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Multiple homoclinic orbits for a class of Hamiltonian systems

Eric Paturel

CEREMADE, UMR CNRS 7534, Université Paris-Dauphine, Place du Maréchal de Lattre de Tassigny, F-75775 Paris Cedex 16, France (e-mail: paturel@pi.ceremade.dauphine.fr)

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Abstract. In this paper, we obtain the existence of at least two nontrivial homoclinic orbits for a class of second order autonomous Hamiltonian systems. This multiplicity result is obtained by a new variational method based on the relative category: to overcome the lack of compactness of the problem, we first solve perturbed nonautonomous problems and study the limit of the solutions as the nonautonomous perturbation goes to 0. This method allows to get rid of some assumptions on the potential used in the work of Ambrosetti and Coti-Zelati.

1 Introduction

The goal of this paper is to prove a multiplicity result on homoclinic orbits, solutions of the following autonomous second order Hamiltonian system, for $q : \mathbb{R} \longrightarrow \mathbb{R}^N$,

(1.1) $$\ddot{q} + V'(q) = 0\,,$$

where the potential V satisfies

(1.2) $$V(q) = -\frac{1}{2}|q|^2 + W(q)\,,\ W \in C^2(\mathbb{R}^N, \mathbb{R}),$$

(1.3) $$\forall x \in \mathbb{R}^N \,/\, V(x) = 0 \text{ and } x \neq 0,\ \nabla V(x) \neq 0\,,$$

and the attractive potential W satisfies the following *pinching* condition

(1.4)
$$\exists \alpha > 2, \exists c_1, c_2 \,/\, \frac{c_2}{c_1} < 2^{\frac{\alpha-2}{2}} \text{ and } \forall\, x \in \mathbb{R}^N,\ c_1|x|^\alpha \leq W(x) \leq c_2|x|^\alpha,$$

Let us recall that an orbit *homoclinic to* 0 is a solution of (1.1) which moreover satisfies the following limit conditions:

$$\lim_{t\to\pm\infty} q(t) = 0\,, \quad \lim_{t\to\pm\infty} |\dot{q}(t)| = 0\,. \tag{1.5}$$

Many results ensure the existence of at least one nontrivial homoclinic orbit for first order ([6], [9], [13]) or second order ([1], [4], [12]) Hamiltonian systems. But there exists few multiplicity results in the autonomous case and these results often require many technical assumptions on the potential V. This work improves a result by Ambrosetti-Coti-Zelati [2], who proved the existence of two homoclinic orbits under a pinching assumption (1.4), a *superquadraticity* condition

$$\forall x \in \mathbb{R}^N,\ W'(x).x \geq \alpha W(x)\,, \tag{1.6}$$

and the following second order conditions

$$W''(0) = 0 \text{ and } \forall x \in \mathbb{R}^N,\ x \neq 0\,, W'(x).x < W''(x).x.x\,. \tag{1.7}$$

Ambrosetti-Coti-Zelati's method is variational and based on the use of a topological tool: the *Lyusternik-Schnirelman category* and its application in critical point theory. Our aim is to generalize their result with the use of a *relative category*, which allows us to get rid of the second order conditions and to weaken condition (1.6) to the local condition (1.3). We then obtain:

Theorem 1.1 *Let V be a potential satisfying (1.2), where W satisfies the local first-order condition (1.3) and the pinching condition (1.4). Then (1.1) admits at least two nontrivial homoclinic orbits.*

The paper is organized as follows. For the reader's convenience, we show in Sect. 2 a multiplicity result for Hamiltonian systems whose potential satisfies conditions (1.4) and (1.6). The main difficulty, the construction of a deformation necessary to calculate a lower bound to the number of critical points, is postponed in Sect. 4. Finally, Theorem 1.1 is proved in Sect. 3, by a new method: we solve suitable perturbed nonautonomous problems, and study the limit of the solutions when the nonautonomous perturbation goes to 0, using the concentration-compactness principle (see [10], [11]).

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2 Multiplicity with superquadraticity

2.1 Presentation and sketch of the proof

The aim of this section is the proof of the following result:

Theorem 2.1 *Let V a potential satisfying*

$$V(q) = -\frac{1}{2}|q|^2 + W(q)\,, \;\; W \in C^2(\mathbb{R}^N, \mathbb{R}), \tag{2.1}$$

where W satisfies pinching and superquadraticity conditions (1.4),(1.6). Then system (1.1) admits at least two nontrivial homoclinic orbits.

In our autonomous case (i.e. V does not depend explicitly on t), the notion of distinct solutions is ambiguous: any time translation of a solution is also a solution. To avoid this problem, we will study functionals defined on spaces of *even* functions.

Let us now introduce the variational framework associated to the homoclinic problem.

Let $E = H^1_{even}(\mathbb{R}, \mathbb{R}^N)$ be the Sobolev space of even L^2 functions defined on $\mathbb{R}$ and taking values in $\mathbb{R}^N$, whose derivatives are in L^2. It is a Hilbert space, when endowed with the following scalar product:

$$(q, q') = \int_{\mathbb{R}} (\langle \dot{q}(t), \dot{q}'(t)\rangle + \langle q(t), q'(t)\rangle) dt\,,$$

where $\langle ., .\rangle$ is the standard euclidian scalar product in $\mathbb{R}^N$, and $\dot{q}$ is the time derivative of q. The notation for the induced norm in E is

$$||q||^2 = \int_{\mathbb{R}} (|\dot{q}(t)|^2 + |q(t)|^2) dt\,.$$

This space is continuously embedded in $C^{0,\frac{1}{2}}(\mathbb{R}, \mathbb{R}^N)$, and $q \in E$ will always be considered as a continuous function.

We define the following *action* functional, for $q \in E$:

$$F(q) = \int_{\mathbb{R}} \left(\frac{1}{2}|\dot{q}(t)|^2 - V(q(t))\right) dt = \frac{1}{2}||q||^2 - \int_{\mathbb{R}} W(q(t)) dt\,. \tag{2.2}$$

It is well-known that the non-trivial critical points of F are the homoclinic solutions of (1.1). The next definition of reference functionals, corresponding with radial potentials in (1.4), will be useful:

$$F_i(q) = \frac{1}{2}||q||^2 - c_i \int_{\mathbb{R}} |q(t)|^\alpha dt \;\; \text{for} \;\; i = 1, 2. \tag{2.3}$$

The pinching property (1.4) implies

(2.4) $$\forall q \in E,\ F_2(q) \leq F(q) \leq F_1(q)\,.$$

In all the paper, we use the notation

$$\int q = \int_{\mathbb{R}} q(t)dt\,.$$

As in every variational proof, we have to prove a compactness result. In fact, there is a lack of compactness due to the invariance of the action functional under time translations. In order to prove a compactness property at the right level, we suppose, by contradiction, the uniqueness of a non-trivial critical point for F, whose existence has been already proved by Bolotin [5], Ambrosetti-Bertotti [1] and Rabinowitz-Tanaka [12]. We then use a concentration-compactness method to prove a Palais-Smale (PS) property for F.

Once we have obtained this property, we can use a topological tool of critical point theory: the *relative category*, which is an extended notion of the well-known *Lyusternik-Schnirelman category*, or (L-S) category. We refer to [3] for an extensive definition and description of this notion. As a difference with (L-S) category, we obtain, with the relative category, critical point theorems for functionals which are *unbounded from below*, so that we don't have here to restrict the problem to a submanifold where F is bounded from below, as Ambrosetti-Coti-Zelati [2] do. The category of a level set of a functional, relatively to a smaller one, if (PS) holds between these levels, is closely related to the critical set of this functional:

Proposition 2.2 *Let F be a functional defined on E, which satisfies $(PS)_c$ property for $c \in [a-\varepsilon, b+\varepsilon]$, with $a \leq b$, $\varepsilon > 0$. Then, with the following notation*

$$K_F^{[a,b]} = \{q \in E \;/\; F'(q) = 0\,,\ a \leq F(q) \leq b\}\,,$$

we have

$$\mathrm{cat}_{(F)^b,(F)^a}(F)^b \leq \# K_F^{[a,b]}\,.$$

We recall that the $(PS)_c$ property for F is the precompactness of each $(PS)_c$ sequence, i.e. sequences (q_n) such that $F(q_n) \to c \in \mathbb{R}$ and $F'(q_n) \to 0$ in E'. We use the following notation for level sets:

$$(F)^a = \{q \in E \;/\; F(q) \leq a\,\}\,.$$

Since the level sets of the functional F are difficult to handle with, we will use the following property of relative categories:

Proposition 2.3 *Let $Y \subseteq X$ et $Y' \subseteq X'$ closed subsets of E. Suppose there exists maps:*

$$(X,Y) \xrightarrow{h} (X',Y') \xrightarrow{g} (X,Y)\,,$$

and a deformation $j_t : X \to X$, with $t \in [0,1]$ such that

$$j_1 = g \circ h \text{ and } \forall t \in [0,1],\ j_t(Y) \subseteq Y\,.$$

Then

$$\operatorname{cat}_{X,Y}(X) \leq \operatorname{cat}_{X',Y'}(X')\,.$$

To efficiently use Proposition 2.3, we have to find spaces X et Y for which $\operatorname{cat}_{X,Y}(X)$ will be easier to compute. The study of functionals F_1 and F_2 will give us good candidates. Then, we build the maps and deformation which occur in the Proposition 2.3. These constructions are done in Sect. 2.3, and the construction of the deformation j, rather technical, is postponed in Sect. 4. Eventually, the computation of $\operatorname{cat}_{X,Y}(X)$ gives a lower bound for the number of non-trivial critical points: as that bound is 2, this is in contradiction with our uniqueness assumption. Since existence is proved, we then obtain the multiplicity result that we claimed.

2.2 Compactness properties

The superquadraticity assumption (1.6) implies the boundedness of $(PS)_c$ sequences at every level:

Lemma 2.4 *Let (q_n) be a $(PS)_c$ sequence, with $c \in \mathbb{R}$. Then there exists $M \in \mathbb{R}$ so that, for all n, we get*

$$||q_n|| \leq M.$$

Proof. Since (q_n) is a $(PS)_c$ sequence, , there exists M_1 such that

$$\frac{1}{2}||q_n||^2 - \int W(q_n) \leq M_1\,,$$

hence, with (1.6),

$$\begin{aligned}\frac{1}{2}||q_n||^2 &\leq M_1 + \frac{1}{\alpha}\int W'(q_n).q_n\,,\\ &\leq M_1 + \frac{1}{\alpha}(||q_n||^2 - F'(q_n).q_n)\,,\end{aligned}$$

and it follows

$$\left(\frac{1}{2} - \frac{1}{\alpha}\right)||q_n||^2 \leq M_1 + \frac{1}{\alpha}||F'(q_n)||_{E'}||q_n||,.$$

Since $||F'(q_n)||_{E'}$ is bounded and $\alpha > 2$, we obtain that $||q_n||$ is bounded.$\diamond$

It is straightforward to see that each critical value for F is nonnegative. This follows from the following computation, for q critical point at the level c:

$$c = F(q) - F'(q).q = \int \frac{1}{2} W'(q).q - W(q) \geq \left(\frac{\alpha}{2} - 1\right) \int W(q) \geq 0\,.$$

This property implies that F satisfies $(PS)_c$ for $c < 0$. Now, let (q_n) be a $(PS)_0$ sequence. Then $F(q_n) - \frac{1}{2}F'(q_n).q_n \to 0$, so we have $\int W(q_n) \to 0$, and, since $F(q_n) \to 0$, we finally get

$$\lim_{n \to +\infty} q_n = 0 \text{ in } E.$$

Thus all $(PS)_0$ sequences converge strongly in E to 0, the trivial critical point of F: F satisfies $(PS)_0$. For positive levels, we prove this lemma:

Lemma 2.5 *The critical value 0 is isolated in the set of critical values of F, i.e. there exists $\varepsilon > 0$ such that every critical value $\kappa \neq 0$ of F satisfies*

$$\kappa \geq \varepsilon > 0\,.$$

Proof. Let (q_n) be a sequence of critical points for F such that

$$F(q_n) \overset{>}{\to} 0\,.$$

Such a sequence is $(PS)_0$, so we have $q_n \to 0$ in E. But, by Sobolev embedding, there exists C_s such that, for all n, we get:

$$||q_n||_{L^\infty} \leq C_s ||q_n||\,.$$

Hence we have $||q_n||_{L^\infty} \to 0$ when $n \to +\infty$.

Moreover, thanks to the assumption (1.6), there exists $\delta > 0$ such that for every $x \in \mathbb{R}^N,\ x \neq 0,\ |x| < \delta$, we get

$$2V(x) + \langle V'(x), x\rangle < 0\,.$$

Since q_n is a critical point for F, we know its regularity, by an elementary *bootstrap* argument: $q_n \in C^\infty(\mathbb{R}, \mathbb{R}^N)$. At points t_n where $|q_n|^2$ reaches his global maximum value, we get then $\frac{d^2}{dt^2}|q_n(t_n)|^2 \leq 0$. But

$$\begin{aligned}
\frac{d^2}{dt^2}|q_n(t)|^2 &= |\dot{q}_n(t)|^2 + \langle q_n(t), \ddot{q}_n(t)\rangle \\
&= |\dot{q}_n(t)|^2 - \langle V'(q_n(t)), q_n(t)\rangle \\
&= -2V(q_n(t)) - \langle V'(q_n(t)), q_n(t)\rangle\,,
\end{aligned}$$

where the last equality is obtained by an energy argument:

$$e_n = \frac{1}{2}|\dot{q}_n(t)|^2 + V(q_n(t)) = 0.$$

For n great enough, we get, for every $t \in \mathbb{R}$, $|q_n(t)| < \delta$. We then have proved that 0 is the only maximum value for $|q_n|$, so $q_n(t) \equiv 0$ for n great enough, which is contradictory with $F(q_n) \stackrel{>}{\to} 0$. □

By a result of Ambrosetti-Bertotti [1], under very large conditions, containing our assumption (1.4) and (1.6), existence of a non-trivial critical point $\bar{q} \in E$ for F is proved, with critical value $\bar{\kappa} > 0$. This result also implies the existence of a critical value κ_1 for F_1 and κ_2 for F_2. Inequality (2.4) then implies (since critical values are obtained in [1] by min-max arguments, conserving potential inequalities),

$$\kappa_2 \leq \bar{\kappa} \leq \kappa_1 \,. \tag{2.5}$$

On the other hand, κ_i is the only non-trivial critical value for F_i, $i = 1, 2$. This is due to the reduction to the following differential equation, possible because of the radial potential:

$$-\ddot{r} + r + \alpha c_i |r|^{\alpha-2} r = 0,$$

$$\lim_{t\to\pm\infty} r(t) = 0\,, \quad \lim_{t\to\pm\infty} \dot{r}(t) = 0\,,$$

which has an unique even and positive solution r_0. Of course, for $N > 1$, F_i admits an infinity of non-trivial critical points of the form $r_0(t).e$, with $e \in S^{N-1}$. In the case $N = 1$, F_i has exactly two critical points.

In order to prove that F admits also two non-trivial critical points at least, we suppose, by contradiction, that $\bar{q}$ is the unique non-trivial critical point for F. This assumption allows us to find compactness properties at the right level, by the means of a concentration-compactness result. We thus obtain:

Lemma 2.6 *Suppose that $\bar{q}$ is the unique non-trivial critical point for F. Then F satisfies $(PS)_c$ property with $c \in (0, 2\bar{\kappa})$.*

Proof. Let (q_n) be a $(PS)_c$ sequence for F, with $c \in (0, 2\bar{\kappa})$. We make use of the well-known concentration-compactness alternative due to P.-L. Lions ([10], [11]) on the following density:

$$\rho_n(t) = \frac{|\dot{q}_n(t)|^2 + |q_n(t)|^2}{||q_n||^2}\,,$$

which is well defined and normed in L^1 since $c > 0$. Then the proof follows from straightforward computations: we first show that the *vanishing* situation is impossible and that the *concentration* case leads to the precompactness of (q_n), according to the fact that every q_n is even. Finally, thanks our uniqueness assumption, we can deduce that there is no *dichotomy* phenomenon, and the proof is over.$\diamond$

2.3 Looking for critical points

This section is devoted to the study of the relative category

$$\mathrm{cat}_{(F)^{2\kappa_2-\varepsilon},(F)^{\kappa_0}}(F)^{2\kappa_2-\varepsilon}\,,$$

where $\varepsilon > 0$ and $\kappa_0 > 0$ will be defined later. Thus we will obtain a lower bound of the number of non-trivial critical points for F, with values in $[\kappa_0, 2\kappa_2 - \varepsilon]$. Following Proposition 2.3, we deform level sets of F into sets whose category will be easier to compute. We then have to study functionals of the form F_1 or F_2. We define the following class of functionals, defined on E, for $c > 0$:

$$F_c(q) = \frac{1}{2}||q||^2 - c\int_{\mathbb{R}} |q(t)|^\alpha dt\,.$$

We get, of course, $F_{c_1} = F_1$ and $F_{c_2} = F_2$. Let

$$\mathcal{M}_c = \{q \in E,\ q \neq 0 \,/\, {F_c}'(q).q = 0\}\,.$$

We check easily that $\mathcal{M}_c$ is a hilbertian submanifold in E, of codimension 1. Indeed, if we define

$$G_c(q) = F_c'(q).q\,,$$

then we have

$$\begin{aligned} G'_c(q).q &= 2||q||^2 - \alpha^2 c\int |q|^\alpha \\ &= (2-\alpha)||q||^2 \neq 0 \text{ for } q \in \mathcal{M}_c\,. \end{aligned} \tag{2.6}$$

Moreover, if we denote by $S = \{q \in E \,/\, ||q|| = 1\}$ the unit sphere in E, we obtain, for all $q \in S$,

$$F_c(\lambda q) = \frac{1}{2}\lambda^2 - c\lambda^\alpha \int |q|^\alpha\,,$$

and

$$F'_c(\lambda q).\lambda q = \lambda^2 - \alpha c\lambda^\alpha \int |q|^\alpha\,.$$

Hence $\lambda q \in \mathcal{M}_c$ if and only if

$$\lambda = \left(\frac{1}{\alpha c}\right)^{\frac{1}{\alpha-2}} .||q||_{L^\alpha}^{\frac{-\alpha}{\alpha-2}} .$$

This dilation gives a bijection between $\mathcal{M}_c$ and S. This yields that $\mathcal{M}_c$ is diffeomorphic to the unit sphere and star-shaped relatively to 0.

Concerning critical points for $F_c|_{\mathcal{M}_{c'}}$, with $c, c' > 0$, we have the straightforward result:

Lemma 2.7 *Let $c > 0$. Then, with our notations, all non-trivial critical points for F_c stay in $\mathcal{M}_c$ and they are the same as critical points for the restricted functional $F_c|_{\mathcal{M}_c}$.*

In other words, the constraint $q \in \mathcal{M}_c$ is artificial and the functional $F_c|_{\mathcal{M}_c}$ does not have more critical points than F_c.

For the correspondence between critical points for F_c and $F_{c'}$, we get this result:

Lemma 2.8 *Let $c, c' > 0$. Critical points for F_c and $F_{c'}$ are in correspondence by the following dilation, centered at 0 and with coefficient*

$$\lambda = \left(\frac{c}{c'}\right)^{\frac{1}{\alpha-2}} .$$

The critical set of F_c is isomorphic to the unit sphere S^{N-1} in $\mathbb{R}^N$, it is the set of functions of the form $q(t) = r_c(t).e$, with $e \in S^{N-1}$ and r_c is the unique positive and even solution of the following differential equation:

$$-\ddot{r} + r + \alpha c|r|^{\alpha-2} r = 0,$$

$$\lim_{t\to\pm\infty} r(t) = 0\,, \quad \lim_{t\to\pm\infty} \dot{r}(t) = 0.$$

We denote by Σ_c the set of non-trivial critical points for F_c, isomorphic to S^{N-1}.

Proof. Let q a non-trivial critical point for F_c. Then, for all $h \in E$, we get

$$F_c{}'(q).h = (q,h) - \alpha c \int |q|^{\alpha-2} \langle q, h\rangle = 0\,.$$

The definition of λ given in the lemma implies

$$\begin{aligned} F_{c'}{}'(\lambda q).h &= (\lambda q, h) - \alpha c' \int |\lambda q|^{\alpha-2} \langle \lambda q, h\rangle \\ &= \lambda((q,h) - \alpha c \int |q|^{\alpha-2}\langle q, h\rangle) \\ &= 0\,. \end{aligned}$$

Therefore λq is a critical point for $F_{c'}$. □

These results allow us to define the following *cylinder*, with $c \le c'$:

$$\Lambda(c,c') = \bigcup_{d=c}^{c'} \Sigma_d$$
$$= \left\{ \lambda q,\ \lambda \in \left[\left(\frac{c}{c'}\right)^{\frac{1}{\alpha-2}}, 1 \right],\ q \in \Sigma_c \right\} .$$

This cylinder is a major element in the construction of tools for the Proposition 2.3. With the notations of that proposition, we define:

$$\begin{cases} X' = (F)^{2\kappa_2 - \varepsilon} \\ Y' = (F)^{\kappa_0}\,, \end{cases}$$

and

$$\begin{cases} X = (F_1)^{\kappa_0} \cup \Lambda(\gamma_1,\gamma_2) \\ Y = (F_1)^{\kappa_0}\,, \end{cases}$$

where $0 < \kappa_0 < \kappa_2$ and $\gamma_1 < c_1 < c_2 < \gamma_2$ satisfy $F_1(\Sigma_{\gamma_1}) = \kappa_0$ and $F_1(\Sigma_{\gamma_2}) = \kappa_0$. The real number $\varepsilon > 0$ will be defined more precisely later.

From (2.4), it follows that X' and Y' are respectively included in

$$\begin{cases} X'' = (F_2)^{2\kappa_2 - \varepsilon} \\ Y'' = (F_2)^{\kappa_0}\,. \end{cases}$$

We show now that we may choose $h = Id$, with notations of Proposition 2.3.

Lemma 2.9 *$X \subset X'$ and $Y \subset Y'$.*

Proof. From (2.4), we get

$$\max_{q \in \Lambda(\gamma_1,\gamma_2)} F(q) \le \max_{q \in \Lambda(\gamma_1,\gamma_2)} F_1(q) = \kappa_1\,.$$

Moreover, according to the pinching assumption (1.4), we claim that

$$\kappa_1 < 2\kappa_2\,. \tag{2.7}$$

This strict inequality defines $\varepsilon > 0$ such that $\kappa_1 \le 2\kappa_2 - \varepsilon$.

We prove (2.7) as follows: let $q_1 \in \Sigma_{c_1}$. Then,

$$F_1(q_1) = c_1 \left(\frac{\alpha}{2} - 1\right) \int |q_1|^\alpha = \kappa_1.$$

Taking $\lambda = \left(\frac{c_1}{c_2}\right)^{\frac{1}{\alpha-2}}$, we have $\lambda q_1 \in \Sigma_{c_2}$, and then

$$F_2(\lambda q_1) = \kappa_2 = \left(\frac{\lambda^2}{2}\alpha c_1 - \lambda^\alpha c_2\right) \int |q_1|^\alpha ,$$

thus we get

$$\frac{\kappa_2}{\kappa_1} = \left(\frac{c_1}{c_2}\right)^{\frac{2}{\alpha-2}} > \frac{1}{2} .$$

We have proved that $\Lambda(\gamma_1, \gamma_2) \subset (F)^{2\kappa_2-\varepsilon}$. Finally, (2.4) and $\kappa_0 < 2\kappa_2 - \varepsilon$ directly infer that $(F_1)^{\kappa_0} \subset (F)^{\kappa_0}$, and the proof is over. □

In order to use Proposition 2.3, we have to build a deformation $(X', Y') \to (X, Y)$ which preserves X and Y globally. The proof of the following lemma, rather technical, is postponed in Sect. 4.

Lemma 2.10 *There exists a deformation $g_t : X' \to E$, with $t \in [0, 1]$, satisfying the following properties:*

- *$t \mapsto g_t$ maps continuously $[0, 1]$ to the set of continuous maps in X';*
- *$g_0 = Id$ and $g_1(X') \subseteq X$, $g_1(Y') \subseteq Y$;*
- *for all $t \in [0, 1]$, we have $g_t(X) \subseteq X$ and $g_t(Y) \subseteq Y$.*

Proof. cf Sect. 4. □

From Proposition 2.3, we infer

(2.8) $$\mathrm{cat}_{X,Y}(X) \le \mathrm{cat}_{X',Y'}(X') \le \# K_F^{[\kappa_0, 2\kappa_2-\varepsilon]} .$$

It remains to compute $\mathrm{cat}_{X,Y}(X)$. From the excision property of relative category, we find that

(2.9) $$\mathrm{cat}_{X,Y}(X) \ge \mathrm{cat}_{\Lambda(\gamma_1,\gamma_2),\partial\Lambda(\gamma_1,\gamma_2)}(\Lambda(\gamma_1, \gamma_2)) .$$

The computation of the category of a cylinder relatively to its boundary $\partial\Lambda(\gamma_1, \gamma_2) = \Sigma_{\gamma_1} \cup \Sigma_{\gamma_2}$ is an easy task, and may be found, for example, in [8]. We get

$$\mathrm{cat}_{\Lambda(\gamma_1,\gamma_2),\partial\Lambda(\gamma_1,\gamma_2)}(\Lambda(\gamma_1, \gamma_2)) = 2 .$$

Using Proposition 2.2, we prove that F admits at least two non-trivial critical points, whose critical values are in $[\kappa_0, 2\kappa_2 - \varepsilon]$. This is contradictory with our uniqueness assumption. Hence, this assumption is false and Theorem 2.1 is proved.

3 Multiplicity without superquadraticity

3.1 Presentation and notations

In order to prove the most general result of this paper, Theorem 1.1, we have to get rid of assumption (1.6). The difficulty here is that (1.6) implies the boundedness of (PS) sequences, and no other assumption here gives the same result. To overcome this problem, we build special (PS) sequences for the new functional, i.e. sequences of critical points for functionals whose related Hamiltonian system is *no more autonomous*, with potentials satisfying a weaker property, called *superquadraticity at infinity*. Convergence of such sequences will require, as in Sect. 2, the combination of a uniqueness assumption with a concentration-compactness method. Topological properties of the relative category will finally ensure the contradiction and prove Theorem 1.1. This part is organized as follows: in 3.2, we solve the non-autonomous problems, in order to build special (PS) sequences; convergence of these sequences, up to subsequences, is proved in 3.3; we show the contradiction and conclude in 3.4.

Let $W \in C^2(\mathbb{R}^N, \mathbb{R})$ satisfy the pinching assumption (1.4). We first modify this attractive potential far from the origin: given $R > 0$, there exists $\tilde{W} \in C^2(\mathbb{R}^N, \mathbb{R})$ such that:

- for all $x \in \mathbb{R}^N$, we have $c_1|x|^\alpha < \tilde{W}(x) < c_2|x|^\alpha$;
- for all $|x| \leq R$, we have $\tilde{W}(x) = W(x)$;
- for all $|x| \geq 2R$, we have $\tilde{W}'(x).x \geq \alpha\tilde{W}(x)$.

We point out that strict pinching inequality is obtained by slightly modifying coefficients c_1 and c_2, and we choose R great enough, such that for all $|x| > R$, we get $-\frac{1}{2}|x|^2 + \tilde{W}(x) > 0$ and $-\frac{1}{2}|x|^2 + W(x) > 0$.

We define now the following non-autonomous potential: given $T > 0$, let $W_T \in C^2(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ satisfy the following conditions:

- for $|t| \leq T$, we have $W_T(x,t) = \tilde{W}(x)$, $\forall x \in \mathbb{R}^N$;
- for $|t| \geq T+1$, we have $W_T(x,t) = c_2|x|^\alpha$, $\forall x \in \mathbb{R}^N$;
- potential W_T is even relatively to t and for all $x \in \mathbb{R}^N \setminus \{0\}$, $t \in (T, T+1)$, we have
$$\frac{\partial}{\partial t} W_T(x,t) > 0\,,$$
and, for all $t \in (T, T+1)$ and x such that $|x| \geq 2R$, we get
$$\frac{\partial}{\partial x} W_T(x,t).x \geq \alpha W_T(x,t)\,.$$

We define the following class of functionals, for $q \in E$:

$$G_T(q) = \frac{1}{2}||q||^2 - \int_{\mathbb{R}} W_T(q(t),t)dt\,,$$

$$G_\infty(q) = \frac{1}{2}||q||^2 - \int_{\mathbb{R}} \tilde{W}(q)\,.$$

The problem is now to find critical points for G_T and show that these approximate solutions converge to critical points for G_∞ as T goes to infinity.

3.2 Resolution of the approached problems

Let $T > 0$. As in Sect. 2, we have to find two non-trivial critical points for a functional, here G_T. This case is not very different from the precedent one, since superquadraticity at infinity still yields the boundedness of (PS) sequences.

Lemma 3.1 *Let $c \in \mathbb{R}$ and (q_n) be a (PS) sequence at level c for the functional G_T. Then there exists $M \in \mathbb{R}$ such that, for all n, we have*

$$||q_n|| \le M\,.$$

Proof. It is a direct computation:

$$\begin{aligned}
\frac{1}{2}||q_n||^2 &= G_T(q_n) + \int W_T(q_n(t),t)dt \\
&= G_T(q_n) + \int_{I_n \cup J_n \cup K} W_T(q_n(t),t)dt \\
&\le G_T(q_n) + \frac{1}{\alpha}\int_{J_n \cup K} \frac{\partial W_T}{\partial x}(q_n(t),t).q_n(t)dt \\
&\quad + \int_{I_n} W_T(q_n(t),t)dt \\
&\le G_T(q_n) + \frac{1}{\alpha}\int_{\mathbb{R}} \frac{\partial W_T}{\partial x}(q_n(t),t).q_n(t)dt \\
&\quad + \int_{I_n} [W_T(q_n(t),t) - \frac{1}{\alpha}\frac{\partial W_T}{\partial x}(q_n(t),t).q_n(t)]dt \\
&\le G_T(q_n) + \frac{1}{\alpha}[||q_n||^2 - G_T'(q_n).q_n] + C_0\,.
\end{aligned}$$

with the following notations:

- $K = (-\infty, -T-1] \cup [T+1, +\infty)$;
- $I_n = \{t \in [-T-1, T+1], |q_n(t)| \le 2R\}$;
- $J_n = \{t \in [-T-1, T+1], |q_n(t)| \ge 2R\}$.

The last inequality easily yields that q_n is bounded. □

Definition of W_T implies that, for $q \in E$:

$$F_2(q) \leq G_T(q) \leq F_1(q)\,.$$

Hence it is possible to use exactly the same topological argument for G_T as for F in Sect. 2. Indeed, if we denote by

$$\begin{cases} X'_T = (G_T)^{2\kappa_2-\varepsilon} \\ Y'_T = (G_T)^{\kappa_0}\,, \end{cases}$$

we also have, with the notations of Sect. 2,

$$\begin{cases} X \subset X'_T \subset X'' \\ Y \subset Y'_T \subset Y''\,. \end{cases}$$

Now, we can apply Lemma 2.10 and its following computation to find two non-trivial critical points for G_T, provided that we show a compactness result, i.e. $(PS)_c$ condition, for $c \in [\kappa_0, 2\kappa_2 - \varepsilon]$. First, by Lemma 3.1, it is possible to show a concentration-compactness property for $(PS)_c$ sequences, for $c > 0$:

Lemma 3.2 *Let $c > 0$ and $(q_n) \in E$ be a $(PS)_c$ sequence for G_T. Then, there exists a subsequence of (q_n), still denoted by (q_n), a set of p non-zero functions $Q^1, ..., Q^p$ in $H^1(\mathbb{R}, \mathbb{R}^N)$, distinct or not, and p sequences of real numbers $(\tau_n^1), ..., (\tau_n^p)$, such that*

(i) $||q_n(.) - \sum_{i=1}^p Q^i(. - \tau_n^i)||_{H^1} \longrightarrow 0$;
(ii) $\forall i \in \{1, ..., p\}, \forall t \in \mathbb{R},\ \ Q^i(t) = Q^{p-i+1}(-t)$;
(iii) $\forall n,\ \ \tau_n^i + \tau_n^{p-i+1} = 0$;
(iv) $\forall i \in \{1, ..., p-1\},\ \ \tau_n^{i+1} - \tau_n^i \to +\infty$.

Moreover, we get

$$c = \lim_{n\to+\infty} \sum_{i=1}^{p} G_T(Q^i(. - \tau_n^i))\,.$$

Proof. It is the same proof as in Lemma 2.6, with the difference that dichotomy is allowed. Then, parity of q_n implies properties (ii) and (iii). □

The norm $||.||_{H^1}$ is the standard norm of space $H^1(\mathbb{R}, \mathbb{R}^N)$. We recall that we choosed the same norm for E. The concentration-compactness method leads to work in $H^1(\mathbb{R}, \mathbb{R}^N)$, because functions Q^i do not have to be even. In the only case $p = 1$, i.e. concentration case, we can conclude that Q^1 is even.

This concentration-compactness property yields a (PS) result:

Lemma 3.3 *The functional G_T satisfies $(PS)_c$ property, for $c \in [\kappa_0, 2\kappa_2 - \varepsilon]$.*

Proof. Let $c \in [\kappa_0, 2\kappa_2 - \varepsilon]$ and (q_n) be a $(PS)_c$ sequence for G_T. We can apply Lemma 3.2, and see q_n as a succession of bumps moving away from each other to infinity as n goes to $+\infty$.

Suppose there exists $1 \leq j \leq p$ such that, up to a subsequence, $(\tau_n^j) \to +\infty$. Then, by Lemma 3.2 (iii), $(\tau_n^{p-j+1}) \to -\infty$, and we may write q_n as the sum of p functions:

$$q_n(t) = \sum_{i=1}^{p} Q_n^i(t)$$

where Q_n^i are (PS) sequences for G_T, representing these bumps. So we have

$$G_T'(Q_n^j) \overset{H^{-1}}{\longrightarrow} 0\,.$$

We claim that Q_n^j converges in $H^1_{loc}(\mathbb{R}, \mathbb{R}^N)$, up to subsequence, to a non trivial critical point for F_2. We get indeed, for all $h \in H^1(\mathbb{R}, \mathbb{R}^N)$,

$$F_2'(Q_n^j).h = G_T'(Q_n^j).h + \int_{-T-1}^{T+1} \left[\frac{\partial W_T}{\partial x}(Q_n^j, t).h - \alpha c_2 |Q_n^j|^{\alpha-2} \langle Q_n^j, h \rangle \right],$$

and it is an easy task to find a nondecreasing function $Y : \mathbb{R}^+ \to \mathbb{R}^+$, taking limit 0 in 0, such that:

$$\left| \int_{-T-1}^{T+1} \left[\frac{\partial W_T}{\partial x}(Q_n^j, t).h - \alpha c_2 |Q_n^j|^{\alpha-2} \langle Q_n^j, h \rangle \right] \right| \leq (2T+2)^2 Y(||Q_n^j||_{L^\infty([-T-1,T+1])}) \cdot \|h\|\,.$$

When n goes to $+\infty$, $\tau_n^j \to +\infty$ and $||Q_n^j||_{L^\infty([-T-1,T+1])} \to 0$. Then, taking $\hat{Q}_n^j(t) = Q_n^j(t + \tau_n^j)$, we get a precompact $(PS)_{c'}$ sequence for F_2, with $c' < 2\kappa_2$. Compactness properties of F_2 then implies that $\hat{Q}_n^j(t)$ converges to an element of Σ_2, and the critical value for F_2 is κ_2.

Same arguments work for $p-j+1$. We get finally, jointly with Lemma 3.2,

$$c \geq \lim_{n \to +\infty} \sum_{i=1}^{p} G_T(Q^i(. - \tau_n^i)) \geq F_2(\hat{Q}_j) + F_2(\hat{Q}_{p-j+1}) = 2\kappa_2\,,$$

which is impossible, as $c \leq 2\kappa_2 - \varepsilon$. Thus, it is impossible for a sequence τ_n^j to go to infinity, and this happens only if $p = 1$. It follows that (q_n) is precompact and Lemma 3.3 is proved. □

With this precompactness lemma, we may use Lemma 2.10 and the computation of the relative category, and hence prove the following result:

Lemma 3.4 *The functional G_T admits at least two non-trivial critical points in E, whose critical levels are given by*

$$G_T(q_T^i) = \inf_{P \in C^i} \sup_{q \in P} G_T(q) > \kappa_0 \textit{ for } i = 1, 2,$$

with

$$C^i = \{P \,/\, (G_T)^{\kappa_0} \subset P \subset (G_T)^{2\kappa_2 - \varepsilon},\ \mathrm{cat}_{(G_T)^{2\kappa_2-\varepsilon},(G_T)^{\kappa_0}}(P) \geq i\}\,.$$

3.3 Precompactness and limit of critical point sequences

Let (T_n) be a sequence of real numbers going to $+\infty$ and $q_n = q^i_{T_n}$ a sequence of non-trivial critical points for G_{T_n}, with $i = 1$ or 2. In order to prove precompactness for such sequences, we will combine, as in Sect. 2, a concentration-compactness method with a uniqueness assumption, to have (PS) property at the right levels.

A theorem of Rabinowitz-Tanaka [12] shows the existence of a non-trivial critical point for G_∞ in E. Indeed, potential

$$\tilde{V}(x) = -\frac{1}{2}|x|^2 + \tilde{W}(x)$$

satisfies all conditions of this theorem ($\tilde{V}$ has a local non degenerate maximum in 0, $\tilde{V}(0) = 0$ and (1.3)). We denote by $\tilde{q} \in E$ this critical point for G_∞ and $\tilde{\kappa} = G_\infty(\tilde{q})$. As critical levels are obtained by a minimization framework, conserving potential inequalities, we still have an inequality like (2.5):

$$\kappa_2 < \tilde{\kappa} < \kappa_1\,. \tag{3.1}$$

As in Sect. 2, we will assume that $\tilde{q}$ is the only non-trivial critical point for G_∞. Exactly as in Lemma 2.6, this yields a compactness result for bounded (PS) sequences:

Lemma 3.5 *Assume that $\tilde{q}$ is the only non-trivial critical point for G_∞. Then, all* bounded *$(PS)_c$ sequences for G_∞, with $c \in [\kappa_0, 2\tilde{\kappa} - \varepsilon]$ are precompact.*

Proof. The proof is exactly the same as in Lemma 2.6. □

In order to prove a concentration-compactness result for (q_n), we have to find an a priori estimate of the H^1 norm of this sequence. Assumption (1.3) will play here an important role.

Lemma 3.6 *With the precedent notations , there exists $m > 0$ and $M > 0$ such that, for all n we have*

$$0 < m \leq ||q_n|| \leq M\,.$$

Proof. Since $\kappa_0 > 0$, it is easy to find a lower bound for $||q_n||$:

$$\frac{1}{2}||q_n||^2 \geq G_{T_n}(q_n) \geq \kappa_0 > 0\,.$$

We define now the following sets, with $\varepsilon > 0$:

- $\Omega_t = \{x \in \mathbb{R}^N, V_T(x,t) < 0\}$;
- $\Phi_\varepsilon(q) = \{t \in \mathbb{R} \,/\, d(q(t), \partial\Omega_t) \geq \varepsilon\}$;
- $\Psi_\varepsilon(q) = \{t \in \mathbb{R} \,/\, |q(t)| \leq \varepsilon\}$;
- $\Xi_\varepsilon(q) = \{t \in \mathbb{R} \,/\, d(q(t), \partial\Omega_t \setminus \{0\}) \leq \varepsilon\}$.

From condition (1.3) and definition of W_T, there results that $\partial\Omega_t$ is the reunion of a regular hypersurface contained in $\mathbb{R}^N$, of class C^1 and the point $\{0\}$. Moreover we get the following energy inequality

$$e_{q_n}(t) = \frac{1}{2}|\dot{q_n}(t)|^2 + V_{T_n}(q_n(t), t) \leq 0\,,$$

hence for all $t \in \mathbb{R}$, $q_n(t) \in \bar{\Omega}_t$. We have, of course,

$$\Phi_\varepsilon(q) \cup \Psi_\varepsilon(q) \cup \Xi_\varepsilon(q) = \mathbb{R}\,.$$

We then have to control the H^1 norm of q_n uniformly in n.

Step 1. We claim that the measure of $\Phi_\varepsilon(q_n)$ is finite and uniformly bounded. Indeed, there exists $\delta > 0$ such that for all $t \in \Phi_\varepsilon(q_n)$, we get

$$-V_{T_n}(q_n(t), t) \geq \delta > 0\,,$$

and for $t \in \Phi_\varepsilon(q_n)$, we then have

$$\frac{1}{2}|\dot{q_n}(t)|^2 - V_{T_n}(q_n(t), t) \geq \delta\,.$$

The integral on $\mathbb{R}$ of the left hand side is $G_{T_n}(q_n) < 2\kappa_2$, so the measure of $\Phi_\varepsilon(q_n)$ has to be finite. This gives an explicit upper bound for H^1 norm of q_n on $\Phi_\varepsilon(q_n)$, independent of n:

$$||q_n||_{H^1(\Phi_\varepsilon)} \leq meas(\Phi_\varepsilon) \max_{t\in\mathbb{R}} \max_{x\in\bar{\Omega}_t} [|x|^2 + |V(x,t|^2]\,. \tag{3.2}$$

Step 2. We claim that the measure of $\Xi_\varepsilon(q_n)$ is also uniformly bounded. This is analogous to a result by Rabinowitz-Tanaka [12], for which assumption (1.3) is necessary:

Proposition 3.7 *Let a non-autonomous potential $V \in C^2(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ satisfy the following properties:*

- *for all t, 0 is a non degenerate local maximum for $V(.,t)$, with $V(0,t) = 0$;*
- *there exists K compact in $\mathbb{R}$ such that for all $t \in \mathbb{R} \setminus K$, the set defined by $\{x/V(x,t) \leq 0\}$ is compact and its boundary is the reunion of $\{0\}$ and a regular hypersurface on which*

$$\left|\frac{\partial V}{\partial x}(x,t)\right| \geq \eta > 0\,.$$

Then, for $\varepsilon > 0$ fixed and $M > 0$, there exists a constant L_0 such that for all critical points q of the corresponding action functional with a positive critical value lower than M, we get, with the precedent notations,

$$mes(\varXi_\varepsilon(q)) \leq L_0\,.$$

This result gives an upper bound to the time while a solution of the system remains close to the outer boundary of $\{x/V(x,t) \leq 0\}$. The potential V_T satisfies assumptions of Proposition 3.7: we obtain an upper bound of the H^1-norm of q_n on $\varXi_\varepsilon(q_n)$ in the same way as in (3.2).
Step 3. Let $t \in \varPsi_\varepsilon(q_n)$. For ε close to 0, because of assumption (1.4), there exists a constant C_ε depending only of ε such that:

$$|\dot{q}_n(t)|^2 + |q_n(t)|^2 \leq C_\varepsilon \left[\frac{1}{2}|\dot{q}_n(t)|^2 + \frac{1}{2}|q_n(t)|^2 - W_{T_n}(q_n(t),t)\right]\,.$$

Integrating this last inequality on $\varPsi_\varepsilon(q_n)$, we get:

$$\begin{aligned}||q_n||_{H^1(\varPsi_\varepsilon(q_n),\mathbb{R}^N)} &\leq C_\varepsilon \left[G_{T_n}(q_n) + \int_{\varPhi_\varepsilon(q_n)\cup\varXi_\varepsilon(q_n)} W_{T_n}(q_n(t),t)\right] \\ &\leq C_\varepsilon \left[2\kappa_2 + C'||q_n||_{H^1(\varPhi_\varepsilon(q_n)\cup\varXi_\varepsilon(q_n)),\mathbb{R}^N)}\right]\,.\end{aligned}$$

Gathering these three steps, we obtain that $||q_n||$ is bounded by a constant which does not depend on n, and Lemma 3.6 is proved. □

Lemma 3.6 allows, in the same way as Lemmas 2.6 and 3.2, to state a concentration-compactness result for the sequence (q_n) of critical points for G_{T_n}:

Lemma 3.8 *With the precedent notations, there exists a subsequence of q_n, still denoted by q_n, a set of p non zero functions $Q^1, ..., Q^p$ in $H^1(\mathbb{R}, \mathbb{R}^N)$, distinct or not, and p sequences $(\tau_n^1), ..., (\tau_n^p)$ of real numbers such that*

(i) $||q_n(.) - \sum_{i=1}^p Q^i(. - \tau_n^i)||_{H^1} \longrightarrow 0$;

(ii) $\forall i \in \{1, ..., p\}, \forall t \in \mathbb{R}, \quad Q^i(t) = Q^{p-i+1}(-t)$*;*
(iii) $\forall n, \quad \tau_n^i + \tau_n^{p-i+1} = 0$*;*
(iv) $\forall i \in \{1, ..., p-1\}, \quad \tau_n^{i+1} - \tau_n^i \to +\infty$*.*

Moreover, we get

$$\lim_{n \to +\infty} G_{T_n}(q_n) - \sum_{i=1}^{p} G_{T_n}(Q^i(. - \tau_n^i)) = 0 \,. \tag{3.3}$$

In order to prove that sequence (q_n) is precompact, as in proof of Lemma 3.3, it is sufficient to prove that the only possible case is $p = 1$, i.e. the concentration case. Suppose that $p \geq 2$, and that $\lim_{n\to+\infty} \tau_n^1 = +\infty$ (if the limit is $-\infty$, then $\lim_{n\to+\infty} \tau_n^p = +\infty$). As in proof of Lemma 3.3, we write

$$q_n(t) = \sum_{i=1}^{p} Q_n^i(t)$$

where Q_n^i are these bumps, moving away from each other to infinity. There are two sequences of real numbers whose limit is $+\infty$: T_n and τ_n^1. By comparing these two sequences, we claim that $p \geq 2$ is impossible.

Step 1. Suppose that, up to a subsequence, we have

$$\lim_{n\to+\infty} (\tau_n^1 - T_n) = +\infty \,.$$

This case is then very similar to the proof of Lemma 3.3: we show, by the same computations, that the sequence $Q_n^1(. + \tau_n^1)$ converges in $H^1(\mathbb{R}, \mathbb{R}^N)$ to a non-trivial critical point for F_2. By parity, there exists k such that

$$\lim_{n\to+\infty} (-T_n - \tau_n^k) = +\infty \,,$$

and we prove in the same way that $Q_n^k(. + \tau_n^k)$ converges in $H^1(\mathbb{R}, \mathbb{R}^N)$ to a non-trivial critical point for F_2, i.e. an element of Σ_2. Thanks to (3.3), we get then

$$2\kappa_2 - \varepsilon \geq \lim_{n\to+\infty} G_{T_n}(q_n) \geq 2\kappa_2 \,,$$

which is impossible, since $\varepsilon > 0$.

Step 2. Suppose that, up to a subsequence, we get

$$\lim_{n\to+\infty} (\tau_n^1 - T_n) = l \in \mathbb{R} \,.$$

We can then suppose that $l = 0$ and work in $H^1(\mathbb{R}, \mathbb{R}^N)$. The sequence $Q_n^1(. + \tau_n^1)$ is precompact in $H^1(\mathbb{R}, \mathbb{R}^N)$ and the limit Q_∞ is a non zero function. If we define the following functional, for $q \in H^1(\mathbb{R}, \mathbb{R}^N)$:

$$G^s(q) = \int_{\mathbb{R}} \frac{1}{2}|\dot{q}(t)|^2 + \frac{1}{2}|q(t)|^2 - W^s(q(t), t) \,,$$

with

- $W^s(x,t) = \tilde{W}(x)$ if $t \le 0$;
- $W^s(x,t) = c_2|x|^\alpha$ if $t \ge 1$;
- $\frac{\partial W^s}{\partial t}(x,t) > 0$ for $x \neq 0$ and $t \in (0,1)$,

a direct computation shows that the sequence $Q_n^1(.+\tau_n^1)$ is a (PS) sequence for G^s, then Q_∞ is a non-trivial critical point for G^s. But this functional does not have any non-trivial critical point: by a classical regularity argument, it would belong to $H^2(\mathbb{R},\mathbb{R}^N)$ and we would have:

$$G^s(Q_\infty).\dot{Q}_\infty = 0\,,$$

and this would imply

$$\int_{\mathbb{R}} \frac{\partial W^s}{\partial t}(Q_\infty(t),t) = 0\,.$$

That is possible if and only if $Q_\infty \equiv 0$, and the contradiction follows.
Step 3. Suppose that, up to a subsequence, we obtain

$$\lim_{n\to+\infty}(\tau_n^1 - T_n) = -\infty\,.$$

By a direct computation, as in proof of Lemma 3.3, this implies that q_n is a bounded (PS) sequence for G_∞, which level stays in $[\kappa_0, 2\tilde{\kappa}-\varepsilon]$. Applying Lemma 3.5, we infer that (q_n) is precompact, which is in contradiction with $p \ge 2$.

Taking together these 3 steps, we see that the only possible case is $p=1$, i.e. precompactness of the sequence (q_n).

3.4 Contradiction and multiplicity result

The precedent section stated that the sequences (q_n) of critical points for G_{T_n} found in 3.2 converge in E, up to subsequence, to non-trivial critical points for G_∞ when T_n goes to infinity. Our uniqueness assumption yields then that the limit for sequences (q_n^1) and (q_n^2) is $\tilde{q}$. We will show the contradiction by a topological method, inspired by a work of Esteban-Sere [7]. We recall that, according to our assumption,

$$\tilde{\kappa} = G_\infty(\tilde{q}) = \lim_{n\to+\infty} G_{T_n}(q_n^i) \;\text{ for } i = 1,2\,.$$

Let $\mathcal{V}_r$ be the open ball in E centered on $\tilde{q}$, with radius $r>0$, such that, for all $q \in \mathcal{V}_r$, we have $G_\infty(q) < 2\kappa_2 - \varepsilon$. Last results imply the following one:

Lemma 3.9 *Let q_n, a sequence in E, satisfy the following properties:*

- $G'_{T_n}(q_n) \to 0$ *in* E';

- *$G_{T_n}(q_n) \to c \in [\kappa_0, 2\kappa_2 - \varepsilon]$;*
- *for all n, $q_n \in \mathcal{V}_r$;*

where (T_n) is a sequence of positive real numbers. Then (q_n) is precompact in E, and, if T_n goes to $+\infty$, then, up to a subsequence, $\tilde{q}$ is the limit of q_n.

Proof. It is a direct consequence of Lemmas 3.3 and 3.5, according to the fact that $\mathcal{V}_r$ is bounded. □

Let $0 < r' < r$ and $\beta > 0$. We put $\mathcal{W} = \mathcal{V}_r \setminus \mathcal{V}_{r'}$. There exists $\tilde{T}$ such that for all $T \geq \tilde{T}$, we get $|G_T(q_T^1) - \theta| \leq \frac{\beta}{10}$ and $|G_T(q_T^2) - \theta| \leq \frac{\beta}{10}$.

Lemma 3.10 *Assuming the uniqueness of a non-trivial critical point for G_∞, we get the following results:*

1. *there exists $\mu > 0$ and $\tilde{T}'$ such that for all $q \in \bar{\mathcal{W}}$, and for all $T \geq \tilde{T}'$, we get*

$$||G'_T(q)||_{E'} \geq \mu \,, \tag{3.4}$$

2. *there exists $\tilde{T}''$ such that, if $T \geq \tilde{T}''$, if $G'_T(q) = 0$ and if moreover $G_T(q) \in [\kappa_0, 2\kappa_2 - \varepsilon]$, then $q \in \mathcal{V}_{r'}$.*
3. *for all T, there exists ν_T such that for all $q \notin \bar{\mathcal{V}}_{r'}$ satisfying $G_T(q) \in [\kappa_0, 2\kappa_2 - \varepsilon]$, we get*

$$||G'_T(q)||_{E'} \geq \nu_T \,. \tag{3.5}$$

Proof. 1. et 2. are direct consequences of Lemma 3.9, considering that there is no critical point for G_∞ and G_T, for T great enough, in $\mathcal{W}$. 3. is a straightforward consequence of 2. □

If we call $T_0 = \max(\tilde{T}, \tilde{T}', \tilde{T}'')$, Lemma 3.10 yields the following result (whose standard proof, based on a deformation lemma which uses Lemma 3.10, will be omitted):

Lemma 3.11 *With our uniqueness assumption, there exists $\beta' > 0$ such that for all $T > T_0$, there exists $s(T) > 0$ satisfying*

$$G_T(\Psi_T(s(T), q)) \leq \theta - \beta' \,, \tag{3.6}$$

for all $q \notin \mathcal{V}_r$ such that $G_T(q) \leq \theta + \beta'$, while $\Psi_T(., q)$ stands for the decreasing flow of G_T.

Topological properties of the relative category allow to conclude. Indeed, Lemma 3.11 builds a deformation $(G_T)^{\theta+\beta}\backslash\mathcal{V}_r \to (G_T)^{\theta-\beta}$ letting $(G_T)^{\kappa_0}$ globally invariant. This implies that

(3.7)
$$\mathrm{cat}_{(G_T)^{2\kappa_2-\varepsilon},(G_T)^{\kappa_0}}((G_T)^{\theta+\beta}\setminus\mathcal{V}_r) \le \mathrm{cat}_{(G_T)^{2\kappa_2-\varepsilon},(G_T)^{\kappa_0}}((G_T)^{\theta-\beta})\,.$$

The sub-additivity property of relative category then yields

$$\begin{aligned}&\mathrm{cat}_{(G_T)^{2\kappa_2-\varepsilon},(G_T)^{\kappa_0}}((G_T)^{\theta+\beta})\\ &\quad\le \mathrm{cat}_{(G_T)^{2\kappa_2-\varepsilon},(G_T)^{\kappa_0}}((G_T)^{\theta+\beta}\setminus\mathcal{V}_r) + \mathrm{cat}_{(G_T)^{2\kappa_2-\varepsilon}}(\mathcal{V}_r)\,.\end{aligned}\tag{3.8}$$

Moreover, definition of $\tilde\kappa$ and $T > T_0$ yield

$$\mathrm{cat}_{(G_T)^{2\kappa_2-\varepsilon},(G_T)^{\kappa_0}}((G_T)^{\theta-\beta}) = 0\,,\tag{3.9}$$

$$\mathrm{cat}_{(G_T)^{2\kappa_2-\varepsilon},(G_T)^{\kappa_0}}((G_T)^{\theta+\beta}) \ge 2.\tag{3.10}$$

Indeed, $(G_T)^{\theta-\beta} \notin C^1$ implies (3.9) and, considering a sequence $(P_n) \in C^2$ realizing the min-max $G_T(q_T^2)$, we obtain, for n great enough, that $P_n \subset (G_T)^{\theta+\beta}$, which directly implies (3.10). Combining (3.7), (3.8), (3.9) and (3.10), we find

$$\mathrm{cat}_{(G_T)^{2\kappa_2-\varepsilon}}(\mathcal{V}_r) \ge 2\,,$$

which is impossible. Thus, the uniqueness assumption leads to a contradiction and Theorem 1.1 is proved.

4 Construction of the deformation

In this section, we will handle X'' and Y'' in place of X' and Y'. This slight modification makes the construction easier, because X'' and Y'' do not depend on any choice. The following results come from straightforward computations and enlighten the geometry of the problem:

Lemma 4.1 *Let $c, c' > 0$. Then*

(i) the critical set for the restricted functional $F_c|_{\mathcal{M}_{c'}}$ is $\Sigma_{c'}$;

(ii) non-trivial critical points for F_c, i.e. elements of Σ_c are mountain pass points:
- *for $q \in \Sigma_c$, $F_c(q) = \min_{q\in\mathcal{M}_c} F_c(q)$, i.e. Σ_c is the set of minimizers of the restricted functional $F_c|_{\mathcal{M}_c}$;*
- *for $q \in \Sigma_c$, $F_c(q) = \max_{\lambda\in\mathbb{R}} F_c(\lambda q)$ and the maximum is obtained for $\lambda = 1$.*

(iii) relatively to the functional $F_c|_{\mathcal{M}_{c'}}$, elements of $\Sigma_{c'}$ are
- *minimizers for $c' > \frac{2c}{\alpha}$ and we have $F_c|_{\mathcal{M}_{c'}} > 0$;*

- *regular points for* $c' = \frac{2c}{\alpha}$ *since then* $F_c|_{\mathcal{M}_{c'}} \equiv 0$*;*
- *maximizers for* $c' < \frac{2c}{\alpha}$ *and we get* $F_c|_{\mathcal{M}_{c'}} < 0$*.*

(iv) $\forall c > 0,\ X'' \cap \mathcal{M}_c \neq \emptyset$*;*
(v) $Y'' \cap \mathcal{M}_c = \emptyset$ *if and only if* $c \in [d_1, d_2]$*, with* $\frac{2c_2}{\alpha} < d_1 < c_2 < d_2$*;*
(vi) $Y \cap \mathcal{M}_c = \emptyset$ *if and only if* $c \in [d'_1, d'_2]$*, with* $\frac{2c_1}{\alpha} < d'_1 < c_1 < d'_2$*;*
(vii) $d'_1 < d_1 < d_2 < d'_2$*.*

The main tool of this construction is the deformation along vector field flows in E. Vector fields are built from F_1 and F_2 gradients, by projection on tangent spaces of manifolds $\mathcal{M}_c$. All vector fields used here will be locally Lipschitz continuous, by the C^2 regularity of the application

$$q \in E \backslash \{0\} \mapsto c(q) = \frac{||q||^2}{\alpha \int_{\mathbb{R}} |q|^\alpha} ,$$

thus the flows are well-defined for any time. We will also extend the vector fields by 0 in 0.

The construction of the deformation - and proof of Lemma 2.10 - will be achieved in several steps. In the first one, we deform X'' in order to bring a part of X'' closer to the cylinder Λ. The problem is now divided into two parts:

- make a projection of the closest part of X'' on the cylinder Λ: that will be the goal of Step 2;
- build a deformation of the part of X'' situated on $\mathcal{M}_c$ manifolds with small and great c's: that will be made in Step 3, which finishes the construction.

Of course, we have to check at each step that sets X and Y are globally invariant, in order to respect conditions of Lemma 2.10.

Step 1. We define the following vector field on E:

$$\begin{cases} e_1(0) = 0 , \\ e_1(q) = -F_2'|_{\mathcal{M}_{c(q)}}(q)\theta_\delta(c - c_2) , \end{cases}$$

with $\theta_\delta \in C^\infty(\mathbb{R}, [0,1])$ an even function being such that $\mathrm{Supp}\theta_\delta = [-2\delta, 2\delta]$, θ_δ increasing on $[-2\delta, -\delta]$ and $\theta_\delta(c) = 1$ for $c \in [-\delta, 0]$. We define $0 < 2\delta < c_2 - c_1$. Let $\Psi_1(q,t)$ be the associate flow, defined as follows:

$$\begin{cases} \Psi_1(q,0) = q \\ \frac{\partial}{\partial t}\Psi_1(q,t) = e_1(\Psi_1(q,t)) . \end{cases}$$

First properties of this flow are:

- We get $c(\Psi_1(q,t)) = c(q)$ since e_1 is tangent to the manifold $\mathcal{M}_{c(q)}$;
- Sets $\Lambda(\gamma_1, \gamma_2)$ is invariant by this flow, since $e_1(q) = 0$ for $q \in \Lambda(\gamma_1, \gamma_2)$;

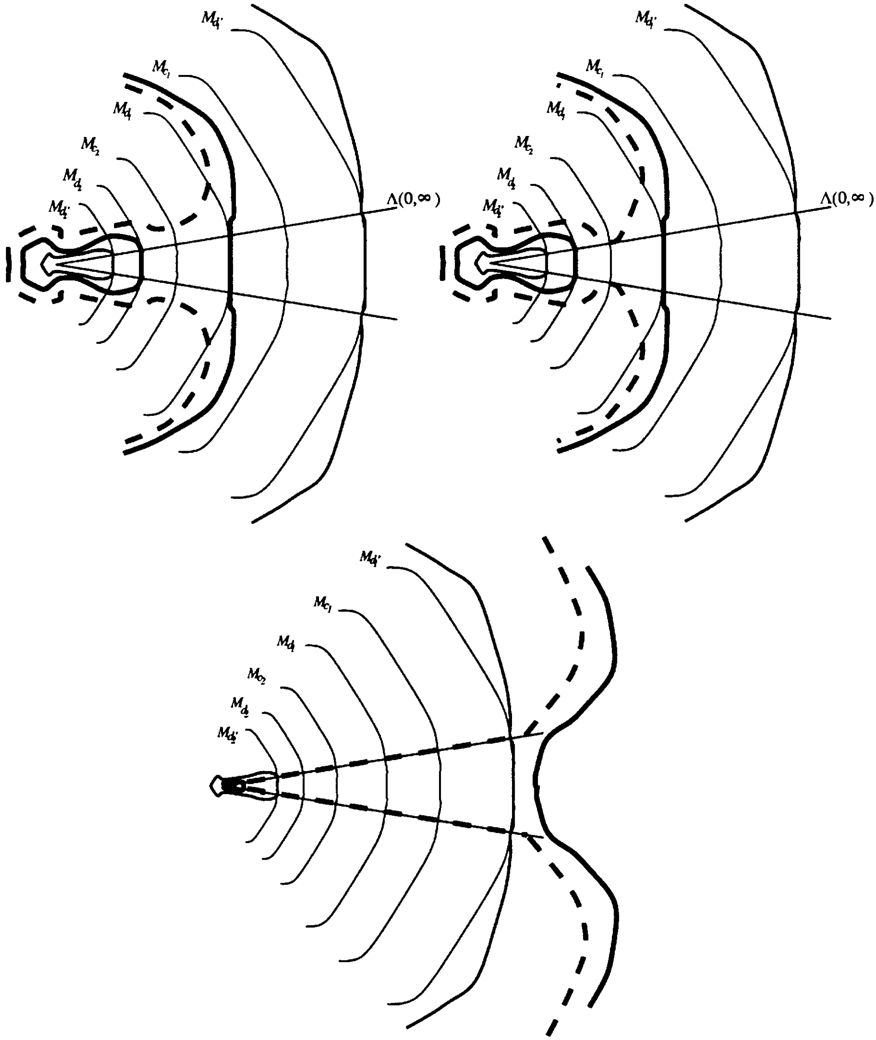

Fig. 1. The deformation: large lines are for X'' (dotted) and Y'' (plain); middle lines represent Y; small lines represent manifolds $\mathcal{M}_c$ and cylinder $\Lambda(0,\infty)$. The first picture represents the level sets before deformation. The second one shows the effect of Steps 1 and 2. The last one represents the final situation

– According to Lemma 4.1 (vi), given any $q \in Y$, $\theta(c(q) - c_2) = 0$, so X and Y are invariant by the flow Ψ_1.

Moreover, given $\nu > 0$ small enough, there exists $\delta < \frac{\alpha-2}{\alpha}$ such that, given any $c \in [c_2 - \delta, c_2 + \delta]$, we get

$$\mathcal{M}_c \cap (F_2)^{2\kappa_2 - \varepsilon} \subset (F_c)^{2\kappa_c - \nu} .$$

Since ${F_2}'|_{\mathcal{M}_c}$ is proportional to F'_c and F_c satisfies the $(PS)_d$ condition for every $d \in [\kappa_c, 2\kappa_c - \nu]$, we obtain:

$$\text{(4.1)} \quad \forall \eta > 0,\ \exists T_1 \,/ \sup_{q \in X'' \cap \bigcup_{c=c_2-\delta}^{c_2+\delta} \mathcal{M}_c} d_E(\Lambda(d_1, d_2), \Psi_1(q, T_1)) < \eta\,,$$

where d_E is the distance in E induced by the norm $||.||$. We put $g^1 = \Psi_1(., T_1)$.

Step 2. Precedent step allows to bring a part of X'' closer to $\Lambda(\gamma_1, \gamma_2)$. This second step consists in projecting that part on to $\Lambda(\gamma_1, \gamma_2)$. To this end, we define the following projection:

Let P_c the orthogonal projection from E on to the linear subspace $\mathcal{F}_c$ spanned by Σ_c, of finite dimension N. Let ${P'}_c$ be the "radial projection" from $\mathcal{F}_c$ on to Σ_c, which is a simple dilation of the unit sphere in $\mathcal{F}_c$. This projection is not well-defined in 0, but, thanks to (4.1), elements of $P_c(g^1(X'') \cap \mathcal{M}_c)$ have a norm close to the norm of elements of Σ_c. Thus, for η small enough, $0 \notin P_c(g^1(X'') \cap \mathcal{M}_c)$. Let

$$\Psi_2(q, t) = (1 - \nu(t, c(q))).q + \nu(t, c(q)).{P'}_{c(q)}(P_{c(q)}(q))\,,$$

where $\nu \in C([0, 1] \times]0, +\infty[, [0, 1])$ satisfies, with small $\rho > 0$:

- $\nu(., c) \equiv 0$ for all $c \le c_2 - 2\delta$ or $c \ge c_2 + 2\delta$;
- $\nu(t, c) = t$ for all $c_2 - \delta < c < c_2 + \delta$;
- $\nu(t, c) = \frac{c - c_2 + 2\delta}{\delta} t$ for all $c_2 - 2\delta < c < c_2 - \delta$;
- $\nu(t, c) = \frac{c_2 + 2\delta - c}{\delta} t$ for all $c_2 + \delta < c < c_2 + 2\delta$.

It is clear that application $g^2 = \Psi_2(q, 1)$ is continuous, that X and Y are invariant (here we just have to see that $\Lambda(\gamma_1, \gamma_2)$ is invariant), and:

$$\text{(4.2)} \qquad g^2\left(g^1\left(X'' \cap \bigcup_{c=c_2-\delta}^{c_2+\delta} \mathcal{M}_c\right)\right) \subset \Lambda(c_2 - \delta, c_2 + \delta)\,.$$

Step 3. This step consists of a dilation, which allows to deform parts of $g^2(g^1(X''))$ not yet contracted on $\Lambda(c_2 - \delta, c_2 + \delta)$. With this dilation, of course, $c(q)$ will not be conserved, and thus we deform these parts on to manifolds $\mathcal{M}_c$ having good properties. We define the following vector field:

$$e_3(q) = (1 - \theta_{\frac{\delta}{2}}(c - c_2))\mathrm{sgn}(c_2 - c)q\,,$$

and denote by $\Psi_3(., t)$ its flow. It is straightforward that $\Lambda(0, +\infty)$ is globally invariant by this flow. In order to state that X and Y are also globally invariant, it is sufficient to show that $(F_1)^{\kappa_0} = Y$ is globally invariant.

According to Lemma 4.1 (iv)-(vii) , Y has a non-empty intersection with $\mathcal{M}_c$ if and only if $c \notin ({d_1}', {d_2}')$. Given $c \geq {d_2}' > c_1$, a dilation by a coefficient lower than 1 yields a decrease for F_1, as this computation shows, for $q \in \mathcal{M}_c$:

$$F_1(\eta q) = \frac{\eta^2}{2}||q||^2 - c_1\eta^\alpha \int |q|^\alpha$$

$$= ||q||^2 \left(\frac{\eta^2}{2} - \frac{c_1\eta^\alpha}{\alpha c} \right) , \tag{4.3}$$

and this function of η is increasing for $0 \leq \eta \leq (\frac{c}{c_1})^{\frac{1}{\alpha-2}} > 1$. Given $c \leq {d_1}' < c_1$, and by the same way, a dilation by a coefficient greater than 1 still yields a decrease for F_1. Thus X and Y are globally invariant.

For $c > c_2 + \delta$, the vector field e_3 is the gradient of a very simple functional (indeed $q \mapsto -\frac{1}{2}||q||^2$) which obviously satisfies the (PS) condition at any level. Moreover, given $q \in \mathcal{M}_c$, we get:

$$F_1(\lambda q) = \lambda^2 \frac{\frac{1}{2} - \frac{c_1\lambda^{\alpha-2}}{\alpha c}}{\frac{1}{2} - \frac{c_2}{\alpha c}} F_2(q) \, . \tag{4.4}$$

Thus there exists $\bar{\lambda} < 1$ such that, for all $q \in X'' \cap \bigcup_{c > c_2+\delta} \mathcal{M}_c$, we get $F_1(\bar{\lambda} q) \leq \kappa_0$. So we infer that there exists T_3^1 such that

$$\Psi_3 \left(\bigcup_{c > c_2+\delta} \mathcal{M}_c \cap X'', T_3^1 \right) \subset (F_1)^{\kappa_0} \, .$$

For $c < c_2 - \delta$, (4.3) implies that there exists T_3^2 such that for every $q \in \mathcal{M}_c \cap X''$, $F_1(\Psi_3(q, T_3^2)) < 0$. Taking $T_3 = \max(T_3^1, T_3^2)$, we put $g^3 = \Psi_3(., T_3)$, and $g = g^3 \circ g^2 \circ g^1$. Properties of g are the following ones:

- $g(X'' \cap \bigcup_{c \leq c_2-\delta} \mathcal{M}_c) \subset (F_1)^{\kappa_0} \subset X$,
- $g(X'' \cap \bigcup_{c=c_2-\delta}^{c_2+\delta} \mathcal{M}_c) \subset \Lambda \subset X$,
- $g(X'' \cap \bigcup_{c \geq c_2+\delta} \mathcal{M}_c) \subset (F_1)^{\kappa_0} \subset X$,
- $g(Y'' \cap \bigcup_{c \leq c_2-\delta} \mathcal{M}_c) \subset (F_1)^{\kappa_0} = Y$,
- $g(Y'' \cap \bigcup_{c \geq c_2+\delta} \mathcal{M}_c) \subset (F_1)^{\kappa_0} = Y$.

Applications g^i, for $1 \leq i \leq 3$ have been constructed as deformation of the identity. By reparametrization and composition, we find $g_t : X'' \to X''$, for $t \in [0,1]$, with g_0 =Id and $g_1 = g$. The mapping $t \mapsto g_t$ is continuous from $[0,1]$ on to the space of continuous mappings from X'' to E, and for all $t \in [0,1]$, $g_t(X) \subset X$, $g_t(Y) \subset Y$. Finally, the last results imply $g_1(X'') = g(X'') \subset X$ and $g_1(Y'') = g(Y'') \subset Y$. Restriction of g_t to (X', Y') satisfies the same properties: Lemma 2.10 is proved.

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