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On linguistic approximation in the frame of fuzzy logic deduction

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Abstract This paper presents a new linguistic approximation algorithm and its implementation in the frame of fuzzy logic deduction. The algorithm presented is designed for fuzzy logic deduction mechanism implemented in Linguistic Fuzzy Logic Controller (LFLC).

Key words Linguistic approximation, inference mechanism, expert systems.

1 Introduction

This article deals with the problem of linguistic approximation. By definition, linguistic approximation is a procedure which assigns linguistic expression to a given fuzzy set.

The usual method of obtaining results in fuzzy systems is called defuzzification. It means that as soon as we perform one step of inference mechanism, we defuzzify the obtained fuzzy set and receive one crisp number. However, it is not much in the spirit of fuzzy logic. It is the only possibility in fuzzy control, because we need control action, e.g. electric current or gas flow, as a crisp number in order to perform the next step of the control process.

In other kinds of fuzzy systems, e.g. fuzzy expert systems, decision support systems, etc., we need something different. It is desirable to obtain a linguistic expression better than the crisp number. The former have similar form as the linguistic expressions which enter the inference mechanism. We must use the linguistic approximation for this purpose.

There are several approaches to this problem, e.g. in [4, 13]. In [7], a good overview of the approaches presented so far is given. In our approach, we will suppose that fuzzy sets which enter the linguistic approximation algorithm are outputs of a fuzzy logic deduction. If it is the case, then we are allowed to construct the linguistic expression which corresponds to these fuzzy sets by using atomic terms and linguistic modifiers used in the fuzzy logic deduction.

This paper is organized as follows: In Sect. 2 we present basic theory of fuzzy logic deduction, Sect. 3 contains description of the linguistic approximation procedure, Sect. 4 presents two examples, and Sect. 5 contains some conclusions.

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2 Fuzzy logic deduction

There are several approaches to fuzzy inference, which is the implementation of the modus ponens inference rule in which the implication and possibly the premise are also given vaguely. They differ mainly in the interpretation of implication. One approach, known as Mamdani–Zadeh inference or fuzzy interpolation or Max-t-norm inference, is essentially an approximation of an unknown function [5]. The second approach, which we call *fuzzy logic deduction*, uses the Łukasiewicz implication operator as a basis for the inference mechanism. The basic scheme known as *generalized modus ponens* is the following:

Condition:

$$\mathscr{R}_1 := \text{IF } X_1 \text{ is } \mathscr{A}_{11} \text{ AND} \dots \text{ AND } X_n \text{ is } \mathscr{A}_{n1} \text{ THEN } Y \text{ is } \mathscr{B}_1$$

....

$$\mathscr{R}_r := \text{IF } X_1 \text{ is } \mathscr{A}_{1r} \text{ AND} \dots \text{ AND } X_n \text{ is } \mathscr{A}_{nr} \text{ THEN } Y \text{ is } \mathscr{B}_r$$

Observation: X_1 is $\mathscr{A}'_1$ AND … AND X_n is $\mathscr{A}'_n$

Conclusion: Y is $\mathscr{B}'$,

where $X_1, \dots, X_n$ are the *antecedent variables*, Y is the *succedent variable*, and $\mathscr{A}'_1, \dots, \mathscr{A}'_n$ are expressions of natural language, which may be slightly different from $\mathscr{A}_{1j}, \dots, \mathscr{A}_{nj}$. Hence, the conclusion $\mathscr{B}'$ can be slightly different from all $\mathscr{B}_j$, $j = 1, \dots, r$.

Natural language expressions $\mathscr{A}_{ij}$, $\mathscr{B}_j$ in the generalized modus ponens scheme are assumed to have the following form [12]:

$$\langle \text{linguistic modifier} \rangle \langle \text{atomic term} \rangle. \quad (1)$$

The atomic terms characterize various properties of objects, and the linguistic modifiers specify various nuances of properties. To describe some qualitative property linguistically we need an ordered linguistic scale with three atomic terms *small*, *medium* and *big*.

The linguistic expressions $\mathscr{A}$ are generally assigned the fuzzy sets $A \underset{\sim}{\subset} U$. In [12], it is shown in detail how the membership functions for the meanings of the atomic terms can be derived using the concept of the horizon. In Fig. 1, typical membership functions of the atomic terms are depicted.

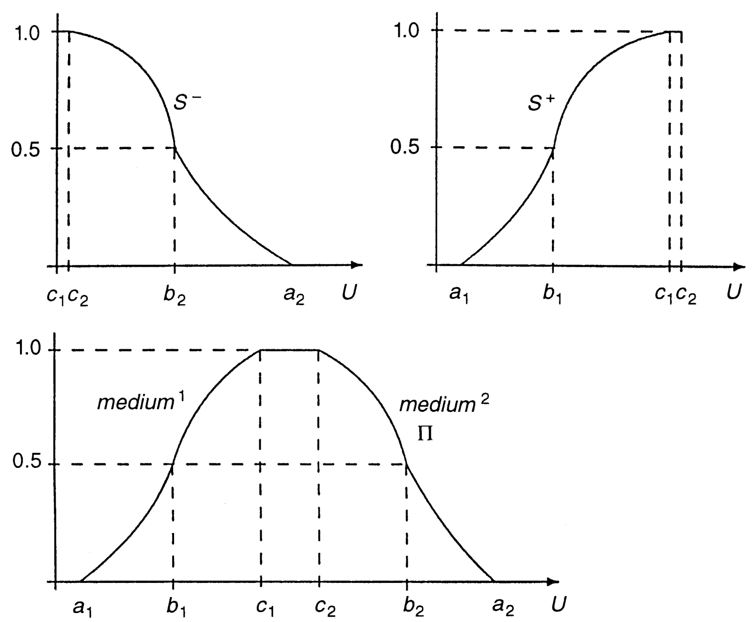

Fig. 1. Membership functions S^-, S^+ and π fuzzy sets. They usually correspond to the meaning of the atomic terms *small*, *big* and *medium*, respectively

In [12], it is also proposed to understand the atomic term *medium* as a compound term of the form

$$medium = medium^1 \ or \ medium^2,$$

where *medium*^{1} is the increasing and *medium*^{2} decreasing part of the membership function for *medium* (cf. Fig. 1).

The membership functions of the meanings of atomic terms can have linear, quadratic or exponential form. In our opinion, the quadratic shape of the membership functions (see Fig. 1) best fits the meaning of the atomic terms as used by the people.

The linguistic modifiers [12, 11] are derived by means of the concept of the *horizon shift*. They can be divided into two categories, namely, linguistic modifiers with *narrowing effect* (very, extremely, etc.) and with *widening effect* (more or less, roughly). The linguistic modifiers with widening effect only are permitted for use with the atomic term *medium*.

The meanings of the linguistic expressions (1) can be modeled using fuzzy sets with membership functions taken from some class of functions $\mathscr{Z}(U)$, [12]

$$\mathscr{Z}(U) = \{Z, S \mid Z, S\colon U \to [0, 1]\},$$

where members of the class of non-decreasing functions Z model the meanings of linguistic expressions with atomic terms *small* and *medium*^{1}, and members of the class of non-increasing functions S model the meanings of linguistic expressions with atomic terms *medium*^{2} and *big*.

Let the fuzzy sets A_{ij}, $i = 1, \ldots, n$, $j = 1, \ldots, r$ and B_j, $j = 1, \ldots, r$ represent the meanings of all the natural language expressions $\mathscr{A}_{ij}$ and $\mathscr{B}_j$, respectively, which occur in the generalized modus ponens scheme above. Then the fuzzy set B' representing the meaning of the conclusion $\mathscr{B}'$ can be computed using the formula [3, 8]

$$B'y = \bigvee_{x_1 \in U_1, \ldots, x_n \in U_n} ((A'_1 x_1 \wedge \cdots \wedge A'_n x_n) \otimes \bigwedge_{j=1}^{r} ((A_{1j} x_1 \wedge \cdots \wedge A_{nj} x_n) \to B_j y)), \quad (2)$$

where $U_1, \ldots, U_n$ are universes of discourse of the antecedent variables, A'_i are fuzzy sets which represent the observations, $\otimes$ and $\to$ are Łukasiewicz conjunction and implication, respectively.

3 Linguistic approximation algorithm

The linguistic approximation is, in general, a method, which assigns linguistic expression to the given fuzzy set. Formally, we can describe this method as follows [9]:

First, we define a similarity relation between fuzzy sets:

Definition 1 A fuzzy relation

$$\mathscr{R} \underset{\sim}{\subseteq} \mathscr{F}(U) \times \mathscr{F}(U),$$

where $\mathscr{F}(U) = [0, 1]^U$ is the set of all fuzzy sets on U, is called *similarity relation*, if for all $A, B \in \mathscr{F}(U)$ the following conditions hold:

1. $\mathscr{R}\langle A, A\rangle = 1$,
2. $\mathscr{R}\langle A, B\rangle = \mathscr{R}\langle B, A\rangle$,
3. $\mathrm{Supp}(A) \cap \mathrm{Supp}(B) = 0$ implies $\mathscr{R}\langle A, B\rangle = 0$,

where

$$\mathrm{Supp}(A) = \{x \in U \mid Ax > 0\}$$

is the support of the fuzzy set A.

Definition 2 Let $\langle \mathscr{X}, \mathscr{T}(\mathscr{X}), U, G, \mathscr{M} \rangle$ be a linguistic variable [14], where $\mathscr{X}$ is name of the variable, $\mathscr{T}(\mathscr{X})$ is the term set, U is the universe, G and $\mathscr{M}$ are the syntactic and semantic rules, respectively. Let $A_0 \underset{\sim}{\subseteq} U$ be a fuzzy set and $\mathscr{R} \underset{\sim}{\subseteq} \mathscr{F}(U) \times \mathscr{F}(U)$ be a fuzzy relation defined above. Then the term $\mathscr{A} \in \mathscr{T}(X)$, for which is $\mathscr{R}\langle M(\mathscr{A}), A_0 \rangle$ maximal, is called the *linguistic approximation* of the fuzzy set A_0.

If there are more terms $\mathscr{A}_i$ with the same value of $\mathscr{R}\langle M(\mathscr{A}_i), A_0 \rangle$, then we should use different similarity measure (possibly more sensitive in this individual situation). Definition 2 states that we search for the term, whose meaning is most similar to the approximated fuzzy set. There are several problems encountered:

- we have to search not only among atomic and modified atomic terms, but also among composed terms, e.g. *roughly small* **or** *medium*,
- the fuzzy set which enters linguistic approximation can be subnormal,
- the information included in the fuzzy set which enters linguistic approximation can be distorted or noisy.

The algorithm which performs linguistic approximation has to take the above-mentioned problems into consideration. The algorithms of the linguistic approximation can be divided into two categories [7]:

1. Algorithm performing the entire check of the term set,
2. Algorithm based on the piecewise decomposition of the fuzzy set.

In the first method, we simply compare a given fuzzy set with all the fuzzy sets corresponding to the terms from the term set. Some similarity relation is used for the determination of similarity between these fuzzy sets. The term whose meaning has the smallest distance from the given one is taken as the result.

The Piecewise Decomposition method, in general, decomposes the given fuzzy set into several pieces or segments, determines the linguistic term for each of them, and then constructs the result by means of logical connectives [4].

In our approach, we want the resulting expression to have the form

$\langle$linguistic modifier$_1\rangle$ $\langle$*small*$\rangle$ **or** $\langle$linguistic modifier$_2\rangle$

$\langle$*medium*$\rangle$ **or** $\langle$linguistic modifier$_3\rangle$ $\langle$*big*$\rangle$,

where $\langle$linguistic modifier$_i\rangle$, $i=1, 2, 3$ are linguistic modifiers described in Sect. 2 including special modifiers *empty* and *ignored*. Modifier *empty* does not change the membership function which models the meaning of the atomic term. Modifier *ignored* forces the membership function which model the meaning of modified atomic term to be identically equal to zero. Thus we can obtain linguistic expression with one, two or three linguistic expressions (1) connected by the connective **or**. We divide approximated fuzzy set into several segments and determine the atomic term and the modifier to each of them.

Fuzzy sets which enter linguistic approximation are outputs B' of fuzzy logic deduction described by formula (2), which is in real situations always performed on discretized universes $U_1, U_2, \ldots, U_n$. Thus fuzzy set B' is also discrete and we can in the following consider only fuzzy sets defined on the linearly ordered discrete universe $U=\{x_1, x_2, \ldots, x_m\}$.

The linguistic approximation algorithm can be described in the following steps:

1. Divide the membership function of the given fuzzy set into segments.
2. Discard "horizontal" segments, which bear no information.
3. Decide, for each segment, whether segment is "peak" or "section".
4. For all segments, if the segment is "peak" then normalize it.
5. Compare segments with meanings of terms from the term set and find term with the highest membership degree of similarity relation.
6. Compose resulting linguistic expression by means of connectives.

Ad 1: We divide membership function into segments whose boundaries are defined as follows:

Definition 3 Let A be a fuzzy set on universe $U=\{x_1, x_2, \ldots, x_m\}$. The point $x_i \in U$ is *boundary* if $x_i=x_0$ or $x_i=x_m$ or

$$\operatorname{sgn}(x_{i+1}-x_i) \neq \operatorname{sgn}(x_i-x_{i-1}),$$

where

$$\operatorname{sgn}(x)=\begin{cases} -1, & x<0, \\ 0, & x=0, \\ 1, & x>0. \end{cases}$$

Definition 4 Let A be a fuzzy set from Definition 3 and

$$B=\{b^1, b^2, \ldots, b^k\}, \quad k<m$$

be a set of all boundary points. Then sets $S_i \subseteq U$

$$S_i=\{x_j=b^i, x_{j+1}, \ldots, x_{j+l}=b^{i+1}\}, \quad i=1, 2, \ldots, k-1$$

are *segments of A*.

Ad 2: Segment S_i which satisfies $Ab^i=Ab^{i+1}$ is called *horizontal*. Horizontal segments bear, in our opinion, no information about resulting linguistic expression, and therefore, they are discarded.

Ad 3: Other kinds of segments are: sections and peaks.

Definition 5 Segment S_i is a *section* if its first boundary point b^i satisfies

$$\left|\frac{Ax_{j+1}-2Ab^i+Ax_{j-1}}{x_{j+1}-x_{j-1}}\right|>\kappa, \tag{3}$$

where x_{j-1}, x_{j+1} are the two closest neighbours of b^i in U, or its second boundary point b^{i+1} satisfies

$$\left|\frac{Ax_{j+1}-2Ab^{i+1}+Ax_{j-1}}{x_{j+1}-x_{j-1}}\right|>\kappa, \tag{4}$$

where x_{j-1}, x_{j+1} are the two closest neighbour points of b^{i+1} in U. If formulae (3) and (4) are not fulfilled then the segment S_i is a *peak*.

The value of the constant κ is important for the proper distinguishing between sections and peaks. If κ is too big, then the segment of type peak can be recognized as section and vice versa. We obtained the best results with $\kappa = 0.5$. Informally, the peak is the segment which corresponds to "upper" part of membership function, i.e. the part in which the global maximum of membership function is included, and the section is the segment which corresponds to that part of the membership function which does not include global maximum.

Ad 4: If the segment is of the type "peak" and is subnormal, i.e. the maximal membership degree is smaller than 1, then we have to normalize it. The reason is that the meanings of terms from our term set are all normal fuzzy sets. The simplest way to normalize the "peak" is to find out the maximal membership degree $M = \bigvee_{x \in S_i} Ax$ in the segment S_i and to modify all membership degrees using the formula

$$A_{\mathrm{mod}}\, x = \frac{Ax}{M}, \quad \forall x \in S_i.$$

Ad 5: The term most appropriate to the given segment can be obtained as follows: we generate fuzzy sets which correspond to all the possible linguistic terms with the same atomic term. As a atomic terms we consider *small*, *big*, *medium*^{1} and *medium*^{2}, as described in Sect. 2. The term we determined as the most appropriate is that which has a membership function most similar to the given segment.

The following formula can be used for computation of the similarity relation [9]:

$$\mathcal{R}\langle A, B\rangle = 1 - \frac{(1/\#(\mathrm{Supp}A \cup \mathrm{Supp}B)) \sum_{x \in \mathrm{Supp}A \cup \mathrm{Supp}B} f(Ax - Bx)}{(1/\#\mathrm{Supp}A) \sum_{x \in \mathrm{Supp}A} f(Ax) + (1/\#\mathrm{Supp}B) \sum_{x \in \mathrm{Supp}B} f(Bx)}, \tag{5}$$

where $\#(C)$ denotes the number of elements of the set C, and

$$f: [-1, 1] \to [0, 1]$$

is an even continuous measurable function increasing on [0, 1].

As the most appropriate term we choose that with the maximal value of $\mathcal{R}$. Functions $f(x)$ are chosen from the family of functions $\mathcal{F}_e = \{|x|, x^2, x^4, \ldots\}$. We start with $f(x) = |x|$ and if there are more than one term with the same maximal value of $\mathcal{R}$, we use the subsequent $f \in \mathcal{F}_e$.

Ad 6: In the last step of algorithm we compose segments with assigned atomic terms *medium*^{1} and *medium*^{2} to one *medium* term, and then compose all partial terms by means of the connective **or**.

4 Example

In this section, we present our linguistic approximation algorithm working on two fuzzy sets, depicted in Figs. 2 and 3.

The shape of the membership function of fuzzy set in Fig. 2 suggests that the corresponding linguistic expression should be composed of two partial expressions with atomic terms *medium* and *big* connected by **or**. Linguistic approximation algorithm indeed gives us the expected result.

At first, we perform Step 1 of our algorithm, division to segments. Result is shown in Fig. 2 and labeled as S1, S2, S3, S4.

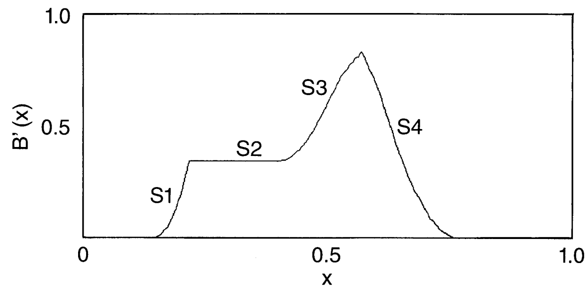

Fig. 2. Example 1 fuzzy set

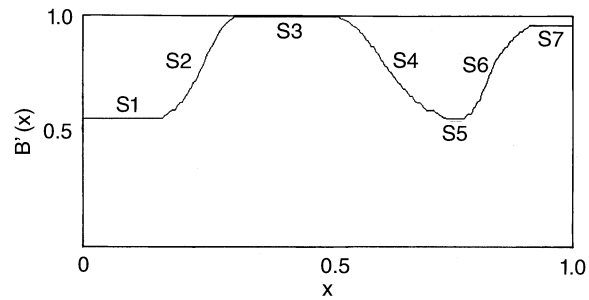

Fig. 3. Example 2 fuzzy set

Then, horizontal segment S2 is discarded (Step 2). All remaining segments S1, S3 and S4 are of type "section" (Step 3), i.e. they do not include the global maximum of the membership function. Since the normalization (Step 4) is performed only for segments of type "peak", it is not used here.

In Step 5, we compare segments S1 and S3, whose membership function is increasing with fuzzy sets which model the meaning of linguistic expressions (1) with atomic terms *small* and *medium*^{1}, and segment S4 whose membership function is decreasing with fuzzy sets which model the meaning of linguistic expressions (1) with atomic term *medium*^{2} and *big*. The comparison is performed by means of formula (5). We obtained the following results: S1 corresponds to *more or less medium*^{1}, S3 to *roughly big* and S4 to *more or less medium*^{2}. As the last step, we compose the final linguistic expression: *more or less medium* **or** *roughly big*.

Linguistic expression describing fuzzy set depicted in Fig. 3 should also be composed of expressions with atomic terms *medium* and *big*, but segments S2, S4 and S6 are of type "peak". As all these segments are normal, no normalization is necessary. Segments S1, S3, S5 and S7 are horizontal. Steps 5 and 6 of linguistic approximation algorithm gives us the final result *quite roughly medium* **or** *big*.

5 Conclusion

Applications of linguistic approximation can be found whenever we need linguistic description of results given by fuzzy systems. The easy interpretability of fuzzy systems is one of their crucial features, and linguistic approximation allows to interpret results given by them.

The most promising is the field of expert systems and decision support systems, where we do not need crisp (defuzzified) value, but rather linguistic expression which can be easily understood by humans. Other fields of application can be found in fuzzy data analysis and fuzzy modeling [13], various kinds of medical systems, and even in some types of fuzzy control [2, 1].

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