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On Geometric Graphs with No Two Edges in Convex Position*

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Abstract. A geometric graph is a graph $G = G(V, E)$ drawn in the plane, where its vertex set V is a set of points in general position and its edge set E consists of straight segments whose endpoints belong to V. Two edges of a geometric graph are in convex position if they are disjoint edges of a convex quadrilateral. It is proved here that a geometric graph with n vertices and no edges in convex position contains at most $2n - 1$ edges. This almost solves a conjecture of Kupitz. The proof relies on a projection method (which may have other applications) and on a simple result of Davenport–Schinzel sequences of order 2.

1. Introduction

A *geometric graph* is a graph $G = G(V, E)$ drawn in the plane whose vertex set V consists of points in general position (i.e., no three are collinear) and whose edge set E consists of straight segments whose endpoints belong to V. Consult [4] for recent results on geometric graphs. Two segments are in *convex position* if they are disjoint edges of a convex quadrilateral. A geometric graph is called *proper* if it has two edges in convex position. Otherwise, it is called *improper*.

The systematic study of geometric graphs was initiated by Kupitz and Perles [1]. Kupitz [2] constructed improper graphs with n vertices and $2n-2$ edges for $n \geq 4$. (See Fig. 1 for an improper geometric graph on 7 vertices and 12 edges.)

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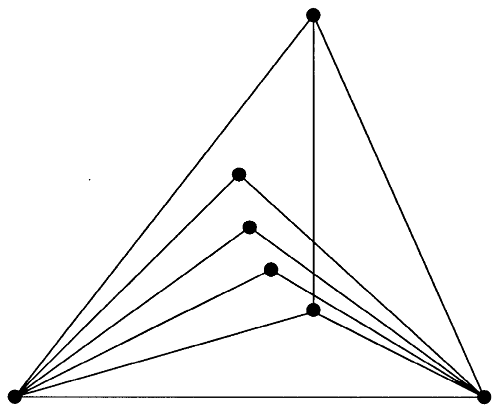

Fig. 1. An improper geometric graph on 7 vertices and 12 edges.

He made the following conjecture:

Kupitz Conjecture. *An improper geometric graph with n vertices has at most* $2n - 2$ *edges*.

Here the conjecture is almost solved:

Theorem. *An improper geometric graph with n vertices has at most* $2n - 1$ *edges*.

This result has appeared in [3]. The proof relies on a projection method and on a simple property of Davenport–Schinzel sequences of order 2.

The second section deals with definitions and two lemmas, followed by the proof of the theorem in the third section.

2. Definitions and Two Lemmas

For distinct points x and y let xy denote the segment with endpoints x and y, let $\ell(x, y)$ denote the line through these points, and let $\overrightarrow{xy}$ denote the ray (half-line) with apex x and containing y. An edge xy of a geometric graph is said to be to the *right* (*left*) of edge xz if the ray $\overrightarrow{xy}$ is obtained from the ray $\overrightarrow{xz}$ by a clockwise (counterclockwise) rotation about x by a positive angle less than π. If there is no edge incident to x to the right (left) of xy, then xy is called the *rightmost* (*leftmost*) edge of x. If xy is not the rightmost or the leftmost edge of x, then it is called an *interior* edge of x.

A *circular* sequence is a sequence whose first and last term are considered adjacent. A circular sequence from a set of n symbols shall be called a *circular Davenport–Schinzel* sequence of order 2 if no two adjacent terms are identical and if it does not contain a circular subsequence of type $abab$.

For more advanced results on Davenport–Schinzel sequences (which are not needed here), applications, etc., we may consult [5]. We shall need a known upper bound on the length of such circular sequences of order 2 which is easily proved by induction on n, the number of symbols, see [5].

Lemma 1. *The length of a circular Davenport–Schinzel sequence of order* 2 *on* n *symbols for* $n \geq 2$ *is at most* $2n - 2$.

A *convex curve* is the boundary of a compact convex planar set with nonempty interior. The next technical lemma is essential to the proof of the theorem.

Lemma 2. *Let* A_i, A_j, A_k, A_l *be four points appearing in this order on a convex curve* γ. *Let* P, Q *be two points inside* γ.

Consider the four (*closed*) *segments*

$$PA_i,\ QA_j,\ PA_k,\ QA_\ell, \tag{1}$$

and assume that among them:

> *there is no segment* s *such that* s *contains only one of the points* P, Q *and the line supporting* s *contains both of them*. (2)

Then two of the four segments are in convex position.

Proof. Observe that:

> If points A, B, C, D in general position lie in this order on a convex curve, then the segments AB and CD are in convex position. (3)

Let $\ell = \ell(P, Q)$ and let l^+ and l^- be two half-planes such that $l^+ \cap l^- = \ell$. Let γ_1 (γ_2) be the boundary of the convex hull of $l^+ \cap \gamma$ (of $l^- \cap \gamma$). It is easy to check, using (2) and (3) that if one of the four points A_i, A_j, A_k, A_ℓ lies on ℓ, then two of the segments are in convex position. Assume therefore without loss of generality that either

$$A_i,\ A_j, \text{ and } A_k \text{ lie in the interior of } l^+ \tag{4}$$

or

$$A_i \text{ and } A_j \text{ lie in the interior of } l^+$$

and

$$A_k \text{ and } A_\ell \text{ lie in the interior of } l^-. \tag{5}$$

In case (4), by (3) applied to γ_1, either PA_i and QA_j are in convex position or PA_k and QA_j are in convex position.

In case (5) if PA_i, QA_j are not in convex position and PA_k, QA_ℓ are not in convex position, then by (3) the order of the points on γ is A_i, A_j, A_ℓ, A_k, a contradiction. □

3. Proof of the Theorem

Let $v_1, \ldots, v_n$ be the vertices of G and e the number of its edges. Assume that G is improper with $e \geq 1$. Let C be a circle containing $v_1, \ldots, v_n$ in its interior. For any two vertices v_i and v_j joined by an edge $v_i v_j$ define two points on C:

$$\alpha_{ij} = \overrightarrow{v_i v_j} \cap C \qquad \text{and} \qquad \alpha_{ji} = \overrightarrow{v_j v_i} \cap C.$$

Arrange the $2e$ points on C in a circular sequence according to the order of their appearance on C. Let $D(G)$ be the circular sequence thus obtained.

Color the points of $D(G)$ with n colors such that

$$\alpha_{ij} \text{ receives the color } i.$$

Point α_{ij} has a *dark* color i if $v_i v_j$ is an interior edge of v_j. Otherwise, α_{ij} has a *light* color i. Divide the sequence $D(G)$ into *arcs* where an arc is a maximal subsequence of consecutive points of $D(G)$ having the same color i. Note that a dark i and a light i may belong to the same arc.

The circular sequence obtained from $D(G)$ by contracting each arc to one of its points and then replacing the point by its color i is called the *pattern sequence* of G, or $PS(G)$. See Fig. 2. for a geometric graph with four vertices and the corresponding sequences $D(G)$ and $PS(G)$. Behind each point α_{ij} is shown its color with a superscript d if it is dark or ℓ if it is light.

Here $D(G) = (\alpha_{41}, \alpha_{42}, \alpha_{43}, \alpha_{23}, \alpha_{12}, \alpha_{14}, \alpha_{24}, \alpha_{34}, \alpha_{32}, \alpha_{21})$. The arcs of G are $(\alpha_{41}, \alpha_{42}, \alpha_{43})$, (α_{23}), $(\alpha_{12}, \alpha_{14})$, (α_{24}), $(\alpha_{34}, \alpha_{32})$, (α_{21}). The pattern sequence is $PS(G) = (4, 2, 1, 2, 3, 2)$.

We need the following two assertions:

Lemma 3. *$PS(G)$ is a circular Davenport–Schinzel sequence of order* 2.

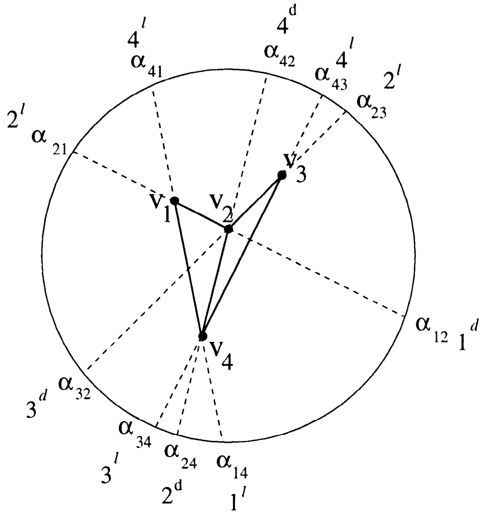

Fig. 2. A geometric graph on four vertices and the derived sequence.

Lemma 4. *An arc of $D(G)$ contains at most one point with dark color.*

We have

$$|D(G)| = 2e = \sharp \text{ of light colored points } + \sharp \text{ of dark colored points.} \qquad (6)$$

Each vertex of G has at most one rightmost edge and at most one leftmost edge incident to it, so that the number of light colored points in $D(G)$ is bounded by $2n$. By Lemmas 1, 3, and 4,

$$\sharp \text{ of dark colored points } \le |PS(G)| \le 2n - 2.$$

Substitute all of the above in (6) to obtain

$$e \le 2n - 1. \qquad \square$$

It remains to prove Lemmas 3 and 4.

Proof of Lemma 3. If $PS(G)$ is not a circular Davenport–Schinzel sequence of order 2, then there are four points, $\alpha_{au_1}, \alpha_{bu_2}, \alpha_{au_3}, \alpha_{bu_4}$ appearing in that order on C. Since $v_1, \ldots, v_n$ are in general position there are no two disjoint segments $v_x v_y$, $v_z v_t$ such that a line through one of them contains an endpoint of the other. Therefore by Lemma 2, two of the segments

$$v_a\alpha_{au_1}, v_b\alpha_{bu_2}, v_a\alpha_{au_3}, v_b\alpha_{bu_4}$$

are in convex position. Since every edge $v_x v_y$ is contained in the segment $v_x\alpha_{xy}$, two of the segments

$$v_a v_{u_1}, v_b v_{u_2}, v_a v_{u_3}, v_b v_{u_4}$$

are in convex position, a contradiction. $\square$

Proof of Lemma 4. Suppose that the points α_{ab}, α_{ac} belong to the same arc and are dark colored. Assume without loss of generality that $v_a v_c$ is to the right of $v_a v_b$. The edges $v_a v_b$, $v_a v_c$ are interior edges of v_b and v_c, respectively. So let $v_b v_x$ be to the right of $v_b v_a$ and let $v_c v_y$ be to the left of $v_c v_a$. Let st be the chord of C that contains $v_b v_c$, so that v_b lies in $v_c s$ and v_c in $v_b t$. Let $\overrightarrow{r}$ be the ray with apex v_b and parallel to $\overrightarrow{v_c v_a}$.

Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ be the following angles:

$$\alpha_1 = \operatorname{conv}\left(\overrightarrow{v_b\alpha_{ba}} \cup \overrightarrow{r}\right), \qquad \alpha_2 = \operatorname{conv}\left(\overrightarrow{r} \cup \overrightarrow{v_b s}\right),$$
$$\alpha_3 = \operatorname{conv}\left(\overrightarrow{v_b s} \cup \overrightarrow{v_b\alpha_{ab}}\right) \quad \text{and} \quad \alpha_4 = \operatorname{conv}\left(\overrightarrow{v_c t} \cup \overrightarrow{v_c\alpha_{ac}}\right).$$

See Fig. 3.

Since $v_b v_x$ is to the right of $v_b v_a$, v_x must lie in at least one of the angles α_1 or α_2 or α_3. If $v_x \in \alpha_1$, then the points $\alpha_{ac}, \alpha_{xb}, \alpha_{ab}, \alpha_{bx}$ are in that order on C. Therefore α_{ab}, α_{ac} are not on the same arc, a contradiction. If $v_x \in \alpha_2$, then $v_b v_x$ and $v_c v_a$ are in convex position, a contradiction.

Therefore $v_b v_x$ is in α_3 and by symmetry $v_c v_y$ is in α_4, implying $v_b v_x$ and $v_c v_y$ are in convex position, a contradiction. $\square$

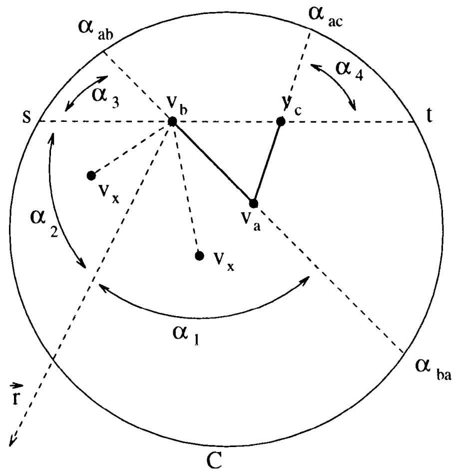

Fig. 3. Two dark points in an arc of an improper geometric graph are impossible.

The conjecture of Kupitz has been proved by Pavel Valtr, consult [6]. The major contribution of [6] is that for any fixed $k \geq 3$, any geometric graph on n vertices with no k edges such that any two of them are in convex position contains at most $O(n)$ edges. Valtr also proves that any geometric graph on n vertices with no k pairwise crossing edges contains at most $O(n \log n)$ edges.

Acknowledgments

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