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LOG-CONCAVE FUNCTIONS AND POSET PROBABILITIES

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For x,y elements of some (finite) poset P, write $p(x<y)$ for the probability that x precedes y in a random (uniform) linear extension of P. For $u,v\in[0,1]$ define

$$\delta(u,v) = \inf\{p(x<z) : p(x<y) \ge u, \qquad p(y<z) \ge v\},$$

where the infimum is over all choices of P and distinct $x,y,z\in P$.

Addressing an issue raised by Fishburn [6], we give the first nontrivial lower bounds on the function δ. This is part of a more general geometric result, the exact determination of the function

$$\gamma(u,v) = \inf\{\Pr(X_1 < X_3) : \Pr(X_1 < X_2) \ge u,\ \Pr(X_2 < X_3) \ge v\},$$

where the infimum is over $X=(X_1,\dots,X_n)$ chosen uniformly from some compact convex subset of a Euclidean space.

These results are mainly based on the Brunn–Minkowski Theorem and a theorem of Keith Ball [1], which allow us to reduce to a 2-dimensional version of the problem.

1. Introduction

1.1. Posets

For a finite partially ordered set (*poset*) P, denote by $p(x<y)$ the fraction of linear extensions of P in which x precedes y; in other words, $p(x<y)=\Pr(f(x)<f(y))$ where f is drawn uniformly from the set of linear extensions of P. (Following a standard notational abuse, we identify a poset P with its element set. A *linear extension* of P is then an order-preserving bijection $f : P \to \{1,\dots,n\}$, where $n=|P|$.)

These probabilities are the subject of a number of fascinating problems and results; see for example the list of references in [2].

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The present work was motivated by a class of questions raised by Peter Fishburn [6] (in turn suggested by [3], [4] and, according to [6], by [15], [14] and [5]). Before discussing the general problem, let us single out the case which people (ourselves included) seem mostly to have considered. (As sometimes happens, we cannot point to written evidence that the problem has received much attention; we can only say that a number of conversations over the last 10 years suggest that the absence of progress on the problem was not due to absence of effort.)

Conjecture 1.1. *There is a positive constant* δ *such that if* x,y,z *are (distinct) elements of a poset* P *satisfying* $p(x<y)\geq 1/2$ *and* $p(y<z)\geq 1/2$*, then* $p(x<z)>\delta$*.*

That this is less obvious than it seems is suggested by the fact—whose verification we leave to the reader (or see [6])—that it is *not* true if we replace $1/2$ by $1/2-\varepsilon$ with $\varepsilon>0$.

As observed in [6], δ in Conjecture 1.1 cannot exceed $1/e$, which in fact seems likely to be the correct value; here we show

Theorem 1.2. *Conjecture* 1.1 *is true with* $\delta=1/4$*.*

As will appear shortly (Section 1.2), Theorem 1.2 is actually true in a more general geometric setting, where the value $1/4$ is best possible.

For the general Fishburn question we define, for any $u,v\in[0,1]$,

$$\delta(u,v)=\inf\{p(x<z):p(x<y)\geq u,\quad p(y<z)\geq v\} \tag{1}$$

(More formally, the infimum is over choices of P and distinct $x,y,z\in P$ satisfying the conditions in (1).) Fishburn's question is: what can we say about these numbers?

The following are easy or trivial (see [6]): (i) δ is symmetric; (ii) $\delta(u,v)=0$ if $u+v<1$; (iii) $\delta(1,v)=v$; (iv) δ is nondecreasing in each of its arguments; (v) $u+v-1\leq\delta(u,v)\leq\min\{u,v\}$. (The last inequality follows from (iii) and (iv).)

Some nontrivial upper bounds for δ were given in [6], but the question of lower bounds—i.e. any improvement of the lower bound in (v) for any u,v with $u+v\geq 1$—has remained open. Here we give such lower bounds.

Let us mention, even before stating these, that what we prove will again be true at a more general geometric level, where the unsightly function g we are about to define is actually optimal.

Let $T=\{(u,v)\in[0,1]^2:u+v\geq 1\}$. We assume henceforth (because of (ii)) that all pairs (u,v) considered lie in T. Let

$$\widetilde{T}=\{(u,v)\in T:v\leq u^2-u+1,\ u\leq v^2-v+1\}$$

(the shaded region in Figure 1). For $(u,v)\in T$, set

$$g(u,v)=\begin{cases}(1-u)(1-v)/(u+v-2\sqrt{u+v-1}) & \text{if } (u,v)\in\widetilde{T}\\ \min\{u,v\} & \text{if } (u,v)\in T\setminus\widetilde{T}.\end{cases}$$

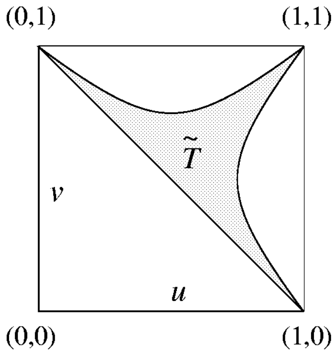

Figure 1

It is easy to check that $\partial g/\partial u, \partial g/\partial v > 0$ on the interior of $\widetilde{T}$, and that $g(u,v)$ is continous on T. So $g(u,v)$ is monotone, and in particular, $g(u,v) \le g(u,1) = u$ (and symmetrically $g(u,v) \le g(1,v) = v$).

Theorem 1.3. *For all* $(u,v) \in T$, $\gamma(u,v) \ge g(u,v)$.

Since $g(1/2,1/2) = 1/4$, this includes Theorem 1.2 except for the strictness of the inequality (that is, $p(x < z) > 1/4$); the latter will follow from the second assertion of Theorem 1.4 below, which allows us to add to Theorem 1.3:

If $(u,v) \notin \operatorname{int}(\widetilde{T}) \cup \{(0,1),(1,0),(1,1)\}$, *then for* x, z *as in* (1), $p(x<z) > g(u,v)$.
Most likely the only possibilities for equality here are the trivial ones with $u, v \in \{0,1\}$. This could perhaps be proved by showing that posets never give rise (via the reductions of Section 2) to the extreme (geometric) examples given at the end of Section 3; but we have not really considered this.

1.2. Geometry

Here and throughout we use *body* to mean a full-dimensional, compact convex subset of $\mathbf{R}^n$, and use $|\cdot|$ for Euclidean volume (with dimension given by context), in particular for Euclidean length.

As mentioned above, Theorems 1.2 and 1.3 are actually true in a more general geometric setting. With K ranging over bodies and $X = (X_1, \ldots, X_n)$ drawn uniformly from K, set

$$(2) \qquad \gamma(u,v) = \inf\{\Pr(X_1 < X_3) : \Pr(X_1 < X_2) \ge u,\ \Pr(X_2 < X_3) \ge v\}.$$

Theorem 1.4. *For all* $(u,v) \in T$, $\gamma(u,v) = g(u,v)$. *The infimum in* (2) *is attained iff* $(u,v) \in \operatorname{int}(\widetilde{T}) \cup \{(0,1),(1,0),(1,1)\}$.

To see that this contains Theorem 1.3, just recall (see [11]) that for $V=(V_x : x\in P)$ drawn uniformly from $\mathcal{O}(P)=\{v\in[0,1]^P : x<_P y \;\Rightarrow\; v_x\le v_y\}$ (the *order polytope* of P), we have $\Pr(V_x<V_y)=p(x<y)$.

Remarks. The value $1/e$ mentioned preceding Theorem 1.2 recalls the following surprising result of Grünbaum [7]. (See also [9], [8] for application of similar arguments to posets. All these results are based on the Brunn–Minkowski Theorem (e.g. [13]), which will again play an important role below.)

Theorem 1.5. *Let K be a body in $\mathbf{R}^n$ and $\{x : v\cdot x=a\}$ any hyperplane through the centroid of K. Then $|K\cap\{x : v\cdot x\ge a\}|\ge(n/(n+1))^n|K|>e^{-1}|K|$.*

For example, this implies that if x,y,z are as in (1) with $u+v\ge 1$ and y is an *isolated* element of P, then $p(x<z)>1/e$. For in this case $p(y<w)$ is (for any w) simply the w-th coordinate, c_w, of the centroid $c=c(\mathcal{O}(P))$. So we have $c_x\le 1-u\le c_z$, and Theorem 1.5 then implies that $p(x<z)>1/e$. (Take $v=e_z-e_x$ with $(e_w : w\in P)$ the standard basis for $\mathbf{R}^P$.)

So it may be useful to think of $p(y<x)$ as a "generalized centroid" $c_y(x)$, and consider Fishburn's Conjecture from this point of view.

The key to the proof of Theorem 1.4 is a beautiful result of Keith Ball [1], which, in conjunction with the Brunn–Minkowski Theorem, allows us to reduce to a statement in dimension 2, *viz.*

Theorem 1.6. *Suppose l_1, l_2, l_3 are (distinct) concurrent lines in $\mathbf{R}^2$ and that l_i^+ is a half-plane bounded by l_i ($i=1,2,3$) with $l_1^+\cap l_2^+\subseteq l_3^+$. Suppose further that K is a body with*

$$|K\cap l_1^+|\ge u|K| \text{ and } |K\cap l_2^+|\ge v|K|. \tag{3}$$

Then $|K\cap l_3^+|\ge g(u,v)|K|$. Moreover this bound is best possible for all u,v, and equality can hold iff $(u,v)\in int(\widetilde{T})\cup\{(0,1),(1,0),(1,1)\}$.

Ball's result (Theorem 2.2) and its application in the present context are given in Section 2, and the proof of Theorem 1.6 is given in Section 3. In Section 4 we give our original proof (more or less) of Theorem 1.6 in the case $u=v=1/2$. One nice point here is Lemma 4.1, one case of which is: if there is a line l bisecting a body K, and K' is the reflection of K through the midpoint of $K\cap l$, then $|K\cap K'|\ge|K|/2$.

2. Log-concave functions

We write $\mathbf{R}^+$ for $[0,\infty)$. Recall that $f:\mathbf{R}^n\to\mathbf{R}^+$ is *logarithmically concave* (or *log-concave*) if $\log f:\mathbf{R}^k\to[-\infty,\infty)$ is concave (with the natural convention regarding $-\infty$).

As mentioned above, we use two known results to reduce Theorem 1.4 to Theorem 1.6. The first, which, as noted in [1] (see Lemma 2), is an easy consequence of the the Brunn–Minkowski Theorem, is

Lemma 2.1. *For an arbitrary body K and subspace H in $\mathbf{R}^n$, $f: H^\perp \to \mathbf{R}^+$ given by*

$$f(w) = |(H+w)\cap K| \tag{4}$$

is log-concave.

(So here $|\cdot|$ is volume in dimension $\dim H$.)

The second is the basic result of [1] (see Theorem 5):

Theorem 2.2. *Let $p \geq 1$ and suppose $f: \mathbf{R}^k \to \mathbf{R}^+$ is log-concave and positive on some neighborhood of $\underline{0}$, and that $\int f < \infty$. Then $\|\cdot\|$ given by*

$$\|x\| = \begin{cases} [\int_0^\infty f(rx) r^{p-1} dr]^{-1/p} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \tag{5}$$

is a (not necessarily symmetric) norm on $\mathbf{R}^k$.

(That is, $\|\alpha x\| = \alpha \|x\|$ for $\alpha \in \mathbf{R}^+$, and $\|x+y\| \leq \|x\| + \|y\|$. The statement of the theorem in [1] assumes that f is even, and concludes that $\|\cdot\|$ is a symmetric norm, but the proof gives the version stated here.)

Reduction to Theorem 1.6. Suppose K is a body in $\mathbf{R}^n$ and that for $X = (X_1, \ldots, X_n)$ drawn uniformly from K we have

$$\Pr(X_1 < X_2) \geq u, \ \Pr(X_2 < X_3) \geq v.$$

For the lower bounds of Theorem 1.4 we must show that

$$\Pr(X_1 < X_3) \geq g(u,v)$$

and that the inequality is strict when it is supposed to be.

(The remaining assertions of Theorem 1.4—that the infimum in (2) is not more than $g(u,v)$, and is actually a minimum for appropriate u,v—will follow easily from the last sentence of Theorem 1.6, since it is easy to see that any example achieving $|K \cap l_3^+| = \alpha |K|$ in Theorem 1.6 can be "lifted" to an example achieving $\Pr(X_1 < X_3) = \alpha$ in (2). So we will have no more to say about this part of the reduction.)

The reduction to Theorem 1.6 is achieved in two steps. We first use Lemma 2.1 to reduce to a 2-dimensional problem, but at the cost of replacing uniform distribution on a body by a general log-concave density. We then use a simple consequence (Lemma 2.4) of Theorem 2.2 to replace the log-concave density by uniform distribution on a body (but now in dimension 2).

We may assume for convenience that

$$|K| = 1. \tag{6}$$

We will apply Lemma 2.1 with

$$H^{\perp} = \{x : x_1 + x_2 + x_3 = 0,\ x_4 = \cdots = x_n = 0\},$$

which copy of $\mathbf{R}^2$ we rename S.

Now because of (6), f as in (4) is a (log-concave) probability density on S. Choosing W according to f is equivalent to choosing X uniformly from K and taking $W = \pi(X)$, the projection of X on S; that is, $\pi(X) = (X_1 - \alpha, X_2 - \alpha, X_3 - \alpha, 0, \ldots, 0)$, where $\alpha = (X_1 + X_2 + X_3)/3$. In particular, $\Pr(W_i < W_j) = \Pr(X_i < X_j)$ for each $1 \le i, j \le 3$. Thus Theorem 1.4 will follow from

Theorem 2.3. *Suppose l_1, l_2, l_3 are concurrent lines in $\mathbf{R}^2$ and that l_i^+ is a half-plane bounded by l_i ($i = 1,2,3$) with $l_1^+ \cap l_2^+ \subseteq l_3^+$. Suppose further that $f : \mathbf{R}^2 \to \mathbf{R}^+$ is log-concave with*

$$\int_{l_1^+} f \ge u \int f, \quad \int_{l_2^+} f \ge v \int f. \tag{7}$$

Then $\int_{l_3^+} f \ge g(u,v) \int f$. Equality is possible iff $(u,v) \in \mathrm{int}\,(\widetilde{T}) \cup \{(0,1),(1,0),(1,1)\}$.

Notation. As usual, S^{k-1} is the unit sphere in $\mathbf{R}^k$. We write μ for Lebesgue measure and σ_{k-1} for its restriction to S^{k-1}. For $B \subseteq S^{k-1}$, set $R(B) = \mathbf{R}^+ B (= \{r\theta : \theta \in B,\ r \in \mathbf{R}^+\})$.

Lemma 2.4. *Suppose $f : \mathbf{R}^k \to \mathbf{R}^+$ is log-concave and positive on some neighborhood of $\underline{0}$ and that $\int f < \infty$. Then there is a body $C \subseteq \mathbf{R}^k$ such that*

$$|C \cap R(B)| = \int_{R(B)} f d\mu$$

for every measurable $B \subseteq S^{k-1}$.

Proof. Let $p = k$ and let D be the unit ball of the norm $\|\cdot\|$ given by (5). Then for each measurable $B \subseteq S^{k-1}$,

$$\begin{aligned} |D \cap R(B)| &= \int_B \int_0^\infty 1_D(r\theta) r^{k-1} dr d\sigma_{k-1}(\theta) = \frac{1}{k} \int_B \|\theta\|^{-k} d\sigma_{k-1}(\theta) \\ &= \frac{1}{k} \int_B \int_0^\infty f(r\theta) r^{k-1} dr d\sigma_{k-1}(\theta) = \frac{1}{k} \int_{R(B)} f d\mu. \end{aligned}$$

So $C = k^{1/k}D$ is the desired set. ∎

Modulo one minor point this finishes our reduction: We may assume in Theorem 2.3 that $\underline{0}$ is the common point of the lines l_i, and apply Lemma 2.4 (with $k=2$) to reduce to an instance of Theorem 1.6.

The minor point is the requirement concerning $\underline{0}$ in Lemma 2.4; that is, we cannot apply the lemma if $\underline{0} \notin (supp(f))^o$. But in this case Theorem 2.3 is easily seen directly as follows. Since $g(u,v) \le \min\{u,v\}$, we may assume $supp(f)$ either meets $l_1^+ \setminus l_3^+$ or is disjoint from $l_1^+ \triangle l_3^+$. The latter alternative implies (since $supp(f)$ is convex and full-dimensional) that $u \in \{0,1\}$, in which case Theorem 2.3 is trivial; so we may assume $supp(f)$ meets $l_1^+ \setminus l_3^+$ and (similarly) $l_2^+ \setminus l_3^+$. But this implies (again using convexity of $supp(f)$) that $supp(f) \cap l_1^+ \cap l_2^+ = \emptyset$ and $u+v<1$.

Remarks. 1. We could have avoided the preceding paragraph by dropping the requirement involving $\underline{0}$ in Theorem 2.2, thus allowing $\|x\| = \infty$ in (5), and checking that this does not invalidate our arguments; but it seemed preferable to treat these essentially trivial cases as a side issue.

2. It follows from Lemma 2.4 that if we define γ' to be the right hand side of (2), but with X drawn from a general log-concave density, then $\gamma' = \gamma$; so we have the corresponding generalization of Theorem 1.4. This can also be done a bit more directly using the fact that any projection of a log-concave function is itself log-concave. (Keith Ball tells us this is usually attributed to Prékopa [12] and Leindler [10].) Substituting this for the application of Brunn–Minkowski (Lemma 2.1), we could have started with X drawn from a general log-concave density and used the above arguments to derive determination of γ' from Theorem 1.6.

3. Proof of Theorem 1.6

For convenience set, for $K' \subseteq K$, $p(K') = |K'|/|K|$. We first prove the main assertion of Theorem 1.6, i.e. that

$$p(K \cap l_3^+) \ge g(u,v). \tag{8}$$

Examples to show that g is best possible and discussion of possibilities for equality are given at the end of the section. This order is convenient because in verifying correctness of the examples we will appeal to the proof of (8); but the reader might find an early peek at the examples helpful in motivating the proof.

We require two lemmas.

Lemma 3.1. *Assume in Figure 2 that $A_1A_2B_1B_2$ is a square of area 2 with all its vertices on the boundary of a convex body K and that B_2B_1 is the x-axis and B_2A_1 the y-axis. Let*

$$R_1 = K \cap \{y \ge \sqrt{2}\},\ R_2 = K \cap \{x \le 0\},\ R_3 = K \cap \{y \le 0\},\ R_4 = K \cap \{x \ge \sqrt{2}\},$$
$$r_i = |R_i| \qquad 1 \le i \le 4.$$

Then $(r_1+r_3)(r_2+r_4)\leq 1$.

Proof. Let $K'=\{(a,b):0\leq b\leq|K\cap\{x=a\}|\}$ and $K''=\{(a,b):0\leq a\leq|K'\cap\{y=b\}|\}$. These are both convex under the convention $|\emptyset|<0$. Defining R_i',r_i',R_i'',r_i'' in analogy with R_i,r_i, we have

$$r_1''=r_1'=r_1+r_3,\quad r_2''=0,\quad r_3''=r_3'=0,\quad r_4''=r_2'+r_4'=r_2+r_4.$$

Finally the desired inequality $1\geq r_1''r_4''=(r_1+r_3)(r_2+r_4)$ follows from the fact that R_1'' and R_4'' are bounded by some support line of K'' at A_2. ∎

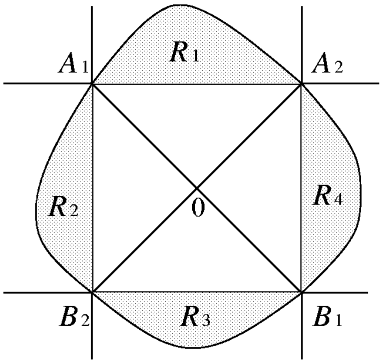

Figure 2

Lemma 3.2. *If* $f(x)$ *and* $g(x)$ *are non-negative, concave functions on* $[0,1]$, *then*

$$\frac{1}{\int_0^1\min\{f(x),g(x)\}dx}\leq\frac{1}{\int_0^1 f(x)dx}+\frac{1}{\int_0^1 g(x)dx}$$

Proof. Assume $f(x)$ attains its maximum at $x=d$. Let I be any measurable subset of $[0,1]$, $a=|I\cap[0,d]|$, $b=|I\cap[d,1]|$, $c=a/(a+b)$. Then $0\leq a\leq c\leq 1-b\leq 1$. Concavity implies

$$\int_0^a f(x)dx\geq\int_0^a\frac{a}{c}f(\frac{cx}{a})dx=|I|^2\int_0^c f(x)dx,$$

and similarly

$$\int_{1-b}^1 f(x)dx\geq|I|^2\int_c^1 f(x)dx.$$

Since $f(x)$ is monotone increasing (resp. decreasing) on $[0,d]$ (resp. $[d,1]$), we have

$$\int_{I\cap[0,d]}f(x)dx\geq\int_0^a f(x)dx\quad\text{and}\quad\int_{I\cap[d,1]}f(x)dx\geq\int_{1-b}^1 f(x)dx.$$

Combining these observations we have

$$\int_I f(x)dx \geq |I|^2 \int_0^1 f(x)dx.$$

(And of course the corresponding inequalities hold for $g(x)$.)

Let $u=\int_0^1 f(x)dx$, $v=\int_0^1 g(x)dx$. Then choosing $I=\{x|f(x)\leq g(x)\}$, we have

$$\int_0^1 \min\{f(x),g(x)\}dx = \int_I f(x)dx + \int_{[0,1]\setminus I} g(x)dx \geq u|I|^2 + v(1-|I|)^2 \geq \frac{uv}{u+v}.$$

(Remark: equality holds iff $f(x)=2ux, g(x)=2v(1-x)$ or $f(x)=2u(1-x), g(x)=2vx$.) ∎

We now turn to the lower bounds, beginning with a few easy reductions. First observe that we may assume equality holds in (3) (otherwise translate l_1,l_2 so that equality does hold, translate l_3 so that the lines are again concurrent, and notice this can only decrease $|K\cap l_3^+|$).

We may also assume (via affine transformation) that l_1,l_2 are the y-axis and x-axis respectively and that $l_1^+\cap l_2^+$ is the first quadrant, Q_1 (so also $Q_1\subseteq l_3^+$). We write $s(l)$ for the slope of a line l (note $-\infty<s(l_3)<0$) and Q_1,Q_2,Q_3,Q_4 for the four quadrants in the usual order. In Figures 3-6, an arrow attached to a line l indicates the half-plane l^+.

Finally, since we have already proved Theorem 2.3 (which contains Theorem 1.6) when $\underline{0}\notin supp(f)^o$ (see the 'minor point' following the proof of Lemma 2.4), we may assume $\underline{0}\in K^o$.

We suppose l_i meets ∂K, the boundary of K, in A_i,B_i $(i=1,2)$ with A_i on the positive axis. We distinguish three cases.

Case 1. $\underline{0}$ is the midpoint of both A_1B_1 and A_2B_2. (This is the main case; the others will be handled by reducing to this one.) We may assume $|\underline{0}A_1|=|\underline{0}A_2|=1$. Let R_1,R_2,R_3,R_4 denote the shaded regions in Figure 2 as indicated, and set $r_i=|R_i|$, $k=|K|$. Thus $k=\sum r_i+2$, $u=(r_1+r_4+1)/k$ and $v=(r_1+r_2+1)/k$. By Lemma 3.1, we have

$$\begin{aligned} k(r_1-r_3) &= r_1^2-r_3^2+(r_2+r_4+2)(r_1-r_3) \leq r_1^2+2r_1+r_1(r_2+r_4) \\ &\leq r_1^2+2r_1+1=(r_1+1)^2 \end{aligned} \tag{9}$$

This implies $u+v-1\leq u^2$ and $u+v-1\leq v^2$, so that $(u,v)\in\widetilde{T}$. Equality holds in (9) iff

$$r_3=0, \quad r_1(r_2+r_4)=1 \tag{10}$$

Moreover again using (9), we have

$$\frac{(r_1+1)(r_2+r_4)+r_2r_4}{r_2+r_4} \geq r_1+1 \geq \frac{r_1+1+\sqrt{k(r_1-r_3)}}{2}$$

$$= \frac{(r_1+1)^2 - k(r_1-r_3)}{2r_1+2-2\sqrt{k(r_1-r_3)}} = \frac{(r_3+1)^2 + (r_3-r_1)(r_2+r_4)}{2r_1+2-2\sqrt{k(r_1-r_3)}}$$

(We can take the square root because $u+v \geq 1$ implies $r_1 \geq r_3$. We have cheated slightly here since the denominator may vanish; but this occurs only if (10) holds, in which case it is easy to see directly that (11) holds with equality.) It follows that

$$1 + r_1 + \frac{r_2 r_4}{r_2+r_4} \geq \frac{(r_3+1)^2 + (r_3-r_1)(r_2+r_4) + (r_1+1)(r_2+r_4) + r_2 r_4}{2r_1+2-2\sqrt{k(r_1-r_3)} + (r_2+r_4)}$$

$$(11) \qquad = \frac{(r_3+r_4+1)(r_3+r_2+1)}{2r_1+r_2+r_4+2-2\sqrt{k(r_1-r_3)}} = \frac{(1-u)(1-v)k}{u+v-2\sqrt{u+v-1}} = g(u,v)k.$$

Let $K' = K \setminus R_1 \setminus R_3$. Note that the convexity of K implies that R_2 and R_4 lie between the lines A_1A_2 and B_1B_2. By Lemma 3.2, we have

$$(12) \qquad |K' \cap (-K')| = 2 + |R_2 \cap (-R_4)| + |R_4 \cap (-R_2)| \geq 2 + \frac{2r_2 r_4}{r_2+r_4}.$$

Thus

$$|K \cap l_3^+| = r_1 + |K' \cap l_3^+| \geq r_1 + |K' \cap (-K') \cap l_3^+|$$

$$(13) \qquad = r_1 + \frac{|K' \cap (-K')|}{2} \geq 1 + r_1 + \frac{r_2 r_4}{r_2+r_4},$$

which with (11) gives (8). ∎

Let $r(\theta)$ be the radius of K in direction θ. In the next case, and again in Section 4, we make repeated use of continuity of $r(\theta)$ and the formula

$$(\star) \qquad |K \cap R(B)| = \frac{1}{2}\int_B r^2(\theta)d\theta$$

($B \subseteq S^1$, $R(B)$ as in Lemma 2.4)

Case 2. $\underline{0}$ is the midpoint of A_1B_1, but not of A_2B_2 (or vice versa). Here we consider two possibilities.

Case 2.1. $|\underline{0}A_2| < |\underline{0}B_2|$ (see Figure 3). That $u+v \geq 1$ implies (is actually equivalent to) $|K \cap Q_1| \geq |K \cap Q_3|$, which in view of $(\star)$ and the assumption $|\underline{0}A_2| < |\underline{0}B_2|$ implies the existence of l_0 with $0 < s(l_0) < \infty$ such that $|\underline{0}B_0| = |\underline{0}A_0|$. (Note we don't need $|\underline{0}A_1| = |\underline{0}B_1|$ here.) Choose such an l_0 with $s(l_0)$ minimum (note the minimum is attained since $|\underline{0}A_2| \neq |\underline{0}B_2|$), let l_0^+ be the half-plane bounded by l_0 that contains Q_2, and set $v' = p(K \cap l_0^+)$. Then our choice of l_0 implies (because of $(\star)$) that $v' > v$. We can now invoke Case 1 (with l_2 replaced by l_0) to finish: $p(K \cap l_3^+) \geq g(u,v') \geq g(u,v)$.

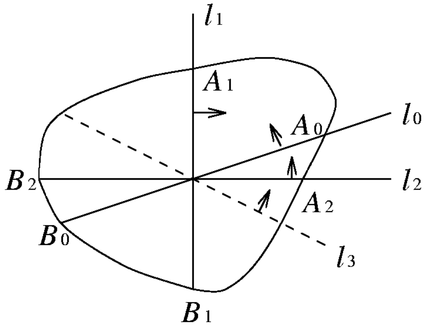

Figure 3

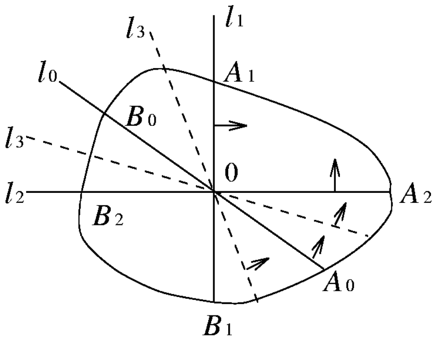

Figure 4

Case 2.2. $|\underline{0}A_2| > |\underline{0}B_2|$ (see Figure 4). If there exists l_0 with $-\infty < s(l_0) < 0$ such that $|\underline{0}B_0| = |\underline{0}A_0|$, then we choose such an l_0 with maximum slope; otherwise let $l_0 = l_1$ and define $s(l_0) = -\infty$. Let l_0^+ be the half-plane bounded by l_0 that contains Q_1, and set $v' = p(K \cap l_0^+)$. Then again using ($\star$) we have $v' > v$.

If $s(l_3) < s(l_0)$, then $l_0 \neq l_1$, and we again use Case 1: $p(K \cap l_3^+) \geq g(u, v') \geq g(u, v)$. If, on the other hand, $s(l_3) \geq s(l_0)$, then ($\star$) implies $|K \cap l_3^+| > |K \cap l_2^+|$, so $p(K \cap l_3^+) > v \geq g(u, v)$. ∎

Case 3. $\underline{0}$ is the midpoint of neither A_1B_1 nor A_2B_2. If $|\underline{0}A_2| < |\underline{0}B_2|$, we can repeat the argument of Case 2.1 to reduce to Case 2 rather than Case 1 (and similarly if $|\underline{0}A_1| < |\underline{0}B_1|$). If instead $|\underline{0}A_1| > |\underline{0}B_1|$ and $|\underline{0}A_2| > |\underline{0}B_2|$, then the continuity of $r(\theta)$ implies that there is l_0 with $-\infty < s(l_0) < 0$ such that $|\underline{0}B_0| = |\underline{0}A_0|$. We may then argue as in Case 2.2, again using Case 2 in place of Case 1. ∎

Finally we need to show that the lower bound g is best possible and that equality in (8) is possible precisely when $(u, v) \in int(\widetilde{T}) \cup \{(0,1), (1,0), (1,1)\}$. Let K

be the triangle CDE of Figure 5, where we retain the assumptions (in particular that $A_1A_2B_1B_2$ is a square) and notation of Case 1. Choose A_3, B_3 so that B_2B_3 and B_1A_3 are parallel to DE and CE respectively. It is easy to see that $A_3, \underline{0}, B_3$ are collinear, and we take the line joining them to be l_3.

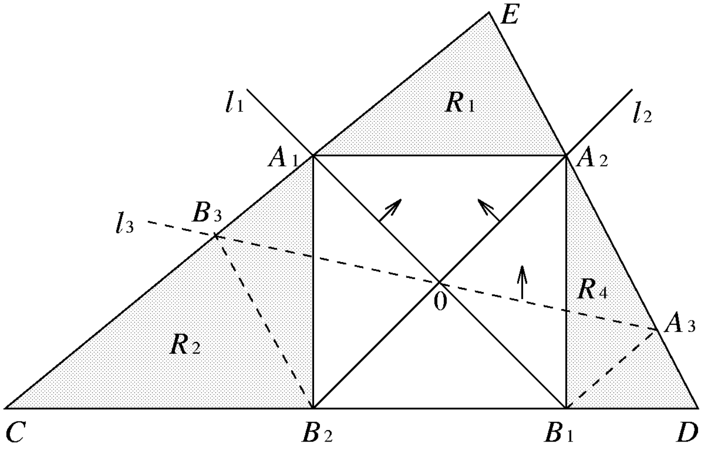

Figure 5

That equality then holds in (8) can, of course, be verified directly; but at this point it is easier to simply observe that the argument of Case 1 gives away nothing here: Since K satisfies (10) we have equality in (9) and (11); equality in (12) is the observation following Lemma 3.2; and our choice of l_3 gives $|K' \cap l_3^+| = |K' \cap (-K') \cap l_3^+|$, and so equality in (13). In view of the requirement (10), it is easy to see that the pairs (u,v) for which the above construction can be carried out are precisely those in $int(\widetilde{T})$; so we have equality in (8) for all such (u,v).

That g is also best possible when $u+v=1$ now follows by continuity (of g). For the cases with $g(u,v) = \min\{u,v\}$, optimality of g is more trivial: just let l_3 approach l_1 (when $g(u,v)=u$) or l_2. Moreover it is easy to see that equality in (8) is impossible here (except in the trivial cases with $u,v \in \{0,1\}$), briefly because: if Case 1 holds then equalities in (9)–(13) require that K and l_3 be constructed as above; on the other hand equality in Case 2 or Case 3 would imply a waste-free reduction to one of the examples above, and this is easily seen to be impossible. ∎

4. Coda: $u=v=1/2$

Before closing we would like to record (something like) the original proof of Theorem 1.6 in the case $u = v = 1/2$, since we think it is a little nicer than the general proof (though the latter also seems reasonably clean considering the form of g). As observed below, this also implies the theorem when $(u,v)=(1/2,3/4)$.

One other case worth mentioning is $u=v=5/9$, for which the fact that $\gamma(u,v)\geq 4/9$ is an immediate consequence of Grünbaum's Theorem 1.5 (and $\gamma(u,v)\leq 4/9$ is easy).

Let K be a body in $\mathbf{R}^2$. For $x\in\mathbf{R}^2$ let $K_x=2x-K$ (the reflection of K through x), and define $f:\mathbf{R}^2\to\mathbf{R}^+$ by $f(x)=|K\cap K_x|/|K|$. In particular, $f(\underline{0})=|K\cap(-K)|/|K|$. We first observe that

$$f \text{ is log-concave.} \tag{14}$$

This is again a consequence of the Brunn–Minkowski Theorem: it is easy to check that $\{(x,y)\in\mathbf{R}^2\times\mathbf{R}^2\mid x\in K, y\in K\cap K_x\}$ is a body in $\mathbf{R}^4$, and then Lemma 2.1 implies (14).

Lemma 4.1. *If l is a line and $\underline{0}$ is the midpoint of $l\cap K$, then $f(\underline{0})\geq 2p(K\cap l^+)p(K\cap l^-)$. Equality holds iff K is a triangle and l passes through a vertex of K.*

Proof. If there is only one line, l, for which the midpoint of $l\cap K$ is $\underline{0}$, then either $(-K)\cap l^+\subseteq K\cap l^+$ or $K\cap l^+\subseteq(-K)\cap l^+$, so

$$f(\underline{0})=2\min\{p(K\cap l^+),p(K\cap l^-)\}\geq 2p(K\cap l^+)p(K\cap l^-).$$

Otherwise, let $l_1=l$ and let l_2 be another such line. Then with notation as in Case 1 of the proof of Theorem 1.6 (and using Lemma 3.1 for the first inequality), we have

$$\begin{aligned}\frac{(1+r_1+r_4)(1+r_2+r_3)}{k}&=1+\frac{(r_1+r_4)(r_2+r_3)-1}{k}\leq\\ 1+&\frac{(r_1+r_4)(r_2+r_3)-(r_1+r_3)(r_2+r_4)}{k}\leq\\ 1+&\frac{r_1r_3+r_2r_4}{k}\leq 1+\frac{r_1r_3}{r_1+r_3}+\frac{r_2r_4}{r_2+r_4},\end{aligned}$$

which with Lemma 3.2 implies $f(\underline{0})\geq 2p(K\cap l^+)p(K\cap l^-)$.

Verification of the second sentence of the lemma is left to the reader. ∎

Lemma 4.2. *If two area bisectors l_1,l_2 of K meet at $\underline{0}$, then $f(\underline{0})>1/2$.*

Proof. We may assume that l_1,l_2 are the y-axis and x-axis respectively as in Figure 6. We work in the 1-dimensional projective space L consisting of all lines through $\underline{0}$. The set S of all bisectors in L is closed, hence compact (since L is compact). We may assume that $\underline{0}$ is not the midpoint of any $l\in S$, since otherwise we are done by Lemma 4.1 (we can't have equality because there is only one bisector through the midpoint of a median of a triangle). But this implies by ($\star$) that for

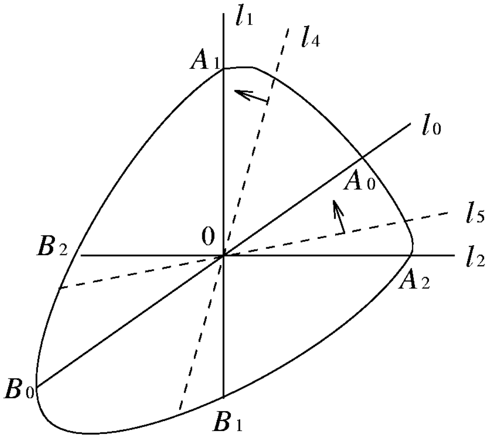

Figure 6

each $l \in S$, some neighbourhood of l contains no other members of S; that is, S is discrete, and hence finite. Suppose l_i meets ∂K at A_i, B_i with A_i on the positive axis, and assume

$$|\underline{0}A_i| > |\underline{0}B_i|. \tag{15}$$

Given $\alpha > 0$ let l_4, l_5 be the members of L with slopes α^{-1} and α respectively and l_4^+, l_5^+ the half-planes defined by l_4, l_5 and containing Q_2. For sufficiently small $\alpha > 0$, (15) implies (again via ($\star$))

$$|K \cap l_4^+| > \frac{|K|}{2} \quad \text{and} \quad |K \cap l_5^+| < \frac{|K|}{2}$$

Thus by continuity, S contains a line of positive slope. Choose such a line l_0 with minimum slope and suppose l_0 meets ∂K at A_0, B_0. If $|\underline{0}A_0| > |\underline{0}B_0|$, then by the preceding argument we can find a line in S between l_0 and l_2 (i.e. with slope in $(0, \alpha)$), contradicting our choice of l_0. So we must have $|\underline{0}A_0| < |\underline{0}B_0|$. Let C_i be the midpoint of A_iB_i, $i = 0, 1, 2$. Then since $\underline{0}$ is clearly inside the triangle $C_0C_1C_2$, Lemma 4.1 and (14) imply $f(\underline{0}) > 1/2$ unless $f(C_i) = 1/2$ for all $i = 0, 1, 2$. But this can only happen if K is a triangle and l_0, l_1, l_2 are its medians (see the last sentence of Lemma 4.1); and then $\underline{0}$ is the centroid of K and $f(\underline{0}) = 2/3$. ∎

The case $u = v = 1/2$ now follows. For if l_1, l_2 are bisectors of K meeting (w.l.o.g.) at $\underline{0}$, and l_3 is any line through $\underline{0}$, then for l_3^+ either of the half-planes bounded by l_3 we have $p(K \cap l_3^+) \geq p(K \cap (-K) \cap l_3^+) = f(\underline{0})/2 > 1/4$.

Finally, for the case $(u, v) = (1/2, 3/4)$, just note that if the desired conclusion, $p(K \cap l_3^+) > 1/2$ fails, then we have $p(K \cap l_3^-) \geq 1/2$, $p(K \cap l_1^+) \geq 1/2$. But according to the case (1/2,1/2) this implies $p(K \cap l_2^-) > 1/4$, which is contrary to assumption. ∎

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