t which have lower bound of zero. The constraints now take the form:

$$\max \sum_{t=1}^{n} p_t v_t - \sum_{t=1}^{n} c_t x_t - \sum_{t=1}^{n} f_t y_t - \sum_{t=1}^{n} h_t \sigma_t,$$

$$(F2) \qquad \begin{aligned} \sigma_{t-1} + x_t &= d_t + v_t + \sigma_t, && \text{for } t = 1, \dots n, && (8)\\ 0 \leq v_t &\leq u_t, && \text{for } t = 1, \dots n, && (9)\\ x_t &\leq M y_t, && \text{for } t = 1, \dots n, && (10)\\ \sigma_0 = 0, \sigma_t &\geq 0, && \text{for } t = 1, \dots n, && (11)\\ x_t \geq 0, 0 \leq y_t &\leq 1, && \text{for } t = 1, \dots n, && (12)\\ y_t &\text{ integral}, && \text{for } t = 1, \dots n. && (13) \end{aligned}$$

We go from the first formulation to the second by taking $\sigma_t = s_t - L_t$ and $d_t = L_t - L_{t-1}$, for all t. To go from the second formulation to the first, we take $L_0 = \max\{0, \max_t[-\sum_{i=1}^{t} d_i]\}$, $L_t = L_0 + \sum_{i=1}^{t} d_i$, and $s_t = L_t + \sigma_t$ for all t. The tradeoff between (F1) and (F2) is between having lower bounds L_t on stocks, or having external demands d_t. Figure 1 presents an instance of this transformation.

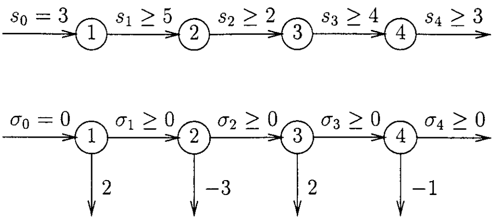

Fig. 1. Transforming lower bounds on stocks into demands

Defining $d_{ij} = \sum_{t=i}^{j} d_t$ and using $\sigma_t = \sum_{i=1}^{t} x_i - \sum_{i=1}^{t} v_i - d_{1t} \geq 0$ from (8) to eliminate the variables σ_t, we obtain a third formulation which will also be useful:

$$\max \sum_{t=1}^{n} p_t v_t - \sum_{t=1}^{n} c_t x_t - \sum_{t=1}^{n} f_t y_t,$$

$$(F3) \qquad \begin{aligned} \sum_{i=1}^{t} x_i \geq \sum_{i=1}^{t} v_i + d_{1t}, &\quad \text{for } t = 1, \ldots n, &(14)\\ 0 \leq v_t \leq u_t, &\quad \text{for } t = 1, \ldots n, &(15)\\ x_t \leq M y_t, &\quad \text{for } t = 1, \ldots n, &(16)\\ x_t \geq 0, 0 \leq y_t \leq 1, &\quad \text{for } t = 1, \ldots n, &(17)\\ y_t \text{ integral}, &\quad \text{for } t = 1, \ldots n. &(18) \end{aligned}$$

Note that the three formulations are equivalent in the sense that there is a 1-1 correspondence between their feasible solutions. Let $X \subseteq R^{3n}$ be the set of feasible solutions of (F3) described by (14)–(18).

To derive an algorithm for ULS[3] we next consider the structure of the optimal solutions. We suppose that $f_t \geq 0$ and $c_t + \sum_{i=t}^{n} h_i \geq 0$ for all t, so that there is always an optimal solution with $x_t < M$. Consider formulation (F2). If $y^* \in \{0, 1\}^n$ is fixed, the remaining problem is a minimum cost flow problem in the network shown in Fig. 2. In a basic optimal solution (x^*, v^*, σ^*), the basic variables form an acyclic graph [16]. Such a basic optimal solution decomposes in a standard way into a sequence of regeneration intervals.

We now look at intervals $[i, i+1, \ldots, j]$ in which $\sigma^*_{i-1} = 0, \sigma^*_i > 0, \ldots, \sigma^*_{j-1} > 0$, and either $\sigma^*_j = 0$, or $j = n$ and $\sigma^*_n > 0$. Consider first an interval with $\sigma^*_j = 0$ called a *regeneration interval of Type 1*, see Fig. 3. Clearly if $0 < x^*_k < M$ and $0 < x^*_l < M$ with $i \leq k < l \leq j$, the set of basic variables forms a cycle. So there is at most one period k in the interval $[i, \ldots, j]$ with $x^*_k > 0$. If $x^*_k > 0$, all variables v^*_l are equal to

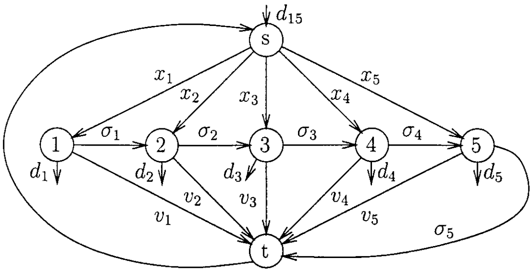

Fig. 2. ULS^3 in a network

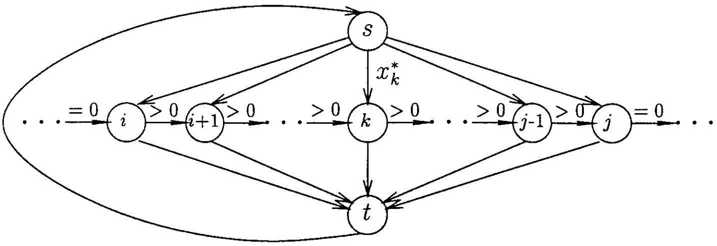

Fig. 3. Regeneration interval of Type 1

0 or u_l to avoid creating a cycle. Alternatively if $x_k^* = 0$ for all $k = i, \ldots, j$, then one variable v_l^* can be basic with $0 < v_l^* < u_l$.

For regeneration intervals of Type 2 with $j = n$ and $\sigma_n^* > 0$, the situation is as shown in Fig. 4. Again (t, s) is basic, and thus $x_k^* = 0$ and $v_k^* = 0$ or u_k for all $k = i, \ldots, j$.

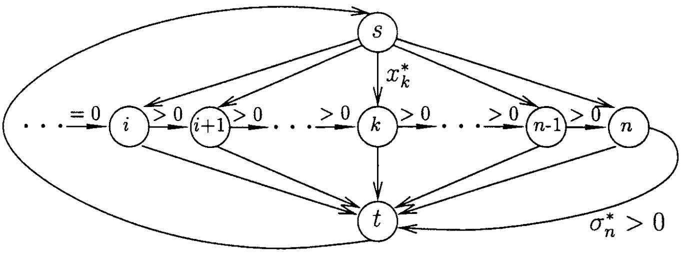

Fig. 4. Regeneration interval of Type 2

We now derive a dynamic program or shortest path problem using regeneration intervals to solve ULS^3.

Consider a regeneration interval $[i, \ldots, j]$ of Type 1. The value β_{ij} of an optimal solution within this interval can be found by solving $j - i + 2$ minimum cost flow problems. There are $j - i + 1$ problems, one for each $l = i, \ldots, j$. In the problem associated with a fixed l, we allow $x_l > 0$ and $y_l = 1$, and $x_t = y_t = 0$ for $t \neq l$.

There is also a final problem in which $x_t = 0$ for $t = i, \dots, j$, which corresponds to the case of no production (this also includes the regeneration intervals of Type 2). For $l \in \{i, \dots, j\}$, problem l is

$$\begin{aligned} \beta^l_{ij} = \max \sum_{t=i}^{j} p_t v_t - c_l x_l - \sum_{t=i}^{j} h_t \sigma_t - f_l, & \\ \sigma_{l-1} + x_l = d_l + v_l + \sigma_l, & \\ \sigma_{t-1} = d_t + v_t + \sigma_t, & \quad \text{for } t = i, \dots j,\ t \neq l, \\ \sigma_{i-1} = 0, & \\ \sigma_j = 0, & \quad \text{if } j \neq n, \\ 0 \le v_t \le u_t, & \quad \text{for } t = i, \dots j, \\ x_t, \sigma_t \ge 0, & \quad \text{for } t = i, \dots j. \end{aligned}$$

β^0_{ij} is defined similarly but with $x_l = 0$ and without the cost term $-f_l$, and $\beta_{ij} = max[\beta^0_{ij}, max_{l=i,\dots,j} \beta^l_{ij}]$. Note that it is not necessary to use a general linear programming algorithm to calculate β^l_{ij}.

An optimal basic solution corresponds to a maximum value sequence of regeneration intervals. If $F(j)$ is defined as the cost of a maximum value sequence of regeneration intervals covering periods from 1 up to j, $F(j)$ can be computed by the recursion $F(j) = \max_{i \le j}\{F(i-1) + \beta_{ij}\}$, starting with $F(0) = 0$. The optimal value of problem ULS^3 is given by $F(n)$. Working backwards leads to an optimal solution.

We now discuss the contents of the paper. In Sect. 2 we give the main result, a family of valid inequalities, called (t, S, R) inequalities, that are shown to provide a complete description of the convex hull of X. In Sect. 3 we consider the special case of ULS^3 in which the lower bounds L_t are nondecreasing over time in formulation (F1), or alternatively $d_t \ge 0$ for all t in formulations (F2) and (F3). We first derive an extended formulation allowing one to solve ULS^3 directly by linear programming, and then give a combinatorial separation algorithm for the family of (t, S, R) inequalities. Finally in Sect. 4 we provide an extended formulation for the case where $d_t = 0$ for all t in formulation (F2) and where we have “Wagner-Whitin” costs. We terminate with a brief discussion of open questions and extensions.

2. The convex hull

To motivate the inequalities developed in this section, consider the small example shown in Fig. 5, where $d = (3, -2, 4, 1)$ and $u = (1, 1, 1, 1)$. Examining periods 3 and 4, the inflow-outflow inequalities from [14], or the (l, S) inequality of [1] with $l = 4$, $S = \{3, 4\}$ give the valid inequality

$$x_3 + x_4 \le 5y_3 + 1y_4 + v_3 + v_4 + \sigma_4$$

where the coefficient $(d_3 + d_4)$ of y_3 is the amount of inflow in x_3 that could escape through the demand nodes d_3, d_4, and not through the arcs v_3, v_4 or σ_4.

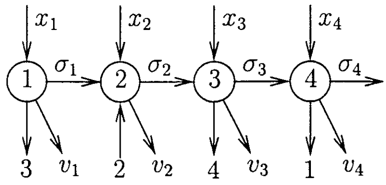

Fig. 5. Small example

However the above inequality does not take into account the fact that d_2 is negative. Because $d_2 = -2 < 0$, $\sigma_2 \geq -d_2 - v_2$, and so $\sigma_2 + x_3 \leq (d_3 + d_4) + v_3 + v_4 + \sigma_4$ implies $x_3 \leq (d_2 + d_3 + d_4) + v_2 + v_3 + v_4 + \sigma_4$. Thus the maximum inflow through x_3 that does not flow out through v_2, v_3, v_4 or σ_4 is $d_2 + d_3 + d_4 = 3$. This leads us to the inequality

$$x_3 + x_4 \leq 3y_3 + 1y_4 + v_2 + v_3 + v_4 + \sigma_4.$$

Now by introducing the complementary variables $\overline{v}_j = u_j - v_j$ for $j \in R = \{1, 3\}$, we convert u_1 and u_3 into fixed demands but with additional inflow $\overline{v}_j$, leading to the situation shown in Fig. 6:

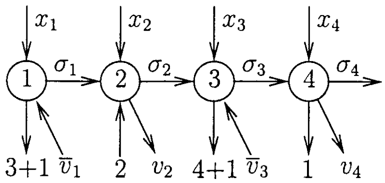

Fig. 6. Small example after substitutions

Now we obtain

$$x_3 + x_4 \leq 4y_3 + 1y_4 + v_2 + v_4 + \sigma_4$$

with the coefficient of y_3 equal to $(d_2 + d_3 + u_3 + d_4)$. Eliminating σ_4 via the equation $\sum_{t=1}^{4} x_t = \sum_{t=1}^{4} v_t + d_{14} + \sigma_4$, obtained by summing (8) for $t = 1, \ldots, 4$, the resulting inequality is

$$x_1 + x_2 + 4y_3 + 1y_4 \geq 6 + v_1 + v_3. \tag{19}$$

Now we describe formally a family of valid inequalities, called (t, S, R) inequalities, generalizing the previous example. We show that they provide all the inequalities missing in formulation (F3) to describe the convex hull of the solutions of ULS^3.

In order to compute the coefficients of the y variables, we define for $R \subseteq \{1, \ldots, n\}$:

1. $d_{ij} = \sum_{i \leq k \leq j} d_k$, for $1 \leq i \leq j \leq n$, $d_{ij} = 0$, if $i > j$;
2. $u^R_{ij} = \sum_{k \in R, i \leq k \leq j} u_k$, for $1 \leq i \leq j \leq n$, $u^R_{ij} = 0$, if $i > j$;

3. $\tilde{b}^R_i = \max_{t=0\ldots i}(u^R_{1t} + d_{1t})$, for $i = 0, \ldots, n$;
4. $\theta(R, i) = \min\{t \in \{0, \ldots, i\} : (u^R_{1t} + d_{1t}) = \tilde{b}^R_i\}$, for $i = 0, \ldots, n$;
5. $\tilde{b}^R_{ij} = \tilde{b}^R_j - \tilde{b}^R_{i-1} \geq 0$, for $1 \leq i \leq j \leq n$.

Note that, if $\tau = \theta(R, i)$, $u^R_{t+1,\tau} + d_{t+1,\tau} \geq 0$, for $0 \leq t < \tau$, and that $u^R_{\tau+1,t} + d_{\tau+1,t} \leq 0$, for $\tau < t \leq i$. For example, in inequality (19) with $R = \{1, 3\}$, we have $\tilde{b}^R_4 = 8$, $\theta(R, 4) = 4$, $\tilde{b}^R_2 = 4$, $\theta(R, 2) = 1$, and $\tilde{b}^R_{34} = 4$.

Observe that $\tilde{b}^R_i$ represents the amount that has to be produced in order to satisfy both the demand for all periods up to i and the maximum amount that can be sold for periods in $R \cap \{1, \ldots, i\}$. Note that, because demands can be negative, this amount must sometimes be produced before period i. This happens in the above example for $\tilde{b}^R_2$ with $R = \{1, 3\}$. $\theta(R, i)$ represents the deadline for producing this amount.

Proposition 1. *The* (t, S, R) *inequalities*

$$\sum_{j \in T\setminus S} x_j + \sum_{j \in S} \tilde{b}^R_{jt} y_j \geq \sum_{j \in R} v_j + d_{1t}, \tag{20}$$

are valid for X *for all* $1 \leq t \leq n$, $T = \{1, \ldots, t\}$, $S \subseteq T$ *and* $R \subseteq T$, *such that* $t = \theta(R, t)$.

Proof. Let $(x^*, y^*, v^*) \in X$. Suppose $y^*_i = 0$ for all $i \in S$. Then as $x^*_i = 0$ for $i \in S$,

$$\sum_{j \in T\setminus S} x^*_j + \sum_{j \in S} \tilde{b}^R_{jt} y^*_j = \sum_{j \in T} x^*_j \geq \sum_{j \in T} v^*_j + d_{1t} \geq \sum_{j \in R} v^*_j + d_{1t}.$$

Otherwise let $k = \min\{i \in S : y^*_i = 1\}$. Let $\tau = \theta(R, k-1)$. As $\tau \leq k-1$, $x^*_j = 0$ for $j \in S$ with $j \leq \tau$. Then

$$\sum_{j \in T\setminus S} x^*_j \geq \sum_{j \in T\setminus S, j \leq \tau} x^*_j = \sum_{j \leq \tau} x^*_j \geq \sum_{j \leq \tau} v^*_j + d_{1\tau} \geq \sum_{j \in R, j \leq \tau} v^*_j + d_{1\tau}. \tag{21}$$

Also,

$$\begin{aligned}\sum_{j \in S} \tilde{b}^R_{jt} y^*_j &\geq \tilde{b}^R_{kt} = \tilde{b}^R_t - \tilde{b}^R_{k-1} = u^R_{1t} + d_{1t} - \left(u^R_{1\tau} + d_{1\tau}\right) \\ &= \sum_{j \in R, \tau+1 \leq j \leq t} u_j + d_{\tau+1,t} \geq \sum_{j \in R, \tau+1 \leq j \leq t} v^*_j + d_{\tau+1,t}.\end{aligned} \tag{22}$$

Adding (21) and (22), the result follows.

□

Theorem 1. *The inequalities (20) together with the inequalities (15)–(17) describe conv(X).*

Let us denote by $M(c, f, p) \subseteq X$ the set of all optimal solutions of the problem $\max\{\sum_{i=1}^{n}(-c_i x_i - f_i y_i + p_i v_i) : (x, y, v) \in X\}$.

To prove the theorem we need two lemmas characterizing the solutions of $M(c, f, p)$, which hold subject to certain conditions. We consider a non-negative cost (c, f, p), an optimal solution $(x^*, y^*, v^*) \in M(c, f, p)$ and an integer $q \in \{1, \ldots, n+1\}$. We define $R = \{j : p_j > 0\}$ and $\tau = \theta(R, q-1)$. The conditions are:

Condition A: $q \in \{1, \ldots, n+1\}$ satisfies Condition A if either

a) $q = n+1$, or

b) $q \neq n+1$, $c_q = 0$ and either $f_q = 0$ or $y_q^* = 1$.

Condition B: $q \in \{1, \ldots, n+1\}$ satisfies Condition B if for each $j \in \{1, \ldots, \tau\}$, either $c_j > 0$ or $y_j^* = 0$.

Given (x^*, y^*, v^*), if q satisfies Condition A, then the production in period q can be increased at no cost. Therefore, as (x^*, y^*, v^*) is optimal, it is never profitable to produce in a period $j \in \{q+1, \ldots, n\}$ in which the cost (c_j or f_j) is positive. Also, it always pays to sell as much as possible in all periods $j \in \{q, \ldots, n\}$ with positive unit price p_j. By definition of $\theta(R, q-1)$, this argument can be extended to periods in $\{\tau+1, \ldots, q-1\}$. This is formalized in Lemma 1.

Lemma 1. *If q satisfies Condition A, then*

1. *for $j = \tau+1, \ldots, n$, $j \neq q$, $x_j^* = 0$ whenever $c_j > 0$;*
2. *for $j = \tau+1, \ldots, n$, $j \neq q$, $y_j^* = 0$ whenever $f_j > 0$; and*
3. *for $j = \tau+1, \ldots, n$, $v_j^* = u_j$ whenever $p_j > 0$.*

Proof. In each case we will assume the contrary and produce a solution (x', y', v') which has higher value than (x^*, y^*, v^*), contradicting the assumption of optimality for (x^*, y^*, v^*). Where not otherwise specified, (x', y', v') coincides with (x^*, y^*, v^*).

1) Suppose we have some $j \neq q$, $\tau+1 \leq j \leq n$, with $c_j > 0$ and $x_j^* > 0$. Make $x_j' = 0$ and $v_k' = 0$ for $k \notin R$. If $q \neq n+1$, make $x_q' = x_q^* + x_j^*$ and $y_q' = 1$. To verify that $(x', y', v') \in X$, we must show that it satisfies inequality (14) for all t. If $t < j$ or $t \geq q$, this fact is immediate as

$$\sum_{k=1}^{t} x_k' = \sum_{k=1}^{t} x_k^* \geq \sum_{k=1}^{t} v_k^* + d_{1t} \geq \sum_{k=1}^{t} v_k' + d_{1t}.$$

Otherwise $j \leq t \leq q-1$, and it follows that

$$\begin{aligned}
\textstyle\sum_{k=1}^{t} x_k' &\geq \textstyle\sum_{k=1}^{\tau} x_k' && (\tau < t)\\
&= \textstyle\sum_{k=1}^{\tau} x_k^* && (\tau < j)\\
&\geq \textstyle\sum_{k=1}^{\tau} v_k^* + d_{1\tau} && ((x^*, y^*, v^*) \text{ is valid})\\
&\geq \textstyle\sum_{k=1}^{\tau} v_k^* + u_{\tau+1,t}^R + d_{1t} && \left(u_{\tau+1,t}^R + d_{\tau+1,t} \leq 0\right)\\
&\geq \textstyle\sum_{k=1}^{\tau} v_k' + u_{\tau+1,t}^R + d_{1t} && \left(v_k' \leq v_k^*, \forall k\right)\\
&\geq \textstyle\sum_{k=1}^{\tau} v_k' + \sum_{k=\tau+1, k\in R}^{t} v_k' + d_{1t} && \left(v_k' \leq u_k, \forall k\right)\\
&= \textstyle\sum_{k=1}^{t} v_k' + d_{1t}. && \left(v_k' = 0, \forall k \notin R\right)
\end{aligned}$$

Solution (x', y', v') is worth $c_j x_j^*$ more than (x^*, y^*, v^*), since if $q \neq n+1$ $c_q = 0$ and either $f_q = 0$ or $y'_q = y_q^* = 1$, and $p_k = 0$ for $k \notin R$.

2) Suppose we have some $j \neq q$, $\tau + 1 \leq j \leq n$, with $f_j > 0$ and $y_j^* = 1$. We construct (x', y', v') in the same way as the above case and in addition we set $y'_j = 0$. Solution (x', y', v') is worth $c_j x_j^* + f_j$ more than (x^*, y^*, v^*). Note that c_j and x_j^* may be zero.

3) Suppose we have some j, $\tau + 1 \leq j \leq n$, with $p_j > 0$ and $v_j^* < u_j$. Make $v'_j = u_j$ and $v'_k = 0$ for $k \notin R$. If $q \neq n+1$, make $x'_q = x_q^* + u_j - v_j^*$ and $y'_q = 1$. Everything shown for case 1) holds also for this case and the reader can verify the validity of the inequalities (14) following the same steps. Solution (x', y', v') is worth $p_j(u_j - v_j^*)$ more than (x^*, y^*, v^*).

□

Similarly, if q satisfies Condition B, then every amount produced in periods $\{1, \ldots, \tau\}$ in the solution (x^*, y^*, v^*) has a positive cost. Therefore, as (x^*, y^*, v^*) is optimal, in case q satisfies also Condition A it is not profitable to produce in periods $\{1, \ldots, \tau\}$ any amount used satisfy demand or sales in $\{q, \ldots, n\}$, because such an amount could be produced at no cost in period q. So the production in $\{1, \ldots, \tau\}$ is used only to satisfy demands or sales in $\{1, \ldots, \tau\}$. Furthermore, because producing in $\{1, \ldots, \tau\}$ has positive cost, sales have to be zero in $\{1, \ldots, \tau\}$ when unit price is zero. This is formalized in Lemma 2.

Lemma 2. *If q satisfies Conditions A and B, then*

1. $\sum_{k=1}^{\tau} x_k^* = \sum_{k=1}^{\tau} v_k^* + d_{1\tau}$, *and*
2. *for* $j = 1, \ldots, \tau$, $v_j^* = 0$ *whenever* $p_j = 0$.

Proof. We proceed in the same way as in Lemma 1.

1) Suppose that $s = \sum_{k=1}^{\tau} x_k^* - \sum_{k=1}^{\tau} v_k^* - d_{1\tau}$ is positive. Then, if $\sum_{k=1}^{\tau} x_k^* = 0$, we have

$$-\sum_{k=1}^{\tau} v_k^* - d_{1\tau} > 0$$

which implies, since $u_{1\tau}^R + d_{1\tau} \geq 0$ and $v_k^* \geq 0$ for all k, that

$$\sum_{k=1, k \in R}^{\tau} v_k^* < u_{1\tau}^R.$$

Hence either $\sum_{k=1}^{\tau} x_k^* > 0$ or $\sum_{k=1, k \in R}^{\tau} v_k^* < u_{1\tau}^R$. It follows that there exists $j \in \{1, \ldots, \tau\}$ with either $x_j^* > 0$, or $j \in R$ and $v_j^* < u_j$. Choose the largest such j. Then for $k = j+1, \ldots, \tau$, $x_k^* = 0$ and $v_k^* = u_k$ when $k \in R$.

Case *i*: Suppose first that $x_j^* > 0$ and take $\epsilon = \min\{s, x_j^*\}$. Make $v'_k = 0$ for $k \notin R$, $x'_j = x_j^* - \epsilon$, and $x'_q = x_q^* + \epsilon$ in case $q \neq n+1$. We need to show that $(x', y', s') \in X$ by showing that it satisfies (14).

For $t = j, \dots, \tau$ we have

$$
\begin{aligned}
\textstyle\sum_{k=1}^{t} x'_k &= \textstyle\sum_{k=1}^{t} x^*_k - \epsilon && \left(j \le t, x^*_j \ge \epsilon, q > j\right)\\
&= \textstyle\sum_{k=1}^{\tau} x^*_k - \epsilon && \left(x^*_k = 0, \forall k = j+1, \dots, \tau\right)\\
&\ge \textstyle\sum_{k=1}^{\tau} v^*_k + d_{1\tau} && (s \ge \epsilon)\\
&\ge \textstyle\sum_{k=1}^{t} v^*_k + \sum_{k=t+1, k\in R}^{\tau} v^*_k + d_{1\tau} && \left(t \le \tau, v^*_k \ge 0\right)\\
&= \textstyle\sum_{k=1}^{t} v^*_k + u^R_{t+1,\tau} + d_{1\tau} && \left(v^*_k = u_k, \forall k \in R,\right.\\
& && \left. k = j+1, \dots, \tau\right)\\
&\ge \textstyle\sum_{k=1}^{t} v^*_k + d_{1t} && \left(u^R_{t+1,\tau} + d_{t+1,\tau} \ge 0\right)\\
&\ge \textstyle\sum_{k=1}^{t} v'_k + d_{1t}. && \left(v'_k \le v^*_k, \forall k\right)
\end{aligned}
$$

Case *ii*: Suppose now that $v^*_j < u_j$, $j \in R$, and take $\epsilon = \min\{s, u_j - v^*_j\}$. Make $v'_j = v^*_j + \epsilon$ and $v'_k = 0$ for $k \notin R$. If $q \ne n+1$, make $x'_q = x^*_q + \epsilon$ and $y'_q = 1$. For $t = j, \dots, \tau$ we have

$$
\begin{aligned}
\textstyle\sum_{k=1}^{t} x'_k &= \textstyle\sum_{k=1}^{t} x^*_k && (t < q)\\
&= \textstyle\sum_{k=1}^{\tau} x^*_k && \left(x^*_k = 0, \forall k = j+1, \dots, \tau\right)\\
&\ge \textstyle\sum_{k=1}^{\tau} v^*_k + d_{1\tau} + \epsilon && (s \ge \epsilon)\\
&\ge \textstyle\sum_{k=1}^{\tau} v'_k + d_{1\tau} && \left(v'_j = v^*_j + \epsilon, v'_k \le v^*_k, \forall k \ne j\right)\\
&= \textstyle\sum_{k=1}^{t} v'_k + \sum_{k=t+1, k\in R}^{\tau} v'_k + d_{1\tau} && \left(v'_k = 0, \forall k \notin R\right)\\
&= \textstyle\sum_{k=1}^{t} v'_k + u^R_{t+1,\tau} + d_{1\tau} && \left(v'_k = v^*_k = u_k, \forall k \in R,\right.\\
& && \left. k = j+1, \dots, \tau\right)\\
&\ge \textstyle\sum_{k=1}^{t} v'_k + d_{1t}. && \left(u^R_{t+1,\tau} + d_{t+1,\tau} \ge 0\right)
\end{aligned}
$$

In both cases we have shown the validity of inequalities (14) for $t = j, \dots, \tau$. For $t < j$ or $t \ge q$, the validity is immediate. Finally, for $t = \tau+1, \dots, q-1$,

$$
\begin{aligned}
\textstyle\sum_{k=1}^{t} x'_k &\ge \textstyle\sum_{k=1}^{\tau} x'_k && (\tau < t)\\
&\ge \textstyle\sum_{k=1}^{\tau} v'_k + d_{1\tau} && \text{(already shown}\\
& && \text{for the case } t = \tau)\\
&\ge \textstyle\sum_{k=1}^{\tau} v'_k + u^R_{\tau+1,t} + d_{1t} && \left(u^R_{\tau+1,t} + d_{\tau+1,t} \le 0\right)\\
&\ge \textstyle\sum_{k=1}^{\tau} v'_k + \sum_{k=\tau+1, k\in R}^{t} v'_k + d_{1t} && \left(v'_k \le u_k, \forall k\right)\\
&= \textstyle\sum_{k=1}^{t} v'_k + d_{1t}. && \left(v'_k = 0, \forall k \notin R\right)
\end{aligned}
$$

In case (i) solution (x', y', v') is worth ϵc_j more, because q satisfies Condition A. As $x^*_j > 0$, then $y^*_j = 1$ and thus, by Condition B, $c_j > 0$. In case (ii), (x', y', v') is worth ϵp_j more, and $p_j > 0$ as $j \in R$.

2) Suppose $v^*_j > 0$ with $1 \le j \le \tau$ and $p_j = 0$. To see that (x^*, y^*, v^*) cannot be an optimal solution it suffices to change it by setting $v^*_j = 0$. The solution is worth the

same, but now $s = \sum_{k=1}^{\tau} x_k^* - \sum_{k=1}^{\tau} v_k^* - d_{1\tau}$ is positive. As we have shown in part 1) of the proof, such a solution cannot be optimal.

□

Proof of Theorem 1. We use a technique due to Lovász [8]. For an arbitrary non-zero objective function max $\sum_{i=1}^{n}(-c_i x_i - f_i y_i + p_i v_i)$ we will show case by case that all points in $M(c, f, p)$ satisfy one of the inequalities (20), (15), (16) or (17) at equality (note that inequalities (14) are special cases of (20)). This proves that the description of the convex hull is complete, since when the objective function is parallel to a facet of the polyhedron the corresponding facet-defining inequality is the only valid inequality that is satisfied at equality by all optimal solutions.

If $c_i < 0$ for some i, then $M(c, f, p) \subseteq \{(x, y, v) : x_i = My_i\}$. If $f_i < 0$ for some i, then $M(c, f, p) \subseteq \{(x, y, v) : y_i = 1\}$. If $p_i < 0$ for some i, then $M(c, f, p) \subseteq \{(x, y, v) : v_i = 0\}$. We suppose next that c, f and p are non-negative.

As $(-c, -f, p) \neq 0$, we can define l as the last period such that c_l, f_l and p_l are not all zero. Let $t = \theta(R, l)$, $T = \{1, \ldots, t\}$, $S = \{i \in T : c_i = 0\}$ and $R = \{i : p_i > 0\}$.

Suppose there is $k \leq l$ such that $c_k = f_k = 0$. Then Lemma 1 can be applied with $q = k$, using l as value of j. If $p_l > 0$, then $M(c, f, p) \subseteq \{(x, y, v) : v_l = u_l\}$. Otherwise $p_l = 0$ and thus $k < l$ and either $c_l > 0$ or $f_l > 0$. It follows that $M(c, f, p) \subseteq \{(x, y, v) : x_l = 0\}$ or $M(c, f, p) \subseteq \{(x, y, v) : y_l = 0\}$. So we assume from now on that there is no such k, and so $f_i > 0$ for all i in S.

Suppose next that $p_i > 0$ for some $i > t$. Then Lemma 1 can be applied with $q = l + 1$ and by 3) $M(c, f, p) \subseteq \{(x, y, v) : v_i = u_i\}$. So from now on we can assume that $R \subseteq T$.

Consider an optimal solution (x^*, y^*, v^*) in $M(c, f, p)$. We now show that the inequality (20) holds at equality.

Suppose first that $y_i^* = 0$ for all $i \in S$. Then

$$\sum_{j \in T \setminus S} x_j^* = \sum_{j \in T} x_j^*, \text{ and} \sum_{j \in S} \tilde{b}_{jl}^R y_j^* = 0.$$

Since $l = n$ or $c_{l+1} = f_{l+1} = 0$, and for $i \in T$ $y_i^* = 1$ implies that $i \notin S$ and hence $c_i > 0$ for $i \leq \theta(R, l)$ such that $y_i = 1$, Lemma 2 can be applied with $q = l + 1$:

$$\text{1) gives } \sum_{j \in T} x_j^* = \sum_{j \in T} v_j^* + d_{1t},$$

$$\text{and 2) gives } \sum_{j \in T} v_j^* = \sum_{j \in R \cap T} v_j^* = \sum_{j \in R} v_j^*.$$

So,

$$\sum_{j \in T \setminus S} x_j^* + \sum_{j \in S} \tilde{b}_{jl}^R y_j^* = \sum_{j \in T} x_j^* = \sum_{j \in T} v_j^* + d_{1t} = \sum_{j \in R} v_j^* + d_{1t},$$

and the inequality (20) holds at equality.

Otherwise take $k = \min\{i \in S : y_i^* = 1\}$. Let $\tau = \theta(R, k-1)$. Applying Lemma 1 with $q = k$, $y_k^* = 1$ and $c_k = 0$,

$$\sum_{j \in T \setminus S, j > \tau} x_j^* = 0 \text{ using 1)}, \tag{23}$$

$$\sum_{j \in S} \tilde{b}_{jt}^R y_j^* = \tilde{b}_{k,t}^R = \tilde{b}_t^R - \tilde{b}_{k-1}^R \text{ using 2), and} \tag{24}$$

$$\sum_{j \in R, j > \tau} v_j^* = u_{\tau+1,t}^R \text{ using 3)}. \tag{25}$$

We have $c_k = 0$, $y_k^* = 1$ and $c_j > 0$ for all $1 \le j \le k-1$ with $y_j^* = 1$ by definition of k. So applying Lemma 2 with $q = k$, 1) gives

$$\sum_{j \le \tau} x_j^* = \sum_{j \le \tau} v_j^* + d_{1\tau}, \tag{26}$$

and by 2),

$$\sum_{j \le \tau} v_j^* = \sum_{j \in R, j \le \tau} v_j^*. \tag{27}$$

Now using (24),

$$\sum_{j \in T \setminus S} x_j^* + \sum_{j \in S} \tilde{b}_{jt}^R y_j^* = \sum_{j \in T \setminus S} x_j^* + \tilde{b}_t^R - \tilde{b}_{k-1}^R.$$

Now, by the fact that $\tilde{b}_{k-1}^R = \tilde{b}_\tau^R$, that $x_j^* = 0$ if $j \in S$ and $j \le \tau \le k-1$, and from (23),

$$\begin{aligned}\sum_{j \in T \setminus S} x_j^* + \tilde{b}_t^R - \tilde{b}_{k-1}^R &= \sum_{j \in T \setminus S, j \le \tau} x_j^* + \sum_{j \in T \setminus S, j > \tau} x_j^* + \tilde{b}_t^R - \tilde{b}_{k-1}^R \\ &= \sum_{j \le \tau} x_j^* + \tilde{b}_t^R - \tilde{b}_\tau^R.\end{aligned}$$

From (26), (27) and the definition of $\tilde{b}_i^R$

$$\begin{aligned}\sum_{j \le \tau} x_j^* + \tilde{b}_t^R - \tilde{b}_\tau^R &= \sum_{j \in R, j \le \tau} v_j^* + d_{1\tau} + u_{1t}^R + d_{1t} - \left(u_{1\tau}^R + d_{1\tau}\right), \\ &= \sum_{j \in R, j \le \tau} v_j^* + u_{\tau+1,t}^R + d_{1t}.\end{aligned}$$

Finally, from (25),

$$\sum_{j \in R, j \le \tau} v_j^* + u_{\tau+1,t}^R + d_{1t} = \sum_{j \in R} v_j^* + d_{1t},$$

and the proof is complete.

$\square$

3. ULS^3 with non-negative demands

When $d_t \geq 0$ for all t, ULS^3 simplifies in a variety of ways. It is natural to fix $\sigma_n = 0$, and it is no longer necessary in a regeneration interval to consider a solution with $0 < v_t^* < u_t$ as this would only be possible if $x_k^* = 0$ for all k in the interval, and then it is impossible to produce $v_t^* > 0$. So we restrict attention to the set $\tilde{X} = X \cap \{(x, v, y, \sigma) : \sigma_n = 0\}$, and the corresponding face conv($\tilde{X}$)= conv(X) $\cap$ $\{(x, v, y, \sigma) : \sigma_n = 0\}$. The inequalities $x_t \leq My_t$ are no longer necessary to describe conv($\tilde{X}$). Also the value $\tilde{b}_{ij}^R$ used in describing facets is given directly by $\tilde{b}_{ij}^R = u_{ij}^R + d_{ij}$ for $1 \leq i \leq j \leq n$.

Proposition 2. *Every extreme point is characterized by three sets* $I, J, K \subseteq \{1, \ldots, n\}$, $I = \{t_1 < t_2 < \ldots < t_q\} \subseteq J$ *where*

$$y_t = \begin{cases} 1, & \text{if } t \in J, \\ 0, & \text{otherwise}, \end{cases}$$

$$v_t = \begin{cases} u_t, & \text{if } t \in K, \\ 0, & \text{otherwise}, \end{cases}$$

$$x_t = \begin{cases} \sum_{i=t_j}^{t_{j+1}-1}(v_i + d_i), & \text{if } t = t_j \in I, \\ 0, & \text{otherwise}. \end{cases}$$

where we take $t_{q+1} = n + 1$.

Now we present an extended formulation for $\tilde{X}$. We let $b_t = u_t + d_t$ for all t, and introduce the 0-1 variables β_{ij}, α_{ij} where $\beta_{ij} = 1$ if the amount d_j is produced in period $i \leq j$ and $\alpha_{ij} = 1$ if the amount $d_j + u_j$ is produced in period $i \leq j$. The resulting formulation is:

$$\max \sum_t p_t v_t - \sum_t f_t y_t - \sum_t c_t x_t,$$

$$(F4) \qquad \begin{aligned} \sum_{j \geq i}(b_j \alpha_{ij} + d_j \beta_{ij}) &= x_i, && i = 1, \ldots, n, \\ \sum_{i \leq j}(b_j \alpha_{ij} + d_j \beta_{ij}) &= v_j + d_j, && j = 1, \ldots, n, \\ \sum_{i \leq j}(\alpha_{ij} + \beta_{ij}) &= 1, && j = 1, \ldots, n, \\ (\alpha_{ij} + \beta_{ij}) &\leq y_i, && 1 \leq i \leq j \leq n, \\ v_i, y_i, x_i, \alpha_{ij}, \beta_{ij} &\geq 0, && 1 \leq i \leq j \leq n, \\ y_i &\leq 1, && i = 1, \ldots, n. \end{aligned}$$

Let P^* be the polytope defined by the constraints of (F4).

It is readily verified that the points in P^* with α, β, y integer are the points of $\tilde{X}$, so P^* is a valid extended formulation for $\tilde{X}$, and conv($\tilde{X}$) $\subseteq$ proj$_{x,y,v}P^*$.

Proposition 3. *proj*$_{x,y,v}P^* =$ *conv*($\tilde{X}$).

Proof. Consider a point (x, y, v, α, β) in P^* and let us show that (x, y, v) is in $\mathrm{conv}(\tilde{X})$. It suffices to show that (x, y, v) satisfies the inequalities which describe $\mathrm{conv}(X)$ and that $\sigma_n = \sum_{t=1}^n x_t - \sum_{t=1}^n v_t - d_{1n} = 0$. Indeed,

$$\sum_{t=1}^n x_t = \sum_{t=1}^n \sum_{j\geq t} (b_j\alpha_{tj} + d_j\beta_{tj}) = \sum_{j=1}^n \sum_{t\leq j} (b_j\alpha_{tj} + d_j\beta_{tj}) = \sum_{j=1}^n v_j + d_{1n},$$

implies that $\sigma_n = 0$. Also,

$$\sum_{k=1}^t x_k = \sum_{k=1}^t \sum_{j=k}^n (b_j\alpha_{kj} + d_j\beta_{kj}) \geq \sum_{j=1}^t \sum_{k=1}^j (b_j\alpha_{kj} + d_j\beta_{kj}) = \sum_{j=1}^t (v_j + d_j),$$

$$v_t = \sum_{i\leq t} (b_t\alpha_{it} + d_t\beta_{it}) - d_t \leq b_t \sum_{i\leq t} (\alpha_{it} + \beta_{it}) - d_t = b_t - d_t = u_t,$$

$$v_t = \sum_{i\leq t} (b_t\alpha_{it} + d_t\beta_{it}) - d_t \geq d_t \sum_{i\leq t} (\alpha_{it} + \beta_{it}) - d_t = d_t - d_t = 0,$$

$$x_t = \sum_{j\geq t} (b_j\alpha_{tj} + d_j\beta_{tj}) \leq \sum_{j\geq t} b_j(\alpha_{tj} + \beta_{tj}) \leq \sum_{j\geq t} b_j y_t \leq M y_t,$$

so the inequalities (14)–(16) are satisfied. Also clearly (17) holds. We complete the proof by checking for the (t, S, R) inequalities (20):

$$\begin{aligned}
\sum_{k\in S} x_k + \sum_{k\in T\setminus S} \tilde{b}_{kt}^R y_k = \\
&= \sum_{k\in S} \sum_{j\geq k} (b_j\alpha_{kj} + d_j\beta_{kj}) + \sum_{k\in T\setminus S} \Big(\sum_{j\geq k, j\in R} b_j y_k + \sum_{j\geq k, j\in T\setminus R} d_j y_k \Big) \\
&= \sum_{k\in S} \sum_{j\geq k, j\in R} (b_j\alpha_{kj} + d_j\beta_{kj}) + \sum_{k\in S} \sum_{j\geq k, j\in T\setminus R} (b_j\alpha_{kj} + d_j\beta_{kj}) + \\
&\quad + \sum_{k\in T\setminus S} \sum_{j\geq k, j\in R} b_j y_k + \sum_{k\in T\setminus S} \sum_{j\geq k, j\in T\setminus R} d_j y_k \\
&\geq \sum_{k\in S} \sum_{j\geq k, j\in R} (b_j\alpha_{kj} + d_j\beta_{kj}) + \sum_{k\in S} \sum_{j\geq k, j\in T\setminus R} d_j(\alpha_{kj} + \beta_{kj}) + \\
&\quad + \sum_{k\in T\setminus S} \sum_{j\geq k, j\in R} b_j(\alpha_{kj} + \beta_{kj}) + \sum_{k\in T\setminus S} \sum_{j\geq k, j\in T\setminus R} d_j(\alpha_{kj} + \beta_{kj}) \\
&\geq \sum_{k\in T} \sum_{j\geq k, j\in R} (b_j\alpha_{kj} + d_j\beta_{kj}) + \sum_{k\in T} \sum_{j\geq k, j\in T\setminus R} d_j(\alpha_{kj} + \beta_{kj})
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j \in R} \sum_{k \le j} (b_j \alpha_{kj} + d_j \beta_{kj}) + \sum_{j \in T \setminus R} d_j \sum_{k \le j} (\alpha_{kj} + \beta_{kj}) \\
&\ge \sum_{j \in R} (v_j + d_j) + \sum_{j \in T \setminus R} d_j \\
&= \sum_{j \in R} v_j + d_{1t}.
\end{aligned}
$$

□

Now we consider the separation problem for conv($\tilde{X}$). Suppose that a point (x^*, y^*, v^*) satisfies (14)–(17), and $\sigma_n^* = 0$, but is not in conv($\tilde{X}$). Then by Theorem 1 at least one of the inequalities

$$
\sum_{j \in T \setminus S} x_j + \sum_{j \in S} \tilde{b}_{jt}^R y_j \ge \sum_{k \in R} v_k + d_{1t}
$$

is violated. Since $d_t \ge 0$ for all t implies $\tilde{b}_{ij}^R = u_{jt}^R + d_{jt}$, for a fixed t and $T = \{1, \dots, t\}$ the separation problem can be solved by minimizing over $R, S \subseteq \{1, \dots, t\}$, the difference

$$
\sum_{j \in T \setminus S} x_j^* + \sum_{j \in S} \big(u_{jt}^R + d_{jt}\big) y_j^* - \sum_{k \in R} v_k^*
$$

which can be rewritten as

$$
\sum_{j \in T \setminus S} x_j^* + \sum_{j \in S} y_j^* (\sum_{k \in R, j \le k \le t} u_k) + \sum_{j \in S} d_{jt} y_j^* - \sum_{k \in R} v_k^*.
$$

To minimize this expression, we take λ and μ respectively as the characteristic vectors of S and $\overline{R} = T \setminus R$. We then minimize over λ and μ

$$
\sum_{j=1}^{t} x_j^* (1 - \lambda_j) + \sum_{j=1}^{t} \sum_{k=j}^{t} u_k y_j^* \lambda_j (1 - \mu_k) + \sum_{j=1}^{t} d_{jt} y_j^* \lambda_j - \sum_{k=1}^{t} v_k^* (1 - \mu_k)
$$

which is equivalent to minimizing

$$
\sum_{j=1}^{t} \big(\sum_{k=j}^{t} (u_k + d_k) y_j^* - x_j^* \big) \lambda_j + \sum_{k=1}^{t} v_k^* \mu_k - \sum_{j=1}^{t} \sum_{k=j}^{t} u_k y_j^* \lambda_j \mu_k.
$$

It is well known that minimizing a quadratic boolean function in which all quadratic terms have non-positive coefficient reduces to a maximum flow problem [13]. Thus solving for each $t = 1, \dots, n$ leads to a polynomial algorithm.

4. ULS^3 with zero demands and Wagner-Whitin costs

With $d_t \geq 0$ for all t as in the previous section, the facet-defining inequalities still depend on subsets S and R. Here, when $d_t = 0$ for all t, we show that the number of facets, though still exponential, decreases by an order of magnitude in the presence of Wagner-Whitin costs. To see this we again introduce an extended formulation.

Assume that the cost functions c_t, h_t satisfy the Wagner-Whitin condition $c_t + h_t \geq c_{t+1}$ for $t = 1, \ldots, n-1$. Alternatively eliminating the production variables from the objective function, the resulting storage costs $h'_t = c_t + h_t - c_{t+1} \geq 0$. This restriction says that, ignoring fixed costs, it is always best to produce as late as possible. Formulations in the presence of Wagner-Whitin costs have been studied in [11].

As the amount sold in period t is either 0 or u_t in an optimal extreme solution, we can define the following 0-1 variables:

$w_t = 1$ if $v_t = u_t$ in period t, and $w_t = 0$ if $v_t = 0$ ($w_t = v_t/u_t$)
$\delta^l_k = 1$ if $v_l = u_l$ and the stock at the end of k contains the corresponding sale u_l.

Clearly $\delta^l_k = \max(0, w_l - \sum_{t=k+1}^{l} y_t)$ due to the Wagner-Whitin property.

The resulting formulation is:

$$\max \sum_{t=1}^{n} p_t u_t w_t - \sum_{t=1}^{n} f_t y_t - \sum_{t=1}^{n} h'_t \sigma_t,$$

$$\sigma_k = \sum_{l=k+1}^{n} u_l \delta^l_k \quad 1 \leq k < t \tag{28}$$

$$\delta^l_k - w_l + \sum_{i=k+1}^{l} y_i \geq 0 \quad 1 \leq k < l \leq t \tag{29}$$

$$\delta^l_k \geq 0 \quad 1 \leq k < l \leq t \tag{30}$$

$$0 \leq w_k \leq 1, 0 \leq y_k \leq 1 \quad 1 \leq k \leq t \tag{31}$$

$$y_k \text{ integer} \quad 1 \leq k \leq t \tag{32}$$

Proposition 4. *The polyhedron Q defined by contraints (29)–(31) is integral.*

Proof. In fact the constraints (29)–(31) define a totally unimodular matrix. Since the constraints (30) and (31) are submatrices of the identity, these rows can be ignored. The same holds for the columns of (29) corresponding to the vector δ. Let $\{a_{ij}\}_{ij}$ be the remaining matrix with the set of columns partitioned into set $Y = \{C_1, \ldots, C_n\}$, corresponding to vector y, and the set $W = \{D_1, \ldots, D_n\}$, corresponding to the vector w.

We base our proof on the theorem [4] which states that $\{a_{ij}\}_{ij}$ is totally unimodular if for any subset of columns J there is a partition (J_1, J_2) of J such that $|\sum_{j\in J_1} a_{ij} - \sum_{j\in J_2} a_{ij}| \leq 1$, for every row i.

Let J be a subset of columns and let $C_{i_1}, \ldots, C_{i_p}$ be the columns of Y in J. For convenience, let $i_0 = 1$ and $i_{p+1} = n+1$.

We allocate alternatively $C_{i_1}, C_{i_2}, \ldots, C_{i_p}$ to J_1 and J_2 so that $C_{i_1} \in J_1$, $C_{i_2} \in J_2$ and so on. The columns D_i in W are allocated to the same set as C_{i_j}, where j is such that $i_j \leq i < i_{j+1}$.

Now let us consider one row of the matrix $\{a_{ij}\}_{ij}$. Suppose it corresponds to variable δ_k^l. In this row the non-zero entries are -1 for D_l and 1 for $C_{k+1}, \ldots, C_l$. Defining $r_Y = \sum_{j \in Y \cap J_1} a_{ij} - \sum_{j \in Y \cap J_2} a_{ij}$ and $r_W = \sum_{j \in W \cap J_1} a_{ij} - \sum_{j \in W \cap J_2} a_{ij}$, we have to show that $|r_W + r_Y| \leq 1$.

Since $\{C_{k+1}, \ldots, C_l\}$ corresponds to an interval of columns of Y, the columns of this set are also alternatively allocated to J_1 and J_2. So $|r_Y| \leq 1$. Clearly $|r_W| \leq 1$ also, since there is only one non-zero entry in W for each row. Suppose now that $r_Y = 1$. Then in particular the last column in J of the interval $C_{k+1}, \ldots, C_l$ is allocated to J_1. So D_l is also allocated to J_1, and thus $r_W = -1$. The case $r_Y = -1$ is analogous. Therefore $|r_W + r_Y| \leq 1$.

□

Now we consider the projection of Q into the space of the original (σ, v, y) variables. Using $v_t = u_t w_t$ for all t, and eliminating the δ_k^l variables in (28) by using (29) or (30) leads directly to the projection, and gives a proof of the next proposition.

Proposition 5. *The polyhedron*

$$\sigma_k \geq \sum_{l \in U} (v_l - u_l \sum_{i=k+1}^{l} y_i), \quad \text{for all } U \subseteq \{k+1, \ldots, n\} \text{ and all } k, \tag{33}$$

$$0 \leq v_k \leq u_k, \quad \text{for } k = 1, \ldots, n, \tag{34}$$

$$0 \leq y_k \leq 1, \quad \text{for } k = 1, \ldots, n, \tag{35}$$

$$0 \leq \sigma_k, \quad \text{for } k = 1, \ldots, n, \tag{36}$$

has y integral at all its extreme points.

Note that the (k, U) inequalities (33) form a special subset of the (t, S, R) inequalities. Specifically taking $t = \max\{i : i \in U\}$, (33) can be rewritten as

$$\sum_{j=1}^{k} x_j + \sum_{i=k+1}^{t} y_i (\sum_{i \leq j \leq t, j \in U} u_j) \geq \sum_{l \in U} v_l + \sum_{j=1}^{k} v_j,$$

or setting $T = \{1, \ldots, t\}$, $R = U \cup \{1, \ldots, k\}$ and $S = \{k+1, \ldots, t\}$ as

$$\sum_{j \in T \setminus S} x_j + \sum_{j \in S} \tilde{b}_{jt}^R y_j \geq \sum_{j \in R} v_j.$$

5. Extensions

Various extensions of the type studied for the classical uncapacitated lot-sizing model appear important. We have initial results for the constant capacity case including a polynomial algorithm and a generalisation of the (t, S, R) inequalities. Results for ULS[3] with backlogging and start-up variables are also needed to treat certain real-life instances.

Theoretically we have only been able to separate the (t, S, R) inequalities in polynomial time when $d_t \geq 0$ for all t. For formulation (F1), this means that the lower bounds $\{L_t\}_{t=0}^n$ are nondecreasing. In practice it is often the case that the initial stock L_0

is arbitrary, and $L_t = L$ constant for $t = 1, \ldots, n$. Thus the only difficulty arises when $L_0 > L$. In terms of formulation (F3), $d_1 = L - L_0$, and $d_t = 0$ for $t = 2, \ldots, n$. It is therefore interesting to investigate the extension of the combinatorial separation algorithm described in Sect. 3 to this case. Practically we plan to develop and test separation heuristics both for ULS^3 and for fixed charge network flows, in which paths with both positive and negative demands are treated, extending the path inequalities developed in [2].

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