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Input-to-State Stability for Nonlinear Time-Varying Systems via Averaging*

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Abstract. We introduce two definitions of an averaged system for a time-varying ordinary differential equation with exogenous disturbances ("strong average" and "weak average"). The class of systems for which the strong average exists is shown to be strictly smaller than the class of systems for which the weak average exists. It is shown that input-to-state stability (ISS) of the strong average of a system implies uniform semi-global practical ISS of the actual system. This result generalizes the result of [TPA] which states that global asymptotic stability of the averaged system implies uniform semi-global practical stability of the actual system. On the other hand, we illustrate by an example that ISS of the weak average of a system does not necessarily imply uniform semi-global practical ISS of the actual system. However, ISS of the weak average of a system does imply a weaker semi-global practical "ISS-like" property for the actual system when the disturbances w are absolutely continuous and $w, \dot{w} \in \mathscr{L}_\infty$. ISS of the weak average of a system is shown to be useful in a stability analysis of time-varying cascaded systems.

Key words. Averaging, Continuous-time, Input-to-state stability, Nonlinear, Time-varying.

1. Introduction

Averaging of ordinary differential equations is an important tool in the stability analysis and synthesis of time-varying systems. Some classical results on averaging and their applications to adaptive control can be found in [ABJ$^+$], [K], [SV], [V] and references therein. In the classical averaging approach one either proves

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the closeness of trajectories of the averaged and the actual system on finite time intervals or, under the assumption of local exponential stability (LES) of the origin of the averaged system, uniform LES of a small periodic trajectory of the actual system. The main proof technique in the classical approach is a change of coordinates in which it becomes apparent that the actual system can be regarded as a small perturbation of the averaged system. Another approach, that is well suited to stability studies, is to compare the solutions of averaged and actual systems at sampling instances. A Lyapunov theorem expressed in terms of solutions at sampling instances was introduced in [AP1] and used to address global and semi-global stability for the cases of exponential stability of averaged system in [AP2] and homogeneous systems in [PA]. A rather general global/semi-global averaging result in the literature was presented in [TPA] where it was proved that global asymptotic stability (GAS) of the averaged system implies uniform semi-global practical stability for the actual system.

We emphasize that the averaging results in the literature are concerned only with stability analysis although very often one needs to analyze the performance of the system under the action of the disturbances. The recently introduced notion of input-to-state stability (ISS) [S1] provides a nice framework for $\mathscr{L}_\infty$ stability analysis of nonlinear systems (see also [S2], [SW1] and [SW2]). The ISS property has proved to be very useful in the analysis and synthesis of nonlinear control systems (see, for instance, [S1], [S2], [SW2] and Sections 2.4, 5.2, 5.3, 6.1, 6.7, 9.1 and 9.8 in [KKK]) and it appears to be important to investigate it in the context of averaging.

It is the purpose of this paper to introduce two definitions of the averaged system for a time-varying system with exogenous disturbances ("the strong average" and "the weak average") and to provide a connection between the ISS properties of such averaged systems with the corresponding ISS properties of the actual system (see also [TN]). Our first main result states that *ISS of the strong average of a system implies uniform semi-global practical ISS of the actual system*, which generalizes the result obtained in [TPA]. We make use of the proof technique presented in [NTS] that relates discrete-time and sampled-data ISS $\mathscr{KL}$-estimates and which can be regarded as a generalization of the stability result in [AP1] to the ISS case. The limitation of our first result is that, even for periodic functions, a strong average may not exist in general. The weak average may exist when a strong average does not but it *does not* allow us to prove in general that ISS of the weak average of a system implies uniform semi-global practical ISS of the actual system. For our second main result we require disturbances to be absolutely continuous, with $w, \dot{w} \in \mathscr{L}_\infty$, and we prove that ISS of the *weak average of a system* implies a semi-global practical "ISS-like" property for the actual system. This property is shown to be useful in the analysis of stability of time-varying cascaded systems, which motivates the use of weak averages. The result for the weak average is similar to the ISS result for singularly perturbed systems in [CT].

The paper is organized as follows. In Section 2 we present definitions of weak and strong averages and preliminary results that are needed in what follows. Several characterizations and relationships between the strong and weak averages are investigated in Section 3. Our main results are stated in Section 4 and several applications of the main results to cascaded time-varying systems are given in Section 5. The proofs are given in Sections 6–8. A summary is given in the last section.

2. Preliminaries

A function γ: $\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class-$\mathcal{G}$ ($\gamma \in \mathcal{G}$) if it is continuous, zero at zero and nondecreasing. It is of class-$\mathcal{K}$ if it is of class-$\mathcal{G}$ and strictly increasing. It is of class-$\mathcal{K}_\infty$ if it is of class-$\mathcal{K}$ and is unbounded. A function β: $\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class-$\mathcal{KL}$ if $\beta(\cdot, t)$ is of class-$\mathcal{K}$ for each $t \geq 0$ and $\beta(s, \cdot)$ is decreasing to zero for each $s > 0$. Given a measurable function $w(\cdot)$, we define its infinity norm $\|w\|_\infty := \operatorname{ess\,sup}_{t\geq 0}|w(t)|$. If we have $\|w\|_\infty < \infty$, then we write $w \in \mathcal{L}_\infty$. If $w(\cdot)$ is absolutely continuous, its derivative is defined almost everywhere and we can write $w(t) - w(t_0) = \int_{t_0}^{t} \dot{w}(\tau)\, d\tau$.

Consider the time-varying system:

$$\dot{x} = f(t, x, w), \tag{1}$$

where $x \in \mathbb{R}^n$ is the state and $w \in \mathbb{R}^m$ is the disturbance. We use the following:

Assumption 1. *System* (1) *is locally Lipschitz in* x, w, *uniformly in* t *and there exists* $c \geq 0$ *such that* $|f(t, 0, 0)| \leq c$, $\forall t \geq 0$.

Hence, we are guaranteed the existence of solutions and can use certain results on continuity of solutions with respect to initial conditions. The solution of system (1) at time t, starting from an initial condition x_o at initial time t_0 and under the action of disturbance $w_{[t_0, t]}$, is denoted as $x(t)$ (since usually $t_0, x_o, w_{[t_0, t]}$ are clear from the context).

We investigate also the time-varying system that depends on a small parameter $\varepsilon > 0$:

$$\dot{x} = f\left(\frac{t}{\varepsilon}, x, w\right). \tag{2}$$

We first recall the standard definition of the average for time-varying systems without disturbances (for instance, see Sections 8.3 and 8.5 in [K]):

Definition 1 (Average). A locally Lipschitz function $F_{\text{av}}(x)$ is said to be the average of $F(t, x)$ if there exist $\beta_{\text{av}} \in \mathcal{KL}$ and $T^* > 0$ such that for all $T \geq T^*$ and $t \geq 0$ we have that

$$\left| F_{\text{av}}(x) - \frac{1}{T}\int_{t}^{t+T} F(\tau, x)\, d\tau \right| \leq \beta_{\text{av}}(\max\{|x|, 1\}, T).$$

The system

$$\dot{x} = F_{\text{av}}(x)$$

is called the average of system $\dot{x} = F(t, x)$.

We are not aware of any references in the literature that deal with averaging of systems with exogenous disturbances and we introduce below two different definitions for an averaged system with exogenous disturbances. Motivation for two different definitions comes from the following:

1. We recall that in the disturbance free case one can show that GAS of the averaged system implies uniform semi-global practical stability of the actual system (see [TPA]). With the first definition of average (strong average) we are able to show that ISS of the strong average of a system implies uniform semi-global practical ISS of the actual system. On the other hand, it is impossible to prove this result with the second definition (weak average). Hence, the strong average appears to be an appropriate generalization of the average in Definition 1 to systems with disturbances if we want to prove the general ISS result.
2. Average in the sense of the second definition (weak average) may exist when the strong average does not. On the other hand, existence of the strong average always implies existence of the weak average. Unfortunately, ISS of the weak average of a systems does not imply uniform semi-global practical ISS of the actual system but only a weaker "ISS-like" property. For this property we require the disturbances to be absolutely continuous and w, $\dot{w} \in \mathscr{L}_\infty$, which is similar to the ISS result for singularly perturbed systems obtained in [CT]. The ISS-like property turns out to be useful for stability analysis of time-varying cascaded systems, which motivates the use of weak averages.

Further motivation for the use of two different definitions is given in the next section.

Definition 2 (Strong Average). A locally Lipschitz function $f_{\text{sa}}: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is said to be the strong average of $f(t,x,w)$ if there exist $\beta_{\text{av}} \in \mathscr{KL}$ and $T^* > 0$ such that $\forall t \geq 0$, $\forall w \in \mathscr{L}_\infty$, $\forall T \geq T^*$ the following holds:

$$\left| \frac{1}{T} \int_t^{t+T} [f_{\text{sa}}(x, w(s)) - f(s, x, w(s))] \, ds \right| \leq \beta_{\text{av}}(\max\{|x|, \|w\|_\infty, 1\}, T). \tag{3}$$

The strong average of system (1) is then defined as

$$\dot{x} = f_{\text{sa}}(x, w). \tag{4}$$

Definition 3 (Weak Average). A locally Lipschitz function $f_{\text{wa}}: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is said to be the weak average of $f(t,x,w)$ if there exist $\beta_{\text{av}} \in \mathscr{KL}$ and $T^* > 0$ such that $\forall T \geq T^*$, $\forall t \geq 0$ we have[1]

$$\left| f_{\text{wa}}(x, w) - \frac{1}{T} \int_t^{t+T} f(s, x, w) \, ds \right| \leq \beta_{\text{av}}(\max\{|x|, |w|, 1\}, T).$$

The weak average of system (1) is then defined as

$$\dot{x} = f_{\text{wa}}(x, w). \tag{5}$$

[1] Note that w in the integral is a constant vector.

We note here that if there exists the strong (respectively, weak) average of system (1), then there exists the strong (respectively, weak) average of system (2) for all $\varepsilon > 0$ and moreover the averages of (1) and (2) are the same.

3. Characterizations of Strong and Weak Averages

In order to understand definitions of strong and weak averages better, we present below several results and examples that illustrate the connections between the two notions and characterize the structure of some systems that allow for strong averages. Moreover, the results of this section provide further motivation for the use of two different definitions of average for systems with inputs.

The following result is trivial to prove and the proof is omitted.

Proposition 1. *Consider system* (1) *where*

$$f(t,x,w) = F(t,x) + g(x,w), \tag{6}$$

and there exists the average for $F(t,x)$, *denoted as* $F_{av}(x)$, *in the sense of Definition* 1. *Then* $f_{sa}(x,w) := F_{av}(x) + g(x,w)$ *satisfies our definition of the strong average for* $F(t,x) + g(x,w)$.

Note that for systems that satisfy (6), computing the strong average is as difficult as computing the average for a system without disturbances. Actually, in this case strong and weak averages coincide, which is not always the case as the following example shows. Indeed, the following example illustrates a system for which the weak average exists whereas the strong average does not.

Example 1. Consider the system

$$\dot{x} = -cx^3 + \cos\left(\frac{t}{\varepsilon}\right)x^3 w, \tag{7}$$

where $x, w \in \mathbb{R}$, $c \in (0, 0.5)$. The weak average of this system is

$$\dot{x} = -cx^3. \tag{8}$$

We now show that there does not exist a strong average for system (7). Pick an arbitrary $\tilde{x} \neq 0$ and note that, given any function $f_{sa}(x,w)$, we have two possibilities:

1. either $f_{sa}(\tilde{x}, w) + c\tilde{x}^3 = 0$, $\forall w$, or
2. $\exists \tilde{w}$ such that $f_{sa}(\tilde{x}, \tilde{w}) + c\tilde{x}^3 \neq 0$.

Suppose that $f_{sa}(x,w)$ is the strong average for $-cx^3 + \cos(t)x^3 w$ and the first case holds. Let $w(t) = \cos(t)$. Then it must hold that

$$\left|\frac{\tilde{x}^3}{T}\int_t^{t+T} \cos^2(s)\, ds\right| \leq \beta_{av}(\max\{|\tilde{x}|, \|w\|_\infty, 1\}, T)$$

for some $\beta_{av} \in \mathcal{KL}$ and for all T sufficiently large. However, it we take $T_k = k\pi$, $k \in \mathbb{N}$, the left-hand side in the above expression is equal to $0.5\tilde{x}^3 \neq 0$ for all $k \in \mathbb{N}$ and it does not converge to zero as $k \to \infty$ (as $T_k \to \infty$).

Suppose now that $f_{\mathrm{sa}}(x, w)$ is the strong average and the second case holds. Pick $w(s) = \tilde{w}$ and then it must be the case that

$$\left|\frac{1}{T}\int_t^{t+T} (f_{\mathrm{sa}}(\tilde{x}, \tilde{w}) + c\tilde{x}^3 - \tilde{w}\tilde{x}^3 \cos(s))\, ds\right| \le \beta_{\mathrm{av}}(\max\{|\tilde{x}|, |\tilde{w}|, 1\}, T)$$

for some $\beta_{\mathrm{av}} \in \mathcal{KL}$ and all T sufficiently large. However, for $T_k = 2k\pi$, $k \in \mathbb{N}$, we have that the left-hand side is equal to $|c\tilde{x}^3 + f_{\mathrm{sa}}(\tilde{x}, \tilde{w})| > 0$ for all $k \in \mathbb{N}$ and it does not converge to zero as $k \to \infty$ (as $T_k \to \infty$). Hence, there does not exist a strong average for system (7).

Note that the system in Example 1 is periodic, it does not have the form (6) given in Proposition 1 and it does not have the strong average. We may ask whether this is always the case for periodic systems. In other words, we may ask if the structure (6) is a *necessary* condition for systems periodic in t to have the strong average. We show next that this is indeed the case. More precisely, we show now that if $f(t, x, w)$ is periodic in t, then systems of the form (6) are the only systems for which the strong average exists. This emphasizes the importance of structure (6) for the existence of the strong average.

Proposition 2. *Suppose that $f(t, x, w)$ is continuous and periodic in t of period $T_1 > 0$. Then there exists a strong average $f_{\mathrm{sa}}(x, w)$ for $f(t, x, w)$ if and only if* (6) *holds for some continuous functions F and g where $F(t, x)$ has a well-defined average in the sense of Definition* 1, *which is denoted as $F_{\mathrm{av}}(x)$. Moreover, we have that $f_{\mathrm{sa}}(x, w) = F_{\mathrm{av}}(x) + g(x, w)$.*

Proof. Sufficiency is trivial to prove and is already stated in Proposition 1. The necessity part is more interesting and is addressed next. Let $f_{\mathrm{sa}}(x, w)$ be the strong average for $f(t, x, w)$ and define $F(t, x, w) := f(t, x, w) - f_{\mathrm{sa}}(x, w)$. By definition there exist $\beta_{\mathrm{av}} \in \mathcal{KL}$ and $T^* > 0$ such that for all $t \ge 0, T \ge T^*$ we have that

$$\left|\frac{1}{T}\int_t^{t+T} F(\tau, x, w(\tau))\, d\tau\right| \le \beta_{\mathrm{av}}(\max\{|x|, \|w\|_\infty, 1\}, T). \tag{9}$$

A consequence of this is that $f_{\mathrm{sa}}(x, w)$ is also the weak average, i.e., for all $t \ge 0$, $T \ge T^*$ we have that

$$\left|\frac{1}{T}\int_t^{t+T} F(\tau, x, w)\, d\tau\right| \le \beta_{\mathrm{av}}(\max\{|x|, |w|, 1\}, T). \tag{10}$$

Note that the bounds (9) and (10) must hold for each entry of the vector-valued function $F(t, x, w)$ and therefore we assume below without loss of generality that $F(t, x, w)$ is a scalar function (i.e., we concentrate on one entry only).

The proposition will be established if we can show that $F(t, x, w)$ is independent of w. Suppose not. Then there exist t^*, x^*, w_1, w_2 with $w_1 \ne w_2$ such that

$$F(t^*, x^*, w_1) \ne F(t^*, x^*, w_2). \tag{11}$$

Since $F(t, x, w)$ is continuous in t, there exists $c_1 \in (0, T_1)$ (recall $f(t, x, w)$ is periodic with period T_1) such that for all $t \in [t^*, t^* + c_1]$ we have that either

$F(t, x^*, w_1) - F(t, x^*, w_2) > 0$ or $F(t, x^*, w_1) - F(t, x^*, w_2) < 0$. Assume, without loss of generality, that the first case holds. Let

$$w^*(\tau) := \begin{Bmatrix} w_1, & \tau \in [t^* + jT_1, t^* + jT_1 + c_1) \\ w_2, & \tau \in [t^* + jT_1 + c_1, t^* + (j+1)T_1), \end{Bmatrix}, \qquad j \in \mathbb{N}.$$

Consider now $T_k = kT_1$. Using (9) we have, for sufficiently large k, that

$$\beta_{\text{av}}(\max\{|x^*|, |w_1|, |w_2|, 1\}, kT_1) \geq \left| \frac{1}{kT_1} \int_{t^*}^{t^*+kT_1} F(\tau, x^*, w^*(\tau))\, d\tau \right|. \tag{12}$$

Add and subtract $\sum_{j=0}^{j=k-1} \int_{t^*+jT_1}^{t^*+jT_1+c_1} F(\tau, x^*, w_2)\, d\tau$ within the absolute value of the right-hand side of (12) and then we can rewrite it as

$$\left| \frac{1}{kT_1} \sum_{j=0}^{j=k-1} \int_{t^*+jT_1}^{t^*+jT_1+c_1} (F(\tau, x^*, w_1) - F(\tau, x^*, w_2))\, d\tau \right.$$
$$\left. + \frac{1}{kT_1} \int_{t^*}^{t^*+kT_1} F(\tau, x^*, w_2)\, d\tau \right|. \tag{13}$$

Since we have that $\mu := \min_{\tau \in [t^*, t^*+c_1]}(F(\tau, x^*, w_1) - F(\tau, x^*, w_2)) > 0$ and F is periodic, we can write, using (10), (12) and (13), that for all $k \in \mathbb{N}$,

$$\begin{aligned} \beta_{\text{av}}(\max\{|x^*|, |w_1|, |w_2|, 1\}, kT_1) &\geq \left| \frac{\mu c_1}{T_1} + \frac{1}{kT_1} \int_{t^*}^{t^*+kT_1} F(\tau, x^*, w_2)\, d\tau \right| \\ &\geq \frac{\mu c_1}{T_1} - \beta_{\text{av}}(\max\{|x^*|, |w_2|, 1\}, kT_1). \end{aligned} \tag{14}$$

Taking the limit as $k \to \infty$ on both sides, we arrive at the contradiction $0 \geq \mu c_1/T_1 > 0$. Hence, $F(t, x, w)$ must not depend on w. ■

Note that for periodic systems that satisfy (6), the strong and weak averages coincide and moreover in this case we can say that the strong average is *equivalent* to the weak average plus the structure given in (6). If this was true in general, then the definition of the strong average would be superfluous since we could use the definition of the weak average plus the structure of the system to obtain the definition of strong average. Hence, an important question is:

Does there exist a (non-periodic) system which is not of the form (6) *for which the strong average exists*?

We show below that this is true.

Example 2. Consider system (1), where

$$f(t, x, w) = -x + \frac{t^2}{1+t^2} w. \tag{15}$$

Note that this system is not periodic and it does not have the structure (6). We show now that the strong average for this system exists and is given by

$$f_{sa}(x, w) = -x + w. \tag{16}$$

Indeed, we can write

$$\begin{aligned}\left|\frac{1}{T}\int_t^{t+T}[f_{sa}(x, w(s)) - f(s, x, w(s))]\,ds\right| &\le \frac{1}{T}\int_t^{t+T}\left|\frac{w(s)}{1+s^2}\right|ds \\ &\le \frac{\|w\|_\infty}{T}\int_t^{t+T}\frac{ds}{1+s^2} \\ &\le \frac{\|w\|_\infty}{T}[\arctan(t+T) - \arctan(t)] \\ &\le \frac{\pi\|w\|_\infty}{T}.\end{aligned}$$

Hence, if we define $\beta_{av}(s, t) := \min\{\pi/t, 1\}s$ and $T^* := \pi$, then we have that inequality (3) used in the definition of strong average holds with the given β_{av} and T^*.

The above example indicates that it is more difficult to characterize the structure of non-periodic systems which are such that the strong average follows from the definition of the weak average. Instead of trying to enumerate all of the system structures when this holds, we use two different definitions of average since they capture the important (but different) properties that the system must have in order to state the main results in the next section.

4. Main Results

In this section we relate the ISS properties of the strong and weak average of a system to the corresponding ISS properties of the actual system. We show that ISS of the strong average of a system implies uniform semi-global practical ISS of the actual system (Theorem 1). Hence, Theorem 1 is a generalization of the main result of [TPA]. On the other hand, ISS of the weak average of a system does not, in general, imply uniform semi-global practical ISS of the actual system and we illustrate this by revisiting Example 1. However, if the weak average of a system is input-to-state stable (ISS) and disturbances are absolutely continuous with $w, \dot{w} \in \mathscr{L}_\infty$, then we can prove a semi-global practical "ISS-like" property for the actual system (Theorem 2). Although this ISS-like property is weaker than uniform semi-global practical ISS (it requires $\dot{w} \in \mathscr{L}_\infty$), it suffices for semi-global stability results involving interconnections, as shown in the next section. The proofs of main results are given in Sections 6 and 7.

In order to put our results in a better context, we recall some results on ISS for time-invariant systems. For the time-invariant system

$$\dot{x} = f_a(x, w), \tag{17}$$

the following definitions follow from those given in [S1] and [SW1]:

Definition 4. Let $\tilde{\gamma} \in \mathcal{G}$. System (17) is said to be Lyapunov-ISS with gain $\tilde{\gamma}$ if there exist a C^1 function $V: \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$, $\gamma \in \mathcal{G}$ such that $\alpha_1^{-1} \circ \alpha_2 \circ \gamma(s) \leq \tilde{\gamma}(s)$ for all $s \geq 0$ and, for all (x, w),

$$\begin{aligned} \alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|), \\ |x| \geq \gamma(|w|) \quad \Rightarrow \quad \frac{\partial V}{\partial x} f_a(x, w) \leq -\alpha_3(|x|). \end{aligned} \tag{18}$$

System (17) is said to be Lyapunov-ISS if, for some $\tilde{\gamma} \in \mathcal{G}$, it is Lyapunov-ISS with gain $\tilde{\gamma}$.

Remark 1. It has been remarked in the literature (see, for example, p. 440 of [S1]) that taking $\alpha_3 \in \mathcal{K}_\infty$ in (18) is equivalent to taking $\alpha_3: \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ continuous and positive definite. Indeed, as shown in [S1], one can construct $q \in \mathcal{K}_\infty$ continuously differentiable so that, with $\tilde{V} := q(V)$ in place of V, (18) holds with $\tilde{\alpha}_1 := q \circ \alpha_1$ in place of α_1, $\tilde{\alpha}_2 := q \circ \alpha_2$ in place of α_2, and $\alpha_3 \in \mathcal{K}_\infty$. We note that, regardless of the particular choice for $q \in \mathcal{K}_\infty$, we have $\tilde{\alpha}_1^{-1} \circ \tilde{\alpha}_2 \circ \gamma = \alpha_1^{-1} \circ \alpha_2 \circ \gamma$ so that this modification does not change the Lyapunov-ISS gain.

Definition 5. Let $\tilde{\gamma} \in \mathcal{G}$. System (17) is said to be ISS with gain $\tilde{\gamma}$ if there exists $\beta \in \mathcal{KL}$ such that, for each $w \in \mathcal{L}_\infty$ and each $x_o \in \mathbb{R}^n$, the solution $x(t)$ of (17) starting at (x_o, t_0) exists for all $t \geq t_0$ and satisfies

$$|x(t)| \leq \max\{\beta(|x_\circ|, t - t_0), \tilde{\gamma}(\|w\|_\infty)\}, \qquad \forall t \geq t_0 \geq 0. \tag{19}$$

System (17) is said to be ISS if, for some $\tilde{\gamma} \in \mathcal{G}$, it is ISS with gain $\tilde{\gamma}$.

The following comes from [SW1]:

Fact 1. *System* (17) *is Lyapunov-ISS if and only if it is ISS.*

The following comes from [S1]:

Fact 2. *If system* (17) *is Lyapunov-ISS with gain* $\tilde{\gamma}$, *then it is ISS with gain* $\tilde{\gamma}$.

To our knowledge, the converse of Fact 2 remains an open question; namely, it is still unknown whether system (17) being ISS with gain $\tilde{\gamma}$ implies that (17) is Lyapunov-ISS with gain $\tilde{\gamma}$.

The main objective of this paper is to determine to what extent Lyapunov-ISS with gain $\tilde{\gamma}$ for the averaged system implies ISS with gain $\tilde{\gamma}$ for the actual system. The main result for the strong average is stated below:

Theorem 1. *If Assumption* 1 *holds and the strong average of* (1) *exists and is Lyapunov-ISS with gain* $\tilde{\gamma}$, *then there exists* $\beta \in \mathcal{KL}$ *and, given any strictly positive real numbers* $\Omega_x, \Omega_w, \delta$, *there exists* $\varepsilon^* > 0$ *such that for all* $\varepsilon \in (0, \varepsilon^*)$ *the solutions of* (2) *satisfy*

$$|x(t)| \leq \max\{\beta(|x(t_0)|, t - t_0), \tilde{\gamma}(\|w\|_\infty)\} + \delta, \qquad \forall t \geq t_0 \geq 0, \tag{20}$$

whenever $|x(t_0)| \le \Omega_x$, $\|w\|_\infty \le \Omega_w$, *i.e., system* (2) *is semi-globally practically* (*in the parameter* ε) *ISS with gain* $\tilde{\gamma}$.

Proof. See Section 6.

In the next corollary, and in Corollary 2, we need a precise definition of uniform semi-global practical asymptotic stability.

Definition 6. The system $\dot{x} = f(t, x, \varepsilon)$ is said to be uniformly semi-globally practically asymptotically stable in ε if there exists $\beta \in \mathcal{KL}$ and, for each pair of strictly positive real numbers δ, Δ, there exists ε^* such that for all $\varepsilon \in (0, \varepsilon^*)$, the solutions of $\dot{x} = f(t, x, \varepsilon)$ satisfy

$$|x(t)| \le \beta(|x_o|, t - t_0) + \delta, \qquad \forall t \ge t_0 \ge 0, \tag{21}$$

whenever $|x_o| \le \Delta$.

A corollary of Theorem 1 is the main result obtained in [TPA]:

Corollary 1. *If the average of system* $\dot{x} = F(t/\varepsilon, x)$ *is globally asymptotically stable, then the actual system* $\dot{x} = F(t/\varepsilon, x)$ *is uniformly semi-globally practically asymptotically stable in* ε.

In general, the semi-global practical (in the parameter ε) ISS property does not follow from the Lyapunov-ISS property for the weak average. We again use Example 1 to illustrate this point.

Example 1 (continued). The weak average for system (7) is given in (8). Since $c > 0$, the weak average is Lyapunov-ISS with gain $\tilde{\gamma} \equiv 0$. Indeed, we can take $V(x) = x^2$.

We will show that the actual system (7) is not uniformly semi-globally practically ISS. In fact, the actual system exhibits finite escape times under the action of bounded disturbances.

Consider the actual system (7) with the disturbance $w_\varepsilon(t) = \cos(t/\varepsilon)$. Note that $\|w_\varepsilon\|_\infty = 1$, $\forall \varepsilon > 0$, and $\|\dot{w}_\varepsilon\|_\infty = 1/\varepsilon$. We recall that $\int_t^{t+T} \cos^2(s)\, ds = 0.5T + 0.25(\sin(2t + 2T) - \sin(2t))$. By direct integration of (7) with this disturbance,

$$\int_{x(t_0)}^{x(t)} \frac{d\chi}{\chi^3} = \int_{t_0}^{t} \left[\cos^2\left(\frac{s}{\varepsilon}\right) - c\right] ds,$$

we obtain

$$x^2(t) = \frac{x^2(t_0)}{1 - 2\psi(\varepsilon, t, t_0)x^2(t_0)},$$

where

$$\psi(\varepsilon, t, t_0) = (0.5 - c)(t - t_0) + 0.25\varepsilon\left(\sin\left(\frac{2t}{\varepsilon}\right) - \sin\left(\frac{2t_0}{\varepsilon}\right)\right).$$

Fix now arbitrary $t_0 \geq 0$, $\varepsilon > 0$ and let $x(t_0) = 1$. It is easy to see that (since $0.5 - c > 0$) there always exists $t^* \geq t_0$ such that $\psi(\varepsilon, t^*, t_0) = 0.5$ and hence there are finite escape times. Hence, the actual system is not uniformly semi-globally practically ISS.

We note that the disturbance that we used to show this result is differentiable and it has arbitrarily large rates. The natural question that one may ask is:

What (strongest) general property can we prove for the solutions of the actual system if the weak average is Lyapunov-ISS with gain $\tilde{\gamma}$?

We note that in the standard definition of average for systems without disturbances (Definition 1) we take fixed x in the integral and then, assuming GAS of $\dot{x} = F_{\text{av}}(x)$, we can establish semi-global practical asymptotic stability for $\dot{x} = f(t/\varepsilon, x)$ (see [TPA]). Note that the solutions $x(t)$ are absolutely continuous and have bounded rates. The disturbances w in the definition of weak average (Definition 3) are also fixed and it is reasonable to expect that in order to prove a semi-global result using the weak average it is sufficient to assume that the disturbances are absolutely continuous and with bounded rates. In fact, a semi-global practical "ISS-like" property of the actual system can be proved if one assumes Lyapunov-ISS with gain $\tilde{\gamma}$ of the weak average of system and that w is absolutely continuous with $w, \dot{w} \in \mathcal{L}_\infty$:

Theorem 2. *If Assumption* 1 *holds and the weak average of* (1) *exists and is Lyapunov-ISS with gain $\tilde{\gamma}$, then there exists $\beta \in \mathcal{KL}$ and, given any strictly positive real numbers $\Omega_x, \Omega_w, \Omega_{\dot{w}}, \nu$, there exists $\varepsilon^* > 0$ such that, $\forall \varepsilon \in (0, \varepsilon^*)$, the solutions of* (2) *satisfy*

$$|x(t)| \leq \max\{\beta(|x(t_0)|, t - t_0), \tilde{\gamma}(\|w\|_\infty)\} + \nu, \qquad \forall t \geq t_0 \geq 0, \tag{22}$$

whenever $|x(t_0)| \leq \Omega_x$, $\|w\|_\infty \leq \Omega_w$, $w(\cdot)$ is absolutely continuous and $\|\dot{w}\|_\infty \leq \Omega_{\dot{w}}$.

Proof. See Section 6.

Note that the continuation of Example 1 does not contradict Theorem 2 since the disturbance that we used in the example has $\|\dot{w}_\varepsilon\|_\infty = 1/\varepsilon$ which becomes arbitrarily large for sufficiently small $\varepsilon > 0$. Hence, this disturbance does not satisfy the condition of Theorem 2: $\|\dot{w}_\varepsilon\|_\infty \leq \Omega_{\dot{w}}$ for some fixed $\Omega_{\dot{w}} > 0$.

Remark 2. To remove the offset δ (respectively, ν) in Theorem 1 (respectively, Theorem 2), it is clearly necessary that $f(t, 0, 0) \equiv 0$. If we also assume local exponential stability for the origin of the average system with disturbances set to zero, well-known results (see, for example, Theorem 8.4 of [K]) provide local exponential stability of the origin of the actual system with disturbances set to zero for $\varepsilon > 0$ sufficiently small. This property of the actual system and the assumption that $f(t, \cdot, \cdot)$ is local Lipschitz uniformly in t implies that the actual system is ISS for $\varepsilon > 0$ sufficiently small (see Lemma 5.4 of [K]). This type of result can be used in conjunction with the main theorems of this section to remove the offsets in the conclusions, at the price of perhaps increasing the ISS gain for small disturbances. We do not pursue this direction further here.

5. Cascaded Time-Varying Systems

The weak average (with the result of Theorem 2) is very useful for the analysis of the stability of several classes of time-varying interconnected systems, where the input of one subsystem is the output of another subsystem, whose derivatives are bounded on compact subsets of the state space. This includes the case where the disturbances enter the system through an ISS filter. In this way, cascaded systems provide an important motivation for the use of weak averages in the analysis of stability properties of time-varying systems. For example, using Theorem 2 on weak averages and the small gain results in [JTP], we can prove the following result:

Proposition 3. *Consider the system*

$$\begin{aligned} \dot{\xi} &= f_1\left(\frac{t}{\varepsilon}, \xi, \eta\right), \\ \dot{\eta} &= f_2(t, \xi, \eta, w, \varepsilon), \end{aligned} \tag{23}$$

where $\xi \in \mathbb{R}^{n_1}$, $\eta \in \mathbb{R}^{n_2}$, $w \in \mathbb{R}^m$ *and suppose* f_1 *satisfies Assumption* 1. *Suppose that the following hold*:

1. *The weak average of the* ξ *subsystem exists and is Lyapunov-ISS with gain* $\gamma_\xi \in \mathcal{G}$.
2. *For each* $r > 0$ *there exist* $R > 0$ *and* $\varepsilon_2^* > 0$ *such that*
$$|(\xi, \eta, w)| \le r, \qquad \varepsilon \in (0, \varepsilon_2^*] \quad \Rightarrow \quad |f_2(t, \xi, \eta, w, \varepsilon)| \le R. \tag{24}$$
3. *There exist* $\beta_\eta \in \mathcal{KL}$ *and* γ_η, $\tilde{\gamma} \in \mathcal{G}$ *such that for any strictly positive* Δ_ξ, Δ_η, Δ_w, δ *there exists* $\varepsilon^* > 0$ *such that for all* $\varepsilon \in (0, \varepsilon^*)$ *the solutions of the* η *subsystem satisfy*
$$|\eta(t)| \le \max\{\beta_\eta(|\eta(t_0)|, t - t_0), \gamma_\eta(\|\xi\|_\infty), \tilde{\gamma}(\|w\|_\infty)\} + \delta, \qquad \forall t \ge t_0 \ge 0,$$
whenever $|\eta(t_0)| \le \Delta_\eta$, $\|\xi\|_\infty \le \Delta_\xi$, $\|w\|_\infty \le \Delta_w$.
4. *The small gain condition* $\gamma_\eta \circ \gamma_\xi(s) < s, \forall s > 0$ *holds.*

Then there exists $\beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}$ *such that for any strictly positive* Ω_x, Ω_w, v_x *there exists* $\varepsilon^* > 0$ *such that for all* $\varepsilon \in (0, \varepsilon^*)$ *we have that the solutions of* (23) *satisfy*

$$|x(t)| \le \max\{\beta(|x(t_0)|, t - t_0), \gamma(\|w\|_\infty)\} + v_x, \qquad \forall t \ge t_0 \ge 0,$$

whenever $|x(t_0)| \le \Omega_x$, $\|w\|_\infty \le \Omega_w$ *and where* $x := (\xi^T \;\; \eta^T)^T$.

Sketch of Proof. See Section 8.

A direct consequence of the above result is:

Corollary 2. *Consider the system*

$$\dot{\xi} = f_1\left(\frac{t}{\varepsilon}, \xi, \eta\right),$$
$$\dot{\eta} = f_2(t, \eta, \varepsilon), \tag{25}$$

where $\xi \in \mathbb{R}^{n_1}$, $\eta \in \mathbb{R}^{n_2}$. *Suppose* f_1 *satisfies Assumption* 1 *and for each* $r > 0$ *there exist* $R > 0$ *and* $\varepsilon^* > 0$ *such that* $|\eta| \le r$ *and* $\varepsilon \in (0, \varepsilon^*)$ *implies* $|f_2(t, \eta, \varepsilon)| \le R$. *If the weak average of the* ξ *subsystem exists and is ISS with respect to* η *and if the* η *subsystem is uniformly semi-globally practically asymptotically stable in* ε, *then system* (25) *is uniformly semi-globally practically asymptotically stable in* ε.

Note that in order to prove uniform semi-global practical asymptotic stability of the cascade, the ξ system does not have to be uniformly semi-globally practically ISS with respect to η, as the following example shows:

Example 3. Consider the cascade

$$\dot{\xi} = -c\xi^3 + \cos\left(\frac{t}{\varepsilon}\right)\xi^3\eta,$$
$$\dot{\eta} = -(1 - 0.5\cos(t))\eta, \tag{26}$$

where $c \in (0, 0.5)$. The weak average of the ξ system was considered in Example 1 and it was shown to be ISS with zero gain (GAS). On the other hand, since the η subsystem does not depend on the small parameter ε, we cannot apply the averaging results of [TPA] to show uniform semi-global practical stability of the cascade. The η system is also uniformly GAS and hence from Corollary 2 we conclude that the cascade (26) is uniformly semi-globally practically asymptotically stable in ε.

6. Proofs of Main Results

The proofs of the main results are based on the approach taken in [AP1], [AP2], [PA], and [TPA]. Namely, we show that, even though the Lyapunov function associated with the average system is not necessarily monotonically decreasing along the trajectories of the time-varying system, it is decreasing along the trajectories of the time-varying system when viewed at certain sampling instances, at least when the size of x is big relative to the size of w and both are bounded, for ε sufficiently small. (See Lemma 2 for Theorem 1 and Lemma 3 for Theorem 2.) We use this observation to construct, for the trajectories of the time-varying system, a "discrete-time" estimate that resembles the continuous-time estimate in the conclusion of each theorem (see Lemma 4). Finally, in Lemma 5, we combine the discrete-time estimate with a standard "inter-sample growth" result (Lemma 1), following the calculations in [NTS], which allows us to conclude.

Since the proofs of Theorems 1 and 2 are quite similar, most of the details of the proof of Theorem 2 are omitted. We do, however, provide some of the details that establish the decrease of the Lyapunov function at sampling instances under the assumptions of Theorem 2.

In this section we state the main technical results that we need to establish the theorems. The proofs of the technical results, unless short, are delayed until the next section.

We start with an "inter-sample growth" result that we need. The proof of this result is standard (see, for instance, Theorems 2.5 and 2.6 in [K]).

Lemma 1. *Under Assumption* 1, *given any strictly positive real numbers* r^b *and* r_1^b, *there exists* $d > 0$ *and* $M > 0$ *such that, for each* $\varepsilon > 0$, *the following property holds*:

Property 00. *For each* $t_0 \geq 0$, *if* $|x(t_0)| \leq r^b$, $\|w\|_\infty \leq r_1^b$, *then the solution* $x(t)$ *of* (2) *exists for all* $t \in [t_0, t_0 + d]$ *and satisfies*

$$|x(t) - x(t_0)| \leq M(t - t_0), \qquad \forall t \in [t_0, t_0 + d]. \tag{27}$$

For notational convenience, we state an obvious corollary:

Corollary 3. *Under Assumption* 1, *given any strictly positive real numbers* r^b *and* r_1^b *and* v_1, *there exists* $d > 0$ *such that for each* $\varepsilon > 0$ *the following property holds*:

Property P0. *For each* $t_0 \geq 0$, *if* $|x(t_0)| \leq r^b$, $\|w\|_\infty \leq r_1^b$, *then the solution* $x(t)$ *of* (2) *exists for all* $t \in [t_0, t_0 + d]$ *and satisfies*

$$|x(t)| \leq |x(t_0)| + v_1, \qquad \forall t \in [t_0, t_0 + d]. \tag{28}$$

Next we state the results on the decrease of the Lyapunov function at sampling instances.

Lemma 2. *Under the assumptions of Theorem* 1, *given any strictly positive real numbers* Δ, δ, Δ_1, *there exists* $d^* > 0$ *and, for any* $d \in (0, d^*)$, *there exists* $\varepsilon^* > 0$ *such that for all* $\varepsilon \in (0, \varepsilon^*)$ *the following property holds*:

Property P1.1. *Given any* $t_0 \geq 0$, *if* $|x(t_0)| \leq \Delta$, $\|w\|_\infty \leq \Delta_1$, *then the solution* $x(t)$ *of* (2) *exists for all* $t \in [t_0, t_0 + d]$ *and satisfies*

$$|x(t_0)| \geq \gamma(\|w\|_\infty) + \delta \quad \Rightarrow$$

$$V(x(t_0 + d)) - V(x(t_0)) \leq -\frac{d}{2}\alpha_3(0.5|x(t_0)|). \tag{29}$$

Proof. See Section 7.1.

The corresponding result for Theorem 2 is similar:

Lemma 3. *Under the assumptions of Theorem* 2, *given any strictly positive real numbers* $\Delta, \delta, \Delta_1, \Delta_2$, *there exists* $d^* > 0$ *and, for any* $d \in (0, d^*)$, *there exists* $\varepsilon^* > 0$ *such that for all* $\varepsilon \in (0, \varepsilon^*)$ *the following property holds*:

Property P1.2. *Given any* $t_0 \geq 0$, *if* $|x(t_0)| \leq \Delta$, $\|w\|_\infty \leq \Delta_1$, $\|\dot{w}\| \leq \Delta_2$, *then the solution* $x(t)$ *of* (2) *exists for all* $t \in [t_0, t_0 + d]$ *and satisfies*

$$|x(t_0)| \geq \gamma(\|w\|_\infty) + \delta \quad \Rightarrow$$

$$V(x(t_0 + d)) - V(x(t_0)) \leq -\frac{d}{2}\alpha_3(0.5|x(t_0)|). \tag{30}$$

Proof. See Section 7.2.

A corollary of Lemmas 1 and 2 is the following.

Corollary 4. *Under the assumptions of Theorem* 1, *given any strictly positive real numbers* $\Delta, \Delta_1, \mu, \mu_1$, *there exists* $d^* > 0$ *and for any* $d \in (0, d^*)$ *there exists* $\varepsilon^* > 0$ *such that for all* $\varepsilon \in (0, \varepsilon^*)$ *the following property holds*:

Property P2. *Given any* $t_0 \geq 0$, *if* $|x(t_0)| \leq \Delta$, $\|w\|_\infty \leq \Delta_1$, *then the solution* $x(t)$ *of* (2) *exists for all* $t \in [t_0, t_0 + d]$ *and satisfies* $V(x(t_0 + d)) \leq V(x(t_0)) + \mu_1$ *and*

$$V(x(t_0)) \geq \alpha_2 \circ \gamma(\|w\|_\infty) + \mu \quad \Rightarrow$$
$$V(x(t_0 + d)) - V(x(t_0)) \leq -\frac{d}{2}\alpha_3(0.5\alpha_2^{-1}(V(x(t_0)))). \tag{31}$$

Proof. Given Δ, Δ_1 and μ_1, it follows from Lemma 1 and the continuity of V that there exists $d^* > 0$ such that for any $d \in (0, d^*)$ if $|x(t_0)| \leq \Delta$, $\|w\|_\infty \leq \Delta_1$, then the solution $x(t)$ of (2) exists for all $t \in [t_0, t_0 + d]$ and satisfies $V(x(t_0 + d)) \leq V(x(t_0)) + \mu_1$.

Given $\Delta_1 > 0$ and $\mu > 0$, let $\delta > 0$ be sufficiently small so that

$$\max_{s \in [0, \gamma(\Delta_1)]} [\alpha_2(s + \delta) - \alpha_2(s)] \leq \mu. \tag{32}$$

Such a δ exists from the continuity of $\alpha_2(\cdot)$. Then note that, for $\|w\|_\infty \leq \Delta_1$,

$$\alpha_2 \circ \gamma(\|w\|_\infty) + \mu \geq \alpha_2(\gamma(\|w\|_\infty) + \delta). \tag{33}$$

Using $V(x) \leq \alpha_2(|x|)$ we get

$$V(x(t_0)) \geq \alpha_2 \circ \gamma(\|w\|_\infty) + \mu \quad \Rightarrow \quad |x(t_0)| \geq \gamma(\|w\|_\infty) + \delta. \tag{34}$$

So the rest of Property P2 follows from Property P1.1. ■

In preparation for stating the next result, given functions α_1, α_2 and α_3 all in class-$\mathcal{K}_\infty$, we define

$$\beta_\alpha(s, t) := \alpha_1^{-1}(\beta_V(\alpha_2(s), t)), \tag{35}$$

where $\beta_V \in \mathcal{KL}$ satisfies

$$\dot{u} \leq -0.5\alpha_3(0.5\alpha_2^{-1}(u)) \quad \Rightarrow \quad u(t) \leq \beta_V(u(0), t), \qquad \forall t \geq 0. \tag{36}$$

Such a β_V exists from standard comparison results (see, for instance, Lemma 4.4 in [LSW]). Note that $\beta_\alpha \in \mathcal{KL}$.

Now we state that Property P2 implies a particular discrete-time ISS estimate.

Lemma 4. *If Property* P2 *holds*, $\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|)$, *and* $\Delta > \alpha_1^{-1}(\alpha_2 \circ \gamma(\Delta_1) + \mu + \mu_1)$, *then there exist strictly positive real numbers* R, r^s, r_1^s (*in particular we can take*

$$\begin{aligned} r^s &:= \alpha_2^{-1} \circ \alpha_1(\Delta), \\ r_1^s &:= \Delta_1, \\ R &:= \max_{s \in [0, \alpha_2 \circ \gamma(\Delta_1)]} (\alpha_1^{-1}(s + \mu + \mu_1) - \alpha_1^{-1}(s)), \end{aligned} \tag{37}$$

such that the following property holds:

Property P3. *For all* $t_0 \geq 0$, $|x(t_0)| \leq r^s$, $\|w\|_\infty \leq r_1^s$, *the solution of* (2) *exists for all* $t \geq t_0$ *and satisfies*

$$|x(t_0 + kd)| \leq \max\{\beta_\alpha(|x(t_0)|, kd), \alpha_1^{-1} \circ \alpha_2 \circ \gamma(\|w\|_\infty)\} + R, \qquad \forall k \geq 0, \quad (38)$$

where β_α *was defined by* (35)–(36).

Proof. See Section 7.3.

The last result we need is the following:

Lemma 5. *If Properties* P0 *and* P3 *hold and* $R < r^b/2$, *then there exist strictly positive* Δ_x, Δ_w, v *and* $\bar{\beta} \in \mathcal{KL}$ (*in particular we can take*

$$\begin{aligned} \bar{\beta}(s,t) &:= 2 \max_{\eta \in [0,t]} 2^{-\eta} \beta_\alpha(s, t-\eta), \\ v &:= R + v_1, \\ \Delta_x &:= \min\left\{ r^s, \beta_0^{-1}\left(\frac{r^b}{2}\right)\right\}, \\ \Delta_w &:= \min\{\chi(r^b), r_1^b, r_1^s\}, \end{aligned} \quad (39)$$

where $\beta_0 \in \mathcal{K}_\infty$ *satisfies* $\beta_0(s) \geq \beta_\alpha(s,0)$ *for all* $s \geq 0$ *and* $\chi(r^b) := \sup\{s\colon \alpha_1^{-1} \circ \alpha_2 \circ \gamma(s) \leq r^b/2\}$) *such that the following property holds*:

Property P4. *If* $|x(t_0)| \leq \Delta_x$, $\|w\|_\infty \leq \Delta_w$, *then*

$$|x(t)| \leq \max\{\bar{\beta}(|x(t_0)|, t - t_0), \alpha_1^{-1} \circ \alpha_2 \circ \gamma(\|w\|_\infty)\} + v, \qquad \forall t \geq t_0 \geq 0. \quad (40)$$

Proof. (The same proof technique was used in [NTS].) Fix an arbitrary $t_0 \geq 0$, let $t_k := t_0 + kd$, and note that with the choice Δ_x, Δ_w given in (39) we have that if $|x(t_0)| \leq \Delta_x$, $\|w\|_\infty \leq \Delta_w$, then $\|w\|_\infty \leq r_1^b$ and $|x(t_k)| \leq r^b$ for all $k \geq 0$. Hence the trajectory exists for all $t \geq t_0$ and moreover we can write

$$|x(t)| \leq |x(t_k)| + v_1, \qquad \forall t \in [t_k, t_{k+1}], \quad \forall k \geq 0.$$

This implies that

$$\begin{aligned} |x(t)| \leq \max\{\beta_\alpha(|x(t_0)|, kd), \alpha_1^{-1} \circ \alpha_2 \circ \gamma(\|w\|_\infty)\} + R + v_1, \\ \forall t \geq t_0 \geq 0. \end{aligned} \quad (41)$$

It was shown in [NTS] that for any $\beta_\alpha \in \mathcal{KL}$, there exists $\bar{\beta} \in \mathcal{KL}$ such that $\beta_\alpha(s, kd) \leq \bar{\beta}(s, (k+1)d)$, $\forall s$, $\forall k$. Moreover, one possible formula for $\bar{\beta}(s,t) := 2 \max_{\eta \in [0,t]} 2^{-\eta} \beta_\alpha(s, t-\eta)$. Since $t - t_0 < (k+1)d$, $\forall t \in [t_k, t_{k+1})$, $k \geq 0$, we can write $\bar{\beta}(s, (k+1)d) \leq \bar{\beta}(s, t - t_0)$, $\forall t \geq t_0 \geq 0$, which completes the proof by letting $v = R + v_1$ in (41). ∎

Proof of Theorem 1. Let α_1, α_2 and α_3 generate the function $\beta_\alpha \in \mathcal{KL}$ according to the definition in (35)–(36). Let $\beta_0 \in \mathcal{K}_\infty$ satisfy $\beta_0(s) \geq \beta_\alpha(s,0)$ for all $s \geq 0$. Let Ω_x, Ω_w and δ be arbitrary strictly positive real numbers. Define $\Delta_1 := \Omega_w$. Let μ and μ_1 be strictly positive real numbers such that

$$\max_{s \in [0, \alpha_2 \circ \gamma(\Delta_1)]} (\alpha_1^{-1}(s + \mu + \mu_1) - \alpha_1^{-1}(s)) \leq \frac{\delta}{2}. \tag{42}$$

Let Δ satisfy

$$\Delta \geq \max\{\alpha_1^{-1} \circ \alpha_2(\Omega_x), \alpha_1^{-1}(\alpha_2 \circ \gamma(\Delta_1) + \mu + \mu_1) + 1\}. \tag{43}$$

Define $v_1 := \delta/2$, $r_1^b := \Omega_w$ and let r^b satisfy

$$r^b \geq \max\{2\beta_0(\Omega_x), 2\delta\} \tag{44}$$

and be such that $\chi(r^b) \geq \Omega_w$.

With these definitions, apply Corollaries 3 and 4 to generate a $d > 0$ and an ε^* such that Properties P0 and P2 hold for all $\varepsilon \in (0, \varepsilon^*(d))$. Next, applying Lemma 4, we get that Property P3 holds with $r^s \geq \Omega_x$, $r_1^s = \Omega_w$ and $R \leq \delta/2$. Finally, we apply Lemma 5 to get that Property P4 holds with $v \leq \delta$, $\Delta_x \geq \Omega_x$ and $\Delta_w \geq \Omega_w$, i.e., the claim of the theorem is established. ■

7. Proofs of Technical Lemmas

In this section we prove technical lemmas that were stated in the previous section.

7.1. *Proof of Lemma 2*

Let Δ, Δ_1 and δ be given strictly positive real numbers. Let $\beta_{\text{av}} \in \mathcal{KL}$ and $T^* > 0$ be such that (3) holds for all $T \geq T^*$. Let $L > 0$ be a (uniform) Lipschitz constant for $\partial V/\partial x, f(t/\varepsilon, x, w), f_{\text{sa}}(x, w)$ over the set where $|x| \leq 2\Delta, |w| \leq \Delta_1$. Let $T \geq T^*$ be such that

$$L\Delta\beta_{\text{av}}(\max\{\Delta, \Delta_1, 1\}, T) \leq 0.5\alpha_3\left(\frac{\delta}{4}\right). \tag{45}$$

Define $r^b := \Delta$ and $r_1^b := \Delta_1$ and $K := 4L\Delta + L\Delta_1 + c$. Throughout the rest of the proof we assume that $|x(t_0)| \leq \Delta$ and $\|w\|_\infty \leq \Delta_1$. Apply Lemma 1 and Corollary 3 to generate d_1^* and M such that Properties P00 and P0 hold for any $d \in (0, d_1^*)$. Use the continuity of α_3 and α_3^{-1} to find $d_2^* > 0$ such that

$$2Md_2^* + 2\alpha_3^{-1}\left(2KLMd_2^* + \alpha_3\left(\frac{\delta}{4}\right)\right) \leq \delta. \tag{46}$$

Define $d^* = \min\{d_1^*, d_2^*\}$. Fix $d \in (0, d^*)$ and define $\varepsilon^*(d) := d/T$. Let $\varepsilon \in (0, \varepsilon^*)$. It follows from Property P00 that

$$|x(t_0)| \geq \gamma(\|w\|_\infty) + Md \quad \Rightarrow \quad |x(t)| \geq \gamma(\|w\|_\infty), \qquad \forall t \in [t_0, t_0 + d]. \tag{47}$$

Therefore, it follows from the assumptions of the lemma that

$$|x(t_0)| \geq \gamma(\|w\|_\infty) + Md$$

implies that for all $t \in [t_0, t_0 + d]$ we have

$$\begin{aligned}
\frac{\partial V}{\partial x}(x(t)) f\left(\frac{t}{\varepsilon}, x(t), w(t)\right) \leq {} & -\alpha_3(|x(t)|) - \frac{\partial V}{\partial x}(x(t_0)) f_{\mathrm{sa}}(x(t_0), w(t)) \\
& + \frac{\partial V}{\partial x}(x(t_0)) f\left(\frac{t}{\varepsilon}, x(t_0), w(t)\right) \\
& + \frac{\partial V}{\partial x}(x(t)) f\left(\frac{t}{\varepsilon}, x(t), w(t)\right) \\
& - \frac{\partial V}{\partial x}(x(t_0)) f\left(\frac{t}{\varepsilon}, x(t_0), w(t)\right) \\
& - \frac{\partial V}{\partial x}(x(t)) f_{\mathrm{sa}}(x(t), w(t)) \\
& + \frac{\partial V}{\partial x}(x(t_0)) f_{\mathrm{sa}}(x(t_0), w(t)).
\end{aligned} \tag{48}$$

Integrate both sides of inequality (48) along the solution $x(t)$ over the interval $[t_0, t_0 + d]$ to obtain

$$\begin{aligned}
& V(x(t_0 + d)) - V(x(t_0)) \\
& \leq \underbrace{-\int_{t_0}^{t_0+d} \alpha_3(|x(s)|)\, ds}_{\mathbf{1}} \\
& \quad \underbrace{- \int_{t_0}^{t_0+d} \left(\frac{\partial V}{\partial x}(x(t_0)) f_{\mathrm{sa}}(x(t_0), w(s)) - \frac{\partial V}{\partial x}(x(t_0)) f\left(\frac{s}{\varepsilon}, x(t_0), w(s)\right) \right) ds}_{\mathbf{2}} \\
& \quad \underbrace{+ \int_{t_0}^{t_0+d} \left(\frac{\partial V}{\partial x}(x(s)) f\left(\frac{s}{\varepsilon}, x(s), w(s)\right) - \frac{\partial V}{\partial x}(x(t_0)) f\left(\frac{s}{\varepsilon}, x(t_0), w(s)\right) \right) ds}_{\mathbf{3}} \\
& \quad \underbrace{- \int_{t_0}^{t_0+d} \left(\frac{\partial V}{\partial x}(x(s)) f_{\mathrm{sa}}(x(s), w(s)) - \frac{\partial V}{\partial x}(x(t_0)) f_{\mathrm{sa}}(x(t_0), w(s)) \right) ds}_{\mathbf{4}}.
\end{aligned} \tag{49}$$

Now we turn to bounding each of the terms on the right-hand side of (49).

1. Using Property P00, we have that

$$-\int_{t_0}^{t_0+d} \alpha_3(|x(s)|)\, ds \leq -d\alpha_3(\max\{|x(t_0)| - Md, 0\}) \tag{50}$$

and thus

$$|x(t_0)| \geq 2Md \quad \Rightarrow \quad -\int_{t_0}^{t_0+d} \alpha_3(|x(s)|)\, ds \leq -d\alpha_3(0.5|x(t_0)|). \tag{51}$$

2. Since $|(\partial V/\partial x)(x(t_0))| \leq L|x(t_0)| \leq L\Delta$ and $d = \varepsilon^* T$ we can over-bound **2** by

$$\varepsilon L\Delta \left| \int_{t_0}^{t_0+T\varepsilon^*} \left(f_{\rm sa}(x(t_0), w(s)) - f\left(\frac{s}{\varepsilon}, x(t_0), w(s)\right)\right) d\left(\frac{s}{\varepsilon}\right) \right|.$$

Introduce the change of variables $\tau = s/\varepsilon$ in the above integral and introduce $w_1(\tau) := w(\varepsilon\tau)$ (note that $\|w_1\|_\infty = \|w\|_\infty \leq \Delta_1$) and $T_1 := \varepsilon^* T/\varepsilon > T$. Then by the definition of strong average we have

$$\begin{aligned} &\left| \frac{1}{T_1} \int_{t_0/\varepsilon}^{t_0/\varepsilon + T_1} (f_{\rm sa}(x(t_0), w_1(\tau)) - f(\tau, x(t_0), w_1(\tau)))\, d\tau \right| \\ &\quad \leq \beta_{\rm av}(\max\{|x(t_0)|, \|w_1\|_\infty, 1\}, T_1) \\ &\quad \leq \beta_{\rm av}(\max\{\Delta, \Delta_1, 1\}, T) \end{aligned} \tag{52}$$

and hence, using (45) and the fact that $\varepsilon T_1 = d$, **2** can be over-bounded by $d0.5\alpha_3(\delta/4)$.

3 and **4.** Using Assumption 1 and the definition for $L > 0$, for all x, w with $\max\{|x_1|, |x_2|\} \leq 2\Delta$, $|w| \leq \Delta_1$ we have

$$\begin{aligned} \left| \frac{\partial V}{\partial x}(x_1) f\left(\frac{s}{\varepsilon}, x_1, w\right) - \frac{\partial V}{\partial x}(x_2) f\left(\frac{s}{\varepsilon}, x_2, w\right) \right| &\leq (4L\Delta + L\Delta_1 + c)L|x_1 - x_2| \\ &= KL|x_1 - x_2|. \end{aligned} \tag{53}$$

The same bound holds if $f(\cdot\,,\cdot\,,\cdot)$ is replaced by $f_{\rm sa}(\cdot\,,\cdot)$. Using Properties P00 and P0, it follows that we can over-bound the sum of terms **3** and **4** by $KLMd^2$.

From the bounds on terms **1**–**4** on the right-hand side of (49), it follows that

$$|x(t_0)| \geq \gamma(\|w\|_\infty) + 2Md + 2\alpha_3^{-1}\left(2KLMd + \alpha_3\left(\frac{\delta}{4}\right)\right) \tag{54}$$

implies

$$V(x(t_0 + d)) - V(x(t_0)) \leq -\frac{d}{2}\alpha_3(0.5|x(t_0)|). \tag{55}$$

From (46) and the fact that $d < d_2^*$, the result follows. ∎

7.2. *Proof of Lemma 3*

Compared with the proof of Lemma 2, the main difference in the proof of Lemma 3 is that the relevant version of (49) is

$$
\begin{aligned}
&V(x(t_0+d)) - V(x(t_0)) \\
&= \underbrace{-\int_{t_0}^{t_0+d} \alpha_3(|x(s)|)\, ds}_{\mathbf{1}} \\
&\quad \underbrace{-\int_{t_0}^{t_0+d} \left(\frac{\partial V}{\partial x}(x(t_0)) f_{\mathrm{wa}}(x(t_0), w(t_0)) - \frac{\partial V}{\partial x}(x(t_0)) f\left(\frac{s}{\varepsilon}, x(t_0), w(t_0)\right)\right) ds}_{\mathbf{2}} \\
&\quad \underbrace{+\int_{t_0}^{t_0+d} \left(\frac{\partial V}{\partial x}(x(s)) f\left(\frac{s}{\varepsilon}, x(s), w(s)\right) - \frac{\partial V}{\partial x}(x(t_0)) f\left(\frac{s}{\varepsilon}, x(t_0), w(t_0)\right)\right) ds}_{\mathbf{3}} \\
&\quad \underbrace{-\int_{t_0}^{t_0+d} \left(\frac{\partial V}{\partial x}(x(s)) f_{\mathrm{wa}}(x(s), w(s)) - \frac{\partial V}{\partial x}(x(t_0)) f_{\mathrm{wa}}(x(t_0), w(t_0))\right) ds}_{\mathbf{4}}. \qquad (56)
\end{aligned}
$$

Term **1** is the same as before. Term **2** is in a form such that the definition of the weak average can be exploited in the same way that the definition of the strong average was exploited in the proof of Lemma 2. Terms **3** and **4** have been modified so that $w(t_0)$ replaces $w(s)$ in some places. Nevertheless, these terms can be over-bounded like in the proof of Lemma 2. The key observations are that

$$
\left|\frac{\partial V}{\partial x}(x_1) f\left(\frac{s}{\varepsilon}, x_1, w_1\right) - \frac{\partial V}{\partial x}(x_2) f\left(\frac{s}{\varepsilon}, x_2, w_2\right)\right| \le KL(|x_1 - x_2| + |w_1 - w_2|)
$$

(similarly for $f_{\mathrm{sa}}(\cdot,\cdot)$ in place of $f(\cdot,\cdot,\cdot)$) and, since we are assuming that w is absolutely continuous and $\|\dot{w}\|_\infty \le \Delta_2$, we have

$$
\int_{t_0}^{t_0+d} |w(s) - w(t_0)|\, ds \le 0.5\Delta_2 d^2. \quad \blacksquare \qquad (57)
$$

7.3. Proof of Lemma 4

Introduce the set $\mathcal{V} := \{x\colon V(x) \le \alpha_1(\Delta)\}$. Define r^s, r_1^s and R as in (37). Consider any $t_0, x(t_0), w(t)$ with $t_0 \ge 0$, $|x(t_0)| \le r^s \le \Delta$, $\|w\|_\infty \le r_1^s$. All the calculations below are given for this (fixed) triple $(t_0, x(t_0), w(t))$. Define $t_k := t_0 + kd$ and, for those k such that $x(t_k)$ is defined, $V_k := V(x(t_k))$. We use the following fact:

Fact 3. *If $V_0 \le \alpha_1(\Delta)$, then $V_k \le \alpha_1(\Delta)$, $\forall k \in \mathbb{N}$.*

Proof of the Fact. Let $j \ge 0$ be such that V_j exists and $V_j \le \alpha_1(\Delta)$. Either $V_j \ge \alpha_2 \circ \gamma(\Delta_1) + \mu$ in which case it follows from the second part of Property P2 that $V_{j+1} \le V_j \le \alpha_1(\Delta)$, or $V_j \le \alpha_2 \circ \gamma(\Delta_1) + \mu$ in which case it follows from the first part of Property P2 and the assumption that $\Delta > \alpha_1^{-1}(\gamma(\Delta_1) + \mu + \mu_1)$ that $V_{j+1} \le \gamma(\Delta_1) + \mu + \mu_1 \le \alpha_1(\Delta)$. The proof follows by induction. $\blacksquare$

Introduce the variable

$$u(s) := V_k + \left(\frac{s}{d} - k\right)(V_{k+1} - V_k), \qquad s \in [kd, (k+1)d), \quad k \geq 0. \tag{58}$$

From Fact 3 it follows that $u(s) \leq \alpha_1(\Delta)$, $\forall s \geq 0$. The function $u(s)$ is a continuous, piecewise linear function of s and hence it is absolutely continuous. Thus, $u(s)$ is differentiable for almost all $s \geq 0$. Note that we have that

$$0 \leq u(s) \leq \max\{V_k, V_{k+1}\}, \qquad \forall s \in [kd, (k+1)d), \quad k \in \mathbb{N}. \tag{59}$$

Hence using the first part of Property P2 we conclude that whenever $|x(t_0)| \leq \Delta$, $\|w\|_\infty \leq \Delta_1$ we have that

$$u(s) \leq V_k + \mu_1, \qquad \forall s \in [kd, (k+1)d), \quad k \in \mathbb{N}. \tag{60}$$

Using (60) and the second part of Property P2, the following implications hold $\forall s \in [kd, (k+1)d)$, $k \in \mathbb{N}$:

$$\begin{aligned} u(s) \geq \alpha_2 \circ \gamma(\|w\|_\infty) + \mu + \mu_1 \quad &\Rightarrow \quad V_k \geq \alpha_2 \circ \gamma(\|w\|_\infty) + \mu \\ &\Rightarrow \quad V_{k+1} - V_k \leq -\frac{d}{2}\alpha_3(0.5\alpha_2^{-1}(V_k)). \end{aligned} \tag{61}$$

Hence, from (58) and (61) we can write $\forall s \in [kd, (k+1)d)$, $k \in \mathbb{N}$,

$$\begin{aligned} u(s) \geq \alpha_2 \circ \gamma(\|w\|_\infty) + \mu + \mu_1 \quad \Rightarrow \quad \frac{d}{ds}u(s) &= \frac{1}{d}(V_{k+1} - V_k) \\ &\leq -0.5\alpha_3(0.5\alpha_2^{-1}(V_k)). \end{aligned} \tag{62}$$

Finally, since whenever $V_k \geq \alpha_2 \circ \gamma(\|w\|_\infty) + \mu$ we have $V_{k+1} \leq V_k$ and hence $u(s) \leq V_k, s \in [kd, (k+1)d), k \in \mathbb{N}$, then from (62) we have that for almost all $s \geq 0$ the following holds:

$$u(s) \geq \alpha_2 \circ \gamma(\|w\|_\infty) + \mu + \mu_1 \quad \Rightarrow \quad \frac{d}{ds}u(s) \leq -0.5\alpha_3(0.5\alpha_2^{-1}(u(s))). \tag{63}$$

Using, for example, Theorem 5.1 of [K], we obtain

$$|u(s)| \leq \max\{\beta_V(|u_0|, s), \alpha_2 \circ \gamma(\|w\|_\infty) + \mu + \mu_1\}, \qquad \forall s \geq 0,$$

where β_V was defined by (36). Since $u(kd) = V_k \geq \alpha_1(|x(t_k)|)$, $u_0 = V_0 \leq \alpha_2(|x(t_0)|)$, for $s = kd$ we obtain

$$|x(t_k)| \leq \max\{\beta_\alpha(|x(t_0)|), kd), \alpha_1^{-1}(\alpha_2 \circ \gamma(\|w\|_\infty) + \mu + \mu_1)\}, \tag{64}$$

where β_α was defined in (35). Using the definition of R in (37) the result follows. ■

8. Sketch of Proof of Proposition 3

Without loss of generality, we take $|x| = \max\{|\xi|, |\eta|\}$. Using the first assumption, let $\beta_\xi \in \mathcal{KL}$ and $\gamma_\xi \in \mathcal{G}$ come from Theorem 2. Let $\beta_\eta \in \mathcal{KL}$, γ_η, $\tilde{\gamma} \in \mathcal{G}$ come from the third assumption. Let the strictly positive real numbers $\Omega_x, \Omega_w, \nu_x$ be given. Define

$$M := \max\{\beta_\xi(\Omega_x, 0), \gamma_\xi(\beta_\eta(\Omega_x, 0)), \beta_\eta(\Omega_x, 0), \gamma_\eta(\beta_\xi(\Omega_x, 0)), \tilde{\gamma}(\Omega_w)\} + \nu_x + 1 \tag{65}$$

and note that $M > \Omega_x$. Let $r > 0$ be such that

$$|\xi| \le M, |\eta| \le M, |w| \le \Omega_w \quad \Rightarrow \quad |(\xi, \eta, w)| \le r. \tag{66}$$

Let r, with the second assumption, generate $R > 0$ and $\varepsilon_1^* > 0$. Let $\nu_1 > 0$ be such that

$$\gamma_\eta(\gamma_\xi(s) + \nu_1) + \nu_1 < \max\left\{s, \frac{\nu_x}{2}\right\}, \qquad \forall s \in [0, M], \tag{67}$$

$$\gamma_\xi(\gamma_\eta(s) + \nu_1) + \nu_1 < \max\left\{s, \frac{\nu_x}{2}\right\}, \qquad \forall s \in [0, M], \tag{68}$$

$$\gamma_\eta(s + \nu_1) \le \gamma_\eta(s) + \frac{\nu_x}{2}, \qquad \forall s \in [0, M], \tag{69}$$

$$\gamma_\xi(s + \nu_1) \le \gamma_\xi(s) + \frac{\nu_x}{2}, \qquad \forall s \in [0, M]. \tag{70}$$

Such a $\nu_1 > 0$ exists since $\nu_x > 0$, γ_η and γ_ξ are continuous, and, with the fourth assumption, since

$$\max_{s \in [\nu_x/2, M]} [s - \max\{\gamma_\eta \circ \gamma_\xi(s), \gamma_\xi \circ \gamma_\eta(s)\}] > 0.$$

Define

$$\delta := \nu := \nu_1 < \frac{\nu_x}{2}. \tag{71}$$

Define $\Omega_\xi := \Omega_\eta := \Omega_x$. Let $\Omega_\xi, \Omega_\eta, \Omega_w, \delta$, with the third assumption, generate $\varepsilon_2^* > 0$. Let $\Omega_\xi, \Omega_\eta, R, \nu$, playing the role of $\Omega_x, \Omega_w, \Omega_{\dot{w}}, \nu$ in Theorem 2, generate $\varepsilon_3^* > 0$. Define

$$\varepsilon^* := \min\{\varepsilon_1^*, \varepsilon_2^*, \varepsilon_3^*\}. \tag{72}$$

Suppose $|x(t_0)| \le \Omega_x$, $\|w\|_\infty \le \Omega_w$ and $\varepsilon \in (0, \varepsilon^*)$. Let $T > 0$ be such that $|x(t)| \le M$ for all $t \in [t_0, t_0 + T]$. Such a $T > 0$ exists since $M > \Omega_x$. Defining

$$\bar{\eta} := \max_{t \in [t_0, t_0 + T]} |\eta(t)|,$$

$$\bar{\xi} := \max_{t \in [t_0, t_0 + T]} |\xi(t)|,$$

and using causality, we have

$$\begin{aligned} \bar{\eta} &\le \max\{\beta_\eta(|\eta(t_0)|, 0), \gamma_\eta(\bar{\xi}), \tilde{\gamma}(\|w\|_\infty)\} + \delta, \\ &\le \max\{\beta_\eta(|\eta(t_0)|, 0), \gamma_\eta(\beta_\xi(|\xi(t_0)|, 0) + \nu), \gamma_\eta(\gamma_\xi(\bar{\eta}) + \nu), \tilde{\gamma}(\|w\|_\infty)\} + \delta. \end{aligned} \tag{73}$$

Together with (67), (69), (71) and the observation that, for nonnegative real numbers (a, b, s),

$$a < s \le \max\{a, b\} \quad \Rightarrow \quad s \le b, \tag{74}$$

it follows that

$$\begin{aligned}\bar{\eta} &\le \max\left\{\beta_\eta(|\eta(t_0)|, 0) + \frac{v_x}{2}, \gamma_\eta(\beta_\xi(|\xi(t_0)|, 0)) + v_x, v_x, \tilde{\gamma}(\|w\|_\infty) + \frac{v_x}{2}\right\} \\ &\le \max\{\beta_\eta(|\eta(t_0)|, 0), \gamma_\eta(\beta_\xi(|\xi(t_0)|, 0)), \tilde{\gamma}(\|w\|_\infty)\} + v_x < M.\end{aligned} \tag{75}$$

Similarly, for $\bar{\xi}$ we can obtain

$$\bar{\xi} \le \max\{\beta_\xi(|\xi(t_0)|, 0), \gamma_\xi(\beta_\eta(|\eta(t_0)|, 0)), \tilde{\gamma}(\|w\|_\infty)\} + v_x < M. \tag{76}$$

It follows that T can be taken to be arbitrarily large, and so the above relations hold when $\bar{\eta} := \sup_{t \ge t_0} |\eta(t)|$ and $\bar{\xi} := \sup_{t \ge t_0} |\xi(t)|$.

Following the same computations as above, we can also show that for each $\rho > 0$ and $r > 0$ there exists $T > 0$ such that

$$\begin{aligned}&|(\xi(t_0), \eta(t_0))| \le r, \|w\|_\infty \le r \quad \Rightarrow \\ &\qquad \sup_{t \ge t_0 + T} \max\{|\xi(t)|, |\eta(t)|\} \le \max\{\rho, \tilde{\gamma}(\|w\|_\infty)\} + v_x.\end{aligned} \tag{77}$$

From here, the construction of $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ as in the conclusion of the corollary follows a combination of the ideas in the proofs of Proposition 2.5 of [LSW] and Theorem 2.1 of [JTP]. ■

9. Conclusions

The ISS property of time-varying nonlinear systems was investigated in the context of the averaging method. We introduced two definitions of the averaged system with disturbances (strong average and weak average). We showed that ISS of the strong average of a system implies uniform semi-global practical ISS of the actual system. This result generalizes a recent result on averaging of time-varying nonlinear systems without disturbances proved in [TPA]. We also showed that ISS of the weak average of a system implies uniform semi-global practical "ISS-like" property for the actual system that also requires the disturbances to be absolutely continuous with $w, \dot{w} \in \mathcal{L}_\infty$. Weak averages were shown to be useful in the stability analysis of cascaded time-varying systems.

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